Convergence of the Adomian Decomposition Method for Initial-Value Problems

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We prove convergence of the Adomian decomposition method for an abstract initial-value problem using the method of majorants from the Cauchy-Kowalevskaya theorem for differential equations with analytic vector fields. Convergence rates of the Adomian method are investigated in the context of the nonlinear Schrödinger equation. Numer Methods Partial Differential Eq 27: 749–766, 2011

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I. INTRODUCTION

In the 1980s, George Adomian (1923–1996) introduced a powerful method for solving nonlinear functional equations. Since then, this method is known as the *Adomian decomposition method* (ADM) [1, 2]. The technique is based on a decomposition of a solution of a nonlinear functional equation in a series of functions. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function. The Adomian method is very simple in an abstract formulation but the difficulty arises in calculating the polynomials and in proving the convergence of the series of functions.

The first proof of convergence of the ADM was given by Cherruault [3], who used fixed point theorems for abstract functional equations. Since then, many articles on convergence of the ADM were published, including the works of Abbaoui and Cherruault [4, 5], Himoun, et al. [6, 7], Hosseini and Nasabzadeh [8], Lesnic [9, 10], and Rach [11, 12]. Furthermore, Babolian and Biazar [13] introduced the order of convergence of the ADM, Boumenir and Gordon [14] discussed the rate of convergence of the ADM, and El-Kalla [15] gave another view on the error analysis of the ADM.

Many authors discussed similarity of the Adomian decomposition method to other analytical and numerical solutions of initial-value problems for differential equations such as the

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Picard method of successive iterations [16, 17], the power series solutions [18], and the Runge–Kutta initial-value solvers [19, 20]. The ADM was also applied to boundary-value problems for differential equations [21, 22] and to partial differential equations [23–26].

In spite of the variety of publications on convergence, computational complexity, improvements, and applications of the ADM, no precise criterion of convergence was formulated in the literature, at least in the context of initial-value problems for ordinary and partial differential equations. Our work addresses this problem with the proof that the Adomian decomposition method always converges for solutions of differential equations with analytic vector fields on small time intervals. Our convergence analysis uses the formalism of the Cauchy-Kowalevskaya theorem that guarantees that solutions of initial-value problems for systems of ordinary differential equations with analytic vector fields are analytic in time for small time intervals. We show that the Adomian series and the series solutions of autonomous differential equations are majored by the same convergent series in powers of time. Moreover, the same majorant problem can be applied for convergence analysis of the ADM in the context of initial-value problems for partial differential equations. We note that Taylor expansions for functions on Banach spaces and the Cauchy–Kowalevskaya theorem were recently discussed in the context of the convergence of the Adomian series by Rach [12].

We illustrate the ADM both for systems of ordinary differential equations and for partial differential equations, with applications to numerical solutions of the nonlinear Schrödinger (NLS) equation. Other numerical procedures such as the Picard method of successive iterations and the initial-value ODE solvers are also discussed in comparison with the decomposition method.

We note in passing that the NLS equation plays an important role in the modeling of several physical phenomena such as the propagation of optical pulses, waves in fluids and plasma, self-focusing effects in lasers, and trapping of atomic gas in Bose-Einstein condensates [27]. The ADM was not previously used in the context of numerical solutions of the NLS equation, to the best of our knowledge, although we are aware of many other techniques applied to numerical solutions of this equation including the finite-difference method [28], the split-step method [29], the symplectic integrations [30], and the relaxation method [31].

The article is organized as follows. The ADM for an abstract initial-value problem is formalized in Section II. The convergence theorem is formulated and proved in Section III. Sections IV and V discuss applications of the ADM to the stationary and time-dependent NLS equations.

II. FORMALISM

Let us consider an abstract initial-value problem

$$\begin{cases} \dot{u} = Lu + N(u), & t > 0, \\ u(0) = f \end{cases} \tag{2.1}$$

where $L : X \to Y$ is a linear operator from a Banach space X to a Banach space Y ($X \subseteq Y$), $N(u) : X \to X$ is a nonlinear function on the Banach space X, $f \in X$ is an initial data, and the dot denotes differentiation in time $t \in \mathbb{R}_+$.

We think about two main examples of the abstract initial-value problem (2.1). For the first example, we consider a system of N autonomous differential equations on $X = Y = \mathbb{R}^N$, so that $L : \mathbb{R}^N \to \mathbb{R}^N$ is a matrix operator and $N(u) : \mathbb{R}^N \to \mathbb{R}^N$ is a vector field.

For the second example, we consider the continuous NLS equation in the form

$$iu_t = -u_{xx} + V(x)u + |u|^2 u, \quad t > 0, \tag{2.2}$$

where $V(x) : \mathbb{R} \to \mathbb{R}$ is an external potential and $u(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$ is an amplitude function. This equation also fits to the abstract formulation (2.1) with

$$L = i\partial_x^2, \quad N(u) = -iV(x)u - i|u|^2 u.$$

We can choose $X = H^2(\mathbb{R})$ and $Y = L^2(\mathbb{R})$ both on an infinite line and in a periodic domain. It follows from properties of the Fourier transform and the Banach algebra of H^2 with respect to poinwise multiplication that $L : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $N : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ if $V \in H^2(\mathbb{R})$.

Assumption 1. *Let $L : X \to Y$ form a continuous semigroup $E(t) = e^{tL} : X \to X$ for $t \in \mathbb{R}_+$ and there is a constant $C > 0$ such that*

$$\|E(t)f\|_X \le C\|f\|_X, \quad \forall f \in X, \quad \forall t \in \mathbb{R}_+. \tag{2.3}$$

Let $N(u) : X \to X$ be analytic near $u = f$ and X be a Banach algebra with the property

$$\|fg\|_X \le \|f\|_X\|g\|_X, \quad \forall f, g \in X. \tag{2.4}$$

Remark 1. Banach algebra property (2.4) may require the constant in the upper bound to be different from one. Since the constant is one for $X = \mathbb{R}^N$ and $X = H^2(\mathbb{R})$ [32], we choose one in the upper bound of (2.4) to simplify some computations. The extension to a general case is straightforward.

