# Bifurcation of Bloch Waves in the Gross-Pitaevskii Equation

PHYSICS 4P06

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# Gross-Pitaevskii Equation

- Let us study the time-independent GPE in a periodic potential in one dimension

$$-\frac{d^2\psi}{dx^2} + V(x)\psi + c|\psi|^2\psi = \mu\psi$$

- ψ is a wave function
- μ is the chemical potential
- c determines the strength of the nonlinear term
- V is the periodic potential, say V(x)=cos(x)

# Bloch Waves

- Bloch waves are plane waves in lattices

$$\psi(x) = e^{ikx}\phi_k(x)$$

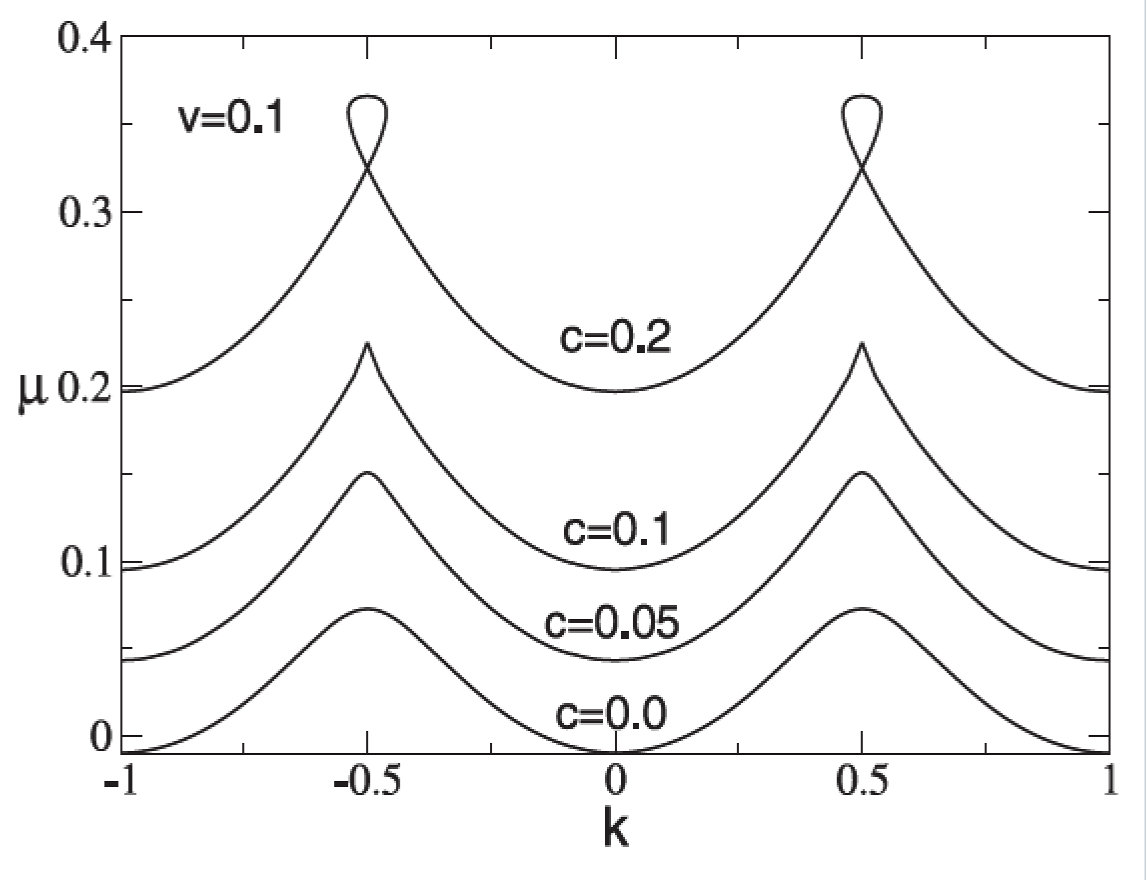
with

$$\phi(x+2\pi) = \phi(x)$$

- k-quasi-momentum
- ψ is periodic for k=0
- ψ is anti-periodic for k=1/2

# Lowest Energy Band

- Bifurcation at $c = c_* = 0.1$
- Brillouin zone $k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$




Biao Wu and Qian Niu (2003) Superfluidity of Bose–Einstein condensate in an optical lattice: Landau–Zener tunnelling and dynamical instability. New Journal of Physics. 5: 104.

