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Bifurcations of Asymmetric Vortices in Symmetric Harmonic Traps

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We show that, under the effect of rotation, symmetric vortices located at the center of a two-dimensional harmonic potential are subject to a pitchfork bifurcation with radial symmetry. This bifurcation leads to the family of asymmetric vortices, which precess constantly along an orbit enclosing the center of symmetry. The radius of the orbit depends monotonically on the difference between the rotation frequency and the eigenfrequency of negative Krein signature associated with the symmetric vortex. We show that both symmetric and asymmetric vortices are spectrally and orbitally stable with respect to small time-dependent perturbations for rotation frequencies exceeding the bifurcation eigenfrequency. At the same time, the symmetric vortex is a local minimizer of energy for supercritical rotation frequencies, whereas the asymmetric vortex corresponds to a saddle point of energy. For subcritical rotation frequencies, the symmetric vortex is a saddle point of the energy.

1 Introduction

In the context of physics of Bose–Einstein condensation, rotating vortices in symmetric harmonic traps have been reported both theoretically [2, 4, 6, 13] and experimentally [1, 7, 27]. Theoretical studies in the physics literature rely on the qualitative approximation obtained from the Rayleigh–Ritz variational method. These approximations have been used to predict frequencies of precession of a single vortex about the center of the

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harmonic trap as well as frequencies of the effective dynamics of dipole, tripole, and quadrupole configurations [26, 28, 31, 34].

Numerical approximations of precessional frequencies for vortex configurations were obtained by Middelkamp *et al.* [29, 30]. Similar numerical results were computed by Kollar and Pego [24] for the spectrum of a single vortex located at the center of the harmonic potential, using shooting methods and Evans function computations.

Recently, vortex-free states and vortex configurations in the symmetric harmonic traps were studied rigorously in the Thomas–Fermi limit of large density states. The Painlevé-II equation for spatially uniform approximation of the vortex-free state was justified by Gallo and Pelinovsky [10]. Minimization of energy was considered by Ignat and Millot [15, 16] with the method of variational calculus. These authors justified earlier computations [2, 4] that the vortex-free state is the global minimizer of energy for small frequencies of the rotating Bose–Einstein condensate, whereas a single vortex of charge 1 is the global minimizer of energy for a frequency above a critical value. Seiringer [36] proved that multiple vortex configurations become the global minimizers of energy for larger rotation frequencies and obtained bounds on the critical values of the rotation frequencies, when a vortex of charge $(n+1)$ becomes energetically favorable to a vortex of charge n. He also proved that radially symmetric vortices cannot be global minimizers of energy for large frequencies (or for fixed frequency but in the Thomas–Fermi limit), which implies that the vortex configurations of charge $n \geqslant 2$ break into a superposition of n individual vortices of charge 1. He refers to this case as the breakdown of the rotational symmetry, although the superposition of n individual vortices can still be rotated on the plane under any angle.

Variational approximations of Castin and Dum [4] (generalized in the anisotropic setting in [28]) suggest that there exist actually two critical frequencies for a vortex of charge 1. When the frequency parameter is increased across the first critical frequency, the single vortex of charge 1 placed at the center of the harmonic potential becomes a local minimizer of energy (it is a saddle point of energy for small frequencies). This vortex becomes the global minimizer of energy when the frequency parameter is increased across the second critical frequency (which is roughly twice of the first one). Figure 4 in [4] also suggests that, for frequencies above the first critical value, there exists a vortex of charge 1 placed at a distance from the center of the harmonic potential, and it is a saddle point of energy. This bifurcation at the first critical frequency and the onset of the asymmetric rotating vortices are considered in the present work.

In particular, we give a rigorous proof of the existence of the first critical frequency for the local bifurcation of the symmetric vortex of charge 1. This critical frequency coincides with the nonzero eigenfrequency of negative Krein signature in the spectrum of linearization of the symmetric vortex in the absence of rotation [24, 29]. This smallest eigenvalue gives a frequency of the vortex precession about the center of symmetry of the harmonic potential at an infinitesimal small distance, studied in our earlier work [34]. It is beyond the linear approximation to decide whether this rotation is free of any radiation and is observed for infinite times. We clarify this question throughout this article, where we prove the birth of an asymmetric vortex that performs steady precession about the center of symmetry of the harmonic potential.

Our results are based on the method of Lyapunov–Schmidt reductions for a local bifurcation problem imposed on the family of symmetric vortices centered at the origin in the rotating coordinate frame. These results do not rely on the Thomas–Fermi limit for large density states or on the variational methods used earlier. (Note that the entire bifurcation at the first critical frequency, as well as the crucial difference between symmetric and asymmetric vortices were missing in previous works [2, 15, 16, 36] that relied on the calculus of variations and functional analysis.) We show that the family of symmetric vortices becomes subject to a pitchfork bifurcation, where the underlying parameter is the precessional frequency. This parameter determines uniquely the radius of the orbit, which the new asymmetric vortex precesses along. The bifurcating asymmetric vortex can be placed at any point along the orbit, hence the pitchfork bifurcation exhibits a radial symmetry.

We also prove that both the symmetric vortex and the new asymmetric vortex are spectrally and orbitally stable with respect to small time-dependent perturbations for supercritical rotation frequencies. This is expected on the basis of the previous theoretical and experimental observations of constantly precessing vortices in the symmetric harmonic potential. Moreover, our results agree with the previous variational approximations [4] suggesting that the symmetric vortex is a local minimizer of energy for supercritical rotation frequencies, whereas the asymmetric vortex is a saddle point of energy (still spectrally and orbitally stable).

For completeness, we also mention recent mathematical works, which are in subjects near the one of our study. The method of Lyapunov–Schmidt reductions has been widely used in a series of recent works of Kapitula *et al.* [17, 18, 20] devoted to similar problems. In [17], rings, multi-poles, soliton necklaces, and vortex necklaces were constructed for the Gross–Pitaevskii equation with a two-dimensional radially symmetric harmonic potential in the weak interaction limit. A superposition of harmonic and small periodic potentials was considered in [18], also in the weak interaction limit. The nonradially symmetric vortices were constructed for a coupled system of two Gross–Pitaevskii equations in [20].

Another but similar bifurcation analysis was developed for a pair of rotating solitary waves in harmonic potentials by Selvitella [37]. Existence and stability of both symmetric and asymmetric vortices were studied by Gallay *et al.* [8, 9] for the Navier–Stokes equations. A similar problem of symmetric and asymmetric vortices in two-component Ginzburg–Landau energy functional was considered by Alama *et al.* [3]. Bifurcations of periodic solutions in the system of relative equilibria of vortex configurations were considered by Garcia–Azpeitia and Ize [11, 12].

The paper is organized as follows. Section 2 reviews the existence and stability of symmetric vortices of charge 1 in the stationary Gross–Pitaevskii equation. In Section 3, we consider symmetric vortices of charge 1 in the rotational coordinate frame and discover a relationship between spectral stability and bifurcation problems in the rotational and nonrotational cases. Section 4 presents the main results on the symmetry-breaking bifurcation of stable vortices of charge 1 in the rotating coordinate frame and offers some numerical illustrations. The normal form for the pitchfork bifurcation with radial symmetry is derived and justified in Section 5. Orbital stability of asymmetric vortices is proved in Section 6. Section 7 concludes the paper.

We set some notations in the rest of this section. The Hilbert–Sobolev space of squared integrable functions on $\mathbb{R}^2$ with square integrable derivatives up to the kth order is denoted by $H^k(\mathbb{R}^2)$. If $f \in H^k(\mathbb{R}^2)$ and there is $m \geqslant 0$ such that $\mathrm{e}^{-\mathrm{i}m\theta} f(x) = \varphi(r)$ is radially symmetric in polar coordinates (r, θ), we say that $\varphi \in H^k_{r,m}(\mathbb{R}_+)$.

The Hilbert–Lebesgue spaces of square integrable functions and their radially symmetric restrictions are denoted by $L^2(\mathbb{R}^2)$ and $L^2_r(\mathbb{R}_+)$, respectively. The corresponding inner products are defined by

$$\forall f, g \in L^2(\mathbb{R}^2): \quad \langle f, g\rangle_{L^2} := \int_{\mathbb{R}^2} f(x,y) g(x,y)\, \mathrm{d}x\, \mathrm{d}y,$$

$$\forall f, g \in L^2_r(\mathbb{R}_+): \quad \langle f, g\rangle_{L^2_r} := \int_0^\infty f(r) g(r) r\, \mathrm{d}r.$$

Similar notations are introduced for Lebesgue spaces $L^p(\mathbb{R}^2)$ and $L^p_r(\mathbb{R}_+)$ for any $p \geqslant 1$.

Finally, we say that $A = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$ if there is $\epsilon_0 > 0$ such that for every $\epsilon \in (-\epsilon_0, \epsilon_0)$ there is a positive constant C such that $|A| \leqslant C|\epsilon|$. If X is a Banach space and $A \in X$, then the notation $A = \mathcal{O}(\epsilon)$ implies that for small ϵ, there is a positive constant C such that $\|A\|_X \leqslant C|\epsilon|$.

2 Symmetric Vortex at the Center of the Harmonic Potential

We consider the Gross–Pitaevskii equation with the symmetric harmonic potential and repulsive nonlinear interactions,

$$\mathrm{i}\epsilon u_t + \epsilon^2(u_{xx} + u_{yy}) + (1 - x^2 - y^2 - |u|^2)u = 0, \tag{1}$$

where parameter $\epsilon > 0$ is inversely proportional to the chemical potential, $(x, y) \in \mathbb{R}^2$ are spatial coordinates, $t \in \mathbb{R}_+$ is the evolution time, and $u(x, y, t) \in \mathbb{C}$ is the wave function.

Let (r, θ) be the polar coordinates on the plane $(x, y) \in \mathbb{R}^2$. We denote the Laplace operator for the mth azimuthal mode by

$$\Delta_m = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}. \tag{2}$$

In particular, $\Delta_0 = \Delta_{m=0}$ and $\Delta_1 = \Delta_{m=1}$.

Two stationary solutions of the Gross–Pitaevskii equation (1) are of our interest. One solution $u(x, y, t) = \eta_\epsilon(r)$ is referred to as the *vortex-free state* if $\eta_\epsilon : \mathbb{R}_+ \to \mathbb{R}$ is a radially symmetric, positive solution of the differential equation,

$$\epsilon^2 \Delta_0 \eta_\epsilon + (1 - r^2 - \eta_\epsilon^2)\eta_\epsilon = 0, \quad \eta_\epsilon(r) > 0, \;\; r \in \mathbb{R}_+. \tag{3}$$

The other solution $u(x, y, t) = \psi_\epsilon(r)\,\mathrm{e}^{\mathrm{i}\theta}$ is referred to as the *symmetric vortex* of charge 1 if $\psi_\epsilon : \mathbb{R}_+ \to \mathbb{R}$ is a radially symmetric, positive solution of the differential equation,

$$\epsilon^2 \Delta_1 \psi_\epsilon + (1 - r^2 - \psi_\epsilon^2)\psi_\epsilon = 0, \quad \psi_\epsilon(r) > 0, \;\; r \in \mathbb{R}_+. \tag{4}$$

The vortex is *symmetric* because its center of symmetry is located at the origin $(0, 0) \in \mathbb{R}^2$ owing to polar coordinates (r, θ). Note that $\psi_\epsilon(0) = 0$ because $r = 0$ is a regular singular point of the differential equation (4).

Let us define the Schrödinger operator for the quantum harmonic oscillator,

$$H(\epsilon) := -\epsilon^2(\partial_x^2 + \partial_y^2) + x^2 + y^2 - 1, \tag{5}$$

with the domain

$$\mathrm{Dom}(H(\epsilon)) := \{u \in H^2(\mathbb{R}^2) : \; |x|^2 u \in L^2(\mathbb{R}^2)\}.$$

The spectrum of $H(\epsilon)$ in $L^2(\mathbb{R}^2)$ is purely discrete. The eigenvalues are known exactly

$$\sigma(H(\epsilon)) = \{\lambda_{n,m}(\epsilon) = -1 + 2\epsilon(n + m + 1), \;\; (n, m) \in \mathbb{N}_0^2\}, \tag{6}$$

where $\mathbb{N}_0$ denotes the set of all natural numbers counting from 0.

When $\epsilon = \frac{1}{2}$, the lowest eigenvalue $\lambda_{0,0}(\epsilon)$ crosses 0 and induces a local bifurcation of the vortex-free state η_ϵ satisfying (3). For small $|\epsilon - \frac{1}{2}|$, the ground state is close to the linear eigenmode $f_{0,0} = \mathrm{e}^{-r^2}$.

When $\epsilon = \frac{1}{4}$, the next double eigenvalue $\lambda_{1,0}(\epsilon) = \lambda_{0,1}(\epsilon)$ crosses 0 and induces a local bifurcation of the symmetric vortex ψ_ϵ satisfying (4). For small $|\epsilon - \frac{1}{4}|$, the symmetric vortex is close to the linear eigenmode $f_{1,0} + \mathrm{i} f_{0,1} = (x + \mathrm{i}y)\,\mathrm{e}^{-2r^2}$. This bifurcation is well known and was recently studied by Kapitula *et al.* [18]. Standard arguments of the local bifurcation theory give the following.

