

Multi-pulse solutions to the focusing NLS equation.



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Non-Linear Schrodinger Equation



$$E\Psi = -\frac{d^2}{dx^2}\Psi + V(x)\Psi + \sigma|\Psi|^2\Psi$$

Energy

Periodic Potential

The non-linear
term

The Defocusing NLS $\sigma = 1$

The Focusing NLS $\sigma = -1$

For our purposes...



- **Periodic Potential**

- The potential that we will use is

$$V(x) = \sin^2(x/2)$$

- **Non-linear term**

- We will be examining the focusing NLS

$$\sigma = -1$$

- All solutions are real

$$E\Psi = -\frac{d^2}{dx^2}\Psi + \sin^2(x/2)\Psi - \Psi^3$$

Spectral Renormalization Method



$$E\Psi(x) = -\partial_x^2\Psi(x) + V(x)\Psi(x) - \Psi(x)^3$$

Define the linear operator L

$$L = -\partial_x^2 + V(x)$$

$$(L - E)\Psi = \Psi^3$$

Consider the Iterations Scheme

$$u_{n+1} = (L - E)^{-1} u_n^3$$

Will This Scheme Converge



$$u_{n+1} = (L - E)^{-1} u_n^3$$

- Lets say u_n is $a_n \Psi$

$$a_{n+1} = a_n^3$$

- So we need to define a renormalization factor

$$M_n = \frac{\langle u_n, (L - E)u_n \rangle}{\langle u_n, u_n^3 \rangle} \quad 1 = \frac{\langle \Psi, (L - E)\Psi \rangle}{\langle \Psi, \Psi^3 \rangle}$$

Renormalized Iteration Scheme

$$u_{n+1} = M_n^{3/2} (L - E)^{-1} u_n^3$$

Numerical Methods



- There are three methods that we will be examining all based on the Spectral Renormalization method

- Finite-Difference

$$\widehat{u_{n+1}} = M_n^{3/2} (L - IE)^{-1} \widehat{u_n^3}$$

- Discrete Fourier

$$\widehat{u_{n+1}} = M_n^{3/2} \frac{\widehat{V(x) u_n + u_n^3}}{k^2 - E}$$

- Bloch-Fourier

$$\tilde{\phi}_n = \int \phi(x) \bar{l}_m(x; k) dx \quad \phi(x) = \int \sum_{m \in \mathbb{N}} \tilde{\phi}_n(k) l_m(x; k) dk$$

How do we measure error?



- Compare with exact solution.

