DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **32**, Number **5**, May **2012**

doi:10.3934/dcds.2012.32.1775

pp. **1775–1799**

# BREATHER CONTINUATION FROM INFINITY IN NONLINEAR OSCILLATOR CHAINS

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(Communicated by Dmitry Treschev)

Abstract. Existence of large-amplitude time-periodic breathers localized near a single site is proved for the discrete Klein–Gordon equation, in the case when the derivative of the on-site potential has a compact support. Breathers are obtained at small coupling between oscillators and under nonresonance conditions. Our method is different from the classical anti-continuum limit developed by MacKay and Aubry, and yields in general branches of breather solutions that cannot be captured with this approach. When the coupling constant goes to zero, the amplitude and period of oscillations at the excited site go to infinity. Our method is based on near-identity transformations, analysis of singular limits in nonlinear oscillator equations, and fixed-point arguments.

1. **Introduction.** Recent studies of spatially localized and time-periodic oscillations (breathers) in lattice models of DNA [7, 16] call for systematic analysis of such excitations in the discrete Klein–Gordon equation

$$\ddot{x}_n + V'(x_n) = \gamma\left(x_{n+1} - 2x_n + x_{n-1}\right), \quad n \in \mathbb{Z}, \tag{1}$$

where $\gamma > 0$ is a coupling constant, $V : \mathbb{R} \to \mathbb{R}$ is a nonlinear potential, and $\mathbf{x}(t) = \{x_n(t)\}_{n\in\mathbb{Z}}$ is a sequence of real-valued amplitudes at time $t \in \mathbb{R}$.

In the classical Peyrard-Bishop model for DNA [17], $V$ is a Morse potential having a global minimum at $x = 0$, confining as $x \to -\infty$ and saturating at a constant level as $x \to \infty$. However, recent studies [15, 20] argued that the Morse potential should be replaced by a potential with a local maximum at $x = a_0 > 0$, which induces a double-well structure, where one of the wells extends to infinity (both kinds of potentials are depicted in Figure 1). The existence of breathers residing in the potential well near $x = 0$ can be proved with classical methods such as the center manifold reduction for maps [6, 8], variational methods [3, 14], and the continuation from the anticontinuum limit $\gamma \to 0$ [2, 10, 18].

2000 *Mathematics Subject Classification.* Primary: 37K50; Secondary: 37K60, 37N25.

*Key words and phrases.* Discrete Klein–Gordon equation, discrete breathers, nonlocal bifurcations, asymptotic methods, fixed-point arguments.

D.P. was supported in part by the Ambassade de France au Canada.

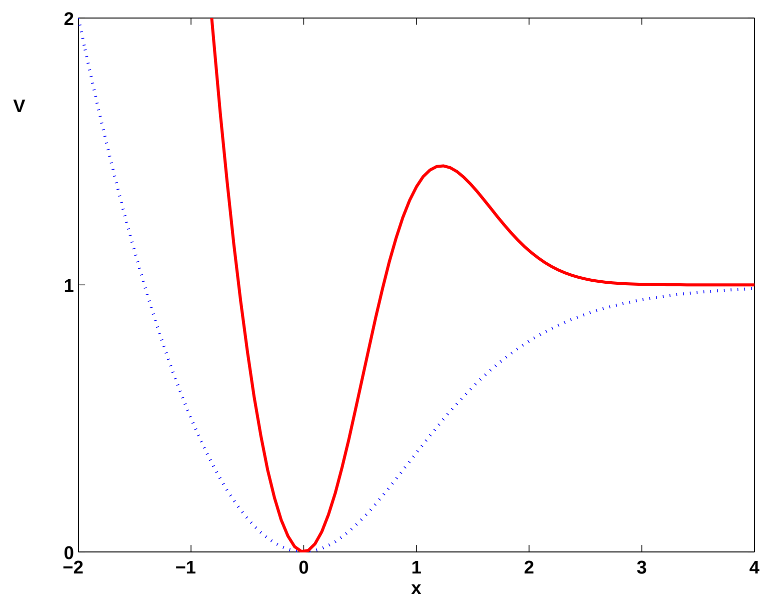


FIGURE 1. Morse potential (dotted line) and modified double-well potential (solid line).

A more delicate problem is the existence of large-amplitude breathers residing in the other potential well that extends to infinity. Large-amplitude stationary solutions bifurcating from infinity as $\gamma \to 0$ have been obtained in [16]. These solutions are localized near a single site, say $n = 0$, and their amplitude diverges as $\gamma \to 0$. Large-amplitude breathers in a finite-size neighborhood of these stationary solutions have been constructed in [7] for small coupling $\gamma$, using the contraction mapping theorem and scaling techniques. These large-amplitude breathers are confined on the other side of the potential barrier of $V$, and their amplitude goes to infinity as $\gamma \to 0$. Existence of large-amplitude breathers oscillating everywhere above the potential barrier of $V$ was left open in [7].

Our goal is to show the existence of large-amplitude breathers oscillating in several potential wells, setting-up a continuation of these solutions from infinity as $\gamma \to 0$. To illustrate some key points of our analysis, let us consider the example

$$V(x) = \frac{1}{4}(1 + e^{-x^2}(x^2 - 1)). \tag{2}$$

Here $V$ has a global minimum at $x = 0$, a pair of symmetric global maxima at $x = \pm a_0$ with $a_0 > 0$, and $\lim_{x\to\pm\infty} V(x) = \frac{1}{4}$.

In the standard anti-continuum limit, one can set $\gamma = 0$, $x_n = 0$ for all $n \in \mathbb{Z}\backslash\{0\}$, and consider a time-periodic solution $x_0(t) \equiv x(t)$ of the nonlinear oscillator equation

$$\ddot{x} + V'(x) = 0. \tag{3}$$

Under a nonresonance condition, this compactly supported time-periodic solution can be continued for $\gamma \approx 0$ into an exponentially localized time-periodic breather solution using the implicit function theorem [10].

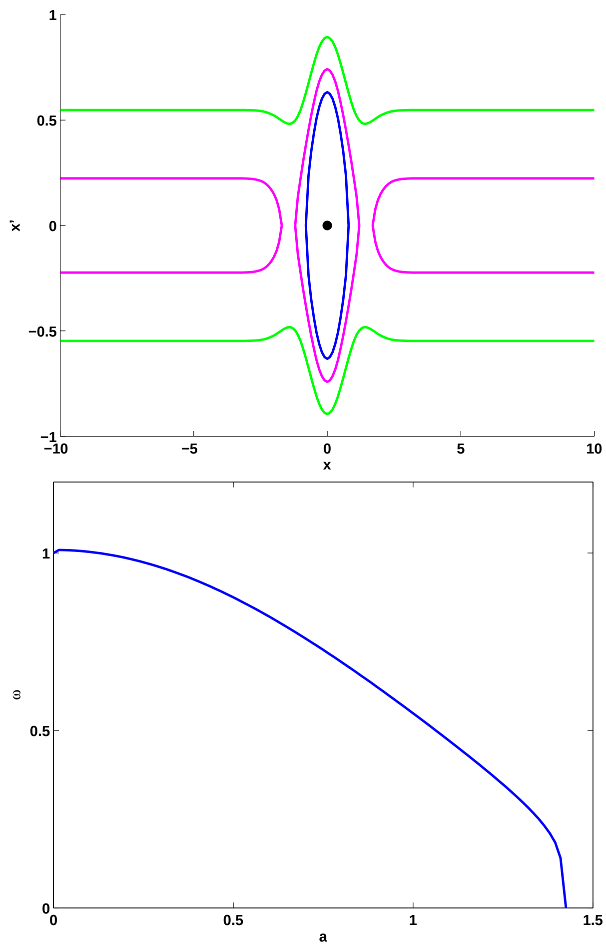


FIGURE 2. The phase plane $(x, \dot{x})$ (top) and the frequency–amplitude diagram $(\omega, a)$ (bottom) for the potential (2).

