McMaster University

Master's Project: Existence and Stability of Stokes Waves in a Fluid of Finite Depth

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Motivation and Historical Background

Surface gravity waves in water of finite depth describe dynamics in coastal regions, laboratory wave tanks, and large-scale oceanographic flows. Finite depth introduces nontrivial bottom effects that fundamentally alter:

- The **dispersion relation** and nonlinear wave interactions
- Wave steepening, breaking, and cross-scale **energy transfer**
- The emergence of **rogue waves** and long-term wave field evolution

Understanding stability of finite-amplitude Stokes waves — periodic traveling waves of permanent form — is central to all of the above.

1 **Saffman (1985)**

Provided rigorous analytical foundation via Zakharov's Hamiltonian formulation. Proved stability exchange at stationary energy; identified a Jordan block of algebraic multiplicity four.

2 **Creedon, Deconinck, and Trichtchenko (2022)**

Developed asymptotic techniques for high-frequency instabilities.

3 **Berti, Maspero, and Ventura (2023)**

Conducted a detailed numerical study of the Benjamin–Feir instability

4 **Pelinovsky and Dyachenko (2025)**

Provided rigorous proof that the instability bifurcation occurs exactly at every extremal point of the wave energy

Main Goal

Develop a systematic conformal formulation for Stokes waves in water of finite depth, properly incorporating all compatibility and normalization constraints that are absent in the deep-water case.

Contributions

- Integrated local (Cauchy–Riemann) and nonlocal (mean-value) constraints into the Lagrangian formulation.
- Derived Babenko's equation in conformal variables together with the integral constraint.
- Obtained the Stokes expansion up to order $O(a^3)$ and reconcile it with existing results in the literature
- Proved that the spectral stability exchange occurs precisely at the extremal points of the wave energy (or horizontal momentum) — extending the classical result of Saffman to finite depth.

Why it matters

This framework provides a consistent analytical and numerical platform for studying wave stability, Benjamin–Feir instability, and high-frequency instabilities in coastal and laboratory settings.

The Conformal Mapping Approach

Key Advantages

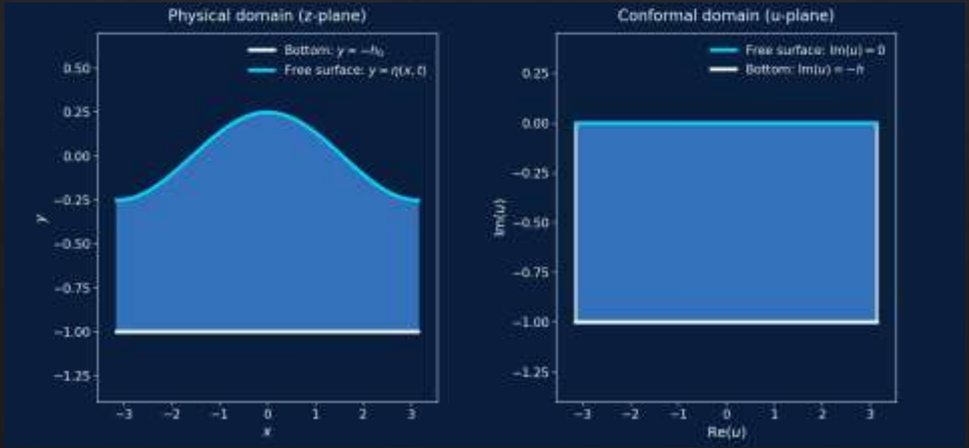
The conformal map **z = z(u,t)** transforms the fluid domain onto the flat strip

$$D_h = \{u \in \mathbb{C} : -\pi \le \mathrm{Re}(u) \le \pi, -h \le \mathrm{Im}(u) \le 0\}$$

reducing the free-surface problem to a system on a **fixed periodic domain**.

Kinematic: $\eta_t + \phi_x \eta_x - \phi_y = 0$

Dynamic: $\phi_t + \frac{1}{2}(\phi_x)^2 + \frac{1}{2}\left(\phi_y\right)^2 + \eta = 0$



Boundary condition

Kinematic: $\eta_t + \phi_x \eta_x - \phi_y = 0$

Dynamic: $\phi_t + \frac{1}{2}(\phi_x)^2 + \frac{1}{2}\left(\phi_y\right)^2 + \eta = 0$

Finite depth introduces additional local (Cauchy–Riemann) and nonlocal (mean-value) constraints not present in the deep-water case.