Lemma 1. Let $\mu := \frac{1}{16} - \epsilon^2$. There exists a positive μ_0 such that for all $\mu \in (0, \mu_0)$, there exist a positive constant C and a solution $\psi_\epsilon \in H^2_{r,1}(\mathbb{R}_+)$ of the differential equation (4) such that

$$\sup_{r\in\mathbb{R}_+} |\psi_\epsilon(r) - (128\mu)^{1/2} r\,\mathrm{e}^{-2r^2}| \leqslant C\mu^{3/2}. \tag{7}$$

The map $(0, \mu_0) \ni \mu \to \psi_\epsilon \in H^2_{r,1}(\mathbb{R}_+)$ is continuously differentiable for any $\mu \in (0, \mu_0)$. □

Proof. We shall derive bound (7) by the standard method of Lyapunov–Schmidt reductions. We rewrite the existence problem (4) as the local bifurcation problem

$$(L_1 + \mu\Delta_1)\psi_\epsilon = -\psi_\epsilon^3, \quad \epsilon^2 = \frac{1}{16} - \mu,$$

where $L_1 = -\frac{1}{16}\Delta_1 + r^2 - 1$. We note that $\mathrm{Ker}(L_1) = \mathrm{span}\{\psi_0\}$, where $\psi_0(r) = r\,\mathrm{e}^{-2r^2}$. Using the orthogonal decomposition,

$$\psi_\epsilon = \mu^{1/2}(a\psi_0 + \varphi_\epsilon), \quad \langle\psi_0, \varphi_\epsilon\rangle_{L^2_r} = 0,$$

the local bifurcation problem is decoupled into a pair of two equations:

$$\langle\psi_0, \Delta_1(a\psi_0 + \varphi_\epsilon) + (a\psi_0 + \varphi_\epsilon)^3\rangle_{L^2_r} = 0 \tag{8}$$

and

$$P_0 L_1 P_0 \varphi_\epsilon = -\mu P_0(\Delta_1(a\psi_0 + \varphi_\epsilon) + (a\psi_0 + \varphi_\epsilon)^3), \tag{9}$$

where P_0 is the orthogonal projection operator from $L^2_r(\mathbb{R}_+)$ to $\mathrm{Ran}(L_1) \subset L^2_r(\mathbb{R}_+)$. By a standard application of the Implicit Function Theorem, for every $a \in \mathbb{R}$ and small $\mu \in \mathbb{R}$, there is a unique $\varphi_\epsilon \in H^2_{r,1}(\mathbb{R}_+)$ that solves Equation (9) and satisfies the bound $\|\varphi_\epsilon\|_{H^2_{r,1}} \leqslant$

$C|a\mu|$ for some $C > 0$. Then, the bifurcation equation (8) gives the root-finding equation:

$$a(\langle\psi_0, \Delta_1\psi_0\rangle_{L_r^2} + a^2\|\psi_0\|_{L_r^4}^4 + \mathcal{O}(\mu)) = 0.$$

Explicit evaluation of the inner products show that this equation is equivalent to

$$a(-\tfrac{1}{4} + \tfrac{1}{512}a^2 + \mathcal{O}(\mu)) = 0,$$

which admits a nonzero positive root $a = \sqrt{128} + \mathcal{O}(\mu)$ as $\mu \to 0$. These brief arguments justify bound (7) of the lemma. ■

Let us consider the reduced energy functional,

$$E_m(\varphi) = \int_0^\infty \left[\epsilon^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2 + \frac{\epsilon^2 m^2\varphi^2}{r^2} + (r^2-1)\varphi^2 + \frac{1}{2}\varphi^4\right] r\,\mathrm{d}r. \tag{10}$$

Euler–Lagrange equations for the reduced energy (10) yield the differential equations (3) for $m = 0$ and (4) for $m = 1$. The energy space is $X_m := \{\varphi \in H^1_{r,m}(\mathbb{R}_+) :\ r\varphi \in L^2_r(\mathbb{R}_+)\}$. The reduced energy functional (10) can be written in the form

$$E_m(\varphi) = Q_m(\varphi) + \tfrac{1}{2}\|\varphi\|_{L_r^4}^4,$$

where $Q_m(\varphi) = \langle\varphi, H(\epsilon)|_{X_m}\varphi\rangle_{L_r^2}$ is the quadratic form associated with the operator $H(\epsilon)$ in (5) restricted on the space X_m, that is, $H(\epsilon)|_{X_m}$ acts on functions in the form $f = \varphi(r)\,\mathrm{e}^{im\theta}$.

Because the smallest eigenvalue of $H(\epsilon)$ is $\lambda_{0,0}(\epsilon) = -1 + 2\epsilon$, the quadratic form $Q_0(\varphi)$ is positive for $\epsilon > \frac{1}{2}$ and sign indefinite for $\epsilon < \frac{1}{2}$. Therefore, the global minimizer of $E_0(\varphi)$ in X_0 is zero for $\epsilon > \frac{1}{2}$ and nonzero for $\epsilon < \frac{1}{2}$. By Ignat and Millot [15, Theorem 2.1], there exists a unique nonzero global minimizer of $E_0(\varphi)$ in X_0 for every $\epsilon \in (0, \frac{1}{2})$ and this minimizer $\varphi = \eta_\epsilon$ is the unique classical solution of the Euler–Lagrange equation (3).

Similarly, because the smallest eigenvalue of $H(\epsilon)|_{X_1}$ is $\lambda_{1,0} = -1 + 4\epsilon$ and using the similar arguments (see [36, Lemma 1]), we deduce the following proposition.

Proposition 1. For every $\epsilon \in (0, \frac{1}{4})$, there exists a unique nonzero global minimizer of $E_1(\varphi)$ in X_1, which yields a unique classical solution $\varphi = \psi_\epsilon$ of the differential equation (4). □

Let us now define the full energy functional,

$$E(u) = \int_{\mathbb{R}^2}\left(\epsilon^2|\nabla u|^2 + (|x|^2 - 1)|u|^2 + \frac{1}{2}|u|^4\right)\mathrm{d}x, \tag{11}$$

in the energy space $X = \{u \in H^1(\mathbb{R}^2) :\ |x|u \in L^2(\mathbb{R}^2)\}$.

Theorems 1.1(i) in [15] states that the vortex-free state $u = \eta_\epsilon(r)$ is the unique global minimizer of $E(u)$ for $\epsilon \in (0, \frac{1}{2})$, up to a complex multiplier of modulus 1. On the other hand, Proposition 1 implies that the vortex solution $u = \psi_\epsilon(r)\,\mathrm{e}^{\mathrm{i}\theta}$ is a critical point of $E(u)$ for $\epsilon \in (0, \frac{1}{4})$. This critical point is actually a saddle point if the functional $E(u)$ is not convex near the vortex solution. To study convexity of $E(u)$ near the vortex solution, we substitute

$$u(x, y) = \psi_\epsilon(r)\,\mathrm{e}^{\mathrm{i}\theta} + U(x, y),$$

and obtain the quadratic form associated with the perturbation vector $\mathbf{U} = [U, \bar{U}]^{\mathrm{T}}$:

$$E(u) - E(\psi_\epsilon\,\mathrm{e}^{\mathrm{i}\theta}) = \langle \mathbf{U}, \mathcal{H}(\epsilon)\mathbf{U}\rangle_{L^2} + \mathcal{O}(\|\mathbf{U}\|^3_{H^1}), \tag{12}$$

where $\mathcal{H}(\epsilon)$ is the matrix Schrödinger operator in the form:

$$\mathcal{H}(\epsilon) = \begin{bmatrix} -\epsilon^2(\partial_x^2 + \partial_y^2) + x^2 + y^2 - 1 + 2\psi_\epsilon^2 & \psi_\epsilon^2\,\mathrm{e}^{2\mathrm{i}\theta} \\ \psi_\epsilon^2\,\mathrm{e}^{-2\mathrm{i}\theta} & -\epsilon^2(\partial_x^2 + \partial_y^2) + x^2 + y^2 - 1 + 2\psi_\epsilon^2 \end{bmatrix}.$$

The matrix Schrödinger operator $\mathcal{H}(\epsilon)$ can be block-diagonalized in polar coordinates [5, 24]. Let us consider the eigenvalue problem $\mathcal{H}(\epsilon)\mathbf{U} = \epsilon\lambda\mathbf{U}$, where the spectral parameter is scaled as $\epsilon\lambda$ for convenience of notations. Using the decomposition in normal modes,

$$U(x, y) = \sum_{m\in\mathbb{Z}} V_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}, \quad \bar{U}(x, y) = \sum_{m\in\mathbb{Z}} W_m(r)\,\mathrm{e}^{\mathrm{i}m\theta},$$

we obtain an uncoupled eigenvalue problem for components (V_m, W_{m-2}):

$$H_m(\epsilon)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \epsilon\lambda \begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z}, \tag{13}$$

where

$$H_m(\epsilon) = \begin{bmatrix} -\epsilon^2\Delta_m + r^2 - 1 + 2\psi_\epsilon^2 & \psi_\epsilon^2 \\ \psi_\epsilon^2 & -\epsilon^2\Delta_{m-2} + r^2 - 1 + 2\psi_\epsilon^2 \end{bmatrix}. \tag{14}$$

We are particularly interested in negative and zero eigenvalues of operators $H_m(\epsilon)$ for $m \in \mathbb{Z}$. The count of negative and zero eigenvalues is given in the following lemma.

Lemma 2. There exists an $\epsilon_0 \in (0, \frac{1}{4})$ such that for every $\epsilon \in (\epsilon_0, \frac{1}{4})$, there exists exactly one negative eigenvalue $\lambda_0(\epsilon)$ of the spectral problems (13), which has algebraic multiplicity two and is associated to the eigenvectors of $H_2(\epsilon)$ and $H_0(\epsilon)$. Moreover, λ_0 is a

C^1 function of ϵ in $(\epsilon_0, \frac{1}{4})$ satisfying

$$\lim_{\epsilon \uparrow \frac{1}{4}} \lambda_0(\epsilon) = -2 \quad \text{and} \quad \lim_{\epsilon \uparrow \frac{1}{4}} \lambda_0'(\epsilon) = -16. \tag{15}$$

The zero eigenvalue of the spectral problems (13) is simple and is associated with the eigenvector of $H_1(\epsilon)$. All other eigenvalues of the spectral problems (13) are strictly positive. □

Proof. Proposition 4.3 in [24] states that eigenvalues of $H_m(\epsilon)$ are strictly positive for $m \geqslant 3$ (and $m \leqslant -1$ by symmetry). For $m = 1$, the spectral problem (13) reduces further to the uncoupled eigenvalue problems for scalar Schrödinger operators:

$$L_+(\epsilon)(V_1 + W_{-1}) = \epsilon\lambda(V_1 + W_{-1}),$$
$$L_-(\epsilon)(V_1 - W_{-1}) = \epsilon\lambda(V_1 - W_{-1}),$$

where

$$L_+(\epsilon) = -\epsilon^2 \Delta_1 + r^2 - 1 + 3\psi_\epsilon^2,$$
$$L_-(\epsilon) = -\epsilon^2 \Delta_1 + r^2 - 1 + \psi_\epsilon^2.$$

Since $L_-(\epsilon)\psi_\epsilon = 0$ and $\psi_\epsilon(r) > 0$ for $r > 0$, Sturm's Oscillation Theorem implies that the operator $L_-(\epsilon)$ has a simple zero eigenvalue and the rest of its spectrum is strictly positive. Since $L_+(\epsilon) = L_-(\epsilon) + 2\psi_\epsilon^2$, the operator $L_+(\epsilon)$ is strictly positive.

By the arguments above, negative and additional zero eigenvalues of the spectral problems (13) may only occur for $m = 2$ (and $m = 0$ by symmetry). For $\epsilon = \frac{1}{4}$, there exists only one negative eigenvalue $-\frac{1}{2}$ of $H_2(\epsilon)$, which corresponds to $\lambda_0 = -2$ and the eigenvector

$$\begin{bmatrix} V_2 \\ W_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \varphi_0 \end{bmatrix}, \quad \varphi_0(r) = e^{-2r^2}. \tag{16}$$

The rest of the spectrum of $H_2(\epsilon)$ is strictly positive for $\epsilon = \frac{1}{4}$. The simple negative eigenvalue $\lambda_0(\epsilon)$ persists as a C^1 function of ϵ for $\epsilon < \frac{1}{4}$ with small $|\epsilon - \frac{1}{4}|$ by the asymptotic perturbation theory [21, Section 8.2.3]. Using the asymptotic expansion (7) in Lemma 1, we write

$$H_2(\epsilon) = \begin{bmatrix} -\frac{1}{16}\Delta_2 + r^2 - 1 & 0 \\ 0 & -\frac{1}{16}\Delta_0 + r^2 - 1 \end{bmatrix} + \mu \begin{bmatrix} \Delta_2 + 256\psi_0^2 & 128\psi_0^2 \\ 128\psi_0^2 & \Delta_0 + 256\psi_0^2 \end{bmatrix} + \mathcal{O}(\mu^2),$$

where $\mu = \frac{1}{16} - \epsilon^2$ is a small positive parameter and $\psi_0(r) = r\,\mathrm{e}^{-2r^2}$. Using the orthogonal decomposition,

$$\begin{bmatrix} V_2 \\ W_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \varphi_0 \end{bmatrix} + \mu \begin{bmatrix} \tilde{V}_2 \\ \tilde{W}_0 \end{bmatrix}, \quad \langle \varphi_0, \tilde{W}_0 \rangle_{L^2_r} = 0,$$

and the rescaling of the eigenvalue $\lambda = \mu\tilde{\lambda}$, we obtain that for small $\mu \in \mathbb{R}$ there is a positive constant C such that

$$\|\tilde{V}_2\|_{H^2_{r,2}} + \|\tilde{W}_0\|_{H^2_{r,0}} \leqslant C|\mu|$$

if and only if $\tilde{\lambda}$ is found from the equation,

$$\epsilon\tilde{\lambda}\|\varphi_0\|^2_{L^2_r} = \langle \varphi_0, (\Delta_0 + 256\psi_0^2)\varphi_0 \rangle_{L^2_r} + \mathcal{O}(\mu).$$

Since $\frac{\mathrm{d}}{\mathrm{d}\epsilon} = -2\epsilon \frac{\mathrm{d}}{\mathrm{d}\mu}$ and $\lambda_0(\epsilon)$ is a C^1 function for $\epsilon < \frac{1}{4}$, this perturbation theory for the eigenvector (16) yields

$$\lim_{\epsilon \uparrow \frac{1}{4}} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\epsilon\lambda_0(\epsilon)) = -\frac{\langle \varphi_0, (\Delta_0 + 256\psi_0^2)\varphi_0 \rangle_{L^2_r}}{2\|\varphi_0\|^2_{L^2_r}} = -6,$$

which is equivalent to (15). The proof of the lemma is complete. ■

By Lemma 2, it follows that the vortex solution is indeed a saddle point of the energy functional $E(u)$ in the energy space X with exactly two directions, for which $E(u) < E(\psi_\epsilon\,\mathrm{e}^{\mathrm{i}\theta})$. Note that in the Thomas–Fermi limit $\epsilon \to 0$, [34, Lemma 2] (similar computations are reported in [2, 4]) shows that

$$E(u_{x_0,y_0}) - E(\psi_\epsilon\,\mathrm{e}^{\mathrm{i}\theta}) = -\pi\epsilon\omega_a(\epsilon)(x_0^2 + y_0^2)(1 + \mathcal{O}(\epsilon^{1/3} + x_0^2 + y_0^2)), \tag{17}$$

where u_{x_0,y_0} is a vortex solution $\psi_\epsilon(r)\,\mathrm{e}^{\mathrm{i}\theta}$ shifted from $(0,0) \in \mathbb{R}^2$ to the point $(x_0, y_0) \in \mathbb{R}^2$ for small (x_0, y_0). The coefficient $\omega_a(\epsilon)$ was found to satisfy the asymptotic expansion,

$$\omega_a(\epsilon) = 2\epsilon \log\left(\frac{1}{\epsilon}\right) + \mathrm{O}(\epsilon) \quad \text{as } \epsilon \to 0. \tag{18}$$

Finally, we address the spectral stability of symmetric vortices of charge 1 in the Gross–Pitaevskii equation (1). Spectral stability is determined by eigenvalues of the non-self-adjoint spectral problem:

$$H_m(\epsilon) \begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \epsilon\gamma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z}, \tag{19}$$

where γ is an eigenfrequency of the perturbation of the vortex. The vortex is spectrally stable if no γ exists with $\mathrm{Im}(\gamma) \neq 0$. Compared with the standard formulation of spectral stability, the spectrum of γ is rotated on the complex plane along 90°.