$$e_{actual}^{(n)} = \sup_{x \in \mathbb{R}} |u_n - \Psi|$$

- Compare M_n with unity

$$e_M^{(n)} = |M_n - 1|$$

- Compare two successive iterations

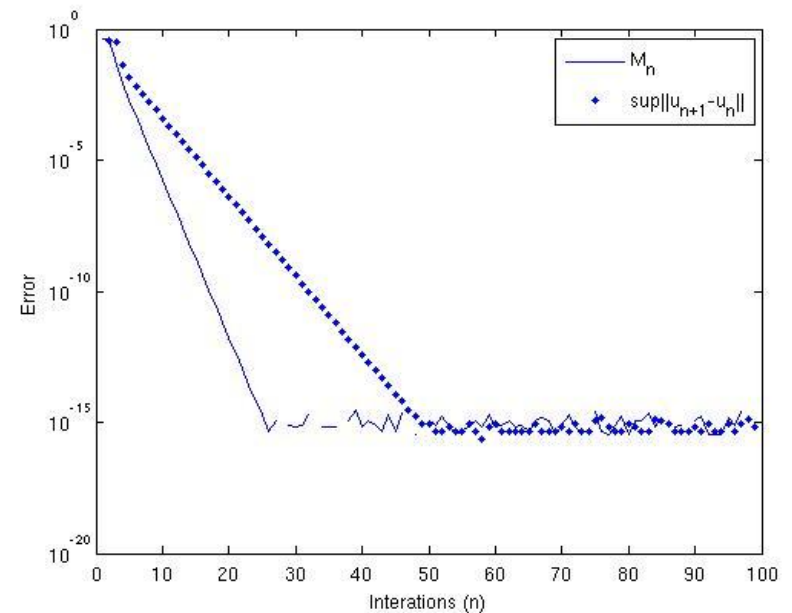
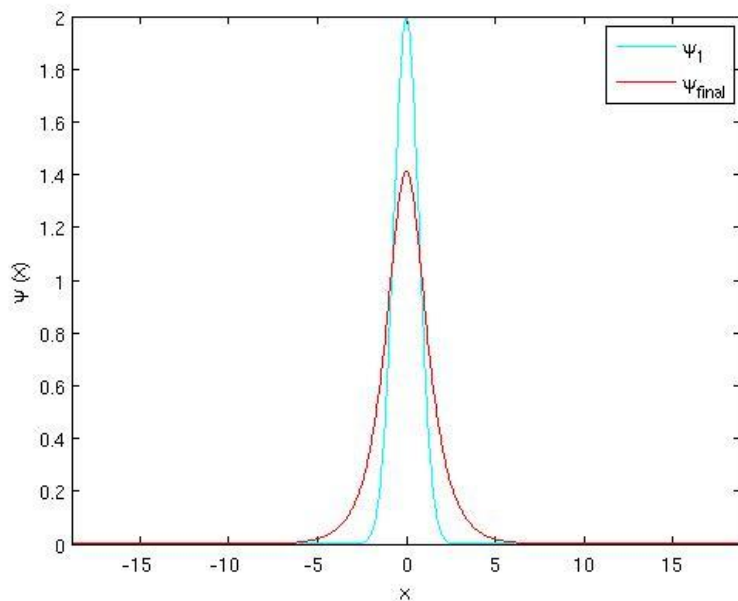
$$e_u^{(n)} = \sup_{x \in \mathbb{R}} |u_n - u_{n-1}|$$