By Duhamel's principle, the initial-value problem (2.1) can be reformulated as an integral equation

$$u(t) = E(t)f + \int_0^t E(t-s)N(u(s))ds, \quad t > 0. \tag{2.5}$$

If $N(u)$ is analytic near f, it satisfies a local Lipschitz condition in the ball $B_\delta(f)$ of a radius $\delta > 0$ centered at f, i.e., there is a constant $K_\delta > 0$ such that

$$\|N(u) - N(\tilde{u})\|_X \le K_\delta\|u - \tilde{u}\|_X, \quad \forall\|u - f| \le \delta, \quad \forall\|\tilde{u} - f| \le \delta.$$

Using Picard's method of successive iterations adopted for partial differential equations by Kato [33], local well-posedness of solutions of the initial-value problem (2.1) with Lipschitz vector field $N(u)$ can be proved for small time intervals. We review some elements of the proof related to the subject of this article.

Theorem 1 (Picard–Kato)**.** *Let L and $N(u)$ satisfy Assumption* 1 *and $f \in X$. There exists a $T > 0$ and a unique solution $u(t)$ of the initial-value problem* (2.1) *on* $[0,T]$ *such that*

$$u(t) \in C([0,T], X) \cap C^1([0,T], Y),$$

and $u(0) = f$. Moreover, the solution $u(t)$ depends continuously on the initial data f.

Proof. The proof is based on the method of successive iterations starting with the free solution $u^{(0)} = E(t)f$. The sequence of functions $\{u^{(n)}(t)\}_{n\in\mathbb{N}}$ is defined from $u^{(0)}(t)$ on a small interval $[0, T]$ according to the following recurrence relation

$$u^{(n+1)}(t) = u^{(0)}(t) + \int_0^t E(t-s)N(u^{(n)}(s))ds, \quad n \geq 0. \tag{2.6}$$

For any $\delta > 0$, there exists a $T > 0$ such that $\|u^{(0)}(t) - f\|_X \leq \frac{1}{2}\delta$ for any $t \in [0, T]$. By the induction method, we obtain for any $n \geq 0$

$$\begin{aligned}\sup_{t\in[0,T]} \|u^{(n+1)}(t) - f\|_X &\leq \sup_{t\in[0,T]} \|u^{(n+1)}(t) - u^{(0)}(t)\|_X + \sup_{t\in[0,T]} \|u^{(0)}(t) - f\|_X \\ &\leq CT \sup_{t\in[0,T]} \|N(u^{(n)}(t))\|_X + \frac{\delta}{2} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,\end{aligned}$$

provided that T satisfies the bound

$$CT \sup_{u\in B_\delta(f)} \|N(u)\|_X \leq \frac{\delta}{2}.$$

Therefore, the iteration operator (2.6) maps $C([0, T], B_\delta(f))$ to $C([0, T], B_\delta(f))$ for small $T > 0$. Also the iteration map is Lipschitz and contraction if $CTK_\delta < 1$. By the Banach fixed point theorem, there exists a unique solution $u(t)$ of the integral equation (2.5) in a complete metric space $C([0, T], B_\delta(f))$. If $u(t) \in C([0, T], X)$, then $Lu(t) + N(u(t)) \in C([0, T], Y)$ so that $u(t)$ is also in $C^1([0, T], Y)$. ■

Remark 2. By the contraction mapping principle, the error of the approximate solution $u^{(n)}(t)$, $n \geq 1$ is estimated by

$$\sup_{t\in[0,T]} \|u(t) - u^{(n)}(t)\|_X \leq \frac{1}{n!}(CK_\delta T)^n \sup_{t\in[0,T]} \|u(t) - u^{(0)}(t)\|_X.$$

where

$$\sup_{t\in[0,T]} \|u(t) - u^{(0)}(t)\|_X \leq \sup_{t\in[0,T]} \|u(t) - f\|_X + \sup_{t\in[0,T]} \|u^{(0)}(t) - f\|_X \leq \frac{3}{2}\delta.$$

To set up the Adomian decomposition method, let us write the solution $u(t)$ and the analytic function $N(u)$ near f in the series form

$$u(t) = u_0(t) + \sum_{n=1}^{\infty} u_n(t), \tag{2.7}$$

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{2.8}$$

where $u_0(t) = E(t)f$ and

$$u_{n+1}(t) = \int_0^t E(t-s)A_n(u_0(s), \ldots, u_n(s))ds, \quad n \geq 0. \tag{2.9}$$

Functions A_n are formally obtained by the rule

$$A_n = \frac{1}{n!}\frac{d^n}{d\varepsilon^n} N\left(\sum_{k=0}^{\infty} \varepsilon^k u_k\right)\Bigg|_{\varepsilon=0}, \quad n \geq 0. \tag{2.10}$$

These functions are polynomials in $(u_1, \ldots, u_n)$, which are referred to as the *Adomian polynomials*. Convergence of series (2.7) is a subject of this article.

In the remainder of this section, we should discuss some controversy in the literature on comparison between the Picard method of successive iterations and the Adomian decomposition method. Golberg [17] claimed that the Adomian method was equivalent to the Picard method but he only considered linear differential equations. This equivalence does not hold for nonlinear differential equations.

Hosseini and Nasabzadeh [8] claimed that the Adomian decomposition method can be formulated as

$$U_{n+1} = u_0 + \int_0^t E(t-s)N(U_n(s))ds, \quad n \geq 0, \tag{2.11}$$

where $U_n = u_0 + \sum_{k=1}^n u_k$. However,

$$\sum_{k=0}^{n} A_k(u_0, u_1, \ldots, u_k) \neq N(U_n(t)), \quad n \geq 1,$$

so that the claim is not justified. Moreover, the iteration formula (2.11) is the same as the iteration formula (2.6) in the Picard method, so that the proof of convergence of the "Adomian method" in [8] repeats the standard proof of the Picard–Kato Theorem and gives no proof of convergence of the Adomian decomposition method.

