Numerical Multi-Pulse Solutions to the Non-linear Schrodinger Equation with a Periodic Potential

James Brown[1], Dmitry Pelinovsky[2]

Department of Physics & Astronomy, McMaster University, Hamilton, Ontario[1],
Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario[2]

Introduction

The non-linear Schrödinger equation is used to model many physical processes. Everything from deep water waves to Bose-Einstein condensate. In general, the form of the Non-Linear Schrödinger (NLS) equation in one dimension is

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + V(x)\psi + \sigma|\psi|^2\psi$$

Where σ=1 is the defocusing case and σ=-1 is the focusing case.

The equation that is being analyzed here is the focusing time-independent NLS equation.

$$E\psi = -\frac{\partial^2\psi}{\partial x^2} + V(x)\psi - |\psi|^2\psi$$

With potential

$$V(x) = \sin^2(x/2)$$

All solutions can be taken to be real due to the gauge symmetry of the system.

Multi-Pulse Localizations

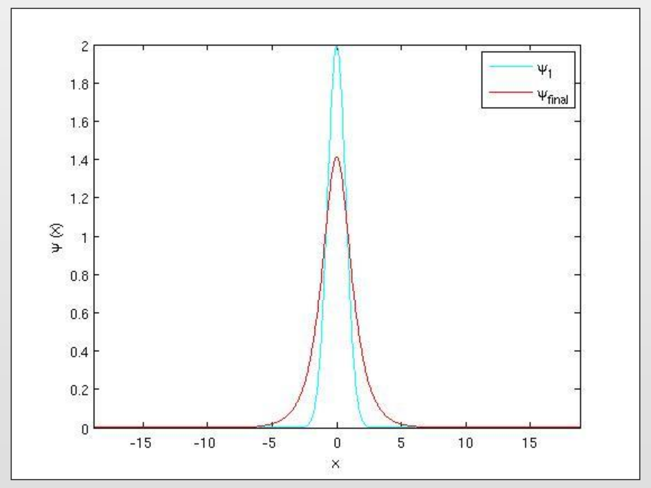
The fundamental solution to the NLS equation is given by a single-pulse function with exponentially decaying tails. Multi-pulse solutions can be thought to be a composition of individual pulses with sufficient separations between them.

Tail-Tail Interactions for NLS solitons when V(x)=0

1. Anti-Phase solitons repel each other
2. In-Phase solitons attract each other

Because of the periodic potential, two pulses do not diverge or collapse but stabilize near the points of equilibrium. This is abalance between the tail-tail interactions and the periodic potential itself.

One Pulse Solution with V(x)=0

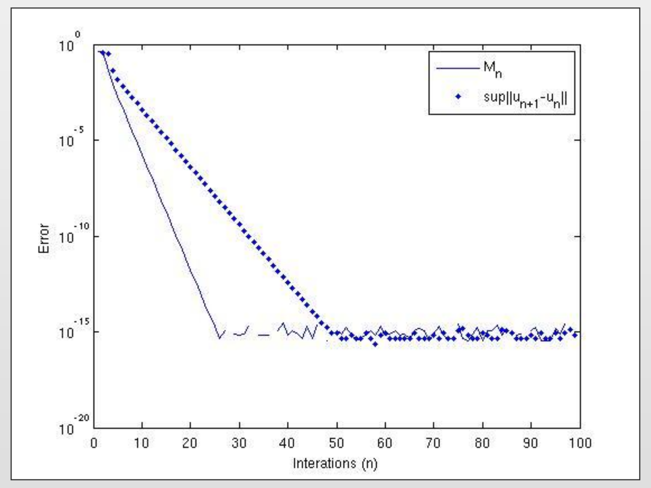


Initial approximation and final solution for localization with no potential

In order to verify the convergence of a method, it is convenient to run it where an exact solution is known. For the NLS equation, this exists.