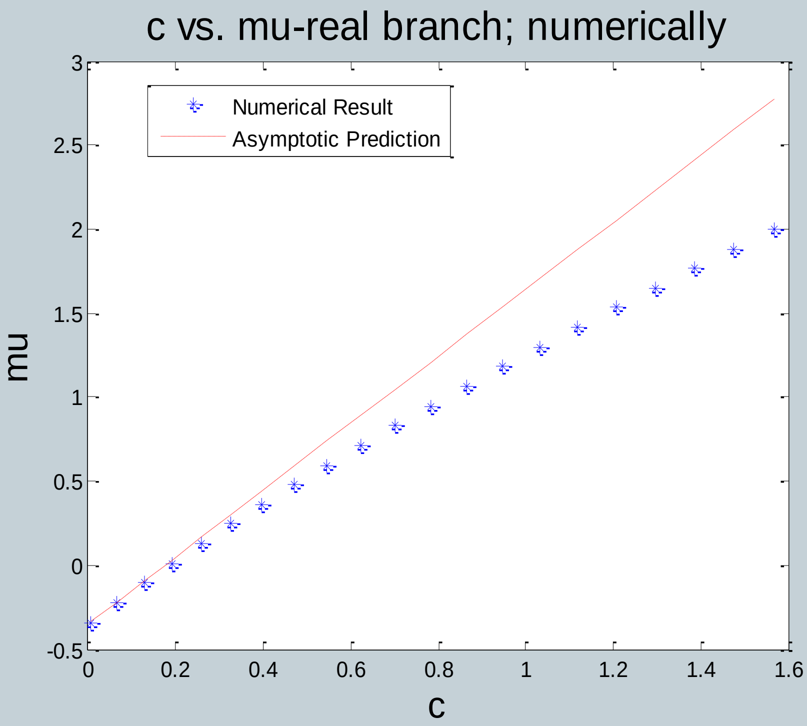
# Goal

- Can we justify qualitative behaviour observed numerically using analytical methods?
- Can we recover the loops in the Bloch bands
- Can we analyse the stability of these steady state solutions

# Stationary Real Branch

- Take k=1/2, so ψ is anti-periodic $\psi(x+2\pi) = -\psi(x)$
- For c>0 we can numerically solve for real ψ