We shall prove that the eigenvalues γ of the spectral problem (19) are all real and semi-simple (i.e., they have equal algebraic and geometric multiplicities) except for the double zero eigenvalue. The main attention is drawn to a pair of real eigenvalues for $m = 2$ and $m = 0$ that correspond to the eigenvectors with negative values of the quadratic form associated with the operators $H_2(\epsilon)$ and $H_0(\epsilon)$. These eigenvalues are known as the eigenvalues of negative Krein signature [5, 32] (see Remark 1). The corresponding result is formulated in the following lemma.

Lemma 3. There exists a $\epsilon_0 \in (0, \frac{1}{4})$ such that for every $\epsilon \in (\epsilon_0, \frac{1}{4})$ and $m \in \mathbb{Z}$, the spectral problem (19) admits only real eigenvalues γ of equal algebraic and geometric multiplicities, in addition to the double zero eigenvalue for $m = 1$.

The smallest nonzero eigenvalue for $m = 2$ is $\gamma = +\omega_0(\epsilon)$ and for $m = 0$ is $\gamma = -\omega_0(\epsilon)$, where $\omega_0(\epsilon) > 0$. These eigenvalues are simple and correspond to the eigenvectors

$$\mathbf{V}_+(\epsilon) = \begin{bmatrix} V_2 \\ W_0 \end{bmatrix}, \quad \mathbf{V}_-(\epsilon) = \begin{bmatrix} V_0 \\ W_{-2} \end{bmatrix} \tag{20}$$

such that $V_2 = W_{-2}$, $V_0 = W_0$, $\|W_0\|_{L^2_r} > \|V_2\|_{L^2_r}$, and

$$\langle \mathbf{V}_+(\epsilon), H_2(\epsilon)\mathbf{V}_+(\epsilon)\rangle_{L^2_r} = \langle \mathbf{V}_-(\epsilon), H_0(\epsilon)\mathbf{V}_-(\epsilon)\rangle_{L^2_r} < 0. \tag{21}$$

Moreover, ω_0 is a C^1 function of ϵ in $(\epsilon_0, \frac{1}{4})$ satisfying

$$\lim_{\epsilon \uparrow \frac{1}{4}} \omega_0(\epsilon) = 2 \quad \text{and} \quad \lim_{\epsilon \uparrow \frac{1}{4}} \omega_0'(\epsilon) = 8. \tag{22}$$

The quadratic form associated with operators $H_m(\epsilon)$ is strictly positive for the eigenvectors corresponding to any other eigenvalue γ of the spectral problems (19). □

Proof. The result follows from the negative index theory [5, 32] and the count of negative eigenvalues of operators $H_m(\epsilon)$ in Lemma 2 (see also [33, Chapter 4.2]). Because operators $H_m(\epsilon)$ for $m \geqslant 3$ and $m \leqslant -1$ are strictly positive, [5, Theorem 6] implies that the spectral stability problem (19) admits only real semi-simple eigenvalues and the quadratic form associated with these operators is strictly positive at the corresponding eigenvectors.

For $m=1$, the algebraic multiplicity of the zero eigenvalue of $H_1(\epsilon)$ is at least two, because of the existence of the eigenvector $\mathbf{V}_1$ and the generalized eigenvector $\tilde{\mathbf{V}}_1$ in the form

$$\mathbf{V}_1(\epsilon)=\begin{bmatrix}\psi_\epsilon\\-\psi_\epsilon\end{bmatrix},\quad \tilde{\mathbf{V}}_1(\epsilon)=\begin{bmatrix}L_+(\epsilon)^{-1}\psi_\epsilon\\L_+(\epsilon)^{-1}\psi_\epsilon\end{bmatrix},$$

where $L_+(\epsilon)$ is strictly positive and thus invertible. We need to show that the algebraic multiplicity is exactly two, which is equivalent to the condition $\langle\psi_\epsilon, L_+(\epsilon)^{-1}\psi_\epsilon\rangle_{L^2_r}\neq 0$. To show this constraint, we let $\Psi_\mu:\mathbb{R}_+\to\mathbb{R}$ be a classical solution of the vortex equation

$$\Delta_1\Psi_\mu+(\mu-r^2-\Psi_\mu^2)\Psi_\mu=0,\quad \Psi_\mu(r)>0,\ r\in\mathbb{R}_+,\ \mu>0. \tag{23}$$

A scaling transformation for Equation (23) implies that

$$\Psi_\mu(r)=\sqrt{\mu}\,\psi_\epsilon(R),\quad \epsilon=\frac{1}{\mu},\quad R=\frac{r}{\sqrt{\mu}}, \tag{24}$$

where ψ_ϵ is a classical solution of Equation (4), existence of which is stated in Proposition 1.

Differentiating equation (23) with respect to μ, we obtain

$$(-\Delta_1+r^2-\mu+3\Psi_\mu^2)\frac{\partial}{\partial\mu}\Psi_\mu=\Psi_\mu,$$

which yields by the scaling transformation (24):

$$(L_+(\epsilon)^{-1}\psi_\epsilon)(R)=\sqrt{\mu}\frac{\partial}{\partial\mu}\sqrt{\mu}\,\psi_{\frac{1}{\mu}}\left(\frac{r}{\sqrt{\mu}}\right)=\frac{1}{2}\psi_\epsilon(R)-\frac{R}{2}\psi_\epsilon'(R)-\epsilon\frac{\partial}{\partial\epsilon}\psi_\epsilon(R).$$

As a result, we have

$$\langle\psi_\epsilon, L_+(\epsilon)^{-1}\psi_\epsilon\rangle_{L^2_r}=\|\psi_\epsilon\|^2_{L^2_r}-\frac{\epsilon}{2}\frac{\partial}{\partial\epsilon}\|\psi_\epsilon\|^2_{L^2_r}.$$

It follows from Lemma 1 that

$$\|\psi_\epsilon\|^2_{L^2_r}=(8-128\epsilon^2)\|\psi_0\|^2_{L^2_r}+\mathcal{O}(8-128\epsilon^2)^2\quad\text{as}\quad \epsilon\uparrow\frac{1}{4},$$

where $\psi_0(r)=re^{-2r^2}$. Therefore, $\|\psi_\epsilon\|^2_{L^2_r}$ is a decreasing function of ϵ near $\epsilon=\frac{1}{4}$ so that $\langle\psi_\epsilon, L_+(\epsilon)^{-1}\psi_\epsilon\rangle_{L^2_r}>0$ at least for small $|\epsilon-\frac{1}{4}|$. Hence the algebraic multiplicity of the zero eigenvalue of $H_1(\epsilon)$ is two.

Finally, for $m=2$ and $m=0$, there is only one negative eigenvalue of operators $H_2(\epsilon)$ and $H_0(\epsilon)$ and no zero eigenvalues for small $|\epsilon-\frac{1}{4}|$ (Lemma 2). At $\epsilon=\frac{1}{4}$, the spectral stability problem (19) admits a double eigenvalue $\gamma=+2$ for $m=2$ and a double eigenvalue $\gamma=-2$ for $m=0$ (by symmetry).

We shall prove that the double eigenvalue $\gamma=2$ of the spectral problem (19) for $m=2$ splits for small $|\epsilon-\frac{1}{4}|\neq 0$ into two real eigenvalues, the smallest of which is denoted as $\gamma=\omega_0(\epsilon)$. From the spectrum of the Schrödinger operator $H(\epsilon)$ in (6), we know that the double eigenvalue $\gamma = 2$ of the spectral problem (19) for $m = 2$ and $\epsilon = \frac{1}{4}$ corresponds to the following two eigenvectors:

$$\mathbf{V}_0 = \begin{bmatrix} 0 \\ \chi_2 \end{bmatrix}, \quad \tilde{\mathbf{V}}_0 = \begin{bmatrix} \chi_1 \\ 0 \end{bmatrix}, \quad \chi_1(r) = r^2\, \mathrm{e}^{-2r^2}, \quad \chi_2(r) = \mathrm{e}^{-2r^2}. \tag{25}$$

Therefore, the double eigenvalue $\gamma = 2$ is semi-simple. By the asymptotic perturbation theory [21, Section 8.2.3], the semi-simple eigenvalues persist as real eigenvalues for $\epsilon < \frac{1}{4}$ with small $|\epsilon - \frac{1}{4}|$ and are C^1 functions of ϵ. The actual values of these eigenvalues for small $|\epsilon - \frac{1}{4}|$ can be approximated from the asymptotic expansions similar to the ones used in the proof of Lemma 2. Using the expansion of $H_2(\epsilon)$ in $\mu = \frac{1}{16} - \epsilon^2$ and the decomposition

$$V_2 = c_1\chi_1(r) + \mu\tilde{V}_2, \quad W_0 = c_2\chi_2(r) + \mu\tilde{W}_0, \quad \gamma = \mu\tilde{\gamma}, \tag{26}$$

we compute the projection equations

$$\epsilon\tilde{\gamma}\begin{bmatrix} \|\chi_1\|^2_{L^2_r} c_1 \\ -\|\chi_2\|^2_{L^2_r} c_2 \end{bmatrix} = \begin{bmatrix} \langle\chi_1, (\Delta_2 + 256\psi_0^2)\chi_1\rangle_{L^2_r} & 128\langle\chi_1, \psi_0^2\chi_2\rangle_{L^2_r} \\ 128\langle\chi_2, \psi_0^2\chi_1\rangle_{L^2_r} & \langle\chi_2, (\Delta_0 + 256\psi_0^2)\chi_2\rangle_{L^2_r} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \mathcal{O}(\mu).$$

Truncating this expansion, using $\frac{\mathrm{d}}{\mathrm{d}\epsilon} = -2\epsilon\frac{\mathrm{d}}{\mathrm{d}\mu}$, and computing the inner products explicitly, we obtain

$$\lim_{\epsilon\uparrow\frac{1}{4}} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\epsilon\gamma(\epsilon))\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{27}$$

Two eigenvalues $\{2, 4\}$ of the reduced eigenvalue problem (27) give two slopes of the eigenvalues γ as a function of ϵ:

$$\lim_{\epsilon\uparrow\frac{1}{4}} \gamma'(\epsilon) = 0: \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \tag{28}$$

and

$$\lim_{\epsilon\uparrow\frac{1}{4}} \gamma'(\epsilon) = 8: \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \tag{29}$$

The first eigenvalue remains at $\gamma = 2$ for all values of $\epsilon < \frac{1}{4}$ by the symmetry of the spectral problem (19) for $m = 2$ (see [24, Appendix] or [22, Equation (28)]). The second eigenvalue detaches from the first eigenvalue as a simple eigenvalue $\gamma = \omega_0(\epsilon) < 2$ for $\epsilon < \frac{1}{4}$. This eigenvalue corresponds to the eigenvector $\mathbf{V}_+(\epsilon)$ in (20). We compute

$$\epsilon = \frac{1}{4}: \quad \|V_2\|^2_{L^2_r} = c_1^2\|\chi_1\|^2_{L^2_r} = \frac{1}{16}, \quad \|W_0\|^2_{L^2_r} = c_2^2\|\chi_2\|^2_{L^2_r} = \frac{1}{8},$$

and hence $\|W_0\|_{L^2_r} > \|V_2\|_{L^2_r}$ and the condition (21) are satisfied for $\epsilon = \frac{1}{4}$ and for small $|\epsilon - \frac{1}{4}|$ by continuity. The symmetry of $H_2(\epsilon)$ and $H_0(\epsilon)$ implies that $V_2 = W_{-2}$ and $V_0 = W_0$.

By Chugunova and Pelinovsky [5, Theorem 6] and Lemma 2, other eigenvalues of the spectral stability problem (19) for $m=2$ and $m=0$ are real and semi-simple, whereas the quadratic form associated with these operators is strictly positive at the corresponding eigenvectors. To show that $\omega_0(\epsilon)$ is the smallest eigenvalue of the spectral problem (19) for $m=2$ and small $|\epsilon - \frac{1}{4}|$, we recall again the spectrum of the Schrödinger operator $H(\epsilon)$ in (6). For $\epsilon = \frac{1}{4}$, the only eigenvalue that has the same absolute value as ω_0 is the simple eigenvalue $\gamma = -2$ that correspond to the eigenvector

$$\hat{\mathbf{V}}_0 = \begin{bmatrix} 0 \\ \hat{\chi}_2 \end{bmatrix}, \quad \hat{\chi}_2(r) = (4r^2 - 1)\,\mathrm{e}^{-2r^2}.$$

The same perturbation theory gives

$$\lim_{\epsilon \uparrow \frac{1}{4}} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(\epsilon\gamma) = \frac{\langle \hat{\chi}_2, (\Delta_0 + 256\psi_0^2)\hat{\chi}_2 \rangle_{L_r^2}}{2\|\hat{\chi}_2\|_{L_r^2}^2} = -2 \quad \Rightarrow \quad \lim_{\epsilon \uparrow \frac{1}{4}} \gamma'(\epsilon) = 0.$$

Hence, this eigenvalue also stays at $\gamma = -2$ by the symmetry described above and $\gamma = \omega_0(\epsilon)$ is the smallest eigenvalue for $\epsilon < \frac{1}{4}$ of the spectral problem (19) for $m=2$. The proof of the lemma is complete. ■

Remark 1. The eigenvalue $\omega_0(\epsilon)$ of the spectral problem (19) for $m=2$ has the negative Krein signature [19, 32], which is determined by the sign of the symplectic 2-form

$$[\mathbf{V}_+(\epsilon), \mathbf{V}_+(\epsilon)] := \|V_2\|_{L_r^2}^2 - \|W_0\|_{L_r^2}^2 \tag{30}$$

and it is the only positive real eigenvalue of the spectral problems (19) with a negative Krein signature. This symplectic 2-form vanishes when a positive eigenvalue of negative Krein signature coalesces with another eigenvalue of positive Krein signature to become the defective eigenvalue that leads ultimately to the instability bifurcations [24]. If the eigenvalue $\omega_0(\epsilon)$ remains the smallest eigenvalue for $m=2$ for $\epsilon \in (0, \frac{1}{4})$ and moves to 0 as $\epsilon \to 0$, such a coalescence does not occur and $[\mathbf{V}_+(\epsilon), \mathbf{V}_+(\epsilon)] < 0$ remains for all $\epsilon \in (0, \frac{1}{4})$. □

Numerical approximations of eigenvalues of the spectral stability problems (19) were performed in [24, Figure 4; 34, Figure 2]. These figures illustrate that all eigenvalues γ are purely real. Among all real eigenvalues, the pair of eigenvalues $\pm\omega_0(\epsilon)$ with negative Krein signature is distinctive because this pair corresponds to the smallest nonzero eigenvalues of the spectral stability problems (19) for sufficiently small ϵ.