One-Pulse Solution with no potential



- Exact Solution for $V(x) = 0$

$$\Psi = \sqrt{2|E|} \operatorname{sech}^2\left(\sqrt{|E|x}\right)$$



One-Pulse Solutions with Potential



- Solutions can only exist where there is an extremum

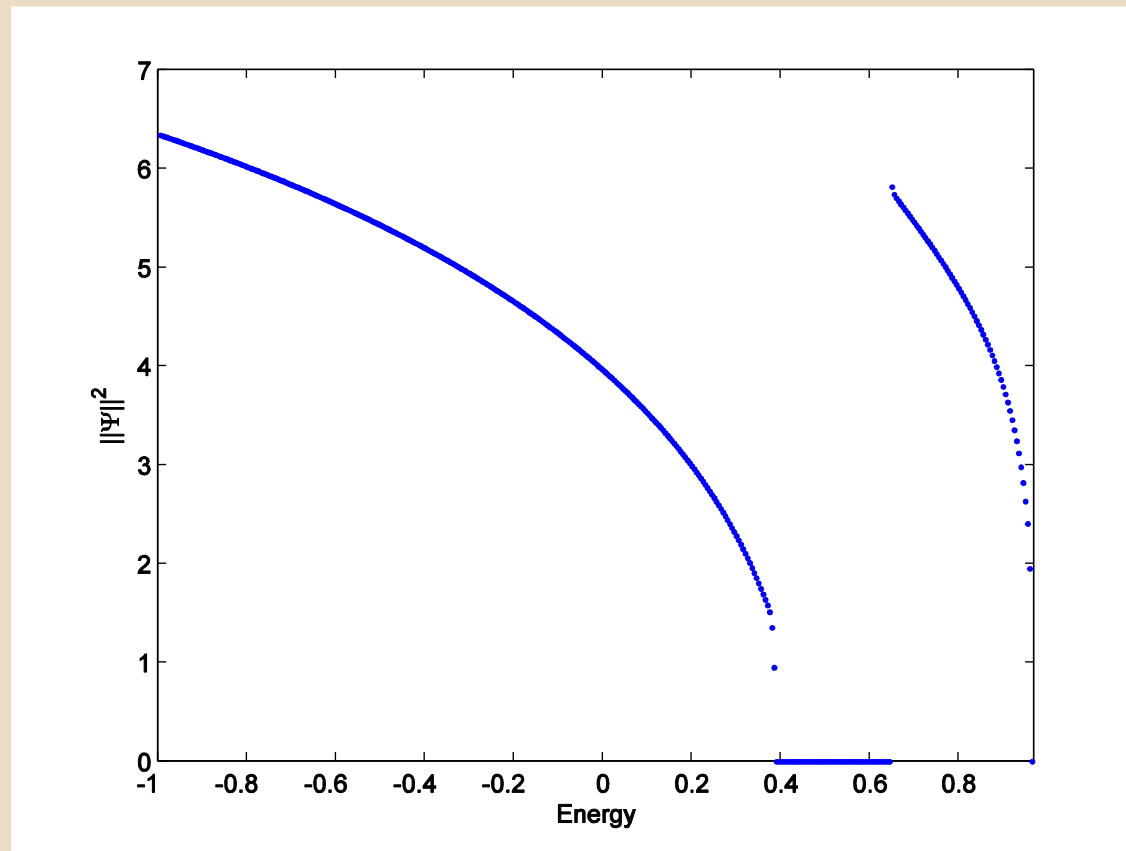
$$V'(x) = 0, \quad V''(x) \neq 0$$

- The solutions exist in $V(x) = \sin^2(x/2)$ at
 - $x = 0$ stable
 - $x = \pi$ unstable

One-Pulse Solutions



- Can only exist in Band Gaps.

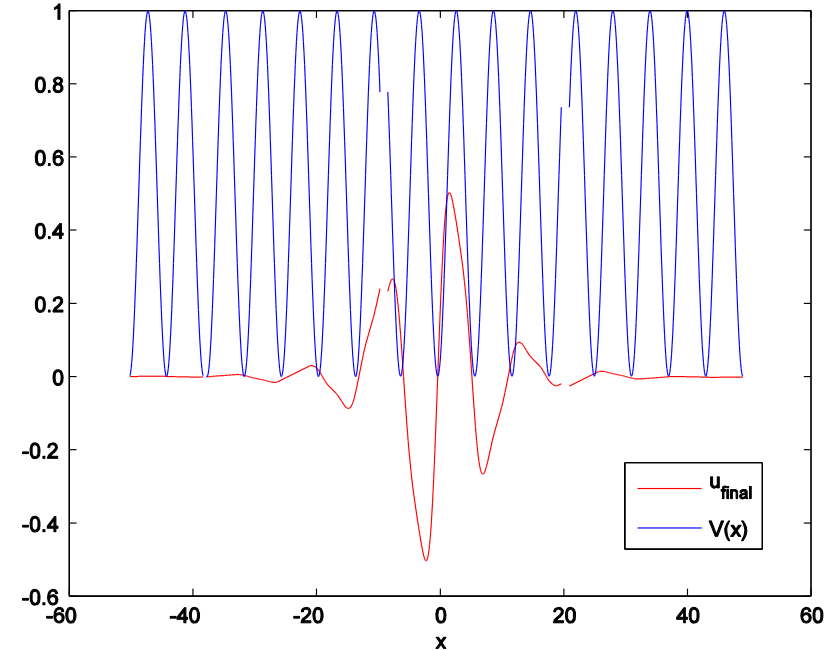
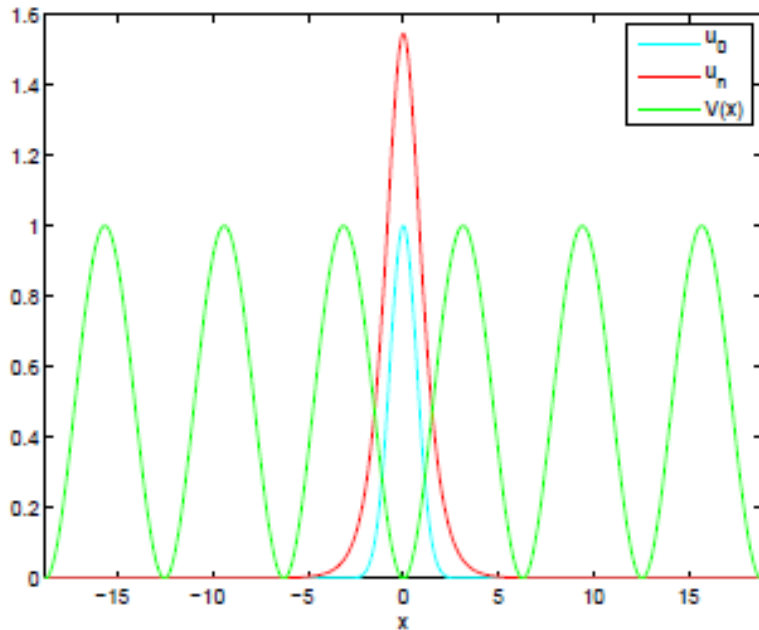