The phase plane $(x, \dot{x})$ and the frequency-amplitude $(\omega, a)$ diagram of the nonlinear oscillator equation (3) with the potential (2) are shown on Figure 2. In this case, the periodic solution $x(t)$ has a cut-off amplitude at $a = a_0$. Only the family of periodic solutions with $a \in (0, a_0)$ can be continued by the anti-continuum technique developed by MacKay and Aubry [10].

In addition, there are two families of unbounded solutions: one corresponds to oscillations on the other side of the potential barrier of $V$ for $|x| > a_0$ and the other one corresponds to oscillations above the potential barrier. Roughly speaking, the new technique developed in [7] allows one to obtain large-amplitude breathers "close" to unbounded solutions of the first family for $\gamma \approx 0$.

The present paper considers large-amplitude breathers near the second family of unbounded solutions. These two families of breathers are obtained by "continuation from infinity" for arbitrarily small values of $\gamma$, but without reaching $\gamma = 0$. In this case, the potential $V$ in the nonlinear oscillator equation (3) can be simply replaced by

$$V_\gamma(x) = V(x) + \gamma x^2. \tag{4}$$

The potential $V_\gamma$ includes a restoring force originating from the nearest-neighbors coupling in the discrete Klein–Gordon equation (1). As $\gamma \to 0$, the amplitudes and periods of the resulting breathers go to infinity. As a result, we need a careful control of nonresonance conditions in order to prove the existence of such breathers.

Although a part of our continuation procedure involving the contraction mapping theorem is close to the one developed in [7], our mathematical analysis is quite different because our breather solutions scale differently in the different potential wells, which induces some singular perturbation analysis and more delicate estimates than in [7]. Note also that the contraction mapping theorem has been used by Treschev [19] to prove the existence of other types of localized solutions (solitary waves) in Fermi-Pasta-Ulam lattices, in which nearest-neighbors are coupled by an anharmonic potential having a repulsive singularity at a short distance. In this case, the existence problem yields an advance-delay differential equation with other kinds of mathematical difficulties.

To simplify our analysis, we assume that $V$ is symmetric and bounded, whereas $V'$ has a compact support. To be precise, the following properties on $V$ are assumed:

- (P1) $V \in C^2(\mathbb{R})$ and $V(-x) = V(x)$ for all $x \in \mathbb{R}$;
- (P2) There is $x_0 > 0$ such that $V \in C^7(-x_0, x_0)$ and the Taylor expansion of $V$ at $x = 0$ is $V(x) = \frac{1}{2}\kappa^2 x^2 + O(x^6)$ with $\kappa > 0$;
- (P3) $0 \leq V(x) \leq V_L$ for all $x \in \mathbb{R}$ and some $V_L > 0$;
- (P4) $V'(x)$ is compactly supported on $[-a_0, a_0]$ for some $a_0 > 0$ such that $V(x) = V_\infty$ for $|x| \geq a_0$ and $V_\infty \in (0, V_L]$.

Assumption (P1) allows us to consider symmetric periodic oscillations, which can be studied on the quarter of the fundamental period. This assumption simplifies the presentation but is not essential, and our analysis could be extended to potentials confining at $-\infty$ (as in Figure 1).

Assumption (P2) allows us to develop a contraction mapping argument for the small-amplitude oscillations on the sites $n \neq 0$, a procedure which cannot be carried out if a quartic term is present in the expansion of $V$ near the origin. It would be useful to relax this condition, which assumes a very weak anharmonicity of small amplitude oscillations. Note that the quartic term in $V(x)$ near $x = 0$ is also excluded in the recent analysis of scattering of small initial data to zero equilibrium by Mielke & Patz [13].

Assumption (P3) allows for large-amplitude oscillations at the central site $n = 0$.

Assumption (P4) allows us to consider linear oscillations of the central site outside the compact support of $V'$. This property is used to solve the singularly perturbed oscillator equation at $n = 0$. This compact support assumption is quite restrictive, and it would be interesting to relax it in a future work, for example, by considering exponentially decaying potentials (as in example (2)) and treating exponential tails as perturbations of the present case.

We note that Fura & Rybicki [5] have proved the existence of periodic solutions bifurcating from infinity for a class of finite-dimensional Hamiltonian systems with asymptotically linear potentials using degree theory. Our analysis is different and consists of two steps. We first reduce the infinite-dimensional Hamiltonian system to a singularly perturbed oscillator equation describing large-amplitude oscillations at the central site, using the contraction mapping theorem. Once this has been achieved, we solve the reduced problem using a topological method (Schauder's fixed point theorem).

Our main result is the existence of the large-amplitude breathers if the potential $V$ satisfies assumptions (P1)–(P4) as well as the technical non-degeneracy condition in equation (11) below. As further problems, it would be interesting to analyze the existence of multibreather solutions bifurcating from infinity, as well as the stability of such solutions, as it was done previously for finite-amplitude breathers near the standard anti-continuum limit (see, e.g., [1, 2, 4, 9, 11, 12]).

The article is organized as follows. Section 2 describes the main result. Large-amplitude oscillations near $n = 0$ are analyzed in Section 3. Small-amplitude oscillations for $n \neq 0$ are considered in Section 4. The proof of the main theorem is given in Section 5. Section 6 gives a proof that the large-amplitude breather decays exponentially in $n \in \mathbb{Z}$.

2. **Main results.** We shall consider the discrete Klein–Gordon equation (1) for small $\gamma > 0$ and assume that the breather is localized near the central site $n = 0$. We consider oscillations in the potential $V_\gamma(x)$ at the energy level $E$:

$$\ddot{x} + V_\gamma'(x) = 0 \quad \Rightarrow \quad E = \frac{1}{2}\dot{x}^2 + V_\gamma(x). \tag{5}$$

Thanks to assumption (P3), the anti-continuum limit $\gamma \to 0$ is singular for $E > V_L$ in the sense that a bounded trajectory of system (5) trapped by the quadratic potential $\gamma x^2$ degenerates into an unbounded trajectory as $\gamma \to 0$.

We would like to select a unique $T$-periodic solution of (5) by fixing its energy $E > V_L$ and choosing $\gamma$ small enough. For a fixed $E > V_L$, we will be working for sufficiently small $\gamma > 0$ to ensure that $V_\gamma(a) = E$ admits a unique positive solution $a(E, \gamma)$. More precisely, thanks to assumptions (P3) and (P4), we obtain a unique solution $a = (E - V_\infty)^{1/2}\gamma^{-1/2}$ for $\gamma < (E - V_L)/a_0^2$. Fixing $\dot{x}(0) = 0$, we can parameterize periodic solutions by $x(0) = a(E, \gamma) > 0$, and their period can be written $T = T(E, \gamma)$. Thanks to assumption (P4), we shall prove (in Section 3) that for any $E > V_L$

$$T(E, \gamma) = \frac{\sqrt{2}\pi}{\gamma^{1/2}} + \lambda(E) + O(\gamma) \quad \text{as} \quad \gamma \to 0 \tag{6}$$

where

$$\lambda(E) = 2\sqrt{2}\left(\int_0^{a_0} \frac{dx}{(E - V(x))^{1/2}} - \frac{a_0}{(E - V_\infty)^{1/2}}\right).$$

Thanks to assumption (P1), the $T$-periodic solution $x(t)$ of the nonlinear oscillator equation (5) with $x(0) = a > 0$ and $\dot{x}(0) = 0$ is symmetric with respect to reflections about the points $t = 0$ and $t = \frac{T}{2}$ and anti-symmetric with respect to reflection about the points $t = \frac{T}{4}$ and $t = \frac{3T}{4}$. Therefore, the $T$-periodic solution satisfies

$$x(-t) = x(t) = -x\left(\frac{T}{2} - t\right), \quad t \in \mathbb{R}. \tag{7}$$