Canonical Variables and the Operator K_h

On the free surface $\mathrm{Im}(u) = 0$, the canonical variables are $z(u,t) = \xi(u,t) + i\eta(u,t)$and $\varphi(u,t) = \psi(u,t)$. The Cauchy–Riemann equations yield the structural constraint:

$$\xi_u = 1 + K_h\eta, \qquad (K_h)_n = n\coth(nh), \qquad n \in \mathbb{Z}$$

Operator T_h

Dirichlet-to-Neumann operator with Fourier symbol $(T_h)_n = i\tanh(nh)$. Bounded, skew-adjoint.

Operator K_h

Self-adjoint, positive operator on L^2_{per}. Defined as $K_h = T_h^{-1}\partial_u$on H^1_{per}.

Depth Constraint

Integrating the Cauchy–Riemann relation over the period links conformal depth h, physical depth h_0, and mean surface elevation:

$$h = h_0 + \bar{\eta}, \qquad \bar{\eta} = \frac{1}{2\pi}\int_0^{2\pi} \eta(u,t), du$$

Both h and $\bar{\eta}$ may depend on time.

Lagrangian Formulation and Equations of Motion

Equations of motion in conformal variables are derived from the Lagrangian functional $\mathcal{L}(\psi, \xi, \eta, h, f, g) = \int_0^{2\pi} \psi(\eta_t \xi_u - \eta_u \xi_t), du + \frac{1}{2}\int_0^{2\pi} \psi, T_h \psi_u, du - \frac{1}{2}\int_0^{2\pi} \eta^2 \xi_u, du + \int_0^{2\pi} f(\eta - h + h_0 - T_h(\xi - u)), du + g\left(\frac{1}{2\pi}\int_0^{2\pi} \eta du - h + h_0\right)$ where f and g are Lagrange multipliers enforcing the structural constraints. Variation yields a coupled system:

δ_ψ: Kinematic condition

$$\eta_t \xi_u - \eta_u \xi_t + T_h \psi_u = 0$$

δ_η: Dynamic condition

$$\psi_u \xi_t - \psi_t \xi_u - \eta \xi_u + f + \frac{1}{2\pi} g = 0$$

δ_ξ: Momentum balance

$$\psi_t \eta_u - \psi_u \eta_t + \eta \eta_u + T_h f = 0$$

δ_h: Depth equation

$$\frac{1}{2}\langle \psi, (\partial_h T_h)\psi_u \rangle - \langle f, (\partial_h T_h)(\xi - u) \rangle - g = 0$$

Mean-value projections of these equations yield three conserved quantities: mass $M(\eta)$, horizontal momentum $P(\psi, \eta)$, and vertical momentum $Q(\psi, \eta)$.

Conservation Laws and the Bernoulli Constant

M

Mass

$M(\eta) = \langle \eta, 1 + \mathrm{k_h}\eta \rangle$ is conserved;

$d/dt\, M \;=\; 0$

P

Horizontal Momentum

$P(\psi, \eta) = -\langle \psi, \eta_{\mathrm{u}} \rangle$ is conserved;

$d/dt\, P \;=\; 0$

Q

Vertical Momentum

$Q(\psi, \eta) = \langle \psi, (1 + K_h \eta) \rangle$ is conserved;

$d/dt\, \mathrm{Q} = 0$

Bernoulli Constant: $\psi = \int B\, dt + \tilde{\psi}; Q\big(\tilde{\psi}, \eta\big)$ conserved iff $g = 2\pi B + M(\eta)$

Babenko's Equation

For steady traveling waves with speed c, the profiles satisfy $\eta(u,t) = \eta(u - ct)$and $\tilde{\psi} = cT_h^{-1}\eta$. Substituting into the equations of motion yields **Babenko's equation** in conformal variables:

$$(c^2 - 2B)K_h\eta - \eta = \frac{1}{2}K_h\eta^2 + \eta, K_h\eta - \frac{1}{2\pi}M(\eta).$$