One-Pulse Solutions



Semi-Infinite Gap

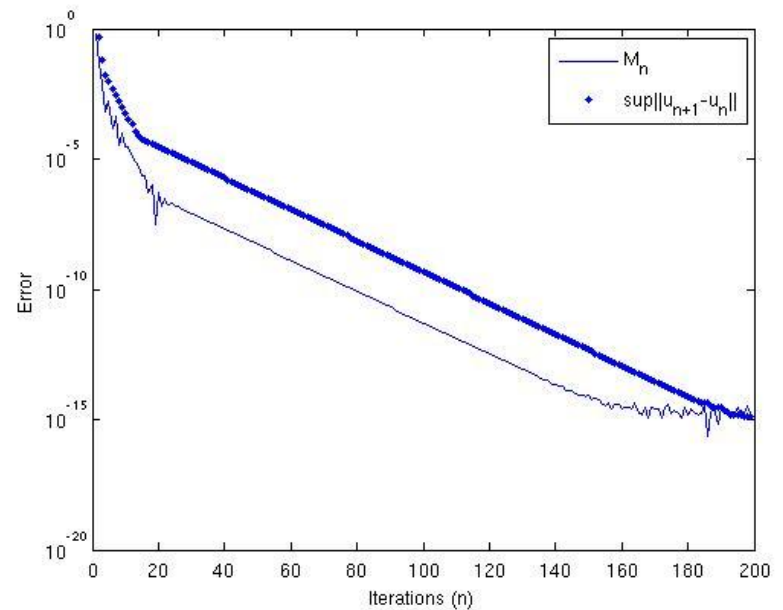
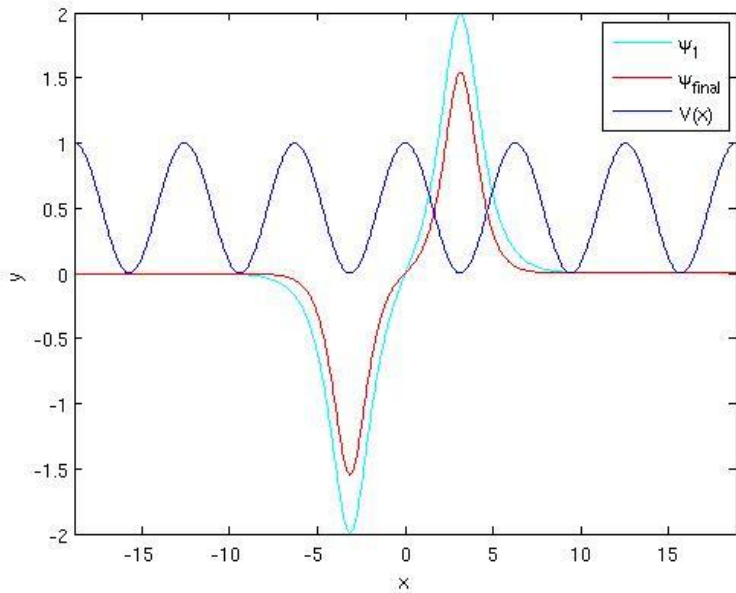
Band-Gap