The normalized frequency of oscillations is defined by

$$\omega_0(E, \gamma) = \frac{2\pi}{T(E, \gamma)}\gamma^{-1/2} \tag{8}$$

such that $\omega_0(E, \gamma) \to \sqrt{2}$ as $\gamma \to 0$ for a fixed $E > V_L$. To avoid resonances of large-amplitude oscillations at the central site $n = 0$ with small-amplitude oscillations

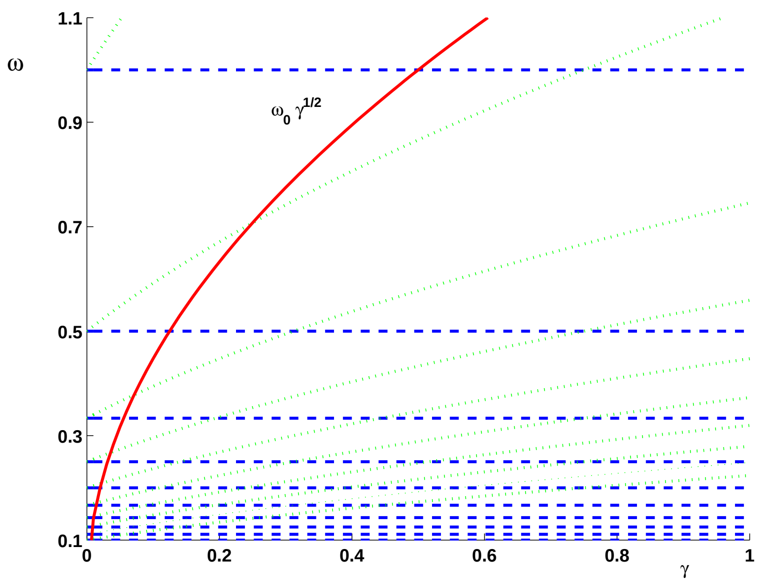


FIGURE 3. Resonance tongues on the plane $(\gamma,\omega)$ (delimited by dotted and dashed lines), and breather frequency curve $\omega=\sqrt{\gamma}\omega_0$ (solid line) for $\kappa=1$. No resonances occur for $\gamma\in\cup_{m\geq m_0}(\Gamma_m,\gamma_m)$, where $\Gamma_m$ and $\gamma_m$ correspond to the intersections of $\omega=\sqrt{\gamma}\omega_0$ with $\omega=\frac{\sqrt{\kappa^2+4\gamma}}{m}$ (dotted lines) and $\omega=\frac{\kappa}{m}$ (dashed lines) respectively.

at the other sites $n\in\mathbb{Z}\backslash\{0\}$, we will show (in Section 4) that the following non-resonance conditions

$$\kappa^2-m^2\gamma\omega_0^2(E,\gamma)+2\gamma(1-\cos(q))\neq 0, \tag{9}$$

must be satisfied for all $m\in\mathbb{Z}$ and all $q\in[-\pi,\pi]$. Here $\kappa$ is the parameter of the expansion of $V$ about $x=0$ in assumption (P2).

For a fixed $E>V_L$, we shall now consider the non-resonant set of parameters (breather frequency $\omega=\frac{2\pi}{T}$ and coupling constant $\gamma$). We plot on Figure 3 the admissible regions between the boundaries of the non-resonant set, given by the curves $\omega=\frac{\sqrt{\kappa^2+4\gamma}}{m}$ and $\omega=\frac{\kappa}{m}$, together with the curve $\omega=\sqrt{\gamma}\,\omega_0(E,\gamma)$. Non-resonance conditions (9) are satisfied if $\gamma$ belongs to the set $C_E=\cup_{m\geq m_0}(\Gamma_m,\gamma_m)$, where $\gamma=\Gamma_m$ and $\gamma=\gamma_m$ correspond to the intersections of the above curves starting with some $m_0\geq 1$. In this case, $\Gamma_m$ and $\gamma_m$ satisfy implicit equations

$$\frac{\sqrt{\kappa^2+4\,\Gamma_m}}{m+1}=\sqrt{\Gamma_m}\omega_0(E,\Gamma_m),\qquad \frac{\kappa}{m}=\sqrt{\gamma_m}\omega_0(E,\gamma_m),\quad m\geq m_0. \tag{10}$$

Equations (10) can be solved for $m$ large enough thanks to expansion (6) and the implicit function arguments, yielding as $m\to\infty$

$$\begin{aligned}\Gamma_m &= \frac{\kappa^2}{2m^2}\Big(1+\frac{\kappa\lambda(E)}{\pi m}-\frac{2}{m}+O(m^{-2})\Big),\\ \gamma_m &= \frac{\kappa^2}{2m^2}\Big(1+\frac{\kappa\lambda(E)}{\pi m}+O(m^{-2})\Big).\end{aligned}$$

In particular, we note that $\Gamma_m < \gamma_m$ and $|\gamma_m - \Gamma_m| = O(m^{-3})$ as $m \to \infty$.

We can now state the main result of this article. Note that the existence of breathers is only obtained for a subset $\tilde{C}_{E,\nu} \subset C_E$ of the non-resonant values of the coupling constant $\gamma$, because our method breaks down near the boundary of $C_E$.

**Theorem 1.** *Assume (P1)–(P4) on $V(x)$ and fix $E > V_L$. Let $x(t)$ be a $T(E,\gamma)$-periodic solution of the nonlinear oscillator equation (5) for small $\gamma > 0$ satisfying symmetries (7) and assume $\lambda'(E) \neq 0$, that is,*

$$\int_0^{a_0} \frac{dx}{(E - V(x))^{3/2}} - \frac{a_0}{(E - V_\infty)^{3/2}} \neq 0. \tag{11}$$

*Fix $\nu \in (0,1)$ and consider the set of coupling constants $\tilde{C}_{E,\nu} = \cup_{m \geq m_0} (\tilde{\Gamma}_m, \tilde{\gamma}_m) \subset C_E$, where $\tilde{\Gamma}_m, \tilde{\gamma}_m$ are defined by the implicit equations*

$$\frac{\sqrt{\kappa^2 + 4\tilde{\Gamma}_m}}{\sqrt{(m+1)^2 - \nu(m+1)}} = \sqrt{\tilde{\Gamma}_m}\,\omega_0(E, \tilde{\Gamma}_m), \qquad \frac{\kappa}{\sqrt{m^2 + \nu m}} = \sqrt{\tilde{\gamma}_m}\,\omega_0(E, \tilde{\gamma}_m), \tag{12}$$

*for $m \geq m_0$, and satisfy as $m \to +\infty$*

$$\begin{aligned} \tilde{\Gamma}_m &= \frac{\kappa^2}{2m^2}\Big(1 + \frac{\kappa\lambda(E)}{\pi m} - \frac{2-\nu}{m} + O(m^{-2})\Big), \\ \tilde{\gamma}_m &= \frac{\kappa^2}{2m^2}\Big(1 + \frac{\kappa\lambda(E)}{\pi m} - \frac{\nu}{m} + O(m^{-2})\Big). \end{aligned}$$

*For all sufficiently small $\gamma$ in $\tilde{C}_{E,\nu}$, there exists a $T$-periodic spatially localized solution $\mathbf{x}(t) \in H^2_{\text{per}}((0,T); l^2(\mathbb{Z}))$ of the Klein–Gordon lattice (1) such that*

$$x_n(t) = x_{-n}(t), \quad n \in \mathbb{Z}; \quad \mathbf{x}(-t) = \mathbf{x}(t) = -\mathbf{x}\left(\frac{T}{2} - t\right), \quad t \in \mathbb{R};$$

*and*

$$\exists C > 0: \quad \sup_{t \in [0,T]} |x_0(t) - x(t)| \leq C\gamma^{-1/4}, \quad \sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} |x_n(t)| \leq C\gamma^{1/4}. \tag{13}$$

**Remark 1.** Since $\|x\|_{L^\infty} = a = O(\gamma^{-1/2})$ as $\gamma \to 0$, the first bound in (13) shows that the relative error $\|x_0 - x\|_{L^\infty} / \|x\|_{L^\infty}$ is as small as $O(\gamma^{1/4})$.