$$-\frac{d^2\psi}{dx^2} + V(x)\psi + c\psi^3 = \mu\psi$$



# Linearization Operators

- Linearization operator with respect to real perturbations

$$L_{+} = -\partial_x^2 + V(x) + 3c\psi^2(x) - \mu$$

- Linearization operator with respect to imaginary perturbations

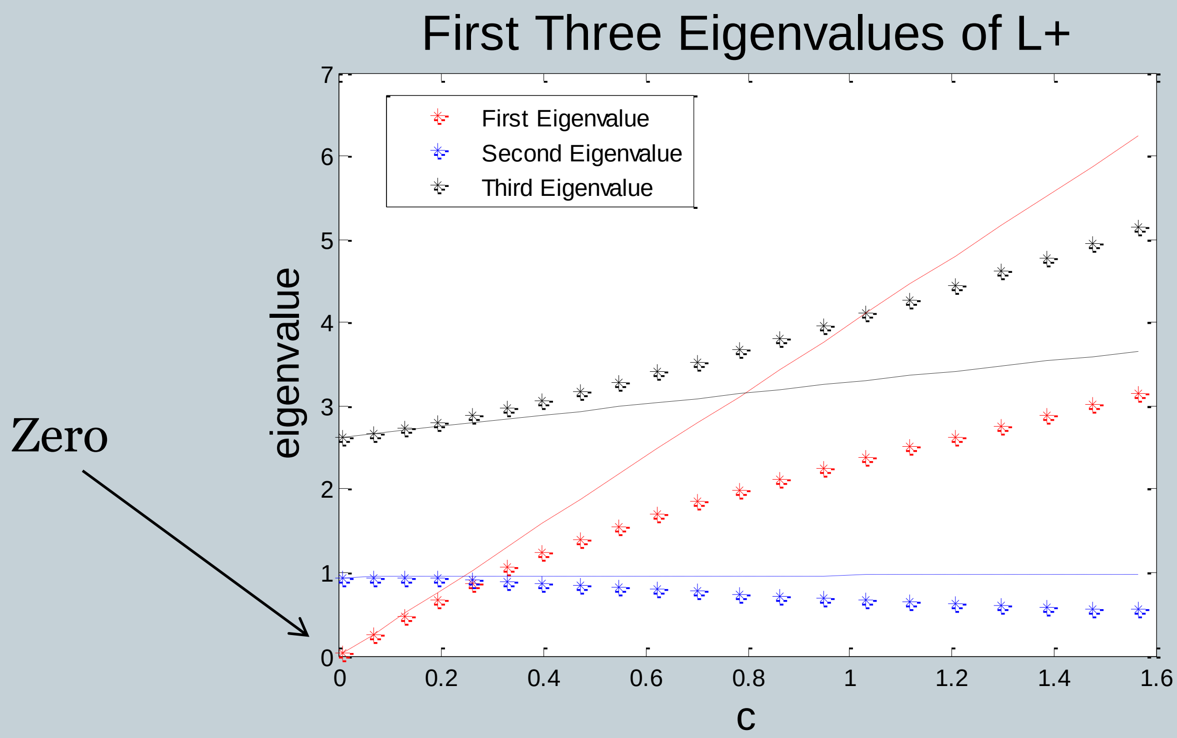
$$L_{-} = -\partial_x^2 + V(x) + c\psi^2(x) - \mu$$

# Finding the Bifurcation

Linearization operators:

$$L_+ = -\partial_x^2 + V(x) + 3c\psi^2(x) - \mu$$

$$L_- = -\partial_x^2 + V(x) + c\psi^2(x) - \mu$$

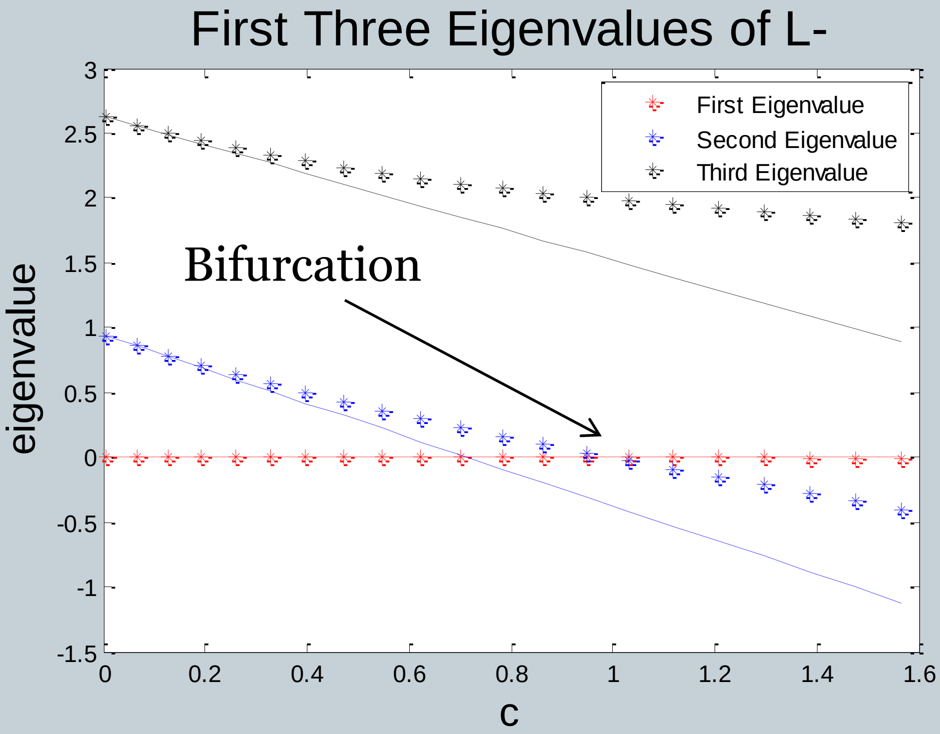


# Finding the Bifurcation

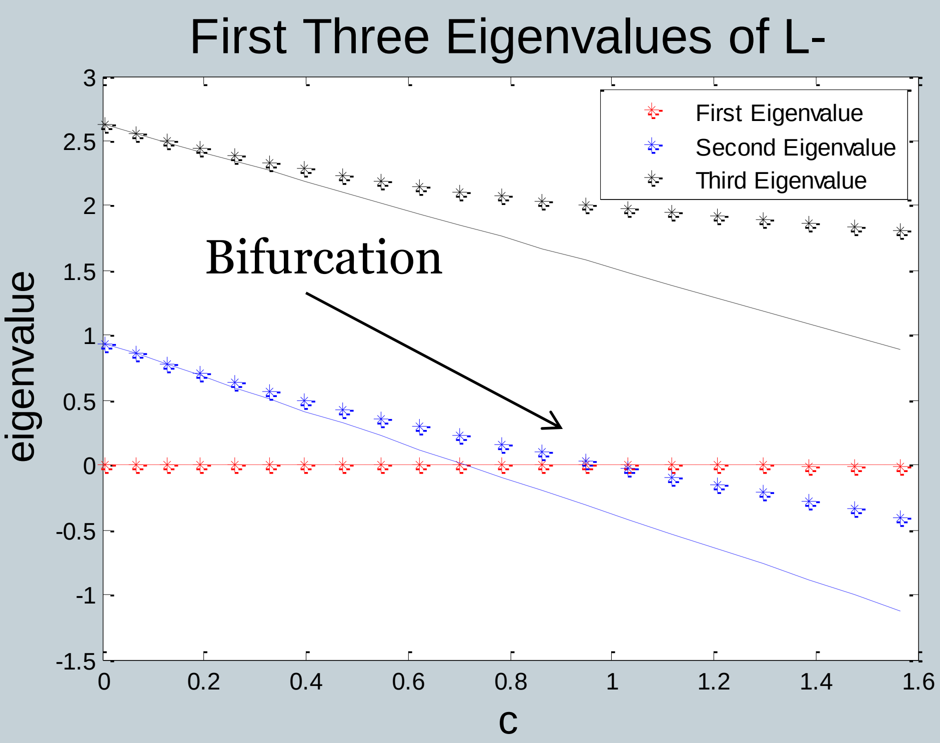
Linearization operators:

$$L_+ = -\partial_x^2 + V(x) + 3c\psi^2(x) - \mu$$

$$L_- = -\partial_x^2 + V(x) + c\psi^2(x) - \mu$$



# Finding the Bifurcation




$$L\_\psi = 0$$

For all ψ by construction

$$L^*\_\varphi_* = 0$$

At the bifurcation point

$$\langle \varphi_*, \psi_* \rangle_{L^2} = 0$$

# Local Bifurcation Analysis

$\psi_*, \mu_*$ Solution at c=c$_*$

$$c = c_* + \varepsilon \qquad \mu = \mu_* + M$$

Let us decompose

$$\psi(x) = \psi_*(x) + ia\varphi_* + u(x) + iw(x)$$

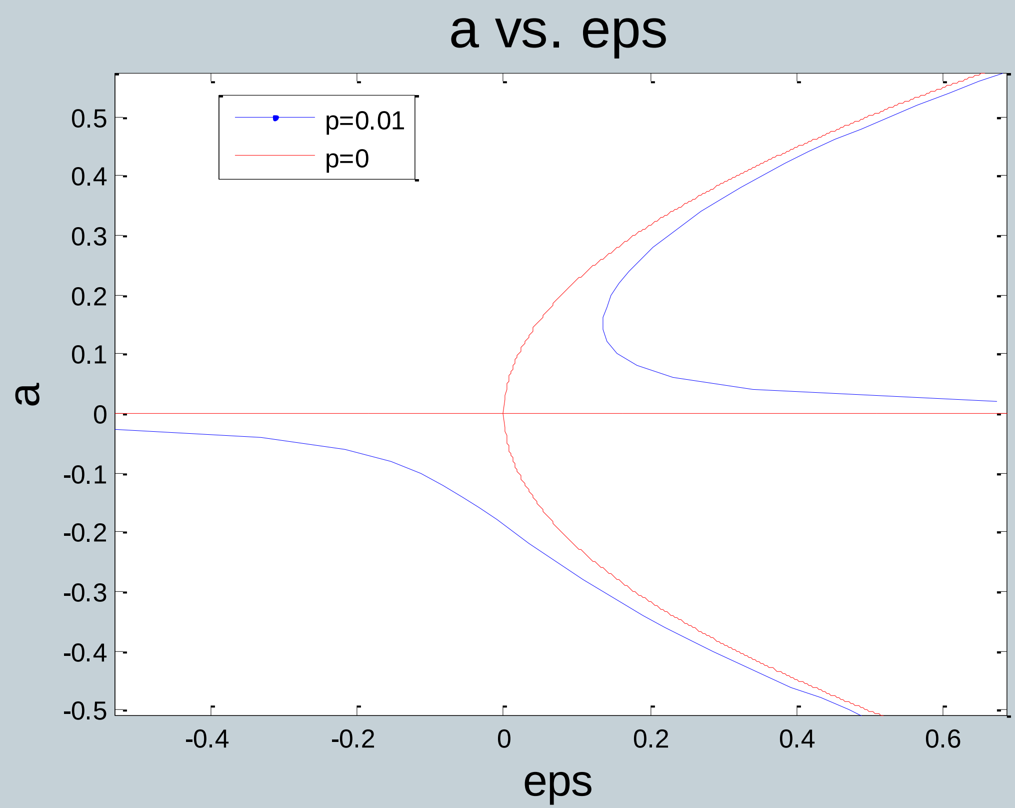
We seek relationships between parameters ε, a and M

# Normal Form Equations

$$\varepsilon a P_0 + a^3 Q_0 + p R_0 = 0$$

$P_o$, $Q_o$, $R_o$ are numerical constants

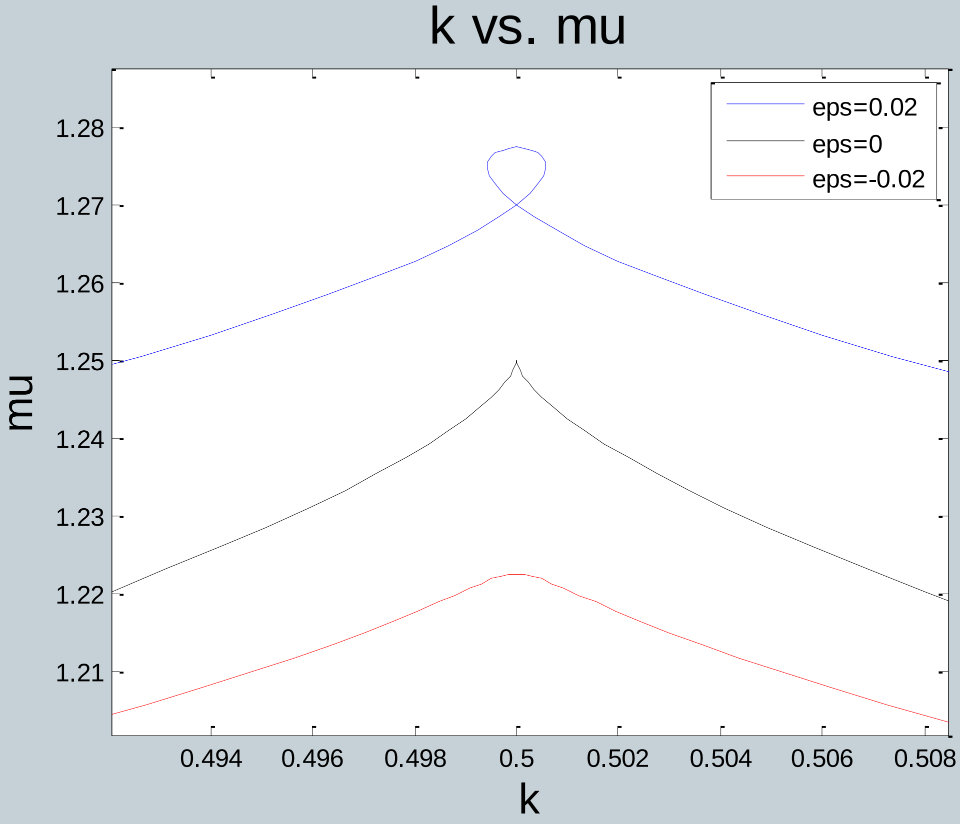
$$p = 1/2 - k$$



# Normal Form Equations

$$\varepsilon a P_0 + a^3 Q_0 + p R_0 = 0 \qquad M = \alpha_0 \varepsilon + \beta_0 a^2$$