Constraint on *B* and *h*

The Bernoulli constant *B* and conformal depth *h* are coupled through the integral constraint:

$$2\pi B + M(\eta) = \frac{1}{2}\langle(c^2 - 2B)\eta - \eta^2, (\partial_h K_h)\eta\rangle$$

Normalization Choice

Two choices are available:

- $B = 0$, allowing
- with $B \neq 0$ **preferred**, as it sets the mean surface elevation to zero in physical coordinates, uniquely defining h_0.

Variational Structure

The traveling wave system admits a variational formulation via the action functional:

$$\Lambda_{c,B}(\eta,h) = \int_0^{2\pi} \left[\frac{1}{2},\eta(c^2K_h - 1)\eta - \frac{1}{2},\eta^2K_h\eta - B\eta(1+K_h\eta)\right],du + g\left(\frac{1}{2\pi}\int_0^{2\pi} \eta du - h + h_0\right)$$

Euler-Lagrange in η

Recovers Babenko's equation. Taking mean values gives $g = 2\pi B + M(\eta)$, consistent with the Bernoulli relation.

Symmetry

The functional is invariant under translation $u \mapsto u + \theta$, generating a one-parameter family of solutions and a zero eigenvalue of translational type.

Stokes Expansion: Order O(a)

Since steady gravity waves are symmetric about their crest, the profile $\eta(u)$ is even in L^2 and expanded in cosine modes.

Wave Profile **η**

$\eta = a \cos u + a^2[A_0 + A_2 \cos 2u] + a^3 A_3 \cos 3u + O(a^4)$

Wave Speed **c²**

$c^2 = c_0^2(h) + a^2 c_2(h) + O(a^4)$

Bernoulli Constant **B**

$B = a^2 B_2(h) + O(a^4)$

At **order O(a)**: the linearized Babenko equation $(c_0{}^2 \mathrm{K_h} - 1)\cos u = 0$ yields the leading-order dispersion relation: **$c_0^2(h) = \tanh(h)$**.

Stokes Expansion: Orders O(a²) and O(a³)

Order O(a²): Coefficients A_2, A_0, B_2

The inhomogeneous equation uniquely determines:

$$A_2(h) = \frac{c_0^4(h) + 3}{4, c_0^6(h)}$$

The coefficient A_0 is *not* fixed by Babenko's equation alone; instead the integral constraint yields the coupled relation:

$$A_0(h) + B_2(h) = -\frac{1}{2}\coth(h) - \frac{1}{2\sinh(2h)}$$

Order O(a³): Coefficients c_2, A_3

Solvability of the $O(a^3)$ inhomogeneous system eliminating the secular cos(u) term fixes the nonlinear frequency correction:

$$c_2(h) = \frac{9 - 6\tanh^2 h + 5\tanh^4 h}{8\tanh^3 h}$$

The third harmonic amplitude is:

$$A_3 = \frac{1}{2} A_2 \frac{\coth(h) + 2\coth(2h) + 3\coth(3h)}{3\tanh(h)\coth(3h) - 1}$$

The integral constraint contributes no new conditions at $O(a^3)$, completing the expansion to $O(a^4)$.

Comparison with Stokes Expansions

Three independent works derive Stokes expansions in the *physical* depth $\mathbf{h_0}$ via distinct analytical methods. Our conformal-coordinate framework reproduces and reconciles these expansions, exposing a normalization inconsistency in [1] and confirming full agreement with [2] and [3].

[1] M. Berti, A. Maspero, P. Ventura

Bernoulli constant $\mathbf{B = 0}$; wave speed expressed via physical depth h_0

[2] R. P. Creedon, B. Deconinck, O. Trichtchenko

Mean-zero surface $\bar{\boldsymbol{\eta}} = \mathbf{0}$; requires careful renormalization of speed c

[3] V. E. Zakharov and A. I. Dyachenko

Conformal depth formulation; nonzero B; full agreement with our expansion

Comparison with [1] and [3]

Agreement with [1]

Setting **B = 0**, the mean-value correction to the surface elevation matches eq. (2.6) of [2]:

$$\eta_2^{[0]} = A_0(h) + \frac{1}{2}\coth(h) = -\frac{1}{2\sinh(2h)} = \frac{c_0^4 - 1}{4c_0^2}$$

The wave speed to order a^2 recovers eq. (2.8) of [1], expressing c in terms of the physical depth h_0 via sech^2 and tanh corrections.