**Remark 2.** If $\nu \in (0,1)$, we still have $\tilde{\Gamma}_m < \tilde{\gamma}_m$ and $|\tilde{\gamma}_m - \tilde{\Gamma}_m| = O(m^{-3})$ as $m \to \infty$. Therefore, the rate of decrease of the interval widths in the set $\tilde{C}_{E,\nu}$ corresponds to the rate of decrease of the widths of the non-resonant intervals in the set $C_E$.

**Remark 3.** Because the proof of Theorem 1 uses Schauder's fixed point theorem, it does not establish uniqueness of the solution. We anticipate that this is a technical difficulty and the solution is unique for fixed $E > V_L$.

**Remark 4.** Although we do not attempt here to deal with non-compact potentials, we believe that assumption (P4) can be relaxed if $V'(x)$ has a sufficiently fast decay to zero as $|x| \to \infty$. In that case, we conjecture that the non-resonance condition (11) would be replaced by

$$Q := \int_0^\infty \left[\frac{1}{(E - V(x))^{3/2}} - \frac{1}{(E - V_\infty)^{3/2}}\right] dx \neq 0,$$

where $V_\infty := \lim_{x \to \infty} V(x)$. Figure 4 illustrates that this condition is satisfied for the particular potential (2), for any finite $E > V_L$ (we note that the value of $Q$ approaches 0 as $E \to \infty$).

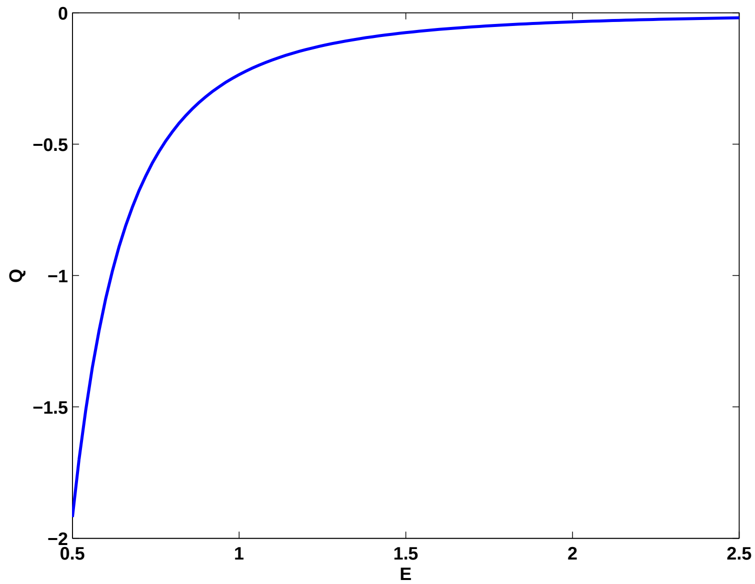


FIGURE 4. Non-resonance coefficient $Q$ versus energy $E$ for the potential (2).

Theorem 1 is proved in Section 5, using intermediate results established for the single oscillator equation (5) with a forcing term (in Section 3) and for the discrete KG equation (1) linearized at zero equilibrium (in Section 4). We finish the article with Section 6, where we prove that the amplitude of breather oscillations decays exponentially in $n$ on $\mathbb{Z}$ in the following sense:

$$\exists D_0 > 0 : \quad \sup_{t\in[0,T]} |x_n(t)| \leq (D_0\gamma)^{(2n-1)/4}, \quad n \in \mathbb{N},$$

for all sufficiently small $\gamma \in \tilde{C}_{E,\nu}$.

## 3. Large-amplitude oscillations at a central site.

We consider here solutions of the nonlinear oscillator equation (5) in the singular limit $\gamma \to 0$. Assumptions (P1)–(P4) on the potential $V$ are used everywhere, without further notes.

**Lemma 1.** *Fix $E > V_L$. There exists $\gamma_0 = \gamma_0(E) > 0$ such that for any $\gamma \in (0, \gamma_0)$, there exist exactly two $T$-periodic solutions of (5) with amplitude*

$$\|x\|_{L^\infty} = (E - V_\infty)^{1/2}\gamma^{-1/2}$$

*satisfying symmetries (7) and the asymptotic expansion as $\gamma \to 0$,*

$$T = \frac{\sqrt{2}\pi}{\gamma^{1/2}} + 2\sqrt{2}\left(\int_0^{a_0} \frac{dx}{(E - V(x))^{1/2}} - \frac{a_0}{(E - V_\infty)^{1/2}}\right) + O(\gamma). \tag{14}$$

*Proof.* Thanks to assumptions (P1) and (P3), for a fixed $E > V_L$ there exists $\gamma_0(E) > 0$ such that for any $\gamma \in (0, \gamma_0)$ equation $V_\gamma(x) = V(x) + \gamma x^2 = E$ admits only two solutions $x = \pm a$ with $a > 0$ such that $V'_\gamma(x) > 0$ for all $x \geq a$. (One can easily compute that $\gamma_0(E) = (E - V_L)/a_0^2$.) The two periodic solutions with symmetries (7) are constructed from the same bounded trajectory on the phase plane $(x,\dot{x})$ departing from either the point $(x(0),\dot{x}(0))=(a,0)$ or the point $(x(0),\dot{x}(0))=(-a,0)$.

Because $E=\gamma a^2+V(a)$, it follows from assumption (P4) that

$$a=(E-V_\infty)^{1/2}\gamma^{-1/2},\quad \gamma\in(0,\gamma_0).$$

Asymptotic expansion of the period $T$ is found from the exact formula

$$T=\sqrt{2}\int_{-a}^{a}\frac{dx}{\sqrt{E-V_\gamma(x)}}=2\sqrt{2}\int_0^a\frac{dx}{\sqrt{E-\gamma x^2-V(x)}}. \tag{15}$$

Thanks to assumption (P4), we know that $V(x)=V_\infty$ for all $x\in[a_0,a]$, so that

$$T=\frac{2\sqrt{2}}{\sqrt{\gamma}}\left(\frac{\pi}{2}-\theta_0\right)+2\sqrt{2}\int_0^{a_0}\frac{dx}{\sqrt{E-\gamma x^2-V(x)}},$$

where $\theta_0$ is the smallest positive root of $a\sin(\theta)=a_0$ satisfying

$$\theta_0=\arcsin\left(\frac{a_0\gamma^{1/2}}{(E-V_\infty)^{1/2}}\right)=\frac{a_0\gamma^{1/2}}{(E-V_\infty)^{1/2}}+O(\gamma^{3/2})\quad\text{as}\quad\gamma\to0.$$

Since $E>V_L$ is fixed, there is $C(E)>0$ such that $E-V(x)\geq C(E)$ for all $x\in[0,a_0]$. As a result, the asymptotic expansion

$$\int_0^{a_0}\frac{dx}{\sqrt{E-\gamma x^2-V(x)}}=\int_0^{a_0}\frac{dx}{(E-V(x))^{1/2}}+O(\gamma)\quad\text{as}\quad\gamma\to0,$$

concludes the proof of the asymptotic expansion (14). □

Let us represent the solution of (5) for $E>V_L$ in the form

$$x(t)=\frac{X(\tau)}{\gamma^{1/2}},\quad \tau=\gamma^{1/2}t. \tag{16}$$

By Lemma 1, we have $\|X\|_{L^\infty}=O(1)$ and $T_0:=\gamma^{1/2}T=O(1)$ as $\gamma\to0$ with precise value $\|X\|_{L^\infty}=(E-V_\infty)^{1/2}$ and the asymptotic expansion,

$$T_0=\sqrt{2}\pi+2\sqrt{2}\gamma^{1/2}\left(\int_0^{a_0}\frac{dx}{(E-V(x))^{1/2}}-\frac{a_0}{(E-V_\infty)^{1/2}}\right)+O(\gamma^{3/2}), \tag{17}$$

as $\gamma\to0$. We shall now derive a series of estimates that will be useful for the proof of Theorem 1.