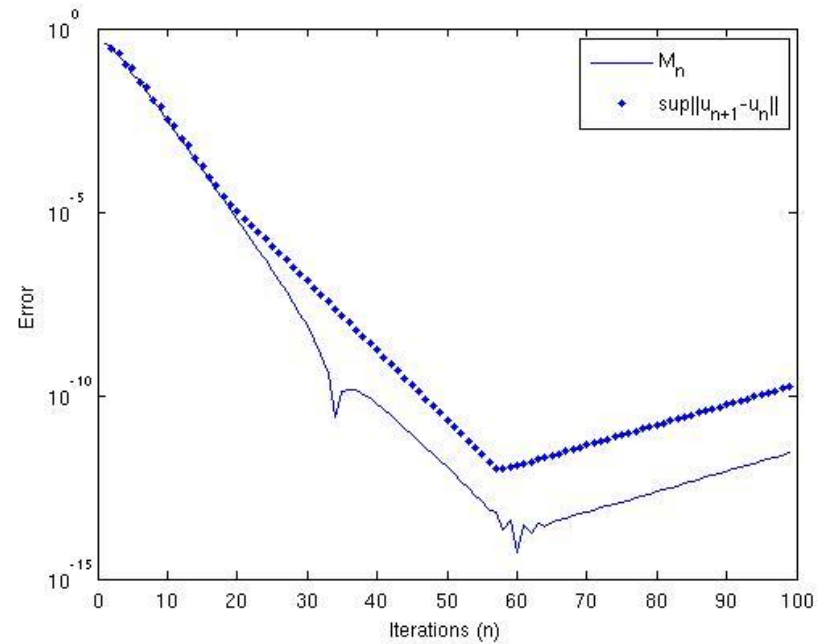
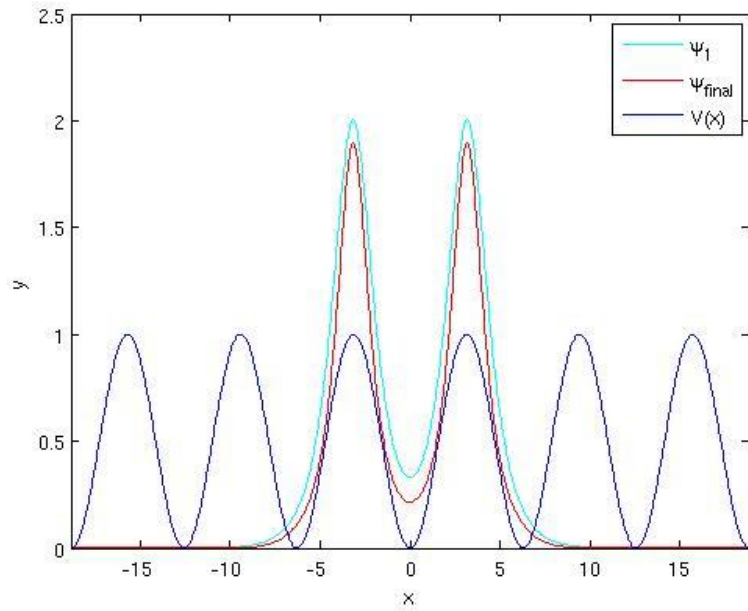
Stable Two-Pulse Solution



- We need to force Symmetry to ensure convergence



Unstable Two-Pulse Solution



Summary



- **Computed One-Pulse Solutions**
 - Semi-Infinite Gap
 - Band Gap
- **Computed Two-Pulse Solutions**
 - Semi-Infinite Gap