Agreement with [3]

The speed correction in conformal depth h matches eq. (4.13) of [3] exactly. Enforcing **M(η) = 0** with nonzero Bernoulli constant B, the conversion to physical depth yields:

$$c = c_0(h_0) + \frac{a^2}{2c_0(h_0)}\left[\frac{9 - 10\tanh^2(h_0) + 9\tanh^4(h_0)}{8\tanh^3(h_0)}\right] + \mathcal{O}(a^4)$$

Matching eq. (4.14) of [3].

Discrepancy with [2] and Its Resolution

Root Cause

Reference [2] sets **B = 0** globally, which is incompatible with our formulation. This causes the velocity potential $\tilde{\psi}$ to acquire a *linear growth* in x rather than remaining periodic.

Expansion (A.2b) in [2] omits the speed renormalization required by the non-periodic $\tilde{\psi}$, yielding an incorrect wave speed at order a^2.

Corrected Expansion

After renormalizing c appropriately using expansion (A.3b) with (A.4e) from [2], one recovers:

$$c = c_0(h_0) + \frac{a^2}{2c_0(h_0)} \left[\frac{9 - 10\tanh^2(h_0) + 9\tanh^4(h_0)}{8\tanh^3(h_0)} \right] + \mathcal{O}(a^4)$$

This agrees with our result. In our solutions, $\tilde{\psi}$ is periodic in u and $B \neq 0$.

Time-Dependent Equations of Motion

The full nonlinear dynamics are cast in conformal coordinates with time-dependent conformal depth h(t) and Bernoulli constant B(t). The system couples surface elevation η and reduced velocity potential ψ̃:

Kinematic Equation (1)

$\eta_t \xi_u - \eta_u \xi_t + \widetilde{T_h} \widetilde{\psi_u} = 0$, encodes the free-surface kinematic condition in conformal variables.

Dynamic Equation (2)

Bernoulli condition rewritten using T_h, K_h operators; the Bernoulli constant B and mean surface elevation M(η) appear explicitly.

Integral Constraint (3)

Describes the evolution of h(t); equivalent to energy conservation when h′(t) ≠ 0 and B′(t)M(η) = 0.

Energy Conservation

Energy Functional

The Hamiltonian is defined as: $H(\tilde{\psi},\eta) = \langle(\eta^2 + 2B\eta),(1 + K_h\eta)\rangle - \langle\tilde{\psi}, T_h\partial_u\tilde{\psi}\rangle$

Dynamic Equations

$$\psi_u\xi_t - \widetilde{\psi_t}\xi_u - (B+\eta)\xi_u - T_h^{-1}(\widetilde{\psi_t}\,\eta_u - \widetilde{\psi_u}\,\eta_t + (B+\eta)\,\eta_u) + B + \frac{1}{2\pi}M(\eta) = 0$$

Integral Constraint

$$2\pi B + M(\eta) = \frac{1}{2}\langle\tilde{\psi},(\partial_h T_h)\,\widetilde{\psi_u}\rangle + \langle(\widetilde{\psi_t}\,\eta_u - \widetilde{\psi_u}\,\eta_t + (B+\eta)\,\eta_u),(\partial_h T_h^{-1})\eta\rangle.$$

Conserved Quantities:

Mass M(η)	Horiz. Momentum P	Vert. Momentum Q
$\langle\eta, 1 + K_h\eta\rangle$	$-\langle\tilde{\psi},,\eta_u\rangle$	$\langle\tilde{\psi},,1 + K_h\eta\rangle$

Proposition 1: H is conserved in time *if and only if* the integral constraint holds and $B'(t)M(\eta) = 0$.