**Corollary 1.** *Let $X(\tau)$ be the $T_0$-periodic function defined by the solution of Lemma 1 in parametrization (16) for any fixed $E>V_L$. Then, $X\in C^3_{\rm per}(0,T_0)$ and for sufficiently small $\gamma>0$, there exists $C(E)>0$ such that $\|X\|_{H^1_{\rm per}}\leq C(E)$. Moreover, $\|X\|_{C^1}\leq(1+\sqrt{2})\sqrt{E}$.*

*Proof.* We recall that $\|X\|_{L^\infty}=(E-V_\infty)^{1/2}\leq\sqrt{E}$. Let us rewrite the energy conservation (5) in parametrization (16):

$$\frac{1}{2}\dot{X}^2+X^2+V(\gamma^{-1/2}X)=E. \tag{18}$$

Since $V\geq0$ we have $\|\dot{X}\|_{L^\infty}\leq\sqrt{2E}$, which gives the bound on $\|X\|_{C^1}$. This also gives the uniform bound on $\|X\|_{H^1_{\rm per}}$ since $T_0=O(1)$ as $\gamma\to0$.

Let us also rewrite the second-order equation (5) in parametrization (16):

$$\ddot{X}(\tau)+2X(\tau)+\gamma^{-1/2}V'(\gamma^{-1/2}X(\tau))=0. \tag{19}$$

Thanks to assumption (P1), the solution $X(\tau)$ is actually in $C^3_{\rm per}(0,T_0)$. □

The potential term of the nonlinear equation (19) is a singular contribution to the linear equation as $\gamma \to 0$. Because of the singular contribution, $\|\ddot{X}\|_{L^\infty}$ grows as $\gamma \to 0$. Nevertheless, thanks to assumption (P4) of the compact support of $V'(x)$, the solution $X(\tau)$ stays in the domain $|X| \geq a_0\gamma^{1/2}$, where $V'(\gamma^{-1/2}X) = 0$ for most of the times $\tau$ in the period $[0, T_0]$. The following lemma estimates the size of the time interval, for which the solution stays in the domain $|X| \leq a_0\gamma^{1/2}$.

**Lemma 2.** *Let $X(\tau)$ be the same as in Corollary 1. Let $\Delta T_0$ be the length of the subset of $[0, T_0]$ in which $|X(\tau)| \leq a_0\gamma^{1/2}$. Then, $\Delta T_0$ admits the asymptotic expansion*

$$\Delta T_0 = 2\sqrt{2}\gamma^{1/2} \int_0^{a_0} \frac{dx}{(E - V(x))^{1/2}} + O(\gamma^{3/2}) \quad \text{as} \quad \gamma \to 0. \tag{20}$$

*Proof.* Consider the splitting of $[0, T_0]$ into

$$\begin{aligned} &\left[0, \tfrac{1}{4}(T_0 - \Delta T_0)\right] \cup \left[\tfrac{1}{4}(T_0 - \Delta T_0), \tfrac{1}{4}(T_0 + \Delta T_0)\right] \cup \left[\tfrac{1}{4}(T_0 + \Delta T_0), \tfrac{1}{4}(3T_0 - \Delta T_0)\right] \\ &\cup \left[\tfrac{1}{4}(3T_0 - \Delta T_0), \tfrac{1}{4}(3T_0 + \Delta T_0)\right] \cup \left[\tfrac{1}{4}(3T_0 + \Delta T_0), T_0\right]. \end{aligned}$$

Thanks to the symmetries (7), we have

$$X\left(\frac{1}{4}T_0\right) = X\left(\frac{3}{4}T_0\right) = 0.$$

In the first, third, and fifth intervals, the second-order equation (19) for sufficiently small $\gamma > 0$ becomes the linear oscillator

$$\ddot{X} + 2X = 0.$$

An explicit solution with $X(0) = (E - V_\infty)^{1/2}$ and $\dot{X}(0) = 0$ has the form

$$X(\tau) = (E - V_\infty)^{1/2} \cos(\sqrt{2}\tau), \quad \tau \in \left[-\frac{1}{4}(T_0 - \Delta T_0), \frac{1}{4}(T_0 - \Delta T_0)\right]. \tag{21}$$

The matching condition $X(\tau_0) = a_0\gamma^{1/2}$ at $\tau_0 = \frac{1}{4}(T_0 - \Delta T_0)$ gives

$$\cos\left(\frac{T_0 - \Delta T_0}{2\sqrt{2}}\right) = \frac{a_0\gamma^{1/2}}{(E - V_\infty)^{1/2}}.$$

Using the asymptotic expansion (17) for $T_0$, we obtain the asymptotic expansion (20) for $\Delta T_0$. ☐

**Corollary 2.** *Let $X(\tau)$ be the same as in Lemma 2 and*

$$Y(\tau) := \gamma^{-1/2} V'(\gamma^{-1/2} X(\tau)). \tag{22}$$

*Then, $Y \in C^1_{\text{per}}(0, T_0)$ and there exists $C(E) > 0$ such that $\|Y\|_{H^1_{\text{per}}} \leq C\gamma^{-3/4}$.*

*Proof.* Using the bounds on $\Delta T_0$ in Lemma 2, Corollary 1, and assumption (P1) on the potential $V(x)$, we obtain

$$\begin{aligned} \int_0^{T_0} |Y(\tau)|^2 d\tau &= \gamma^{-1} \int_0^{T_0} |V'(\gamma^{-1/2} X(\tau))|^2 d\tau \\ &\leq \gamma^{-1} \Delta T_0 \|V'\|^2_{L^\infty} \leq C_1 \gamma^{-1/2}, \end{aligned}$$

and

$$\int_0^{T_0} |\dot{Y}(\tau)|^2 d\tau = \gamma^{-2} \int_0^{T_0} |\dot{X}(\tau)|^2 |V''(\gamma^{-1/2} X(\tau))|^2 d\tau$$
$$\leq \gamma^{-2} \Delta T_0 \|V''\|^2_{L^\infty} \|\dot{X}\|^2_{L^\infty} \leq C_2 \gamma^{-3/2},$$

for some constants $C_1, C_2 > 0$. The bound on $\|Y\|_{H^1_{\rm per}}$ follows from the above computation. □

We shall be working in the space of $T_0$-periodic functions in $H^2$, $H^1$, or $L^2$ satisfying symmetry (7). Therefore, let us denote for all $p \geq 1$

$$H^p_e := \left\{ X \in H^p_{\rm per}(0, T_0) : \ X(-\tau) = X(\tau) = -X\left(\frac{T_0}{2} - \tau\right), \quad \tau \in \mathbb{R} \right\} \tag{23}$$

and use similar notations for $L^2_e$ and $L^\infty_e$.