El-Kalla [15] introduced a new definition of the Adomian polynomials

$$\tilde{A}_0 = N(u_0), \quad \tilde{A}_n = N(U_n) - N(U_{n-1}), \quad n \in \mathbb{N},$$

where U_n is the same as in (2.11). As a result, we obtain

$$\sum_{k=0}^{n} \tilde{A}_k(u_0, \ldots, u_k) = N(U_n),$$

which means that the modified Adomian method from [15] is identical to the Picard method

$$U_{n+1} = u_0 + \sum_{k=0}^{n} \int_0^t \tilde{A}_k(u_0(s), \ldots u_k(s))ds = u_0 + \int_0^t N(U_n(s))ds.$$

Although one of the claims of El-Kalla [15] is that the modified Adomian method converges faster than the original Adomian method, we found in [34] that the errors of the Adomian method are smaller than the errors of the Picard method in the computational examples of [15]. Since the Adomian method requires analyticity of the vector field $N(u)$ while the Picard method only needs Lipschitz continuity of the vector field, we should anticipate generally that the Adomian method converges faster than the Picard method.

III. CONVERGENCE ANALYSIS

The first proof of convergence of the Adomian decomposition method was developed by Cherruault [3] in the context of an abstract functional equation

$$y = y_0 + f(y), \quad y \in X, \tag{3.1}$$

where X is a Banach space and $f(y) : X \to X$ is analytic near y_0. Let

$$Y_n = y_0 + \sum_{k=1}^{n} y_k, \quad f_n(Y_n) = \sum_{k=0}^{n} A_k(y_0, y_1, \ldots, y_k).$$

The Adomian decomposition method is equivalent to determining a sequence $\{Y_n\}_{n\in\mathbb{N}}$ from

$$Y_0 = y_0, \quad Y_{n+1} = y_0 + f_n(Y_n), \quad n \geq 0.$$

If there exist limits

$$Y = \lim_{n\to\infty} Y_n, \quad f = \lim_{n\to\infty} f_n,$$

in the Banach space X, then Y solves the fixed-point equation $Y = y_0 + f(Y)$ in X. The convergence of the Adomian method was proved in [3] under the following two conditions:

$$\|f(y)\|_X \leq 1, \quad \forall y \in X,$$

and

$$\|f_n(Y_n) - f(Y)\|_X \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

These two conditions are rather restrictive. The first condition implies a constraint on the nonlinear function $f(y)$ and the second condition implies convergence of the series of Adomian polynomials to the locally analytic function $f(y)$, which needs to be proven.

In the following, we shall prove convergence of the Adomian method in the context of an abstract initial-value problem (2.1). Our construction uses the method of majorants from the proof of the Cauchy-Kowalevskaya theorem for differential equations with analytic vector fields. Let us review the statement and the proof of this theorem (see, e.g., [35]).

Theorem 2 (Cauchy-Kowalevskaya). *Let Assumption* 1 *be satisfied with $Y = X$. Let $u(t)$ be a unique solution of the initial-value problem* (2.1) *in $C^1([0,T), X)$, where $T > 0$ is the maximal existence time. Then there exists $\tau \in (0,T)$ such that $u : [0,\tau] \to X$ is also a real analytic function.*

Remark 3. Existence and uniqueness of the solution $u(t)$ of the initial-value problem (2.1) in $C^1([0,T), X)$ is proved in Theorem 1 for $Y = X$.

Proof. As $Y = X$, we can set $L \equiv 0$ (or denote $N(u) + Lu$ by $N(u)$). As $N(u)$ is analytic near f, by Cauchy estimates, there exist constants $a, b > 0$ such that

$$\frac{1}{k!}\left\|\partial_u^k N(f)\right\|_X \leq \frac{b}{a^k}, \quad \forall k \geq 0, \tag{3.2}$$

where $\partial_u^k N(u)$ denote operators in the sense of the Fréchet derivative, e.g., $\partial_u N(u) = N'(u)$ is the Jacobian operator. The Taylor series for $N(u)$ at f converges for any $\|u - f\|_X < a$, and moreover, we obtain

$$\|N(u)\|_X \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left\| \partial_u^k N(f) \right\|_X \|(u-f)^k\|_X \leq b \sum_{k=0}^{\infty} \frac{\|u-f\|_X^k}{a^k} = \frac{ab}{a-\rho} =: g(\rho),$$

where $\rho = \|u - f\|_X < a$ and the Banach algebra property (2.4) is used. From the majorant function $g(\rho)$, it is clear that

$$\frac{1}{k!} \left\| \partial_u^k N(f) \right\|_X \leq \frac{1}{k!} \partial_\rho^k g(0), \quad k \geq 0. \tag{3.3}$$

Let us consider the majorant problem

$$\begin{cases} \dfrac{d\rho}{dt} = g(\rho), \quad t > 0, \\ \rho(0) = 0, \end{cases} \tag{3.4}$$

where $\rho \in \mathbb{R}_+$. The majorant problem has an explicit solution

$$\rho(t) = a - \sqrt{a^2 - 2abt},$$

which is an analytic function of t on $(-\infty, \frac{a}{2b})$. By comparison principle, if $u(t)$ solves the integral equation

$$u(t) = f + \int_0^t N(u(s))ds,$$

then

$$\|u(t) - f\|_X \leq \int_0^t \|N(u(s))\|_X ds \leq \int_0^t g(\rho(s))ds = \rho(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} \partial_t^k \rho(0).$$

The majorant Taylor series converges absolutely for any $|t| < \frac{a}{2b}$. To prove that $u(t)$ is also an analytic function in t on $[0, \frac{a}{2b})$, it remains to prove that

$$\left\| \partial_t^k u(0) \right\|_X \leq \partial_t^k \rho(0), \quad k \geq 1.$$

If this is the case, then the Taylor series for $u(t)$ has the majorant series and hence it converges, by the Weierstrass M-Test. To prove the bound above for $k = 1, 2, 3$, we compute

$$\begin{aligned} \partial_t u(t) &= N(u(t)), \\ \partial_t^2 u(t) &= N'(u(t))N(u(t)) \\ \partial_t^3 u(t) &= N''(u(t))N(u(t))N(u(t)) + N'(u(t))N'(u(t))N(u(t)). \end{aligned}$$