The proof multiplies K.E. by $\tilde{\psi}$□ and D.E. by η□, integrates over one period, then applies skew-adjointness of T^{-1}□ and self-adjointness of T□∂_u, K□.

Linearization Near a Traveling Wave Profile

Setup

Perturbation the traveling wave $(\eta, 0)$satisfying Babenko's equation by *(v,w)*. Linearize the conformal depth $h(t) \approx h_0 + \bar{\eta} + \bar{v}(t)$, Bernoulli constant by δB, and mass by δM.

$$Meanvalues: \bar{\eta} = \frac{1}{2\pi}\int_0^{2\pi} \eta, du\,, \qquad \bar{v}(t) = \frac{1}{2\pi}\int_0^{2\pi} v, du$$

Linearized Operators

$\mathcal{M}_{\hbar}$

$(1 + K_h\eta) - \eta' T_h^{-1}$

$\mathcal{M}_{\hbar}^{*}$

$(1 + K_h\eta) + T_h^{-1}(\eta', \cdot,)$

$\mathcal{L}_{\hbar}$

$(c^2 - 2B)K_h - 1 - K_h\eta - \eta K_h - K_h(\eta, \cdot,)$

Conservation: Proposition

The linearized equations **preserve all conservation laws** of the nonlinear system. Specifically, projecting the linearized equations onto distinguished test functions yields the conserved quantities via operator identities:

Project eq. (lin-1) onto *1* →

$$\frac{d}{dt}\delta M = 0, using \mathcal{M}_{\hbar}^{*}1 = (1 + 2K_h\eta)$$

Project eq. (lin-2) onto η' →

$$\frac{d}{dt}\delta P = 0, using \mathcal{L}_{\hbar}\eta' = 0$$

Project eq. (lin-2) onto *1* →

$$\frac{d}{dt}\delta Q = 0, using \mathcal{L}_{\hbar}1 = -(1 + 2K_h\eta)$$

Lin-1 (Kinematic): $\mathcal{M}_{\hbar}v_t = -T_h\partial_u w + \bar{v}\eta'(\partial_h T_h^{-1})\eta$

Lin-2 (Dynamic): $\mathcal{M}_{\hbar}^{*}w_t + 2cT_h^{-1}v_t = \mathcal{L}_{\hbar}v + \bar{v}\partial_h S_h(\eta) - 2(\delta B)K_h\eta + \frac{1}{2\pi}\delta M - c\bar{v}'(\partial_h T_h^{-1})\eta$

Spectral Problem

Separating variables $v = e^{\lambda t} v(u), w = e^{\lambda t} w(u), \bar{v} = e^{\lambda t} \bar{v}, \delta B = e^{\lambda t} b$, and setting $\delta M \;=\; 0$ for $\lambda \neq 0$, the linearized system becomes:

Spectral Equation I

$$-T_h \partial_u w = \lambda\left(\mathcal{M}_\hbar v - \bar{v}\eta'\left(\partial_h T_h^{-1}\right)\eta\right)$$

Spectral Equation II

$$\mathcal{L}_\hbar v + \bar{v}\partial_h S_h(\eta) - 2bK_h\eta =$$
$$\lambda\left(\mathcal{M}_\hbar^* w + 2cT_h^{-1} v + c\bar{v}\left(\partial_h T_h^{-1}\right)\eta\right)$$

Spectral Constraint

$$2\pi b\left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] = \langle v, \partial_h S_h(\eta)\rangle +$$
$$\frac{1}{2}\bar{v}\left\langle (c^2 - 2B)\eta - \eta^2, \left(\partial_h^2 K_h\right), \eta\right\rangle +$$
$$\lambda\left(\left\langle w, \eta'\left(\partial_h T_h^{-1}\right), \eta\right\rangle + c\left\langle v, \left(\partial_h T_h^{-1}\right)\eta\right\rangle\right).$$

Orthogonality and Zero Second Variation

Proposition: For any eigenfunction $(v, w) \in H^1_{\text{per}} \times H^1_{\text{per}}$ *with* $\lambda \neq 0$:

$$\delta M = 0$$

Orthogonality to *1* in spectral eq. I

$$\delta P = 0$$

Orthogonality to η' in spectral eq. II

$$\delta Q = 0$$

Orthogonality to *1* in spectral eq. II

$$\delta^2 H = 0$$

Spectral constraint ≡ zero second variation of energy at traveling wave $(cT_h^{-1}\eta, \eta)$

The spectral constraint is rewritten, via inner products of the spectral equations with *w* and *v* respectively and skew-adjointness of T_h^{-1}, as a purely quadratic expression in the perturbation, confirming equivalence with vanishing second variation of *H*.