This pair of eigenvalues is associated with the *precessional* frequency of the symmetric vortex of charge 1 misplaced from $(0,0) \in \mathbb{R}^2$ to the point $(x_0, y_0) \in \mathbb{R}^2$ for small (x_0, y_0). Using the Rayleigh–Ritz variational method based on the computation (17), it was found in [34] that $\omega_0(\epsilon) \approx \omega_a(\epsilon)$ as $\epsilon \to 0$, where $\omega_a(\epsilon)$ satisfies the asymptotic expansion (18).

We will show that the eigenvalue $\omega_0(\epsilon)$ also determines bifurcations of asymmetric vortices from symmetric vortices of charge 1 in rotating Bose–Einstein condensates. Although the results of Lemmas 2 and 3 are applicable for small $|\epsilon - \frac{1}{4}|$, we will fix the value of ϵ arbitrarily in $(\epsilon_0, \frac{1}{4})$ without further restrictions on ϵ_0. Note that because ϵ is fixed, we will not indicate the explicit dependence of the eigenvalue ω_0 from ϵ in the remainder of this article.

3 Rotating Coordinates and the Symmetry-Breaking Bifurcation

We look at the existence of vortex solutions of the Gross–Pitaevskii equation (1) in the rotating coordinate frame and use the variables,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{31}$$

where $\omega \in \mathbb{R}$ is the rotation frequency in the counterclockwise direction, which is favorable to the vortex compared with the rotation in the clockwise direction. In new coordinates, the Gross–Pitaevskii equation (1) takes the form:

$$\mathrm{i}\epsilon u_t + \epsilon^2 (u_{\xi\xi} + u_{\eta\eta}) + (1 - \xi^2 - \eta^2 - |u|^2)u - \mathrm{i}\epsilon\omega(\xi u_\eta - \eta u_\xi) = 0. \tag{32}$$

In the polar coordinates (r, θ) on the plane $(\xi, \eta) \in \mathbb{R}^2$ (note that we use the same notations for polar coordinates for simplicity), the solution in the vortex form $u = \varphi_{\omega,\epsilon}(r)\, \mathrm{e}^{\mathrm{i}\theta}$ satisfies the differential equation:

$$\epsilon^2 \Delta_1 \varphi_{\omega,\epsilon} + (1 + \epsilon\omega - r^2 - \varphi_{\omega,\epsilon}^2)\varphi_{\omega,\epsilon} = 0, \quad \varphi_{\omega,\epsilon}(r) > 0, \quad r \in \mathbb{R}_+. \tag{33}$$

Assuming that $1 + \epsilon\omega > 0$ and using the scaling transformation,

$$\varphi_{\omega,\epsilon}(r) = \sqrt{1 + \epsilon\omega}\,\psi_\nu(R), \quad r = \sqrt{1 + \epsilon\omega}\,R, \quad \nu = \frac{\epsilon}{1 + \epsilon\omega}, \tag{34}$$

we can rewrite the differential equation (33) in the one-parameter form,

$$\nu^2 \left(\frac{\mathrm{d}^2 \psi_\nu}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}\psi_\nu}{\mathrm{d}R} - \frac{\psi_\nu}{R^2} \right) + (1 - R^2 - \psi_\nu^2)\psi_\nu = 0, \quad \psi_\nu(R) > 0, \ \ R \in \mathbb{R}_+, \tag{35}$$

which is nothing but Equation (4) with the correspondence $\nu = \epsilon$ and $R = r$. Existence of vortices of charge 1 for $\nu \in (0, \frac{1}{4})$ is guaranteed by Proposition 1. The local bifurcation at $\nu = \frac{1}{4}$ corresponds to

$$\epsilon = \left. \frac{\nu}{1 - \nu\omega} \right|_{\nu = \frac{1}{4}} = \frac{1}{4 - \omega},$$

and we require $\omega < 4$ to have the interval $[0, \frac{1}{4}]$ for ν be mapped to the interval $[0, \frac{1}{4-\omega}]$ for ϵ.

To study stability and bifurcations of symmetric vortices in the rotating coordinate frame, we introduce a linearization of the Gross–Pitaevskii equation (32). We substitute the decomposition,

$$u(\xi, \eta, t) = \varphi_{\omega,\epsilon}(r)\, e^{i\theta} + U(\xi, \eta, t),$$

and obtain the linearized evolution problem by neglecting the quadratic terms in U,

$$i\epsilon U_t + \epsilon^2 (U_{\xi\xi} + U_{\eta\eta}) + (1 - \xi^2 - \eta^2 - 2\varphi_{\omega,\epsilon}^2) U - \varphi_{\omega,\epsilon}^2\, e^{2i\theta}\, \bar{U} - i\epsilon\omega(\xi U_\eta - \eta U_\xi) = 0. \tag{36}$$

Using the normal coordinates,

$$U(\xi, \eta, t) = \sum_{m \in \mathbb{Z}} V^{(m)}(r)\, e^{im\theta}\, e^{-i\sigma t}, \quad \bar{U}(\xi, \eta, t) = \sum_{m \in \mathbb{Z}} W^{(m)}(r)\, e^{im\theta}\, e^{-i\sigma t},$$

we obtain uncoupled eigenvalue problems for components $(V^{(m)}, W^{(m-2)})$,

$$H_{\omega,\epsilon}^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \epsilon\sigma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix}, \quad m \in \mathbb{Z}, \tag{37}$$

where

$$H_{\omega,\epsilon}^{(m)} = \begin{bmatrix} -\epsilon^2 \Delta_m + r^2 - 1 - \epsilon\omega m + 2\varphi_{\omega,\epsilon}^2 & \varphi_{\omega,\epsilon}^2 \\ \varphi_{\omega,\epsilon}^2 & -\epsilon^2 \Delta_{m-2} + r^2 - 1 + \epsilon\omega(m-2) + 2\varphi_{\omega,\epsilon}^2 \end{bmatrix}. \tag{38}$$

The eigenvalue problems (37) determine the spectral stability of the symmetric vortex with respect to the time-dependent perturbations. To consider the bifurcation of the symmetric vortex, we need to study the linearization of the time-independent version of the Gross–Pitaevskii equation (32), which results in the self-adjoint version of the spectral problems (37),

$$H_{\omega,\epsilon}^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \epsilon\lambda \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix}, \quad m \in \mathbb{Z}. \tag{39}$$

Using variables (34) and the representation

$$V^{(m)}(r) = V_m(R), \quad W^{(m-2)}(r) = W_{m-2}(R), \quad m \in \mathbb{Z},$$

we rewrite the eigenvalue problems (37) and (39) in the equivalent forms:

$$H_m(\nu)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu(\sigma + \omega(m-1))\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z} \tag{40}$$

and

$$H_m(\nu)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu\lambda\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} + \nu\omega(m-1)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z}, \tag{41}$$

where $H_m(\nu)$ is given by (14) for $\epsilon = \nu$ and $r = R$.

Let us denote $\gamma = \sigma + \omega(m-1)$ and rewrite the spectral stability problems (40) in the form:

$$H_m(\nu)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu\gamma\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z}. \tag{42}$$

The spectral problems (42) coincide with the spectral problems (19) with the correspondence $\epsilon = \nu$. By Lemma 3, all eigenvalues γ of the spectral problems (42) are real and semi-simple except for the double zero eigenvalue. Therefore, all eigenvalues σ of the spectral stability problems (37) are real and semi-simple, except for the double zero eigenvalue. This result implies the spectral stability of symmetric vortices of charge 1 for all values of ω, for which the transformation (34) makes sense, and hence proves the following.

Proposition 2. Fix $\omega < 4$. There exists $\epsilon_0 \in (0, \frac{1}{4})$ such that for every $\epsilon \in (\frac{\epsilon_0}{1-\epsilon_0\omega}, \frac{1}{4-\omega})$, the symmetric vortex of charge 1 is spectrally stable in the sense that all eigenvalues σ of the spectral stability problems (37) are real and semi-simple, except for the double zero eigenvalue. □

We can now address a possibility of bifurcations of the symmetric vortex $u = \varphi_{\omega,\epsilon}(r)\,e^{i\theta}$ in the rotating coordinate frame (31). These bifurcations are determined by zero eigenvalues λ of the self-adjoint eigenvalue problems (39). Equivalently, these bifurcations are determined by zero eigenvalues λ of the self-adjoint eigenvalue problems (41).

Lemma 4. Let $\omega_0 > 0$ be the eigenvalue defined by Lemma 3 with the correspondence $\epsilon \equiv \nu$. There exists $\epsilon_0 \in (0, \frac{1}{2})$ such that for every $\epsilon \in (\epsilon_0, \frac{1}{2})$, the eigenvalue problem (39) for $m = 2$ admits a zero eigenvalue $\lambda = 0$ at $\omega = \omega_0$. Moreover, the smallest eigenvalue λ is a C^1 function of ω near ω_0 with $\lambda'(\omega_0) > 0$. □

Proof. Existence of zero eigenvalue $\lambda = 0$ in the self-adjoint eigenvalue problem (41) for any $m \geqslant 2$ is equivalent to the existence of a positive eigenvalue $\gamma = \omega(m-1)$ in the non-self-adjoint eigenvalue problem (42). By Lemma 3, the spectral problem (42) for $m = 2$ has the smallest nonzero eigenvalue $\gamma = \omega_0(\nu) > 0$ for $\nu < \frac{1}{4}$ at least for small $|\nu - \frac{1}{4}|$. It follows from (22) that

$$\lim_{\nu \uparrow \frac{1}{4}} \omega_0(\nu) = 2 \quad \text{and} \quad \lim_{\nu \uparrow \frac{1}{4}} \omega_0'(\nu) = 8. \tag{43}$$

Using the inverse transformation (34) with $\omega = \omega_0(\nu)$, that is,

$$\epsilon(\nu) = \frac{\nu}{1 - \nu\omega_0(\nu)},$$

we obtain

$$\lim_{\nu \uparrow \frac{1}{4}} \epsilon(\nu) = \frac{1}{2} \quad \text{and} \quad \lim_{\nu \uparrow \frac{1}{4}} \epsilon'(\nu) = 6. \tag{44}$$

The positive slope of ϵ versus ν tells us that the one-sided neighborhood $\nu < \frac{1}{4}$ is located for $\epsilon < \frac{1}{2}$ if $\omega = \omega_0(\nu)$. Therefore, there is $\epsilon_0 \in (0, \frac{1}{2})$ such that the self-adjoint eigenvalue problem (39) for $m = 2$ and $\omega = \omega_0(\nu)$, admits a zero eigenvalue $\lambda = 0$ for every $\epsilon \in (\epsilon_0, \frac{1}{2})$. Once this is established, we shall now omit the argument ν in the notation for ω_0.

Next we show that the eigenvalue λ of the eigenvalue problem (39) for $m = 2$ that crosses zero at $\omega = \omega_0$ becomes positive for $\omega > \omega_0$. By the asymptotic perturbation theory [21, Section 8.2.3], λ is a C^1 function of ω near ω_0 because the zero eigenvalue at $\omega = \omega_0$ is simple. Differentiating the rescaled eigenvalue problem (41) for $m = 2$ with respect to ω at a fixed $\nu < \frac{1}{4}$, we obtain

$$H_2(\nu) \frac{d}{d\omega} \begin{bmatrix} V_2 \\ W_0 \end{bmatrix} = \nu\lambda \frac{d}{d\omega} \begin{bmatrix} V_2 \\ W_0 \end{bmatrix} + \nu\omega \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{d}{d\omega} \begin{bmatrix} V_2 \\ W_0 \end{bmatrix} + \nu \frac{d\lambda}{d\omega} \begin{bmatrix} V_2 \\ W_0 \end{bmatrix} + \nu \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ W_0 \end{bmatrix}.$$

Projecting this derivative problem at $\lambda = 0$ and $\omega = \omega_0$ to the eigenvector $[V_2, W_0]$ of the eigenvalue problem (41), we obtain

$$\lambda'(\omega_0)(\|W_0\|^2_{L^2_r} + \|V_2\|^2_{L^2_r}) = \|W_0\|^2_{L^2_r} - \|V_2\|^2_{L^2_r} > 0, \tag{45}$$

where the positivity follows from Lemma 3. Hence, $\lambda'(\omega_0) > 0$ and the assertion of the lemma is proved. ■

Remark 2. On the basis of numerical evidences, we conjecture that $\epsilon_0 = 0$ in Lemmas 2–4. This statement has been clearly illustrated in the numerical studies of [24, 29, 30] but the mathematical proof of this conjecture is beyond the scope of this article. □

Remark 3. Numerical approximations show that all other eigenvalues of the spectral problem (42) for $m = 2$ and $m \geqslant 3$ are located for $\gamma = \omega(m-1) \geqslant 1$ as $\nu \to 0$ (see [34, Figure 2]). These eigenvalues produce a countable set of possible bifurcations in the self-adjoint eigenvalue problem (39) if $\epsilon = \frac{\nu}{1-\nu\omega} > 0$. These additional bifurcations may be related to the fact that for larger rotation frequencies ω, multiple vortex configurations become local and global minimizers of the energy (see [4, Figure 8; 36, Section 3]). Although we do not study these possible bifurcations in this paper, we note that if these bifurcations occur, they are related with eigenvalues of positive Krein signature that become negative when the frequency parameter ω is increased. □

We shall mention two relevant results in connection to Lemma 4. First, the critical frequency $\omega_0 = 2$ in the limit of weak interactions $\nu \uparrow \frac{1}{4}$ coincides with the critical frequency of the symmetry-breaking bifurcation studied by Seiringer [36, Section 7]. By Seiringer [36, Theorem 4] (modified in our notations), for all $\omega \in (0, 2)$, there is an $\epsilon_0 \in (0, \frac{1}{2})$ such that for all $\epsilon \in (0, \epsilon_0)$, the radially symmetric vortex cannot be a global minimizer of the energy. By Seiringer [36, Theorem 8], all vortices of charge n with $n \geqslant 10$ are orbitally unstable (in fact, the author conjectured that this theorem remains true for all $n \geqslant 2$).