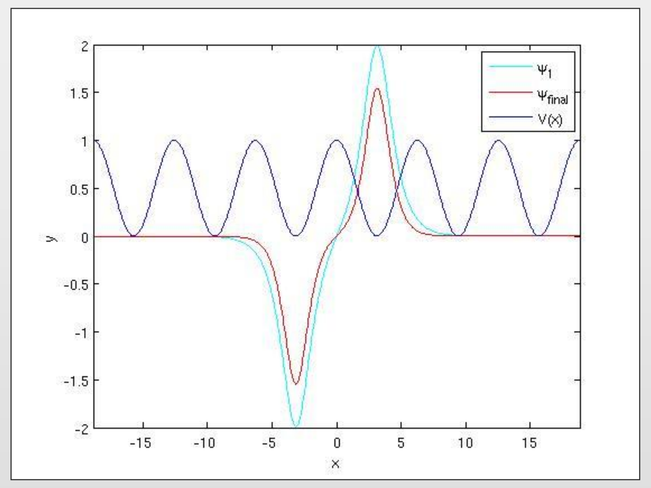
$$\psi = \sqrt{2E}\,sech\left(\sqrt{E}x\right)$$

As you can see from the graph on the right, the method is clearly converging.



Convergence of method for single-pulse solution

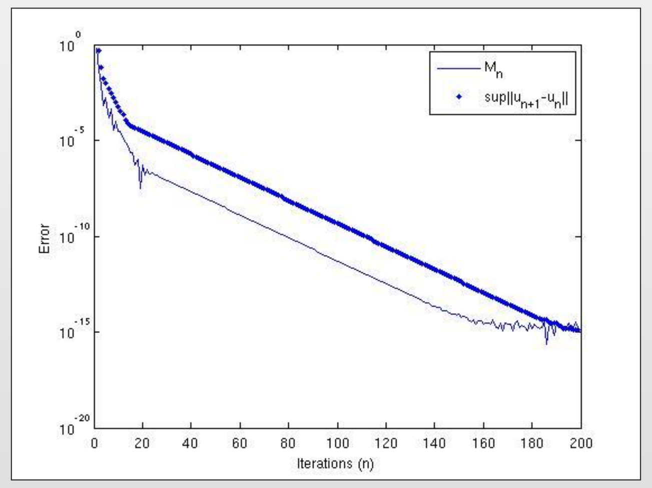
Stable Two-Pulse Solution



Initial approximation and final solution for two-pulse localization with periodic potential.

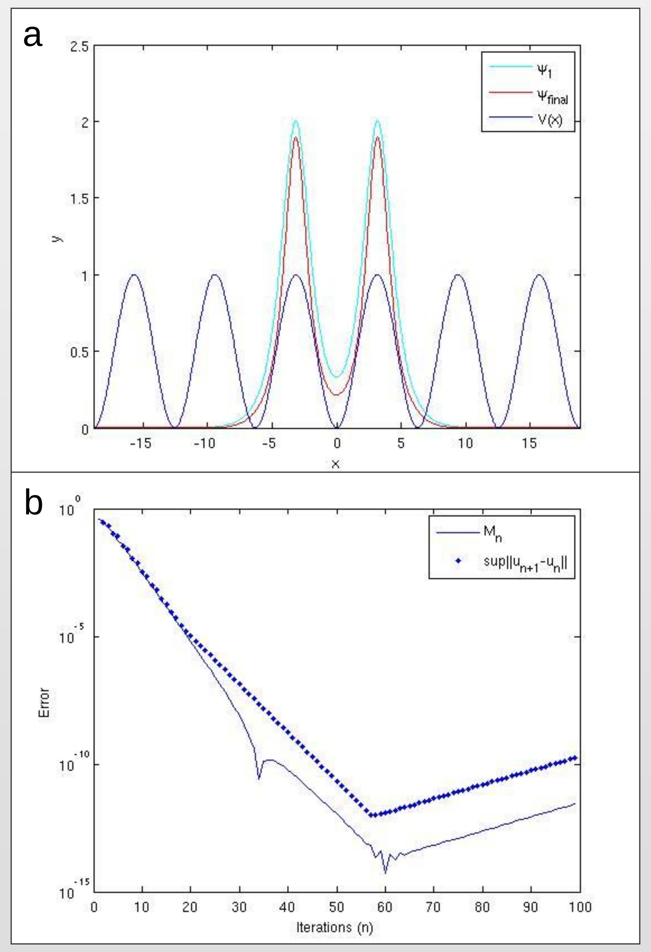
The stable solutions exist at the minimum of the potential. The numerical method converges if we enforce the symmetry (anti-symmetry) of the solution through reducing the Fourier series to either a cosine (sine) series.