As a result,

$$\begin{aligned}
\left\|\partial_t u(0)\right\|_X &\le \|N(u(0))\|_X \le g(\rho(0)) = \partial_t \rho(0),\\
\left\|\partial_t^2 u(0)\right\|_X &\le \|N'(u(0))\|_X \|N(u(0))\|_X \le g'(\rho(0)) g(\rho(0)) = \partial_t^2 \rho(0),\\
\left\|\partial_t^3 u(0)\right\|_X &\le \|N''(u(0))\|_X \|N(u(0))\|_X \|N(u(0))\|_X + \|N'(u(0))\|_X \|N' u(0))\|_X \|N(u(0))\|_X\\
&\le g''(\rho(0)) g(\rho(0)) g(\rho(0)) + g'(\rho(0)) g'(\rho(0)) g(\rho(0)) = \partial_t^3 \rho(0).
\end{aligned}$$

Generally, for any $k \ge 0$,

$$u^{(k+1)}(t) = P_k(N(u(t))),$$

where $P_k(N)$ is a polynomial of N and its Fréchet derivatives up to the kth order with positive coefficients. As a result, we obtain

$$\left\|\partial_t^{k+1} u(0)\right\|_X = \|P_k(N(u(0)))\|_X \le P_k(\|N(u(0))\|_X) \le P_k(g(\rho(0))) = \partial_t^{k+1} \rho(0).$$

This bound concludes the proof of the theorem. ∎

We can now state and prove the main result of this article.

Theorem 3. *Let Assumption 1 be satisfied. Let $u(t)$ be a unique solution of Eq.* (2.5) *in $C([0,T), X)$, where $T > 0$ is the maximal existence time. Let $u_n(t)$ be defined by Eq.* (2.9). *There exist a $\tau \in (0,T)$ such that the nth partial sum $U_n(t) = \sum_{k=0}^n u_k(t)$ of the Adomian series* (2.7) *converges to the solution $u(t)$ in $C([0,\tau], X)$.*

Remark 4. Existence and uniqueness of the solution $u(t)$ of Eq. (2.5) in $C([0,T), X)$ is proved in Theorem 1.

Proof. From Theorem 1, for any given $\delta > 0$, there exists a $t_0 \in (0,T)$ such that

$$\sup_{t\in[0,t_0]} \|u_0(t) - f\|_X \le \frac{1}{2}\delta. \tag{3.5}$$

We choose $\delta < 2a$, where a is the radius of analyticity of $N(u)$ near f. The Cauchy estimates (3.2)–(3.3) are generalized as

$$\begin{aligned}
\frac{1}{k!}\left\|\partial_u^k N(u_0)\right\|_X &\le \sum_{m\ge k} \frac{m(m-1)\dots(m-k+1)}{m!k!} \left\|\partial_u^m N(f)\right\|_X \|u_0 - f\|_X^{m-k}\\
&\le b \sum_{m\ge k} \frac{m(m-1)\dots(m-k+1)}{k!} \frac{\|u_0 - f\|_X^{m-k}}{a^m} = \frac{1}{k!}\partial_\rho^k g(\rho), \quad k \ge 0,
\end{aligned} \tag{3.6}$$

where $\rho = \|u_0 - f\|_X < a$ and $g(\rho) = \frac{ab}{a-\rho}$. Given $g(\rho)$, let $\rho(t)$ satisfy the majorant problem (3.4) for $t \in [0, \frac{a}{2b})$. Then, $\rho(t)$ and all its derivatives with respect to t are increasing functions of

t and so are $g(\rho(t))$ and all its derivatives with respect to ρ. Using the semigroup property (2.3) and the new Cauchy estimates (3.6), we obtain for any $t \in [0, \frac{a}{2b})$,

$$\begin{aligned}\|u_1(t)\|_X &\le \int_0^t \|E(t-s)A_0(u_0(s))\|_X ds \le C\int_0^t \|N(u_0(s))\|_X ds\\ &\le C\int_0^t g(\rho(s))ds \le Ctg(\rho(t)) = Ct\rho'(t),\end{aligned}$$

and

$$\begin{aligned}\|u_2(t)\|_X &\le \int_0^t \|E(t-s)A_1(u_0(s),u_1(s))\|_X ds \le C\int_0^t \|N'(u_0(s))u_1(s)\|_X ds\\ &\le C^2\int_0^t g'(\rho(s))g(\rho(s))sds \le \frac{t^2}{2}C^2 g'(\rho(t))g(\rho(t)) = \frac{C^2t^2}{2}\rho''(t).\end{aligned}$$

By induction, we assume that

$$\|u_k(t)\|_X \le \frac{C^kt^k}{k!}\partial_t^k\rho(t), \quad t \in \left[0, \frac{a}{2b}\right), \quad k \in \{1,2,\ldots,n\},$$

and prove that the same relation remains true at $k = n+1$:

$$\|u_{n+1}(t)\|_X \le \frac{C^{n+1}t^{n+1}}{(n+1)!}\partial_t^{n+1}\rho(t), \quad t \in \left[0, \frac{a}{2b}\right).$$

As $\rho(t)$ is analytic in t for all $t \in [0, \frac{a}{2b})$, for any small $\varepsilon > 0$, there exists a C^∞-function $\tilde{\rho}^\varepsilon(t)$ on $[0, \frac{a}{2b})$ such that

$$\rho((1+\varepsilon C)t) = \sum_{k=0}^n \frac{\varepsilon^kC^kt^k}{k!}\partial_t^k\rho(t) + \frac{\varepsilon^{n+1}C^{n+1}t^{n+1}}{(n+1)!}\tilde{\rho}^\varepsilon(t),$$

for any $\varepsilon > 0$. Let

$$U_n^\varepsilon(t) = \sum_{k=0}^n \varepsilon^k u_k(t),$$

so that

$$\left\|U_n^\varepsilon(t)\right\|_X \le \sum_{k=0}^n \varepsilon^k\|u_k(t)\|_X \le \sum_{k=0}^n \frac{\varepsilon^kC^kt^k}{k!}\partial_t^k\rho(t) = \rho((1+\varepsilon C)t) - \frac{\varepsilon^{n+1}C^{n+1}t^{n+1}}{(n+1)!}\tilde{\rho}^\varepsilon(t).$$