The other relevant result is a connection between the sign of $\lambda'(\omega_0)$ and the Krein signature (30) established in (45). The linear interpolation between self-adjoint and skew-adjoint spectral problems obtained in the rescaled eigenvalue problem (41) was considered long ago by Krein and Ljubarskii [25], where motion of simple eigenvalues was found to be connected to the sign of the Krein signature.

Let us rewrite the full energy functional (11) in the rotating coordinate frame (31),

$$E_{\omega,\epsilon}(u) = \int_{\mathbb{R}^2} \left[\epsilon^2 |\nabla u|^2 + (|\xi|^2 - 1)|u|^2 + \frac{1}{2}|u|^4 + \frac{\mathrm{i}\epsilon\omega}{2}\xi(\bar{u}u_\eta - u\bar{u}_\eta) - \frac{\mathrm{i}\epsilon\omega}{2}\eta(\bar{u}u_\xi - u\bar{u}_\xi) \right] \mathrm{d}\xi. \tag{46}$$

The symmetric vortex $u = \varphi_{\omega,\epsilon}(r)\,\mathrm{e}^{\mathrm{i}\theta}$ is a critical point of the energy $E_{\omega,\epsilon}(u)$. Eigenvalues of self-adjoint operators $H^{(m)}_{\omega,\epsilon}$ determine convexity of $E_{\omega,\epsilon}(u)$ at the critical point. By Lemmas 2 and 4, the symmetric vortex of charge 1 is a saddle point of $E_\omega(u)$ for $\omega < \omega_0$

but becomes a local minimizer of $E_\omega(u)$ for $\omega > \omega_0$ near ω_0. Applying the orbital stability theory [14] (see also [33, Chapter 4.4.2]), we obtain the following proposition.

Proposition 3. For every $\epsilon \in (\epsilon_0, \frac{1}{2})$, let ω_0 be the bifurcation value in Lemma 4. For $\omega > \omega_0$ near ω_0, the symmetric vortex of charge 1 $u_{\omega,\epsilon} = \varphi_{\omega,\epsilon}(r)\, e^{i\theta}$ is orbitally stable in the following sense: for any $\varepsilon > 0$ there is a $\delta > 0$, such that if $\|u(0) - u_{\omega,\epsilon}\|_X \leqslant \delta$, then

$$\inf_{\zeta \in \mathbb{R}} \|u(t) - e^{i\zeta} u_{\omega,\epsilon}\|_X \leqslant \varepsilon, \quad t \in \mathbb{R}_+,$$

where $X = \{u \in H^1(\mathbb{R}^2) : |\xi| u \in L^2(\mathbb{R}^2)\}$ is the energy space of the Gross–Pitaevskii equation (32). □

Remark 4. Although the symmetric vortex is spectrally stable for $\omega < \omega_0$, it is a saddle point of energy and hence, we cannot conclude that it is orbitally stable for $\omega < \omega_0$. □

4 Main Results and Illustrations

We formulate now the main results of this article. The results guarantee the existence and orbital stability of a steadily precessing vortex of charge 1 in the Gross–Pitaevskii equation (1) with the precessional frequency ω slightly exceeding the bifurcation value ω_0. This vortex is different from the symmetric vortex $u = \varphi_{\omega,\epsilon}(r)\, e^{i\theta}$, because it precesses along an orbit enclosing the center of the harmonic potential. Because of the rotational invariance, the vortex can be placed at any point along the orbit, hence it has an additional parameter α for the angle along the precessional orbit. We refer to this solution as to *the asymmetric vortex* because it has no symmetry about the origin $(0,0) \in \mathbb{R}^2$. The asymmetric vortex exists and is orbitally stable, according to the following two theorems.

Theorem 1. For every $\epsilon \in (\epsilon_0, \frac{1}{2})$, let ω_0 be the bifurcation value in Lemma 4. Besides the symmetric vortex $u = \varphi_{\omega,\epsilon}(r)\, e^{i\theta}$, there exists another time-independent vortex solution $u = u_{\omega,\epsilon,\alpha}(\xi)$ of the Gross–Pitaevskii equation (32) for $\omega > \omega_0$ near ω_0, where $|u_{\omega,\epsilon,\alpha}|$ is not radially symmetric. The center of $|u_{\omega,\epsilon,\alpha}|$ is placed on the circle of radius $|a|$ centered at the origin $(0,0) \in \mathbb{R}^2$ at the angle α and there is $C > 0$ such that $|a| \leqslant C\sqrt{\epsilon(\omega - \omega_0)}$. □

Theorem 2. Under the conditions of Theorem 1, the asymmetric vortex $u_{\omega,\epsilon,\alpha}$ is orbitally stable in the following sense: for any $\varepsilon > 0$ there is a $\delta > 0$, such that if $\|u(0) - u_{\omega,\epsilon,0}\|_X \leqslant \delta$, then

$$\inf_{(\zeta,\alpha) \in \mathbb{R}^2} \|u(t) - e^{i\zeta} u_{\omega,\epsilon,\alpha}\|_X \leqslant \varepsilon, \quad t \in \mathbb{R}_+.$$

□

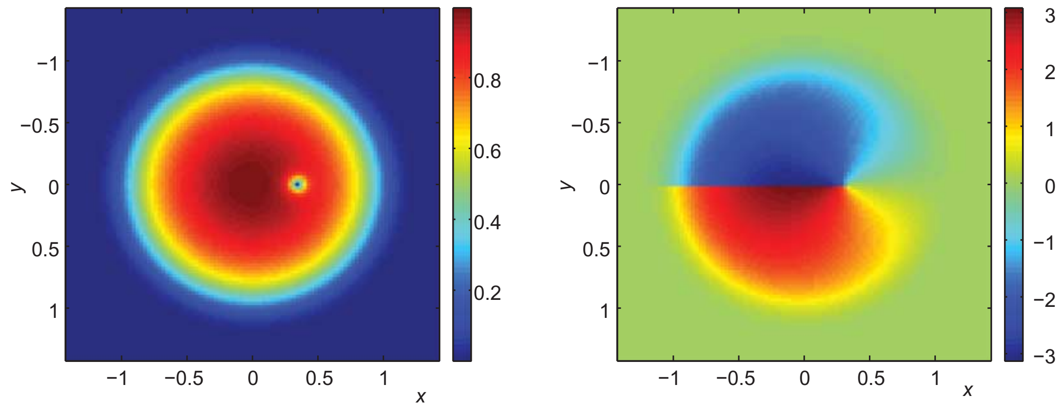

Fig. 1. Spatial contour plots of the amplitude (left) and phase (right) of the asymmetric vortex for $\omega = 0.365$ and $\epsilon = 0.035$.

We shall now illustrate the results of Theorems 1 and 2 numerically. Figure 1 shows the amplitude $|u|$ and the phase $\arg(u)$ (the latter multiplied by $|u|$ for clarity) of the asymmetric vortex $u_{\omega,\epsilon,\alpha}$ as the time-independent solution of the Gross–Pitaevskii equation (32) for $\epsilon = 0.035$ and $\omega = 0.365$. We can see that the center of the asymmetric vortex is already distant from the center of the harmonic potential. Figure 2 displays an approximate measure of the distance (through a smoothed identification of the maximum of the vorticity field) as a function of $\omega - \omega_0$ for $\epsilon = 0.035$. The curve displays the characteristic supercritical pitchfork behavior. For $\omega > \omega_0$, both symmetric and asymmetric vortices coexist as solutions of the Gross–Pitaevskii equation (32).

Figure 3 shows the smallest positive eigenvalue σ of the spectral stability problems (37) as a function of ω. The positive eigenvalue for $m = 2$ crosses zero at $\omega = \omega_0$ and becomes negative for $\omega > \omega_0$. The negative eigenvalue for $m = 0$ crosses zero at $\omega = \omega_0$ and becomes positive for $\omega > \omega_0$. The smallest positive eigenvalue σ has negative Krein signature (30) for $\omega < \omega_0$ and positive Krein signature for $\omega > \omega_0$.

Figure 4 shows eigenvalues of the spectral stability problems (37) for all $m \in \mathbb{Z}$ associated to the symmetric vortex $u = \varphi_{\omega,\epsilon}(r)\, e^{i\theta}$ for $\epsilon = 0.035$ and two values of ω before (left) and after (right) the pitchfork bifurcation. Note that the real eigenvalues σ are rotated to purely imaginary eigenvalues λ in the standard formulation of the spectral stability problem. It is evident that the zero crossing of the smallest eigenvalues does not lead to the creation of a pair of real unstable eigenvalues. The symmetric vortex is spectrally stable for both $\omega < \omega_0$ and $\omega > \omega_0$.

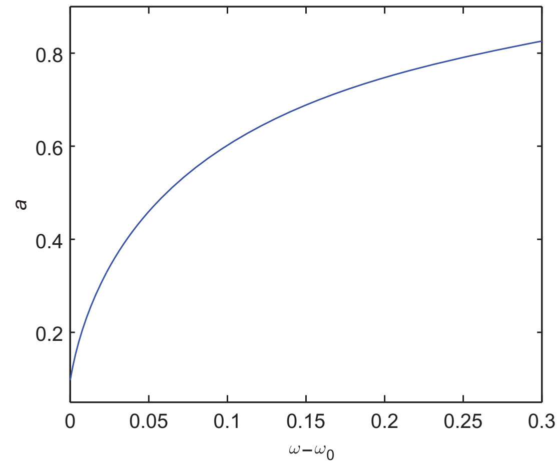

Fig. 2. The distance a of the asymmetric vortex from the center of the trap is shown as a function of $\omega - \omega_0$ (where ω_0 is the frequency at the bifurcation point) for $\epsilon = 0.035$.

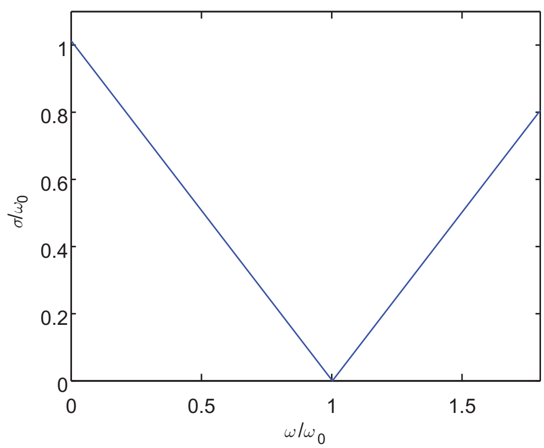

Fig. 3. The smallest positive eigenvalue σ of the spectral stability problems (37) associated with the symmetric vortex for $\epsilon = 0.035$ is shown as a function of the rotation frequency ω. Both frequencies are scaled by the bifurcation frequency ω_0.

Figure 5 shows eigenvalues of a spectral stability problem associated with the asymmetric vortex $u = u_{\omega,\epsilon,\alpha}$ for $\epsilon = 0.035$ and $\omega = 0.365$ past the bifurcation value. The linearization problem has no small nonzero eigenvalues, instead it has a zero eigenvalue of algebraic multiplicity four and geometric multiplicity two. This feature will be proved rigorously within the bifurcation setting in Section 6. The zero eigenvalue is induced by

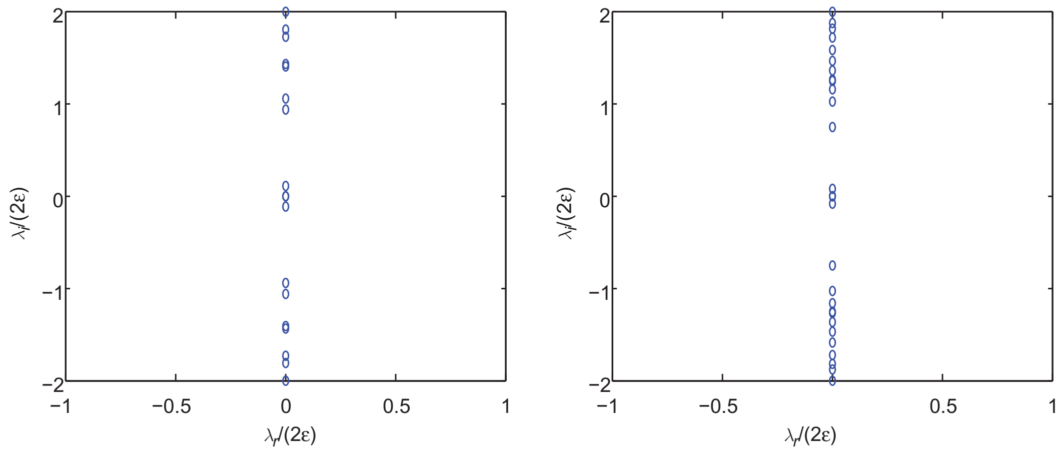

Fig. 4. Eigenvalues $\lambda = -i\sigma = \lambda_r + i\lambda_i$ of the spectral stability problems (37) associated with the symmetric vortex for $\epsilon = 0.035$ for two values of ω before and after the pitchfork bifurcation: $\omega = 0.121$ (left) and $\omega = 0.501$ (right). The axes are scaled by a factor of 2ϵ, so that all eigenvalues but the ones crossing zero are of $\mathcal{O}(1)$.

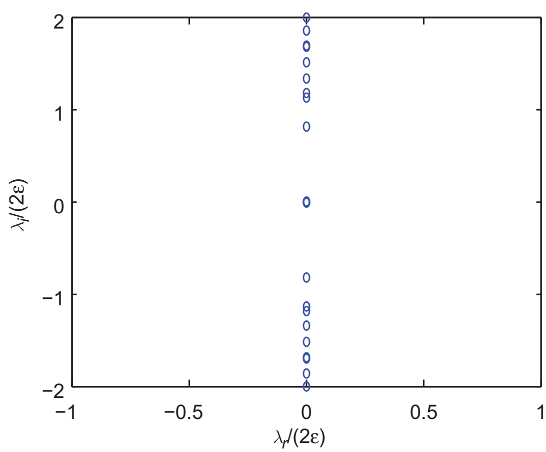

Fig. 5. Eigenvalues of a spectral stability problem associated with the asymmetric vortex for $\epsilon = 0.035$ and $\omega = 0.365$. The zero eigenvalue has algebraic multiplicity four.

the phase invariance of the Gross–Pitaevskii equation (first pair) *and* by the rotational invariance of the position of the vortex along its precession orbit (second pair). Asymmetric vortices can be "pinned" at any point of their precessional orbit parametrized by the rotation frequency or, equivalently, by the distance from the center of the trap. All other eigenvalues are purely imaginary so that the asymmetric vortex is also spectrally stable with respect to the time evolution of the Gross–Pitaevskii equation (32) for $\omega > \omega_0$ near ω_0.