$$p = 1/2 - k$$



# Stability

To expose the stability of the stationary solutions we consider the full time-dependent Gross-Pitaevskii equation,

$$i\frac{d\Psi}{dt} = -\frac{d^2\Psi}{dx^2} + c|\Psi|^2\Psi + V(x)\Psi$$

where,

$$\Psi = \Psi(x,t) = e^{-i\mu t}\psi(x)$$

$$\psi(x) - \text{stationary state}$$

# Stability Analysis

Again consider a neighbourhood of the bifurcation,

$$c = c_* + \varepsilon \qquad \mu = \mu_* + M(t)$$

Parameters are now functions of time,

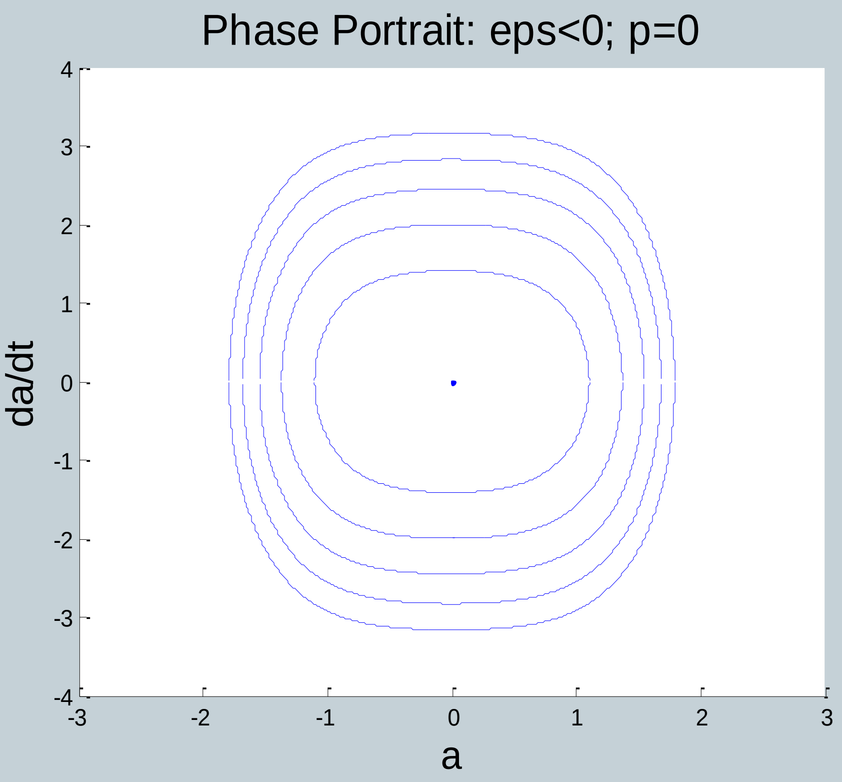
M=M(t)    a=a(t)