We are now prepared to deal with the singularly perturbed linear oscillator under the small source term:

$$\ddot{Z}(\tau) + 2Z(\tau) + \gamma^{-1/2} V'(\gamma^{-1/2} Z(\tau)) = \gamma^{\varepsilon + 1/2} F(\tau), \tag{24}$$

where $\varepsilon > 0$, $F \in L^2_e$, and $\|F\|_{L^2_{\rm per}} = O(1)$ as $\gamma \to 0$. It is necessary to consider the inhomogeneous problem (24) in order to control the effect of small coupling in the discrete Klein–Gordon equation (1) at the central site $n = 0$. Energy for the perturbed oscillator equation (24) can be written in the form

$$H(\tau) = \frac{1}{2} \dot{Z}^2 + Z^2 + V(\gamma^{-1/2} Z). \tag{25}$$

Because the homogeneous equation with $F \equiv 0$ admits a $T_0$-periodic solution $X \in H^2_e$ with $\partial_E T_0(E, \gamma) = O(\gamma^{1/2})$ (equation (24) is a singular perturbation of a linear oscillator), a source term of order one would generate a large output as $\gamma \to 0$, i.e. the output $\|Z - X\|_{H^1_{\rm per}}$ is going to be $\gamma^{-1/2}$ larger than the source term $\gamma^{\varepsilon+1/2} \|F\|_{L^2_{\rm per}}$. This can be intuitively understood by linearizing equation (24) around $X$. Indeed, the linearized operator

$$L_0 := \frac{d^2}{d\tau^2} + 2 + \gamma^{-1} V''\left(\gamma^{-1/2} X(\tau)\right) \tag{26}$$

admits a nontrivial kernel $\mathrm{Ker}(L_0) = \mathrm{span}\{\dot{X}\}$ in $H^2_{\rm per}(0, T_0)$. However, $\mathrm{Ker}(L_0) = \{0\}$ in the subspace $H^2_e$ of even $T_0$-periodic functions, under the condition $\partial_E T_0(E, \gamma) \neq 0$. In this subspace, we have $\|L_0^{-1}\|_{\mathcal{L}(L^2_{\rm per}, H^1_{\rm per})} = O(\gamma^{-1/2})$, due to the fact that $\partial_E T_0(E, \gamma) = O(\gamma^{1/2})$ (this follows from standard computations based on the variation of constants method). Thanks to the scaling of the source term considered in (24), we will get $\|Z - X\|_{H^1_{\rm per}} = O(\gamma^\varepsilon)$ if $\|F\|_{L^2_{\rm per}} = O(1)$ and $\varepsilon \leq \frac{1}{4}$ (see Lemma 3 below). Since $\|X\|_{H^1_{\rm per}} = O(1)$ as $\gamma \to 0$ by Corollary 1, the perturbation $\|Z - X\|_{H^1_{\rm per}}$ is smaller than the unperturbed solution $\|X\|_{H^1_{\rm per}}$ if $\varepsilon > 0$.

There exists an obstacle on the direct application of the Implicit Function Theorem to obtain $T_0$-periodic solutions of the singularly perturbed oscillator equation (24). The obstacle comes from the power series expansion

$$V'(\gamma^{-1/2} Z) = V'(\gamma^{-1/2} X) + \gamma^{-1/2} V''(\gamma^{-1/2} X)(Z - X)$$
$$+ \gamma^{-1} V'''(\gamma^{-1/2} X)(Z - X)^2 + ...,$$

which generate large terms in the limit of small $\gamma > 0$ because of the singular perturbation in the nonlinear potential. To avoid this difficulty, we use the fact that $V'(\gamma^{-1/2}Z)$ has a compact support, and transform the search of periodic solutions of equation (24) to a root-finding problem to which the Implicit Function Theorem can be applied. We shall prove the following.

**Lemma 3.** *Let $X \in H^2_e$ be the solution of Lemma 1 in parametrization (16) for any fixed $E > V_L$. Let $B_\delta$ be a ball of radius $\delta > 0$ in $L^2_e$ centered at 0. Fix $\varepsilon \in \left(0, \frac{1}{2}\right)$. If*

$$\int_0^{a_0} \frac{dx}{(E - V(x))^{3/2}} - \frac{a_0}{(E - V_\infty)^{3/2}} \neq 0, \tag{27}$$

*there exist $\gamma_0(\varepsilon, \delta, E) > 0$, $\eta(\varepsilon, \delta, E) > 0$, and $C(\varepsilon, \delta, E) > 0$ such that the inhomogeneous equation (24) with $\gamma \in (0, \gamma_0)$ and $F \in B_\delta$ admits a unique solution $Z = \mathcal{G}_{\gamma,\varepsilon}(F) \in H^2_e$ satisfying*

$$Z(\tau_0) = a_0 \gamma^{1/2}, \quad |H(\tau_0) - E| < \eta(\varepsilon, \delta, E)\gamma^\varepsilon$$

*for some $\tau_0 \in (0, \frac{T_0}{4})$. Moreover, $Z$ is close to $X$ in $H^1_e$ with*

$$\|Z - X\|_{H^1_{\rm per}} \leq C(\varepsilon, \delta, E)\gamma^{\tilde{\varepsilon}}, \quad \tilde{\varepsilon} = \min\left(\frac{1}{4}, \varepsilon\right) \tag{28}$$

*and satisfies the estimate*

$$\|Z\|_{C^1} \leq 3\sqrt{E}. \tag{29}$$

*In addition, $Z(\tau) \geq a_0\gamma^{1/2}$ for $\tau \in [0, \tau_0]$, $Z(\tau) \in [0, a_0\gamma^{1/2}]$ for $\tau \in [\tau_0, \frac{T_0}{4}]$, and there exists $\theta(\varepsilon, \delta, E) > 0$ such that*

$$\left|\frac{T_0}{4} - \tau_0\right| \leq \theta(\varepsilon, \delta, E)\gamma^{1/2}. \tag{30}$$

*Proof.* We shall use a kind of shooting method to transform the differential equation (24) to a root-finding problem. Let us consider an initial-value problem for the second-order differential equation (24) starting with the initial data $Z(0) = Z_0$ and $\dot{Z}(0) = 0$, where $Z_0$ is a positive $\gamma$-independent parameter.

Thanks to the compact support in assumption (P4), we solve the inhomogeneous linear equation

$$\ddot{Z}(\tau) + 2Z(\tau) = \gamma^{\varepsilon+1/2}F(\tau), \quad \tau \in [0, \tau_0], \tag{31}$$

where $\tau_0$ will be determined from the condition $Z(\tau_0) = a_0\gamma^{1/2}$. We shall prove that for small $\gamma > 0$, a unique $\tau_0 \in \left(0, \frac{\pi}{2\sqrt{2}}\right)$ exists. In what follows we denote $W_0 := \dot{Z}(\tau_0)$ and $\tau_0 := \frac{\pi}{2\sqrt{2}} - \Delta_0$. We shall consider $W_0 \in \mathbb{R}_-$ as a free parameter and express $(Z_0, \tau_0)$ (or equivalently $(Z_0, \Delta_0)$) as a function of $W_0$.

The unique solution of the linear equation (31) with the initial data $Z(0) = Z_0$ and $\dot{Z}(0) = 0$ is given by

$$Z(\tau) = Z_0 \cos(\sqrt{2}\tau) + \frac{1}{\sqrt{2}}\gamma^{\varepsilon+1/2} \int_0^\tau F(\tau') \sin(\sqrt{2}(\tau - \tau'))d\tau'. \tag{32}$$

At $\tau = \tau_0$, we have the system of nonlinear equations

$$\left\{ \begin{array}{l} Z_0 \sin(\sqrt{2}\Delta_0) = a_0\gamma^{1/2} - \frac{1}{\sqrt{2}}\gamma^{\varepsilon+1/2} \int_0^{\tau_0} F(\tau') \sin(\sqrt{2}(\tau_0 - \tau'))d\tau', \\ Z_0 \cos(\sqrt{2}\Delta_0) = -\frac{1}{\sqrt{2}}W_0 + \frac{1}{\sqrt{2}}\gamma^{\varepsilon+1/2} \int_0^{\tau_0} F(\tau') \cos(\sqrt{2}(\tau_0 - \tau'))d\tau', \end{array} \right.$$

which can be rewritten in the equivalent form

$$\begin{cases} Z_0 = -\frac{1}{\sqrt{2}} W_0 \cos(\sqrt{2}\Delta_0) + a_0 \gamma^{1/2} \sin(\sqrt{2}\Delta_0) + \frac{1}{\sqrt{2}} \gamma^{\varepsilon+1/2} \int_0^{\tau_0} F(\tau) \sin(\sqrt{2}\tau) d\tau, \\ \frac{1}{\sqrt{2}} W_0 \sin(\sqrt{2}\Delta_0) = -a_0 \gamma^{1/2} \cos(\sqrt{2}\Delta_0) + \frac{1}{\sqrt{2}} \gamma^{\varepsilon+1/2} \int_0^{\tau_0} F(\tau) \cos(\sqrt{2}\tau) d\tau \end{cases}$$