By definition (2.10) of the Adomian polynomials, we obtain

$$A_n = \frac{1}{n!}\frac{d^n}{d\varepsilon^n}N\left(\sum_{k=0}^\infty \varepsilon^k u_k\right)\Bigg|_{\varepsilon=0} = \frac{1}{n!}\frac{d^n}{d\varepsilon^n}N(U_n^\varepsilon)\Bigg|_{\varepsilon=0},$$

so that

$$\|A_n(t)\|_X \leq \frac{1}{n!} \left\| \frac{d^n}{d\varepsilon^n} N(U_n^\varepsilon(t)) \right\|_X \Big|_{\varepsilon=0} \leq \frac{1}{n!} \frac{d^n}{d\varepsilon^n} g(\rho((1+\varepsilon C)t)) \Big|_{\varepsilon=0}$$
$$\leq \frac{C^n t^n}{n!} \frac{d^n}{d\mu^n} g(\rho(\mu)) \Big|_{\mu=t} = \frac{C^n t^n}{n!} P_n(g(\rho(t))) = \frac{C^n t^n}{n!} \partial_t^{n+1} \rho(t),$$

where $P_n(g)$ is a polynomial of g and its derivatives up to the nth order with positive coefficients (the same as in the proof of Theorem 2). Using the iterative formula (2.9), we obtain

$$\|u_{n+1}(t)\|_X \leq \int_0^t \|E(t-s)A_n(s)\|_X ds \leq \frac{C^{n+1} t^{n+1}}{(n+1)!} \partial_t^{n+1} \rho(t).$$

Therefore, the series solution (2.7) is majorant in X by the power series

$$\rho((1+C)t) = \sum_{k=0}^{\infty} \frac{C^k t^k}{k!} \partial_t^k \rho(t) = a - \sqrt{a^2 - 2ab(1+C)t},$$

which converges for all $|t| < \frac{a}{2b(1+C)}$. Recall the constraint $t_0 \in (0, T)$ in bound (3.5). By the Weierstrass M-test, the series solution (2.7) converges in $C([0,\tau], X)$ for any $\tau \in (0, \tau_0)$, where $\tau_0 = \min\{t_0, \frac{a}{2b(1+C)}\}$, to the unique solution $u(t)$ of Eq. (2.5). ■

Remark 5. If $X \neq Y$, convergence of the Adomian series does not imply that the solution $u(t)$ is analytic in t near $t = 0$.

Remark 6. If $L = 0$ (so that $u_0 = f$, $E(t) = I$, and $X = Y$), the n-th term $u_n(t)$ of the Adomian series (2.7) becomes the power term t^n of the Taylor series of the analytic solution $u(t)$ near $t = 0$.

The exponential rate of convergence of the Adomian decomposition method follows from the bound on the error of the Adomian series.

Corollary 1. *There exists a constant $C_0 > 0$ such that the error of the Adomian series is bounded by*

$$E_n = \sup_{t\in[0,\tau]} \|u(t) - U_n(t)\|_X \leq C_0 \left(\frac{2bC\tau}{a}\right)^{n+1}, \quad n \geq 1,$$

where (a, b, C, τ) are defined in Theorem 3.

Proof. It follows from the bound in the proof of Theorem 3 that

$$\sup_{t\in[0,\tau]} \|u_n(t)\|_X \leq \frac{C^n \tau^n}{n!} \partial_t^n \rho(\tau).$$

It follows from the explicit form for $\rho(t)$ that

$$\rho^{(n)}(\tau) = \frac{(2n-3)!!b^n a^n}{(a^2 - 2ab\tau)^{n-1/2}}.$$

As a result, we obtain

$$
\begin{aligned}
E_n &\le \sum_{k=n+1}^{\infty} \sup_{t\in[0,\tau]} \|u_k(t)\|_X \le \sum_{k=n+1}^{\infty} \frac{\tau^n}{n!} C^n \rho^{(n)}(\tau) \\
&\le \sqrt{a^2 - 2ab\tau} \sum_{k=n+1}^{\infty} \frac{(2k-3)!!}{k!} \left(\frac{abC\tau}{a^2 - 2ab\tau} \right)^k \\
&\le \sqrt{a^2 - 2ab\tau} \left(\frac{2abC\tau}{a^2 - 2ab\tau} \right)^{n+1} \sum_{k=0}^{\infty} \frac{(2k+2n-1)!!}{2^{k+n+1}(k+n+1)!} \left(\frac{2abC\tau}{a^2 - 2ab\tau} \right)^k .
\end{aligned}
$$

As

$$
\frac{(2k+2n-1)!!}{2^{k+n+1}(k+n+1)!} \le \frac{1}{2n+2k} \le 1, \quad n \ge 1, \quad k \ge 1,
$$

we obtain

$$
E_n \le \sqrt{a^2 - 2ab\tau} \left(\frac{2abC\tau}{a^2 - 2ab\tau} \right)^{n+1} \sum_{k=0}^{\infty} \left(\frac{2abC\tau}{a^2 - 2ab\tau} \right)^k = C_0 \left(\frac{2abC\tau}{a^2 - 2ab\tau} \right)^{n+1},
$$

for any $\tau \in (0, \frac{a}{2b(1+C)})$, where $C_0 = \frac{\sqrt{a^2-2ab\tau}}{1-\frac{2abC\tau}{a^2-2ab\tau}}$. ■

Remark 7. Unlike the Picard method of successive iterations (2.6), the Adomian map (2.9) is not a contraction operator in $C([0,\tau], X)$, no matter how small $\tau > 0$ is. If it was, there would exist $Q \in (0, 1)$ and $N \ge 1$ such that

$$
\sup_{t\in[0,\tau]} \|U_{n+1}(t) - U_n(t)\|_X \le Q \sup_{t\in[0,\tau]} \|U_n(t) - U_{n-1}(t)\|_X, \quad \forall n \ge N,
$$

or, equivalently,

$$
\sup_{t\in[0,\tau]} \|u_{n+1}(t)\|_X \le Q \sup_{t\in[0,\tau]} \|u_n(t)\|_X, \quad \forall n \ge N.
$$

However, it is always possible to construct a counter-example, which shows that corrections $u_n(t)$ may vanish identically for some $n \ge 1$ but the Adomian series is not truncated.