Two-Parameter Family of Stokes Waves and Implicit Function Theorem Condition

We consider solutions of Babenko's equation with the normalization $M(\eta) = 0$. Neglecting the depth constraint for a moment, the traveling wave profile η and Bernoulli constant B are regarded as functions of two independent parameters (c, h).

IFT condition

The solvability condition for smooth continuation in (c, h) is:

$$1 + \frac{1}{2\pi}\langle \eta, \partial_h K_h \eta \rangle \neq 0$$

Under this condition, **$\eta = \eta(u; c, h)$** and **$B = B(c, h)$** are smoothly continued.

Depth Constraint and Condition

Adding the depth constraint yields a locally unique one-parameter family ***$\eta = \eta(u; c, h(c))$***, provided the additional condition:

$$1 - \frac{1}{2\pi}\int_0^{2\pi} \partial_h \eta(u; c, h)\, du \neq 0$$

holds.

This two-parameter framework is the foundation for the spectral analysis at $\lambda = 0$.

Spectral Problem at λ = 0: Geometric and Algebraic Multiplicity

The linearized spectral problem, closed with the integral constraint, is studied at the critical value **λ = 0**.

Proposition 1

Assume the non-degeneracy and the solvability conditions. Then, the geometric Multiplicity of $\lambda = 0$ in the spectral problem closed with the integral constraint is equal to two.

Proposition 2

Assume the non-degeneracy and the solvability conditions. Then, the algebraic multiplicity of $\lambda = 0$ in the spectral problem closed with the integral constraint is at lest three.

Geometric Multiplicity = 2

First solution: (v, w) = (0, 1)

Second solution: (v, w) = (v, 0) with arbitrary mean value

Algebraic Multiplicity ≥ 3

By Proposition 1 and 2 Explicit generalized eigenvector (first element of the Jordan chain):

$$v_1 = -\alpha(\partial_c \eta + h'(c)\partial_h \eta), \quad w_1 = -\alpha T_h^{-1} \eta.$$

All statements hold under the non-degeneracy assumption **Ker(L□) = span(η′)** together with the solvability conditions

Threshold Criterion for Stability Exchange

We derive the precise condition under which the algebraic multiplicity of λ = 0 becomes **exactly three**.

> **Proposition 3**
>
> The algebraic multiplicity of λ = 0 is **exactly three if and only if**
>
> $$P'(C) \neq 0$$
>
> where the vertical momentum is
>
> $$P(c,h) = c\langle K_h \eta, \eta \rangle$$

Extremum Condition

The condition **P'(c) = 0** is equivalent to an **extremum** of the total wave energy (or equivalently, of the horizontal momentum).

Stability Exchange

At this critical point the Stokes wave **exchanges stability** and the superharmonic instability bifurcation occurs.

Classical Connection

This theorem recovers the classical **Saffman (1985)** criterion within the conformal formulation for water of finite depth.

Main Results

- Developed a systematic conformal formulation for Stokes waves in finite depth, including both local (Cauchy–Riemann) and nonlocal mean-value constraints.
- Derived Babenko's equation in conformal variables together with the consistent integral constraint on B and h.
- Obtained the Stokes expansion up to $O(a^3)$ and reconciled it with the existing literature, resolving normalization inconsistencies in previous works.
- Proved that the linearized system includes all conservation laws of the nonlinear system.
- Showed that spectral stability exchange occurs precisely at the extremal points of the wave energy H (equivalently, of the horizontal momentum P).

Key novelty

The framework properly accounts for time-dependent conformal depth $h(t)$ and Bernoulli constant $B(t)$, which was missing in earlier conformal approaches for finite depth.