5 The Birth of the Asymmetric Vortex

The zero eigenvalue of the self-adjoint spectral problem (39) for $m = 2$ and $\omega = \omega_0$ exists by Lemma 4 and signals a bifurcation along the family of symmetric vortex solutions. This bifurcation gives birth to the family of asymmetric vortices for $\omega > \omega_0$. To prove Theorem 1, we work with the local bifurcation method and study the pitchfork bifurcation at $\omega = \omega_0$.

To develop the algorithm of Lyapunov–Schmidt reductions, we rewrite the existence problem for the stationary Gross–Pitaevskii equation (32) in the rotating reference frame (31) as the root-finding problem,

$$N(u; \omega) := -\epsilon^2(u_{\xi\xi} + u_{\eta\eta}) + (\xi^2 + \eta^2 - 1 + |u|^2)u + \mathrm{i}\epsilon\omega(\xi u_\eta - \eta u_\xi) = 0. \tag{47}$$

Because the root-finding problem involves $\bar{u}$, we need to add another equation $\overline{N(u; \omega)} = 0$. We will use bolded notations for 2-vectors, for example, $\mathbf{N} = (N, \bar{N})$. Note that parameter ϵ is fixed in the interval $(\epsilon_0, \frac{1}{4})$ to guarantee the validity of Lemmas 1, 2, 3, and 4. Therefore, we do not indicate the explicit dependence of $N(u; \omega)$ on ϵ.

The Jacobian operator of $\mathbf{N} = (N, \bar{N})$ with respect to $\mathbf{u} = (u, \bar{u})$ is given by

$$D_u N(u; \omega) = \begin{bmatrix} -\epsilon^2\Delta + |\xi|^2 - 1 + \mathrm{i}\epsilon\omega(\xi\partial_\eta - \eta\partial_\xi) + 2|u|^2 & u^2 \\ \bar{u}^2 & -\epsilon^2\Delta + |\xi|^2 - 1 - \mathrm{i}\epsilon\omega(\xi\partial_\eta - \eta\partial_\xi) + 2|u|^2 \end{bmatrix}.$$

We note that $N(\varphi_{\omega,\epsilon}\, \mathrm{e}^{\mathrm{i}\theta}; \omega) = 0$ is equivalent to the differential equation (33). Let us denote $\varphi_0 \equiv \varphi_{\omega_0,\epsilon}$ and $H_0^{(m)} \equiv H_{\omega_0,\epsilon}^{(m)}$, $m \in \mathbb{Z}$. At the bifurcation value ω_0 in Lemma 4, we know that the kernel of $D_u N(\varphi_0\, \mathrm{e}^{\mathrm{i}\theta}; \omega_0)$ is three-dimensional thanks to the gauge invariance and the double degeneracy of the bifurcating mode of the self-adjoint problem (39) for $m = 2$ and $m = 0$,

$$\mathrm{Ker}(D_u N(\varphi_0\, \mathrm{e}^{\mathrm{i}\theta}; \omega_0)) = \mathrm{span}\left\{ \begin{bmatrix} \varphi_0(r)\, \mathrm{e}^{\mathrm{i}\theta} \\ -\varphi_0(r)\, \mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix}, \begin{bmatrix} V_0(r)\, \mathrm{e}^{2\mathrm{i}\theta} \\ W_0(r) \end{bmatrix}, \begin{bmatrix} W_0(r) \\ V_0(r)\, \mathrm{e}^{-2\mathrm{i}\theta} \end{bmatrix} \right\}, \tag{48}$$

where (V_0, W_0) is a real-valued solution of the homogeneous system

$$H_0^{(2)} \begin{bmatrix} V_0 \\ W_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{49}$$

It follows from (47) that for every fixed $\epsilon > 0$, the function $N(u;\omega) : H^2(\mathbb{R}^2)\times\mathbb{R}\to L^2(\mathbb{R}^2)$ is smooth with respect to its arguments. We use the decomposition,

$$u = \varphi_0(r)\,\mathrm{e}^{\mathrm{i}\theta} + aV_0(r)\,\mathrm{e}^{2\mathrm{i}\theta} + \bar{a}W_0(r) + U, \quad \omega = \omega_0 + \Omega, \tag{50}$$

where $a\in\mathbb{C}$ and $\Omega\in\mathbb{R}$ are parameters of the decomposition and $\mathbf{U} = (U,\bar{U})$ satisfies the orthogonality conditions,

$$\langle \mathbf{V},\mathbf{U}\rangle := \int_{\mathbb{R}^2} (\bar{V}U + \bar{W}\bar{U})\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \text{for every } \mathbf{V} = \begin{bmatrix} V \\ W \end{bmatrix} \in \mathrm{Ker}(D_u N(\varphi_0\,\mathrm{e}^{\mathrm{i}\theta};\omega_0)). \tag{51}$$

Note that the constraint on the conjugation of a in (50) follows from representation (48) and the fact that $\bar{u}$ is the complex conjugate of u. The root-finding problem is now decoupled into algebraic equations

$$\langle \mathbf{V},\mathbf{N}(\varphi_0\,\mathrm{e}^{\mathrm{i}\theta} + aV_0\,\mathrm{e}^{2\mathrm{i}\theta} + \bar{a}W_0 + U;\omega_0+\Omega)\rangle = 0 \quad \text{for every } \mathbf{V} = \begin{bmatrix} V \\ W \end{bmatrix} \in \mathrm{Ker}(D_u N(\varphi_0\,\mathrm{e}^{\mathrm{i}\theta};\omega_0)) \tag{52}$$

and a differential equation for the error term $\mathbf{U}$. The system of algebraic Equations (52) gives three equations because of three eigenvectors in $\mathrm{Ker}(D_u N(u_0;\omega_0))$. However, because of the gauge invariance, projection to the first eigenvector in (48) is identically zero if $\mathbf{U}$ satisfies (51). Also projection to the third eigenvector in (48) gives a complex conjugation of the projection to the second equation. Therefore, the system of algebraic Equations (52) reduces to the scalar equation,

$$\begin{aligned} F(U;\omega,a) := \int_{\mathbb{R}^2} \big(\,&\mathrm{e}^{-2\mathrm{i}\theta}\,V_0 N(\varphi_0\,\mathrm{e}^{\mathrm{i}\theta} + aV_0\,\mathrm{e}^{2\mathrm{i}\theta} + \bar{a}W_0 + U;\omega_0+\Omega) \\ &+ W_0\overline{N(\varphi_0\,\mathrm{e}^{\mathrm{i}\theta} + aV_0\,\mathrm{e}^{2\mathrm{i}\theta} + \bar{a}W_0 + U;\omega_0+\Omega)}\big)\,\mathrm{d}x\,\mathrm{d}y = 0. \end{aligned} \tag{53}$$

We can see that $F(U;\omega,a) : H^2(\mathbb{R}^2)\times\mathbb{R}\times\mathbb{C}\to\mathbb{C}$ is a smooth function with respect to its variables.

As it often happens for the pitchfork bifurcation [23, 35], we need a near-identity transformation to remove nonresonant quadratic terms and to compute resonant cubic terms in amplitude a. Taking into account the decomposition for $\omega = \omega_0 + \Omega$, we set

$$U = \epsilon\Omega V_1(r)\,\mathrm{e}^{\mathrm{i}\theta} + a^2 V_{20}(r)\,\mathrm{e}^{3\mathrm{i}\theta} + |a|^2 V_{11}(r)\,\mathrm{e}^{\mathrm{i}\theta} + \bar{a}^2 V_{02}(r)\,\mathrm{e}^{-\mathrm{i}\theta} + \tilde{U}, \tag{54}$$

where $\tilde{U}$ is a new error term, and obtain the linear inhomogeneous equations:

$$H_0^{(1)}\begin{bmatrix} V_1 \\ W_1 \end{bmatrix} = \begin{bmatrix} \varphi_0 \\ \varphi_0 \end{bmatrix}, \tag{55}$$

$$H_0^{(3)}\begin{bmatrix} V_{20} \\ W_{20} \end{bmatrix} = -\begin{bmatrix} \varphi_0 V_0(V_0 + 2W_0) \\ \varphi_0 W_0(W_0 + 2V_0) \end{bmatrix}, \tag{56}$$

$$H_0^{(1)}\begin{bmatrix} V_{11} \\ W_{11} \end{bmatrix} = -2\begin{bmatrix} \varphi_0(V_0^2 + V_0 W_0 + W_0^2) \\ \varphi_0(V_0^2 + V_0 W_0 + W_0^2) \end{bmatrix}, \tag{57}$$

$$H_0^{(-1)}\begin{bmatrix} V_{02} \\ W_{02} \end{bmatrix} = -\begin{bmatrix} \varphi_0 W_0(W_0 + 2V_0) \\ \varphi_0 V_0(V_0 + 2W_0) \end{bmatrix}. \tag{58}$$

By the symmetry of $H_0^{(3)}$, $H_0^{(1)}$, and $H_0^{(-1)}$, we have

$$V_1 = W_1, \quad V_{02} = W_{20}, \quad W_{02} = V_{20}, \quad W_{11} = V_{11}. \tag{59}$$

We note that $H_0^{(3)}$ and $H_0^{(-1)}$ are invertible, whereas the Fredholm condition for $H_0^{(1)}$ is satisfied. Therefore, there exist unique solutions of the linear homogeneous Equations (55)–(58) subject to constraints (51).

Using standard fixed-point arguments, it is easy to prove that there exist positive constants a_0, Ω_0, and C such that the differential equation for $\tilde{U}$ admits a unique solution for all $|a| \leqslant a_0$ and $|\Omega| \leqslant \Omega_0$ satisfying the bound

$$\|\tilde{U}\|_{H^2} \leqslant C(|\Omega| + |a|^2)|a|. \tag{60}$$

Substituting decompositions (50) and (54) into the scalar Equation (53), we obtain the normal form for the radially symmetric pitchfork bifurcation,

$$a(2\epsilon\Omega\sigma + \beta|a|^2 + \tilde{F}) = 0, \tag{61}$$

where

$$\sigma = \int_0^\infty [2\varphi_0 V_1(V_0^2 + V_0 W_0 + W_0^2) - V_0^2] r \, dr, \tag{62}$$

$$\beta = \int_0^\infty [V_0^4 + 4V_0^2 W_0^2 + W_0^4 + 4\varphi_0(V_0^2 + V_0 W_0 + W_0^2)V_{11} + 4\varphi_0 V_0 W_0(V_{20} + W_{20}) + 2\varphi_0(V_0^2 V_{20} + W_0^2 W_{20})] r \, dr, \tag{63}$$

and the remainder term is small $\tilde{F} = \mathcal{O}(\Omega^2, |a|^4)$.

Nontrivial solutions of the normal-form Equation (61) for small $\Omega \in \mathbb{R}$ with $|a| \neq 0$ exist for $\mathrm{sign}(\Omega) = -\mathrm{sign}(\sigma\beta)$. We will show in Lemmas 5 and 6 that $\sigma > 0$ and $\beta < 0$.

In this case, the nontrivial solutions of the normal-form Equation (61) exist for $\Omega > 0$, that is, for $\omega > \omega_0$, and satisfy the expansion,

$$|a|^2 = -\frac{2\sigma}{\beta}\epsilon\Omega + \mathcal{O}(\Omega^2). \tag{64}$$

If $a = |a|\,\mathrm{e}^{\mathrm{i}\alpha}$, then α is an arbitrary parameter of the bifurcating solution, whereas $|a|$ is uniquely determined as $|a| = \mathcal{O}(\sqrt{\epsilon|\omega-\omega_0|})$. Note that $|a|$ measures the distance of the vortex center from the center of the harmonic potential, whereas α is an angle along the circle of the radius $|a|$. The new asymmetric vortex can be placed at any angle α. The proof of Theorem 1 is complete.

We prove now the claim that $\sigma > 0$ and $\beta < 0$.

Lemma 5. Under the assumptions of Lemma 4, the following statement holds:

$$\sigma = \tfrac{1}{2}(\|W_0\|^2_{L^2_r} - \|V_0\|^2_{L^2_r}),$$

hence $\sigma > 0$. □

Proof. We look for the small eigenvalue of the self-adjoint spectral problem (39) for $m = 2$ and ω near ω_0. Using the expansion

$$\varphi_{\omega,\epsilon} = \varphi_0(r) + \epsilon\Omega V_1(r) + \mathcal{O}(\Omega^2),$$

we rewrite the eigenvalue problem (39) for $m = 2$ in the perturbation form,

$$\left(H_0^{(2)} - 2\epsilon\Omega\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 2\epsilon\Omega\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\varphi_0 V_1 + \mathcal{O}(\Omega^2)\right)\begin{bmatrix}V^{(2)}\\ W^{(0)}\end{bmatrix} = \epsilon\lambda\begin{bmatrix}V^{(2)}\\ W^{(0)}\end{bmatrix}. \tag{65}$$

Let $\lambda(\omega)$ be an eigenvalue of the perturbed spectral problem (65) such that $\lambda(\omega_0) = 0$ at $\Omega = 0$. The corresponding eigenvector at $\Omega = 0$ is $(V^{(2)}, W^{(0)}) = (V_0, W_0)$ by (49). By the asymptotic perturbation theory [21, Section 8.2.3], λ is a C^1 function of ω near ω_0. Writing the derivative equation,

$$\left(H_0^{(2)} - 2\epsilon\Omega\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 2\epsilon\Omega\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\varphi_0 V_1 + \mathcal{O}(\Omega^2)\right)\frac{\mathrm{d}}{\mathrm{d}\omega}\begin{bmatrix}V^{(2)}\\ W^{(0)}\end{bmatrix}$$
$$\left(-2\epsilon\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 2\epsilon\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\varphi_0 V_1 + \mathcal{O}(\Omega)\right)\begin{bmatrix}V^{(2)}\\ W^{(0)}\end{bmatrix} = \epsilon\lambda\frac{\mathrm{d}}{\mathrm{d}\omega}\begin{bmatrix}V^{(2)}\\ W^{(0)}\end{bmatrix} + \epsilon\frac{\mathrm{d}\lambda}{\mathrm{d}\omega}\begin{bmatrix}V^{(2)}\\ W^{(0)}\end{bmatrix},$$

and computing projections to the eigenvector (49), we obtain

$$2\sigma = \lambda'(\omega_0)(\|V_0\|^2_{L^2_r} + \|W_0\|^2_{L^2_r}) = \|W_0\|^2_{L^2_r} - \|V_0\|^2_{L^2_r}, \tag{66}$$

where the equality (45) of Lemma 4 is used, subject to the change of notations. Because the Krein signature (30) of the relevant eigenvalue is negative, we have $\|W_0\|^2_{L^2_r} > \|V_0\|^2_{L^2_r}$, hence $\sigma > 0$. ■

Lemma 6. There exists $\epsilon_0 \in (0, \frac{1}{2})$ such that $\beta < 0$ for every $\epsilon \in (\epsilon_0, \frac{1}{2})$. □

Proof. Using the inhomogeneous Equations (56)–(58) and the symmetry (59), the coefficient β in (63) can be written in the equivalent form,

$$\beta = \langle \mathbf{V}_0^2, M_0 \mathbf{V}_0^2 \rangle_{L^2_r} - \langle \mathbf{V}_{20}, H_0^{(3)} \mathbf{V}_{20} \rangle_{L^2_r} - \langle \mathbf{V}_{11}, H_0^{(1)} \mathbf{V}_{11} \rangle_{L^2_r} - \langle \mathbf{V}_{02}, H_0^{(-1)} \mathbf{V}_{02} \rangle_{L^2_r}, \tag{67}$$

where

$$\mathbf{V}_{ij} = \begin{bmatrix} V_{ij} \\ W_{ij} \end{bmatrix}, \quad \mathbf{V}_0^2 = \begin{bmatrix} V_0^2 \\ W_0^2 \end{bmatrix}, \quad \text{and} \quad M_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Because operators $H_0^{(3)}$, $H_0^{(1)}$, and $H_0^{(-1)}$ are nonnegative, the last three terms in (67) are strictly negative, whereas the first term is strictly positive. Therefore, it is impossible to establish that $\beta < 0$ generally, but we can develop perturbation expansions for small $|\epsilon - \frac{1}{2}|$ to show that the three negative terms dominate over the positive term as $\epsilon \to \frac{1}{2}$.