A counter-example in Remark 7 can be constructed by the following first-order autonomous differential equation

$$
\begin{cases} \dot{u} = 2u - u^2 \\ u(0) = 1 \end{cases} .
$$

This equation admits the exact solution $u(t) = 1 + \tanh(t)$, which is analytic for any $|t| < \frac{\pi}{2}$. By the Adomian decomposition method, we compute the Adomian polynomials for $N(u) = 2u - u^2$

$$\begin{aligned} A_0 &= 2u_0 - u_0^2, \\ A_1 &= 2u_1 - 2u_0u_1, \\ A_2 &= 2u_2 - 2u_0u_2 - u_1^2, \\ A_3 &= 2u_3 - 2(u_0u_3 + u_1u_2), \\ A_4 &= 2u_4 - 2(u_0u_4 + u_1u_3) - u_2^2. \end{aligned}$$

The first terms of the series solution (2.7) are computed explicitly by

$$u_0 = 1, \quad u_1 = t, \quad u_2 = 0, \quad u_3 = -\frac{t^3}{3}, \quad u_4 = 0, \quad u_5 = \frac{2t^5}{15}.$$

These terms represent the first power terms of the Taylor series of the analytical solution $u(t) = 1 + \tanh(t)$ near 0. As $u_n(t) = 0$ for even $n \in \mathbb{N}$, the Adomian iteration method is not related to a contraction operator.

On the other hand, for the same example, successive iterations of the Picard method are given by

$$u^{(n+1)} = 1 + \int_0^t \left(2u^{(n)}(s) - (u^{(n)}(s))^2\right) ds,$$

or equivalently, by

$$u^{(0)} = 1, \quad u^{(1)} = 1 + t, \quad u^{(2)} = 1 + t - \frac{t^3}{3}, \quad u^{(3)} = 1 + t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{t^7}{63}.$$

These approximations do not correspond to the partial sums of the Taylor series for the analytic solution $u(t) = 1 + \tanh(t)$. Nevertheless, they are generated by a contraction operator for small time intervals $[0, \tau]$ in the sense that there exist $Q \in (0, 1)$ and $N \geq 1$ such that

$$\sup_{t \in [0,\tau]} \|u^{(n+1)}(t) - u^{(n)}(t)\|_X \leq Q \sup_{t \in [0,\tau]} \|u^{(n)}(t) - u^{(n-1)}(t)\|_X, \quad \forall n \geq N.$$

This example explains further the difference between the Picard method of successive iterations and the Adomian decomposition method.

IV. APPLICATIONS TO THE STATIONARY NLS EQUATION

Consider a particular example of the stationary NLS equation,

$$-u''(t) + (1 - 3\operatorname{sech}^2(t))u(t) + u^3(t) = 0, \tag{4.1}$$

with initial conditions $u(0) = 1$ and $u'(0) = 0$. The exact solution of this initial-value problem is $u(t) = \operatorname{sech}(\mathrm{t})$.

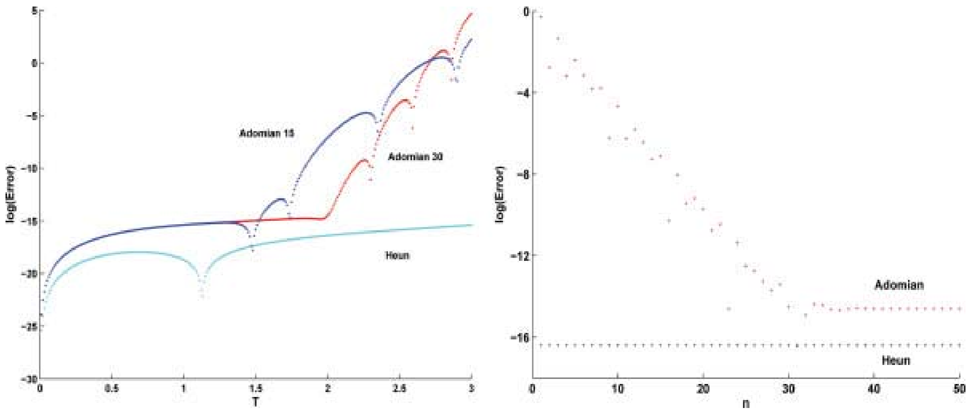

FIG. 1. Left: Errors of the Adomian method with $n = 15$, $n = 30$, and the Heun method versus T. Right: Errors of the Adomian method versus n and the constant error of the Heun method for fixed $T = 2$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

The Adomian polynomials for $N(u) = u^3$ are computed using the generating formula (2.10), or explicitly, by

$$A_k = \sum_{i=0}^{k} \sum_{j=0}^{k-i} u_i u_j u_{k-i-j}.$$

Let $u'(t) = v(t)$ and obtain the recursive formula for the series solution (2.7) in the form

$$\begin{cases} u_{n+1}(t) = \int_0^t v_n(s)ds, \\ v_{n+1}(t) = \int_0^t ([1 - 3\text{sech}^2(s)]u_n(s) + A_n(u_0(s), \dots, u_n(s)))ds, \end{cases} \quad n \geq 0, \tag{4.2}$$

starting with $u_0 = u(0)$ and $v_0 = u'(0)$. The computational error of the ADM is defined by

$$E_n^{\text{ADM}}(T) = \sup_{t \in [0,T]} |U_n(t) - u(t)|, \tag{4.3}$$

where $U_n(t)$ is the n-th partial sum of the series solution (2.7). We compute the time integrals in (4.2) and the absolute error in (4.3) on the equally spaced discrete grid $\{t_m\}_{m=0}^M \in [0, T]$ using the trapezoidal rule. The time step (spacing of grid points) is denoted by h and will be chosen $h = 0.001$.

For comparison, solutions of Eq. (4.1) can be approximated by the Heun method on the same discrete grid $\{t_m\}_{m=0}^M$ based on the same trapezoidal rule. The Heun method for $\dot{y} = f(t, y)$ is

$$\begin{cases} y_{k+1}^* = y_k + hf(t_k, y_k), \\ y_{k+1} = y_k + \frac{1}{2}h(f(t_k, y_k) + f(t_{k+1}, y_{k+1}^*)). \end{cases} \tag{4.4}$$

The computational error of the Heun method is given by

$$E^{\text{HM}}(T) = \sup_{t \in [0,T]} |U^{HM}(t) - u(t)|,$$

where $U^{\text{HM}}(t)$ is the numerical approximation of the Heun method.