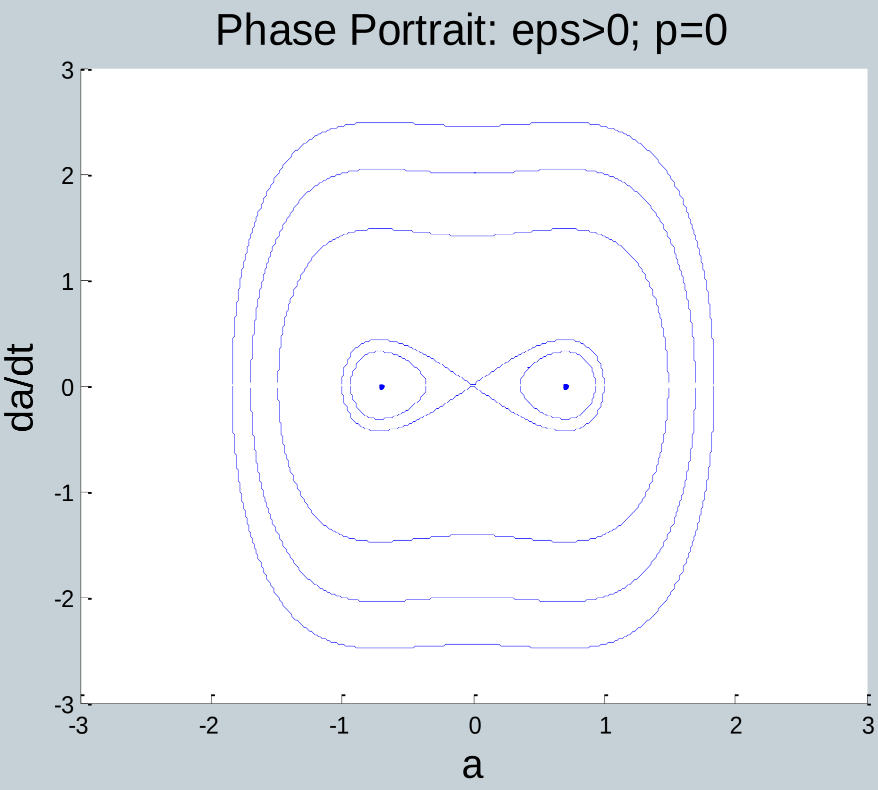
Analysis yields the time-dependent normal form equation,