Let us use the scaling transformation (34) to map the point $\epsilon = \frac{1}{2}$ to the point $\nu = \frac{1}{4}$, for which ω_0 satisfies the limits (43) of Lemma 4. Let $\nu^2 = \frac{1}{16} - \mu$ for small positive μ. It follows from (43) that

$$\omega_0(\nu) = 2 + 8(\nu - \tfrac{1}{4}) + \mathcal{O}(\nu - \tfrac{1}{4})^2 = 2 - 16\mu + \mathcal{O}(\mu^2) \quad \text{as } \mu \to 0. \tag{68}$$

On the other hand, the asymptotic expansions (7), (26), and (29) of Lemmas 1 and 3 imply that

$$\psi_\nu = (128\mu)^{1/2}\psi_0 + \mathcal{O}(\mu^{3/2}), \quad \psi_0(R) = R\,\mathrm{e}^{-2R^2} \tag{69}$$

and

$$V_0 = -2\chi_1 + \mathcal{O}(\mu), \quad W_0 = \chi_2 + \mathcal{O}(\mu), \quad \chi_1(R) = R^2\,\mathrm{e}^{-2R^2}, \quad \chi_2(R) = \mathrm{e}^{-2R^2}. \tag{70}$$

This allows us to compute the first term in (63):

$$\beta_1 = \int_0^\infty (V_0^4 + 4V_0^2 W_0^2 + W_0^4) R\,\mathrm{d}R = \frac{51}{512} + \mathcal{O}(\mu).$$

To compute the next term in (63), we rewrite the linear inhomogeneous Equation (57) after the scaling transformation (34):

$$H_1(\nu)\begin{bmatrix} V_{11} \\ V_{11} \end{bmatrix} = -2\psi_\nu (V_0^2 + V_0 W_0 + W_0^2) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which is equivalent to the following perturbed equation

$$(-\tfrac{1}{16}\Delta_1 + R^2 - 1 + \mu(\Delta_1 + 384\psi_0^2) + \mathcal{O}(\mu^2))V_{11} = -2\psi_\nu(V_0^2 + V_0 W_0 + W_0^2).$$

Because the zeroth-order operator $L_1 = -\frac{1}{16}\Delta_1 + R^2 - 1$ has the one-dimensional kernel spanned by ψ_0, the solution of this linear inhomogeneous equation is singular in the limit $\mu \to 0$ and satisfies the following meromorphic expansion

$$V_{11} = \frac{1}{\mu}(C_{11}\psi_0 + \mathcal{O}(\mu)) \quad \text{as } \mu \to 0,$$

where the projection coefficient C_{11} is computed by

$$C_{11}\langle \psi_0, (\Delta_1 + 384\psi_0^2)\psi_0\rangle_{L_r^2} = -2\langle \psi_0, \psi_\nu(V_0^2 + V_0 W_0 + W_0^2)\rangle_{L_r^2}.$$

Substituting this expansion to the second term in (63), we obtain

$$\begin{aligned}\beta_2 &= 4\int_0^\infty \psi_\nu(V_0^2 + V_0 W_0 + W_0^2)V_{11} R\,\mathrm{d}R \\ &= -\frac{8\cdot 128|\langle \psi_0, \psi_0(V_0^2 + V_0 W_0 + W_0^2)\rangle_{L_r^2}|^2}{\langle \psi_0, (\Delta_1 + 384\psi_0^2)\psi_0\rangle_{L_r^2}} + \mathcal{O}(\mu) \\ &= -\frac{49}{512} + \mathcal{O}(\mu).\end{aligned}$$

To compute the last term in (63), we rewrite the linear inhomogeneous Equation (56) after the scaling transformation (34):

$$H_3(\nu)\begin{bmatrix} V_{20} \\ W_{20}\end{bmatrix} - 2\nu\omega_0 \begin{bmatrix} V_{20} \\ -W_{20}\end{bmatrix} = -\psi_\nu \begin{bmatrix} V_0(V_0 + 2W_0) \\ W_0(2V_0 + W_0)\end{bmatrix}.$$

With the help of (68) and (69), the inhomogeneous equation is equivalent to the following perturbed equation:

$$\begin{aligned}&\left(\begin{bmatrix} -\frac{1}{16}\Delta_3 + R^2 - 2 & 0 \\ 0 & -\frac{1}{16}\Delta_1 + R^2 \end{bmatrix} + \mu \begin{bmatrix} \Delta_3 + 16 + 256\psi_0^2 & 128\psi_0^2 \\ 128\psi_0^2 & \Delta_1 - 16 + 256\psi_0^2\end{bmatrix} + \mathcal{O}(\mu^2)\right) \\ &\times \begin{bmatrix} V_{20} \\ W_{20}\end{bmatrix} = -\psi_\nu \begin{bmatrix} V_0(V_0 + 2W_0) \\ W_0(2V_0 + W_0)\end{bmatrix}.\end{aligned}$$

Again, the zeroth-order operator has the one-dimensional kernel spanned by

$$\begin{bmatrix} \chi_3 \\ 0 \end{bmatrix}, \quad \chi_3(R) = R^3\,\mathrm{e}^{-2R^2}.$$

As a result, the solution of this linear inhomogeneous equation is singular in the limit $\mu \to 0$ and admits the following meromorphic expansion

$$\begin{bmatrix} V_{20} \\ W_{20} \end{bmatrix} = \frac{1}{\mu}\left(C_{20} \begin{bmatrix} \chi_3 \\ 0 \end{bmatrix} + \mathcal{O}(\mu) \right) \quad \text{as } \mu \to 0,$$

where the projection coefficient C_{20} is computed by

$$C_{20}\langle \chi_3, (\Delta_3 + 16 + 256\psi_0^2)\chi_3\rangle_{L_r^2} = -\langle \chi_3, \psi_\nu V_0(V_0 + 2W_0)\rangle_{L_r^2}.$$

Substituting this expansion to the last term in (63), we obtain

$$\begin{aligned}\beta_3 &= \int_0^\infty \psi_\nu (4V_0W_0(V_{20} + W_{20}) + 2V_0^2 V_{20} + 2W_0^2 W_{20}) R\mathrm{d}R \\ &= -\frac{256|\langle \chi_3, \psi_0 V_0(V_0 + 2W_0)\rangle_{L_r^2}|^2}{\langle \chi_3, (\Delta_3 + 16 + 256\psi_0^2)\chi_3\rangle_{L_r^2}} + \mathcal{O}(\mu) \\ &= -\frac{3}{512} + \mathcal{O}(\mu).\end{aligned}$$

Combining all together, we obtain $\beta = \beta_1 + \beta_2 + \beta_3 = -\frac{1}{512} < 0$ and, by continuity in μ, β remains negative for small $|\nu - \frac{1}{4}|$. ∎

Remark 5. As a by-product of asymptotic computations in Lemmas 3, 5, and 6, we obtain from (64) that

$$|a|^2 = -\frac{2\sigma}{\beta}\epsilon\Omega + \mathcal{O}(\Omega^2) = 32\epsilon\Omega + \mathcal{O}(\Omega^2, \mu),$$

which gives a useful approximation of the displacement distance of the asymmetric vortex from the center of the harmonic potential for small $|\omega - \omega_0|$ and $|\epsilon - \frac{1}{2}|$. □

6 Stability of Asymmetric Vortices Past the Bifurcation Point

When the pitchfork bifurcation occurs, there is typically a transition from the stable to unstable solutions [23, 35]. However, we know from Proposition 2 that the symmetric vortex of charge 1 is spectrally stable both for $\omega < \omega_0$ and $\omega > \omega_0$. Moreover, by Lemma 4 and Proposition 3, it is a local minimizer of energy (46) for $\omega > \omega_0$ near ω_0 and hence the symmetric vortex of charge 1 is orbitally stable with respect to perturbations of finite amplitude for $\omega > \omega_0$.

On the other hand, the asymmetric vortex bifurcating for $\omega > \omega_0$ must have a negative eigenvalue in the linearized operator $D_uN(u_{\omega,\epsilon,\alpha}; \omega)$, where $u_{\omega,\epsilon,\alpha}$ is the asymmetric vortex, and hence it is a saddle point of energy (46). This means typically that the critical point is not orbitally stable with respect to perturbations of finite amplitude. Nevertheless, the linearized operator $D_u N(u_{\omega,\epsilon,\alpha};\omega)$ has also a two-dimensional kernel,

$$\mathrm{Ker}(D_u N(u_{\omega,\epsilon,\alpha};\omega)) = \mathrm{span}\{\mathbf{V}_g, \mathbf{V}_r\}, \quad \omega > \omega_0, \tag{71}$$

where

$$\mathbf{V}_g = \begin{bmatrix} u_{\omega,\epsilon,\alpha} \\ -\bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix}, \quad \mathbf{V}_r = \begin{bmatrix} \partial_\alpha u_{\omega,\epsilon,\alpha} \\ \partial_\alpha \bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix}, \tag{72}$$

thanks to the gauge and rotational symmetries of the Gross–Pitaevskii equation (32). In what follows, we shall prove Theorem 2 that states orbital stability of the asymmetric vortex of charge 1 by incorporating the constraints related to the gauge and rotational symmetries in (71).

First, let us prove the following lemma.

Lemma 7. For every $\epsilon \in (\epsilon_0, \frac{1}{2})$, let $\omega_0 > 0$ be the bifurcation value in Lemma 4 and $u_{\omega,\epsilon,\alpha}$ be the asymmetric vortex in Theorem 1. There exists $\omega_1 > \omega_0$ such that for every $\omega \in (\omega_0, \omega_1)$, the linearized operator $D_u N(u_{\omega,\epsilon,\alpha};\omega)$ has exactly one negative eigenvalue and a double zero eigenvalue associated with the eigenvectors (71). □

Proof. We use perturbation theory for the triple zero eigenvalue of $D_u N(\varphi_0\, e^{i\theta};\omega_0)$ associated with the eigenvectors (48). By Lemmas 2 and 4, we know that the only negative eigenvalue of $D_u N(\varphi_{\omega,\epsilon}\, e^{i\theta};\omega)$ for $\omega < \omega_0$ becomes zero at $\omega = \omega_0$ so that the operator $D_u N(\varphi_0\, e^{i\theta};\omega_0)$ is nonnegative.

The following asymptotic expansion for the asymmetric vortex $u_{\omega,\epsilon,\alpha}$ is obtained in the proof of Theorem 1 for small a and $\Omega = \omega - \omega_0$:

$$\begin{aligned} u_{\omega,\epsilon,\alpha} &= \varphi_0\, e^{i\theta} + aV_0\, e^{2i\theta} + \bar{a}W_0 + +\epsilon\Omega V_1(r)\, e^{i\theta} \\ &\quad + a^2 V_{20}(r)\, e^{3i\theta} + |a|^2 V_{11}(r)\, e^{i\theta} + \bar{a}^2 V_{02}(r)\, e^{-i\theta} + \mathcal{O}(|a|^3), \end{aligned}$$

where a and Ω are related by the normal form equation (61). As a result, we write

$$\begin{aligned} \mathcal{H}_{\omega,\epsilon} := D_u N(u_{\omega,\epsilon,\alpha};\omega) &= D_u N(\varphi_0\, e^{i\theta};\omega_0) + a\mathcal{H}_{10} + \bar{a}\mathcal{H}_{01} + \epsilon\Omega\mathcal{H}_1 \\ &\quad + a^2\mathcal{H}_{20} + |a|^2\mathcal{H}_{11} + \bar{a}^2\mathcal{H}_{02} + \mathcal{O}(|a|^3), \end{aligned}$$

where correction terms are uniquely computed. We approximate now small eigenvalues of the self-adjoint eigenvalue problem

$$\mathcal{H}_{\omega,\epsilon} \begin{bmatrix} V \\ W \end{bmatrix} = \epsilon\lambda \begin{bmatrix} V \\ W \end{bmatrix}. \tag{73}$$