For the anti-symmetric solution on the left, the sine series was used.



Convergence of method for the stable two-pulse localizations.

Unstable Two-Pulse Solution



The unstable solution exists at the maximum of the potential. In general, the numerical method diverges for this solution. However, with proper starting points, the error can approach machine precision before diverging.

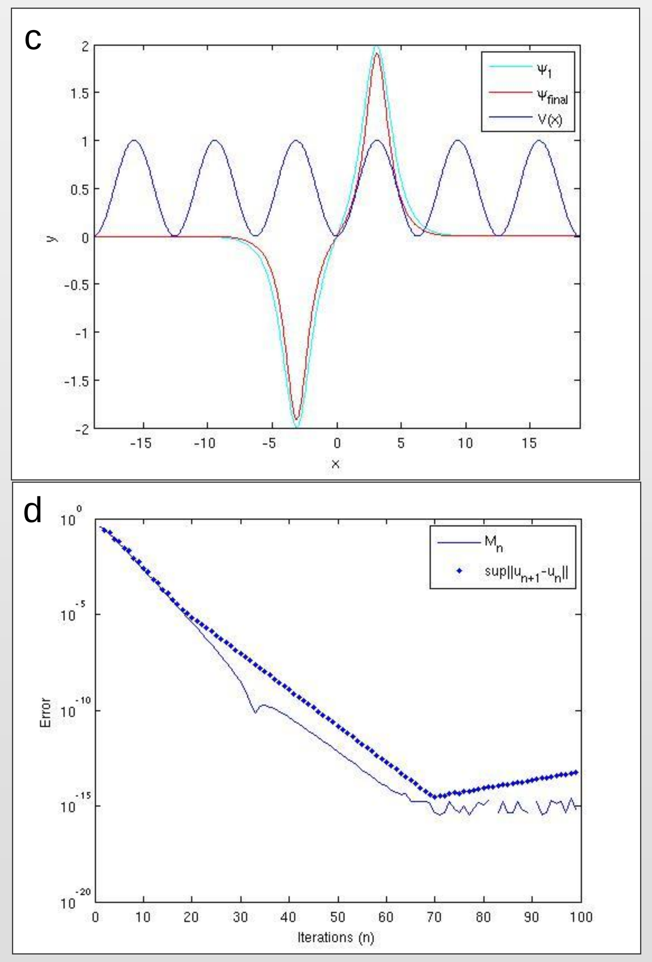
Starting Points

•1.3×10^{-6} outward from potential maximum for anti-symmetric localizations (right)

•4.5×10^{-5} inward from potential maximum for symmetric localizations (left)

Left: Solution (a) and convergence (b) for in-phase pulses.

Right: Solution (c) and convergence (d) for anti-phase pulses.



Numerical Method #1

We rewrite the focusing time-independent Schrödinger equation in the Fourier transform form.

$$E\psi + k^2\psi = \widehat{V(x)\psi} + \widehat{\psi^3}$$

Where the modulus square is removed as the solution is real. The iteration scheme is then defined as

$$\widehat{u_{n+1}} = M_n^{3/2}\frac{\widehat{u_n^3}}{E+k^2}$$

Where M_n is introduced for convergence purposes.

$$M_n = \frac{\left\langle \widehat{u_n}, (E+k^2)\widehat{u_n} \right\rangle}{\left\langle \widehat{u_n}, \widehat{u_n^3} \right\rangle}$$

Numerical Method #2

Instead of using the Fourier space for iterations, we use the finite difference approximation for the L,

$$L = -\partial_{xx} + V(x)$$

with periodic boundary conditions.

The convergence factor M_n is now defined as

$$M_n = \frac{\langle u_n, (L-IE)u_n \rangle}{\langle u_n, u_n^3 \rangle}$$

Such that I is the identity matrix.

Future Work

1. Compare the two numerical methods for convergence and efficiency.
2. Extend numerical solutions to N-Pulses
3. Apply the two methods to sign-varying periodic non-linearities.