$$\ddot{a}N_0 + \varepsilon a P_0 + a^3 Q_0 + p R_0 = 0$$

# Phase Portraits

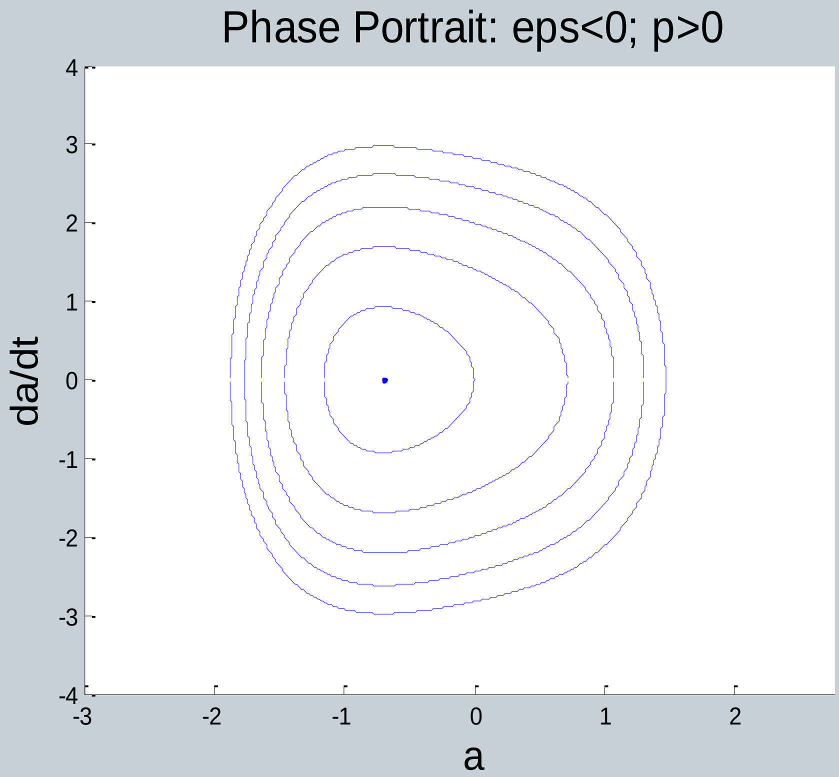
- Phase portraits for p=0, k=1/2,

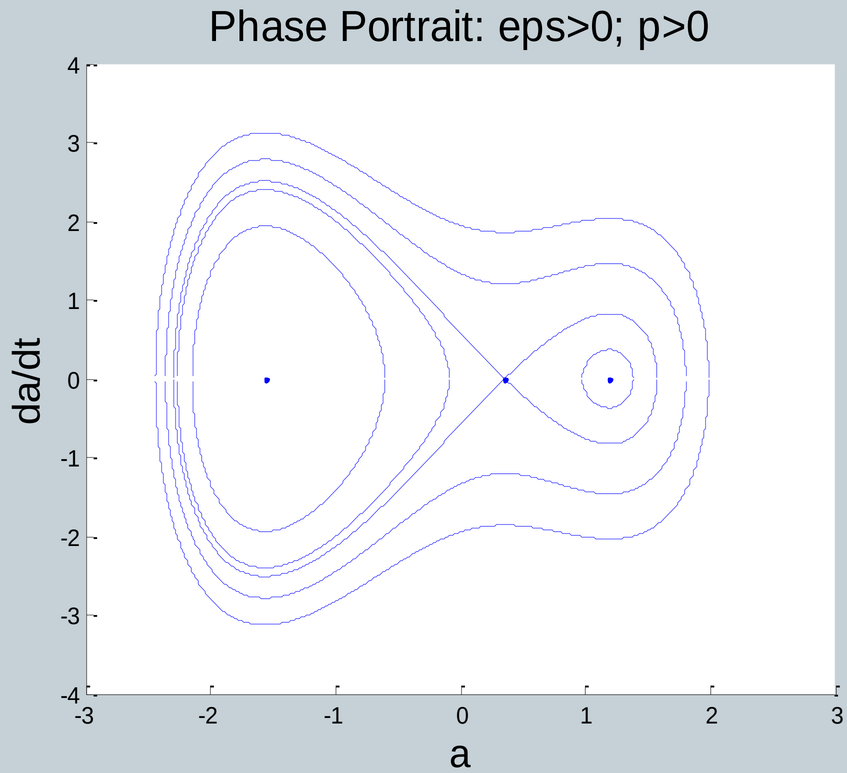


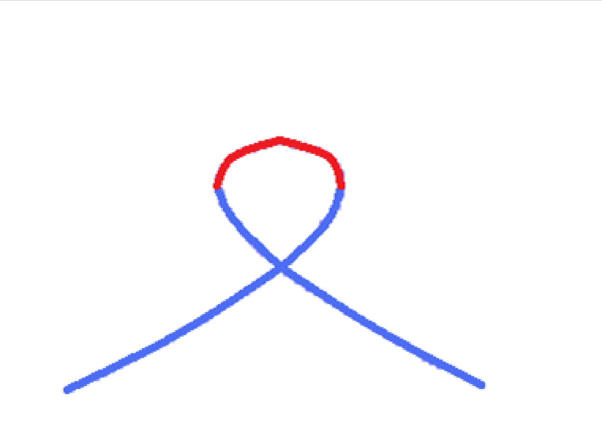
# Phase Portraits

- Phase portraits for p≠0 (small), k=1/2-p



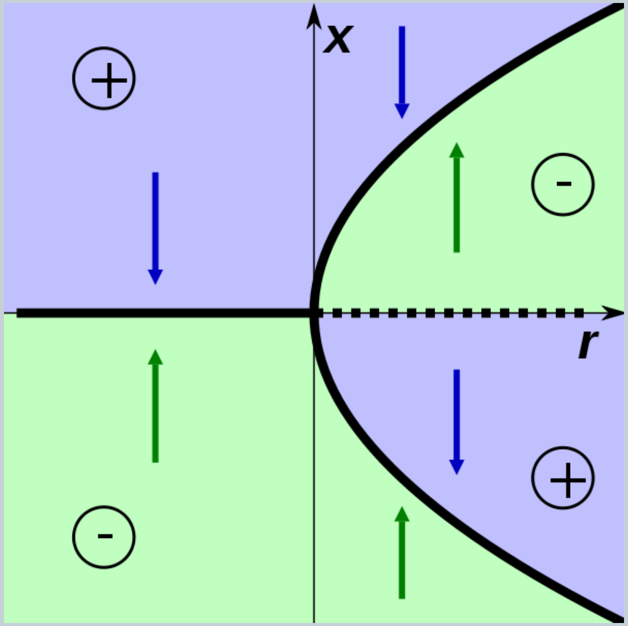



# Stability



- Unstable solutions are shown in red
- Stable solutions are shown in blue

Bifurcation is a supercritical pitchfork bifurcation.



# Summary

- Bifurcation analysis recovers qualitative behaviour of solutions
- Analysis is valid for any excited state and both for c>0 and c<0