We recall that the eigenvector $\mathbf{V}_g$ in (72) persists for all a because of the gauge symmetry. Therefore, we are looking for the splitting of the double zero eigenvalue associated with the perturbation expansion

$$\begin{bmatrix} V \\ W \end{bmatrix} = b_1 \begin{bmatrix} V_0(r)\,\mathrm{e}^{2\mathrm{i}\theta} \\ W_0(r) \end{bmatrix} + b_2 \begin{bmatrix} W_0(r) \\ V_0(r)\,\mathrm{e}^{-2\mathrm{i}\theta} \end{bmatrix} + \begin{bmatrix} \tilde{V} \\ \tilde{W} \end{bmatrix}, \tag{74}$$

where (b_1, b_2) are coordinates of the projections and $\tilde{\mathbf{V}}$ is orthogonal to the eigenvectors in (48). We now substitute perturbation expansion (74) to the eigenvalue problem (73). At the order of $\mathcal{O}(|a|)$, Fredholm conditions (52) for the residual terms are trivially satisfied, so that a solution exists, in fact, in an explicit form. Therefore, we incorporate this explicit solution to continue the expansion (74) as follows:

$$\begin{aligned} \begin{bmatrix} \tilde{V} \\ \tilde{W} \end{bmatrix} &= b_1 \left(2a \begin{bmatrix} V_{20}(r)\,\mathrm{e}^{3\mathrm{i}\theta} \\ W_{20}(r)\,\mathrm{e}^{\mathrm{i}\theta} \end{bmatrix} + \bar{a} \begin{bmatrix} V_{11}(r)\,\mathrm{e}^{\mathrm{i}\theta} \\ W_{11}(r)\,\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix} \right) \\ &\quad + b_2 \left(a \begin{bmatrix} V_{11}(r)\,\mathrm{e}^{\mathrm{i}\theta} \\ W_{11}(r)\,\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix} + 2\bar{a} \begin{bmatrix} V_{02}(r)\,\mathrm{e}^{-\mathrm{i}\theta} \\ W_{02}(r)\,\mathrm{e}^{-3\mathrm{i}\theta} \end{bmatrix} \right) + \begin{bmatrix} \hat{V} \\ \hat{W} \end{bmatrix}, \end{aligned}$$

where $V_{20} = W_{02}$, $V_{11} = W_{11}$, $V_{02} = W_{20}$ are determined from solutions of the linear inhomogeneous Equations (56)–(58) and $\hat{\mathbf{V}}$ is a new correction term of the order of $\mathcal{O}(|a|^2)$. Computing projections of the eigenvalue problem (73) to the eigenvectors in (48) at the order of $\mathcal{O}(|a|^2)$, we obtain the following reduced eigenvalue problem:

$$\begin{bmatrix} 2\epsilon\Omega\sigma + 2\beta|a|^2 & \beta a^2 \\ \beta\bar{a}^2 & 2\epsilon\Omega\sigma + 2\beta|a|^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \epsilon\lambda(\|V_0\|^2_{L^2_r} + \|W_0\|^2_{L^2_r}) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{75}$$

Taking into account the normal form Equation (61) that gives $2\epsilon\Omega\sigma + \beta|a|^2 = 0$ at the order of $\mathcal{O}(|a|^2)$ and the parametrization $a = |a|\,\mathrm{e}^{\mathrm{i}\alpha}$, we obtain two eigenvalues of the reduced eigenvalue problem (75):

$$\lambda = 0: \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\alpha} \\ -\mathrm{e}^{-\mathrm{i}\alpha} \end{bmatrix}$$

and

$$\lambda = \frac{2\beta|a|^2}{\epsilon(\|V_0\|^2_{L^2_r} + \|W_0\|^2_{L^2_r})}: \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\alpha} \\ \mathrm{e}^{-\mathrm{i}\alpha} \end{bmatrix}.$$

The zero eigenvalue corresponds to the second eigenvector $\mathbf{V}_r$ in (72) and is induced by the rotational symmetry. The nonzero eigenvalue is actually negative because $\beta < 0$ by Lemma 6. It persists as a negative eigenvalue for small $\omega > \omega_0$ by the asymptotic perturbation theory [21, Section 8.2.3]. ■

To prove Theorem 2, we will show that the linearized operator $D_u N(u_{\omega,\epsilon,\alpha};\omega)$ is nonnegative in the constrained space

$$L^2_c(\mathbb{R}^2) = \left\{ U \in L^2(\mathbb{R}^2): \quad \langle \mathbf{V}, \sigma_3 \mathbf{U} \rangle := \int_{\mathbb{R}^2} (\bar{V}U - \bar{W}\bar{U})\, \mathrm{d}x\, \mathrm{d}y = 0, \right.$$
$$\left. \text{for every } \mathbf{V} = \begin{bmatrix} V \\ W \end{bmatrix} \in \mathrm{Ker}(D_u N(u_{\omega,\epsilon,\alpha};\omega)) \right\}, \tag{76}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$ is the third Pauli matrix. The orthogonality conditions in (76) incorporate the symplectic structure of the Gross–Pitaevskii equation (32) resulting in the non-self-adjoint spectral stability problem,

$$\mathcal{H}_{\omega,\epsilon} \begin{bmatrix} V \\ W \end{bmatrix} = \mathrm{i}\epsilon\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}. \tag{77}$$

The zero eigenvalue of the spectral stability problem (77) has geometric multiplicity two and algebraic multiplicity at least four because of the presence of the generalized eigenvectors

$$\tilde{\mathbf{V}}_g = \mathcal{H}^{-1}_{\omega,\epsilon} \begin{bmatrix} u_{\omega,\epsilon,\alpha} \\ \bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix}, \quad \tilde{\mathbf{V}}_r = \mathcal{H}^{-1}_{\omega,\epsilon} \begin{bmatrix} \partial_\alpha u_{\omega,\epsilon,\alpha} \\ -\partial_\alpha \bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix}. \tag{78}$$

Note that $\tilde{\mathbf{V}}_g$ and $\tilde{\mathbf{V}}_r$ exist because $\langle \mathbf{U}, \sigma_3 \mathbf{U} \rangle = 0$ for any $\mathbf{U} \in L^2(\mathbb{R}^2)$ and $\|u_{\omega,\epsilon,\alpha}\|^2_{L^2}$ is α-independent. Also note that the projection algorithm in Lemma 7 applies to the non-self-adjoint eigenvalue problem (77) and produces the reduced eigenvalue problem:

$$\begin{bmatrix} 2\epsilon\Omega\sigma + 2\beta|a|^2 & \beta a^2 \\ \beta \bar{a}^2 & 2\epsilon\Omega\sigma + 2\beta|a|^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathrm{i}\epsilon\lambda(\|V_0\|^2_{L^2_r} - \|W_0\|^2_{L^2_r}) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{79}$$

which has zero eigenvalue of geometric multiplicity one and algebraic multiplicity two because $\|W_0\|^2_{L^2_r} > \|V_0\|^2_{L^2_r}$ by Lemma 3 for the values of ϵ, where the pitchfork bifurcation of Theorem 1 is considered.

The algebraic multiplicity of the zero eigenvalue is exactly four if the matrix of symplectic projections

$$D_{\omega,\epsilon} = \begin{bmatrix} \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_g \rangle & \langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_g \rangle \\ \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_r \rangle & \langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_r \rangle \end{bmatrix}$$

is invertible. The same matrix also determines the number of negative eigenvalues of $\mathcal{H}_{\omega,\epsilon}$ restricted to $L^2_c(\mathbb{R}^2)$ [19, 32] (see also [33, Chapter 4.1.1]). To be precise, let $n(D_{\omega,\epsilon})$ be the number of negative eigenvalues of $D_{\omega,\epsilon}$ and $n(\mathcal{H}_{\omega,\epsilon})$ be the number of negative eigenvalues of $\mathcal{H}_{\omega,\epsilon}$ in $L^2(\mathbb{R}^2)$. Then, the number of negative eigenvalues of $\mathcal{H}_{\omega,\epsilon}$ in $L^2_c(\mathbb{R}^2)$ is $n(\mathcal{H}_{\omega,\epsilon}) - n(D_{\omega,\epsilon})$. By Lemma 7, $n(\mathcal{H}_{\omega,\epsilon}) = 1$, hence $n(D_{\omega,\epsilon}) \leqslant 1$ and we need to prove that

$n(D_{\omega,\epsilon}) = 1$ to ensure that $\mathcal{H}_{\omega,\epsilon}$ is nonnegative in the constrained space $L^2_c(\mathbb{R}^2)$. This is proved in the following lemma.

Lemma 8. Under the conditions of Lemma 7, for every $\omega \in (\omega_0, \omega_1)$, $D_{\omega,\epsilon}$ has one positive and one negative eigenvalue. □

Proof. We use perturbation expansions of Lemma 7 for small $a \in \mathbb{C}$ and $\Omega = \omega - \omega_0 \in \mathbb{R}_+$ and approximate the eigenvectors $\mathbf{V}_g$ and $\mathbf{V}_r$ as follows:

$$\mathbf{V}_g = \begin{bmatrix} \varphi_0(r)\,\mathrm{e}^{\mathrm{i}\theta} \\ -\varphi_0(r)\,\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix} + a \begin{bmatrix} V_0(r)\,\mathrm{e}^{2\mathrm{i}\theta} \\ -W_0(r) \end{bmatrix} + \bar{a} \begin{bmatrix} W_0(r) \\ -V_0(r)\,\mathrm{e}^{-2\mathrm{i}\theta} \end{bmatrix} + \mathcal{O}(|a|^2) \tag{80}$$

and

$$-\mathrm{i}\mathbf{V}_r = a \begin{bmatrix} V_0(r)\,\mathrm{e}^{2\mathrm{i}\theta} \\ W_0(r) \end{bmatrix} - \bar{a} \begin{bmatrix} W_0(r) \\ V_0(r)\,\mathrm{e}^{-2\mathrm{i}\theta} \end{bmatrix} + 2a^2 \begin{bmatrix} V_{20}(r)\,\mathrm{e}^{3\mathrm{i}\theta} \\ W_{20}(r)\,\mathrm{e}^{\mathrm{i}\theta} \end{bmatrix} - 2\bar{a}^2 \begin{bmatrix} V_{02}(r)\,\mathrm{e}^{-\mathrm{i}\theta} \\ W_{02}(r)\,\mathrm{e}^{-3\mathrm{i}\theta} \end{bmatrix} + \mathcal{O}(|a|^3). \tag{81}$$

To compute $\langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_g \rangle$, we define a solution of the elliptic problem:

$$-(\partial_\xi^2 + \partial_\eta^2) U_{\omega,\mu,\alpha} + (\xi^2 + \eta^2 - \mu + |U_{\omega,\mu,\alpha}|^2) U_{\omega,\mu,\alpha} + \mathrm{i}\omega(\xi\partial_\eta - \eta\partial_\xi) U_{\omega,\mu,\alpha} = 0. \tag{82}$$

Using the scaling transformation, we represent

$$U_{\omega,\mu,\alpha}(\xi) = \sqrt{\mu}\, u_{\omega,\epsilon,\alpha}(\Xi), \quad \epsilon = \frac{1}{\mu}, \quad \Xi = \frac{\xi}{\sqrt{\mu}}, \tag{83}$$

where $u_{\omega,\epsilon,\alpha}$ is the asymmetric vortex in Theorem 1. Differentiating (82) with respect to μ, we obtain

$$(-\partial_\xi^2 - \partial_\eta^2 + \xi^2 + \eta^2 - \mu + 2|U_{\omega,\mu,\alpha}|^2 + \mathrm{i}\omega(\xi\partial_\eta - \eta\partial_\xi)) \frac{\partial}{\partial\mu} U_{\omega,\mu,\alpha} + U_{\omega,\mu,\alpha}^2 \frac{\partial}{\partial\mu} \bar{U}_{\omega,\mu,\alpha} = U_{\omega,\mu,\alpha},$$

hence

$$\tilde{\mathbf{V}}_g = \mathcal{H}_{\omega,\epsilon}^{-1} \begin{bmatrix} u_{\omega,\epsilon,\alpha} \\ \bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u_{\omega,\epsilon,\alpha} \\ \bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix} - \frac{1}{2} \xi \cdot \nabla \begin{bmatrix} u_{\omega,\epsilon,\alpha} \\ \bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix} - \epsilon \frac{\partial}{\partial\epsilon} \begin{bmatrix} u_{\omega,\epsilon,\alpha} \\ \bar{u}_{\omega,\epsilon,\alpha} \end{bmatrix} \tag{84}$$

and

$$\begin{aligned} \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_g \rangle &= 2\|u_{\omega,\epsilon,\alpha}\|_{L^2}^2 - \epsilon \frac{\partial}{\partial\epsilon} \|u_{\omega,\epsilon,\alpha}\|_{L^2}^2 \\ &= 2\pi \left(2\|\varphi_0\|_{L_r^2}^2 - \epsilon \frac{\partial}{\partial\epsilon} \|\varphi_0\|_{L_r^2}^2 \right) + \mathcal{O}(|a|^2) > 0, \end{aligned}$$

where we use the symmetries in the expansion (80) and the previous argument that $\|\varphi_0\|^2_{L^2_r}$ is a decreasing function of ϵ near the local bifurcation threshold that corresponds to $\epsilon = \frac{1}{2}$.

From perturbation expansions (80) and (81), we infer that

$$\langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_g \rangle = \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_r \rangle = \mathcal{O}(|a|^2) \quad \text{as } a \to 0.$$

On the other hand, we have

$$\langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_r \rangle = \langle \sigma_3 \mathbf{V}_r, \mathcal{H}^{-1}_{\omega,\epsilon} \sigma_3 \mathbf{V}_g \rangle = \mathcal{O}(|a|^2) < 0 \quad \text{as } a \to 0$$

because $\mathcal{H}_{\omega,\epsilon}$ is nonpositive on the leading-order term of the perturbation expansion (81) by Lemma 7 and it cannot be zero. Because the off-diagonal terms have the same order of $\mathcal{O}(|a|^2)$, one eigenvalue of $D_{\omega,\epsilon}$ is positive of the order of $\mathcal{O}(1)$ and the other eigenvalue is negative of the order of $\mathcal{O}(|a|^2)$ as $a \to 0$. This computation yields the assertion of the lemma for small $a \in \mathbb{C}$, or equivalently, for small $|\omega - \omega_0|$. ■

By Lemma 8, the linearized operator $\mathcal{H}_{\omega,\epsilon}$ is nonnegative in the constrained space $L^2_c(\mathbb{R}^2)$. By Lemma 7, it has a two-dimensional kernel (71) induced by the gauge and rotational symmetries. Conditions of the orbital stability theory [14] are satisfied and the result of Theorem 2 follows from this theory [33, Chapter 4.4.2].

7 Conclusion

We have shown that the rotating symmetric vortices at the center of the harmonic potential become subject to a pitchfork bifurcation with radial symmetry. This bifurcation occurs at the rotation frequency coinciding with the eigenfrequency of negative Krein signature in the spectral stability problem associated with the symmetric vortices in the nonrotating case. As a result of this bifurcation, for supercritical rotation frequencies, the symmetric vortex becomes an orbitally stable local minimizer of the energy functional in the rotational reference frame. At the same time, a new family of asymmetric vortices is born and the asymmetric vortices are placed at a circle of small radius around the center of the harmonic potential. The asymmetric vortices are saddle points of the energy functional, but nevertheless, they are orbitally stable in the time evolution of the Gross–Pitaevskii equation. Although our rigorous results are obtained far from the Thomas–Fermi limit, where the vortex state is close to the linear eigenstate of the quantum harmonic oscillator, our numerical evidence indicates that the result holds true for the entire range of chemical potentials, including the large-density vortex states near the Thomas–Fermi limit.

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