(multiply each equation by $\sin(\sqrt{2}\Delta_0)$ and $\cos(\sqrt{2}\Delta_0)$ and sum the resulting equations). This problem can be rewritten

$$\begin{cases} Z_0 = -\frac{1}{\sqrt{2}} W_0 \cos(\sqrt{2}\Delta_0) + a_0 \gamma^{1/2} \sin(\sqrt{2}\Delta_0) \\ \qquad\qquad + \frac{1}{\sqrt{2}} \gamma^{\varepsilon+1/2} \int_0^{\frac{\pi}{2\sqrt{2}}-\Delta_0} F(\tau) \sin(\sqrt{2}\tau) d\tau, \\ \Delta_0 = -\frac{a_0}{W_0 \operatorname{sinc}(\sqrt{2}\Delta_0)} \gamma^{1/2} \cos(\sqrt{2}\Delta_0) \\ \qquad\qquad + \frac{1}{\sqrt{2} W_0 \operatorname{sinc}(\sqrt{2}\Delta_0)} \gamma^{\varepsilon+1/2} \int_0^{\frac{\pi}{2\sqrt{2}}-\Delta_0} F(\tau) \cos(\sqrt{2}\tau) d\tau, \end{cases} \tag{33}$$

where $\operatorname{sinc}(x) = \sin(x)/x$. Assuming that $W_0 \in (-\infty, W_{\max})$ is bounded away from 0 and $\gamma \in (0, \gamma_{\max})$ is small enough, the second equation can be solved by the contraction mapping theorem for $\Delta_0 \in \left[0, \frac{\pi}{2\sqrt{2}}\right]$, and then the first equation determines $Z_0$. This yields finally

$$Z_0 = -\frac{1}{\sqrt{2}} W_0 + \frac{1}{\sqrt{2}} \gamma^{\varepsilon+1/2} \int_0^{\frac{\pi}{2\sqrt{2}}} F(\tau) \sin(\sqrt{2}\tau) d\tau + O(\gamma) \tag{34}$$

and

$$\Delta_0 = -\frac{a_0 \gamma^{1/2}}{W_0} + \frac{1}{\sqrt{2} W_0} \gamma^{\varepsilon+1/2} \int_0^{\frac{\pi}{2\sqrt{2}}} F(\tau) \cos(\sqrt{2}\tau) d\tau + O(\gamma^{3/2}). \tag{35}$$

Then $Z_0 = O(1)$ and $\Delta_0 = O(\gamma^{1/2})$ as $\gamma \to 0$, so that existence of a unique $\tau_0 \in \left(0, \frac{\pi}{2\sqrt{2}}\right)$ follows.

We can now continue solution $Z(\tau)$ to $\tau > \tau_0$ starting with the initial conditions $Z(\tau_0) = a_0 \gamma^{1/2}$ and $\dot{Z}(\tau_0) = W_0 < 0$. Our aim is to solve the inhomogeneous differential equation

$$\ddot{Z}(\tau) + 2Z(\tau) + \gamma^{-1/2} V'(\gamma^{-1/2} Z(\tau)) = \gamma^{\varepsilon+1/2} F(\tau), \quad \tau \in [\tau_0, \tau_*], \tag{36}$$

where $\tau_* > \tau_0$ is the first time where $Z(\tau_*) = 0$. For this purpose the first step is to show that $\tau_* > \tau_0$ exists.

Taking the derivative of the energy $H$ in (25) with respect to $\tau$ and using equation (36), we infer that

$$\dot{H} = \gamma^{\varepsilon+1/2} \dot{Z} F. \tag{37}$$

Let $E^* := H(\tau_0) = \frac{1}{2} W_0^2 + a_0^2 \gamma + V_\infty$ and assume that $E^* > V_L \geq V_\infty$. Note that $E^*$ becomes now the parameter of the solution family in place of $W_0$.

Let us denote by $\tau_1$ the maximal time such that $|Z(\tau)| < a_0 \gamma^{1/2}$ and $\dot{Z}(\tau) < 0$ for all $\tau \in (\tau_0, \tau_1)$. We have

$$\dot{Z}(\tau) = -\sqrt{2(H(\tau) - V(\gamma^{-1/2} Z(\tau)) - Z^2(\tau))}, \quad \tau \in [\tau_0, \tau_1]. \tag{38}$$

Using formula (38) and integrating equation (37), we obtain for all $\tau \in [\tau_0, \tau_1]$,

$$|H(\tau) - E^*| \leq \gamma^{\varepsilon+1/2} \int_{\tau_0}^{\tau} \sqrt{2(H(\tau') - E^* + E^* - V(\gamma^{-1/2} Z(\tau')) - Z^2(\tau'))} |F(\tau')| d\tau'.$$

Using the triangle inequality, we find that there exist $\gamma_0 > 0$ and a $\gamma$-independent constant $C(E^*, \varepsilon) > 0$ such that for all $\gamma \in (0, \gamma_0)$,

$$|H(\tau) - E^*| \leq C(E^*, \varepsilon) \gamma^{\varepsilon+1/2} \|F\|_{L^2_{\mathrm{per}}}, \quad \tau \in [\tau_0, \tau_1]. \tag{39}$$

Now it follows that

$$\frac{1}{2}\dot{Z}^2(\tau_1) = H(\tau_1) - Z^2(\tau_1) - V(\gamma^{-1/2}Z(\tau_1)) \geq E^* - V_L + O(\gamma^{\varepsilon+1/2}),$$

hence $\dot{Z}(\tau_1) \neq 0$ for $\gamma$ small enough since $E^* > V_L$. Consequently one has $Z(\tau_1) = -a_0\gamma^{1/2}$ and $\dot{Z}(\tau) < 0$ for all $\tau \in [\tau_0, \tau_1]$. By the intermediate value theorem, this yields the existence of $\tau_* \in (\tau_0, \tau_1)$ such that $Z(\tau_*) = 0$ and $Z(\tau) \in (0, a_0\gamma^{1/2})$ for all $\tau \in (\tau_0, \tau_*)$.

We now express the distance $|\tau_* - \tau_0|$ from the energy (25) controlled by bound (39). From equations (37) and (38), we obtain

$$\tau_* - \tau_0 = \int_{\tau_0}^{\tau_*} \frac{\dot{Z}(\tau)}{\dot{Z}(\tau)} d\tau = -\int_{\tau_0}^{\tau_*} \frac{\dot{Z}(\tau) d\tau}{\sqrt{2(H(\tau) - V(\gamma^{-1/2}Z(\tau)) - Z^2(\tau))}}.$$

Using bound (39), we have for all $\tau \in [\tau_0, \tau_*]$ as $\gamma \to 0$,

$$\frac{1}{\sqrt{2(H(\tau) - V(\gamma^{-1/2}Z(\tau)) - Z^2(\tau))}} = \frac{1}{\sqrt{2(E^* - V(\gamma^{-1/2}Z(\tau)) - Z^2(\tau))}} + O(\gamma^{\varepsilon+1/2}),$$

so that the change of variables $x = \gamma^{-1/2}Z(\tau)$ gives

$$\tau_* - \tau_0 = \gamma^{1/2} \int_0^{a_0} \frac{dx}{\sqrt{2(E^* - V(x) - \gamma x^2)}} + O(\gamma^{\varepsilon+1/2}) \quad \text{as} \quad \gamma \to 0. \tag{40}$$

Combining this expansion of $\tau_* - \tau_0$ with the expansion of $\tau_0 = \frac{\pi}{2\sqrt{2}} - \Delta_0$ obtained in (35), and using the fact that $W_0 = -\sqrt{2(E^* - V_\infty - a_0^2\gamma)}$, we end up with the expansion

$$\begin{aligned}&\tau_*(E^*, \gamma, F)\\ =&\frac{\pi}{2\sqrt{2}} - \frac{a_0\gamma^{1/2}}{\sqrt{2(E^* - V_\infty)}} + \gamma^{1/2} \int_0^{a_0} \frac{dx}{\sqrt{2(E^* - V(x))}} + O(\gamma^{\varepsilon+1/2}),\end{aligned} \tag{41}$$

as $\gamma \to 0$, uniformly in $F \in B_\delta$ for fixed $\varepsilon > 0$ and $\delta > 0$.