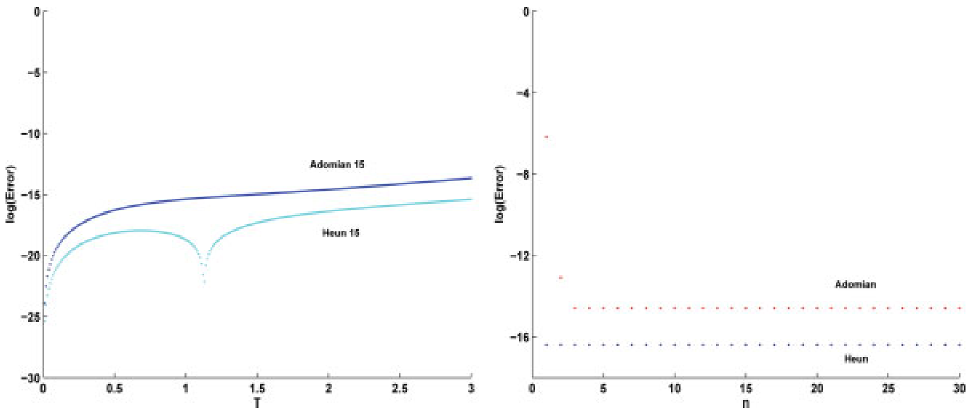

FIG. 2. The same as in Figure 1 but for the single-step Adomian method. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Figure 1 shows the errors $E_n^{\mathrm{ADM}}(T)$ and $E^{\mathrm{HM}}(T)$ as functions of T for $n = 15$ and $n = 30$ (left) and as functions of n for $T = 2$ (right). We can see from Figure 1 (left) that the Adomian method has a slightly higher error at the same time step h compared with the Heun method. The errors gradually increase in time until a sudden jump in the loss of accuracy occurs in the Adomian method. The smaller n is, the faster the jump in accuracy occurs. The accuracy of the Heun method declines more gradually. For a fixed T, the error $E_n^{\mathrm{ADM}}(T)$ quickly reduces with larger $n \geq 1$ to a constant level that depends on the time step h, see Figure 1 (right).

Figure 2 shows a similar picture for the Adomian method implemented as a single-step iterative scheme. This means that the numerical solution is constructed using the Adomian recurrent rule (4.2) on the elementary time interval $[t_m, t_{m+1}]$ for $m \geq 0$ starting with the values of $u_0 = U_n(t_m)$ and $v_0 = U_n'(t_m)$ obtained from the nth partial sum of the Adomian series at the previous time interval. We can see from Figure 2 (left) that no sudden loss of accuracy occurs in the single-step Adomian method and that the errors of the ADM approximations for $n = 15$ and $n = 30$ coincide for this time step h. Figure 2 (right) illustrates that the same level of accuracy can be achieved already with $n \geq 3$.

Figure 3 shows how the errors of the Adomian method with $n = 15$ and the Heun method converge with smaller time steps h for a fixed time $T = 2$. We can see that the error of the Adomian method is always larger than that of the Heun method but their convergence follows the same pattern.

V. APPLICATIONS TO THE TIME-DEPENDENT NLS EQUATION

Consider the nonlinear Schrödinger equation (2.2) with the potential $V(x) = -3\mathrm{sech}^2(x)$ on the symmetric interval $[-L, L]$ with $L = 10$ subject to the periodic boundary conditions. We solve the initial-value problem (2.1) for Eq. (2.2) in spaces $X = H^2_{\mathrm{per}}(-L, L)$ and $Y = L^2_{\mathrm{per}}(-L, L)$. By Parseval's identity, the fundamental solution operator $E(t) = e^{it\partial_x^2}$ is unitary in H^s, so that the semigroup bound (2.3) holds with $C = 1$. On the other hand, $N(u) = u^2\bar{u}$ is analytic in variables $(u, \bar{u})$ if $V \in H^2_{\mathrm{per}}(-L, L)$, so that Assumption 1 is satisfied thanks to the Banach algebra of $H^2_{\mathrm{per}}(-L, L)$ with respect to poinwise multiplication.

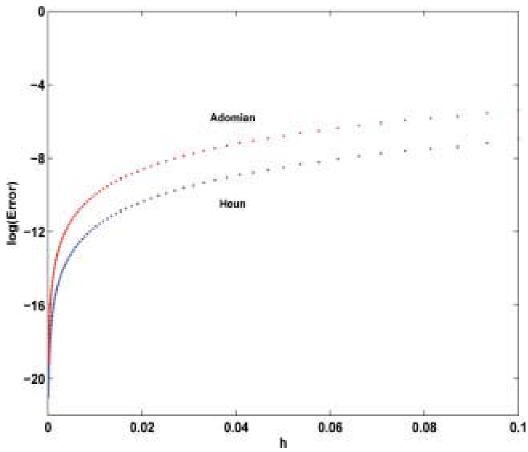

FIG. 3. Convergence of the Adomian and Heun methods for $n = 15$ and $T = 2$ with respect to the time step h. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

The series solution (2.7) is truncated at the partial sum $U_n = u_0 + \sum_{k=1}^{n} u_k$. The Adomian polynomials for $N(u) = u^2\bar{u}$ are computed using the explicit formula

$$A_n = \sum_{k=0}^{n} \sum_{j=0}^{k-i} \bar{u}_k u_j u_{n-k-j},$$

so that the terms of the series solutions are found for any $n \geq 0$ from

$$u_{n+1}(x,t) = i \int_0^t E(t-s)[3\text{sech}^2(x)u_n(x,s) - A_n(u_0(s), \ldots, u_n(s))]ds, \quad n \geq 0,$$

starting with $u_0 = E(t)f$. We use the discrete Fourier transform on an equally spaced discrete grid $\{x_k\}_{k=1}^{K} \in [-L, L]$ with $K/2$ Fourier harmonics and the trapezoidal rule to approximate the time integrals on the equally spaced discrete grid $\{t_m\}_{m=0}^{M} \in [0, T]$ with the time step h. For numerical computations, we use $K = 400$ and $h = 0.005$.

Consider the initial data in the form $f(x) = \text{sech}(x)$. Eq. (2.2) with these $V(x)$ and $f(x)$ admits the exact solution $u(x,t) = e^{it}\text{sech}(x)$. This allows us to compute the computational error by

$$E_n(T) = \sup_{t\in[0,T]} \sup_{x\in[-L,L]} |U_n - u|.$$

Figure 4 shows the first few approximations of the Adomian method (namely, U_0, U_1, and U_{10}) and the absolute errors $E_n(T)$ for $n = 0, 1, \ldots, 10$ versus T. The starting approximation $U_0 = u_0(x,t)$ decays in time with dispersive oscillations according to solutions of the linear Schrödinger equation

$$\begin{cases} i\partial_t u_0 + \partial_x^2 u_0 = 0, & t > 0, \quad x \in (-L, L), \\ u_0(x, 0) = f(x). \end{cases}$$

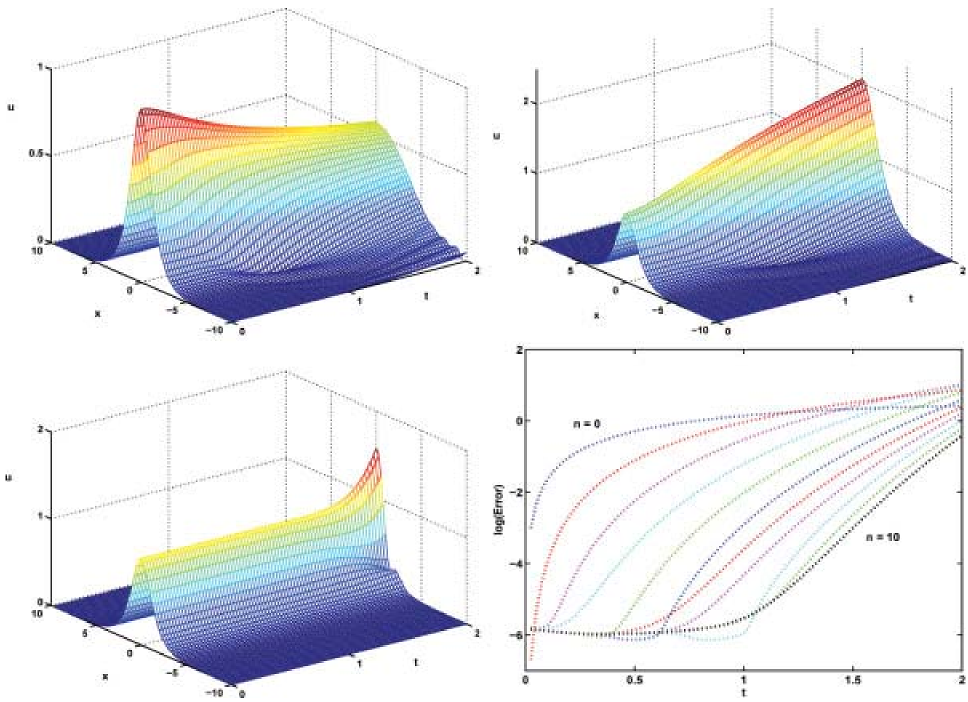

FIG. 4. Surfaces of numerical approximations $|U_0|$ (top left), $|U_1|$ (top right), U_{10} (bottom left) and the absolute errors $E_n(T)$ versus T for $n = 0, 1, \ldots, 10$ (bottom right). [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

The approximation $U_1 = u_0 + u_1$ grows gradually in time. The approximation U_{10} remains nearly constant in amplitude as time evolves until $T = 1.5$ after which it grows. We note from Figure 4 (bottom right) that the absolute errors $E_n(T)$ decrease with increasing n.

Figure 5 shows a similar picture for the single-step Adomian method applied recursively on the time intervals $[t_m, t_{m+1}]$, $m \geq 0$ starting with the initial data $f(x) = U_{10}(x, t_m)$, where U_{10}

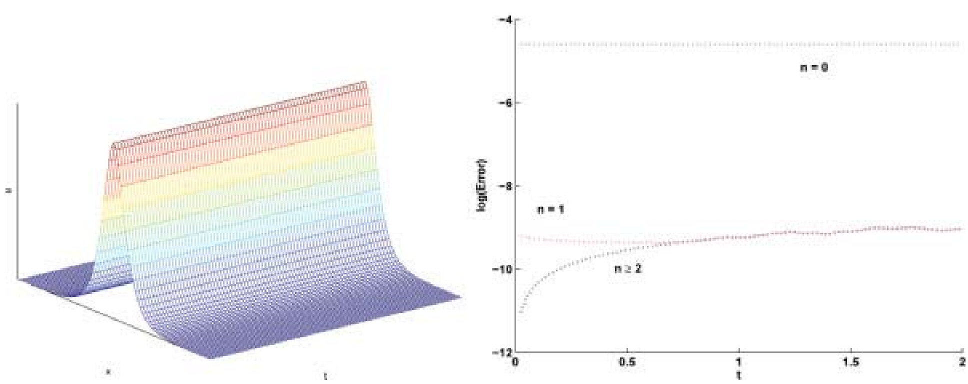

FIG. 5. The surface of numerical approximation U_{10} (left) and the absolute errors $E_n(T)$ versus T for $n = 0, 1, \ldots, 10$ (right) for the single-step Adomian method. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

is obtained at the previous time interval. The surface of $U_{10}(x,t)$ resembles the exact solution $u(x,t) = e^{it}\text{sech}(x)$. Figure 5 (right) shows that the error quickly drops to a nearly constant function, which remains the same for all $n \geq 2$. The magnitude of the error depends on the time step and the number of Fourier harmonics.

We note that the algorithms based on the Adomian decomposition method are not optimized and the computational times are rather long. The cpu time for computations of Figure 4 is 448.8 and that for computations of Figure 5 is 198.2.

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