Let us now examine the regularity of $\tau_*$ with respect to the variable $(E^*, \gamma, F)$ including the functional parameter $F$. From the standard fixed-point reformulation of the Cauchy problem for differential equations, it follows that the map $(W_0, \gamma, F) \mapsto Z$ is $C^1$ from $(-\infty, W_{\max}) \times (0, \gamma_{\max}) \times L_e^2$ into $C^1([0, \frac{\pi}{\sqrt{2}}])$. Consequently, the map $(E^*, \gamma, F, \tau) \mapsto Z(\tau)$ is $C^1$ from $(E_{\min}, +\infty) \times (0, \gamma_{\max}) \times L_e^2 \times \left(0, \frac{\pi}{\sqrt{2}}\right)$ into $\mathbb{R}$, where we have denoted $E_{\min} = \frac{1}{2}W_{\max}^2 + V_\infty$. Since $Z(\tau_*) = 0$ and $\dot{Z}(\tau_*) \neq 0$, the implicit function theorem ensures that $\tau_*$ is $C^1$ with respect to $(E^*, \gamma, F) \in (E_{\min}, +\infty) \times (0, \gamma_{\max}) \times B_\delta$.

Now we come back to equation (24). If we manage to find a value of $E^*$ such that

$$\tau_*(E^*, \gamma, F) = \frac{1}{4} T_0(E, \gamma), \tag{42}$$

then the symmetry constraints satisfied by $F \in L_e^2$ in (23) imply that $Z \in H^2(0, T_0)$ also satisfies the symmetry conditions in (23). Indeed, $Z(\tau)$ and $Z(-\tau)$ are solutions of the same Cauchy problem, hence $Z(\tau) = Z(-\tau)$ by Cauchy's theorem. Using the same argument one obtains $Z(\tau + T_0/4) = -Z(-\tau + T_0/4)$ (this requires in addition the evenness of $V$). These two equalities imply $Z(0) = -Z(T_0/2) = -Z(-T_0/2) = Z(T_0)$ and in the same way $\dot{Z}(0) = \dot{Z}(T_0) = 0$. Consequently, $Z(\tau)$ and $Z(\tau + T_0)$ are solutions of the same Cauchy problem, hence $Z$ is $T_0$-periodic. Therefore, $Z \in H_e^2$ if there is $E^*$ solving equation (42).

Using expansions (17) and (41), equation (42) can be rewritten

$$R(E^*, \gamma^\varepsilon) = 0, \tag{43}$$

where

$$\begin{aligned} R(E^*, \gamma^\varepsilon) &= \int_0^{a_0} \frac{dx}{\sqrt{E - V(x)}} - \frac{a_0}{\sqrt{E - V_\infty}} \\ &\quad - \int_0^{a_0} \frac{dx}{\sqrt{E^* - V(x)}} + \frac{a_0}{\sqrt{E^* - V_\infty}} + O(\gamma^\varepsilon) \end{aligned} \tag{44}$$

as $\gamma \to 0$, uniformly in $F \in B_\delta$ (we omit the dependency of $R$ with respect to $E$ and $F$ in notations). Now let us consider the non-degeneracy condition (27), which is equivalent to $\frac{\partial R}{\partial E^*}(E, 0) \neq 0$. Due to the non-differentiability of $R$ at $\gamma = 0$, we cannot solve (43) directly by the implicit function theorem, but rewrite the problem in the form

$$E^* - E = O(\gamma^\varepsilon + (E^* - E)^2) \tag{45}$$

using the fact that $\frac{\partial R}{\partial E^*}(E, 0) \neq 0$. By the contraction mapping theorem, if $E > V_L$, $\varepsilon \in \left(0, \frac{1}{2}\right)$, $\delta > 0$, $F \in B_\delta$ are fixed, there exist constants $\gamma_0(\varepsilon, \delta, E) > 0$ and $C_0(\varepsilon, \delta, E)$ such that equation (45) admits a unique solution $E^*$ near $E$ for all $\gamma \in (0, \gamma_0)$, with

$$|E^* - E| \leq C_0(\varepsilon, \delta, E)\gamma^\varepsilon. \tag{46}$$

Moreover, the map $(\gamma, F) \mapsto E^*$ is $C^1$ on $(0, \gamma_0) \times B_\delta$ by the uniform contraction principle.

This completes the proof of existence and local uniqueness of solution $Z$. Estimate (30) follows directly from expansion (40).

It remains to prove bounds (28) and (29). For this purpose, we first consider a time interval in which $Z$ and $X$ are both given by the explicit solution of a linear equation. Recall that

$$X(\tau) = \sqrt{E - V_\infty}\, \cos(\sqrt{2}\,\tau), \quad 0 \leq \tau \leq \tilde{\tau}_0, \tag{47}$$

where

$$\tilde{\tau}_0 = \frac{1}{\sqrt{2}} \arccos\left(\frac{a_0 \gamma^{1/2}}{(E - V_\infty)^{1/2}}\right).$$

Now consider $\hat{\tau}_0 = \min(\tau_0, \tilde{\tau}_0) = \frac{T_0}{4} + O(\gamma^{1/2})$. By combining the explicit expressions of $X$ and $Z$ (see equation (32)), estimates (34) and (46), and observing that

$$W_0 = -\sqrt{2(E^* - V_\infty)} + O(\gamma),$$

one obtains

$$\|Z - X\|_{L^\infty(0,\hat{\tau}_0)} \leq C(\varepsilon, \delta, E)\gamma^\varepsilon, \quad \|\dot{Z} - \dot{X}\|_{L^\infty(0,\hat{\tau}_0)} \leq C(\varepsilon, \delta, E)\gamma^\varepsilon, \tag{48}$$

for some $C(\varepsilon, \delta, E) > 0$ for all $\gamma \in (0, \gamma_0)$.

Since $\hat{\tau}_0 = \frac{T_0}{4} + O(\gamma^{1/2})$, it follows from equation (32) and (47) that

$$\|X\|_{L^\infty(\hat{\tau}_0, T_0/4)} + \|Z\|_{L^\infty(\hat{\tau}_0, T_0/4)} = O(\gamma^{1/2}) \quad \text{as} \quad \gamma \to 0. \tag{49}$$

As a result, we extend the first bound in (48) to

$$\|Z - X\|_{L^\infty} \leq C(\varepsilon, \delta, E)\gamma^\varepsilon,$$

because $\varepsilon \in \left(0, \frac{1}{2}\right)$. To extend the second bound in (48), we write for all $\tau \in \left[\hat{\tau}_0, \frac{1}{4}T_0\right]$,

$$|\dot{Z}(\tau)| = \sqrt{2(H(\tau) - V(\gamma^{-1/2}Z(\tau)) - Z^2(\tau))} \leq \sqrt{2H(\tau)} = \sqrt{2E} + O(\gamma^{\varepsilon}), \quad (50)$$

$$|\dot{X}(\tau)| = \sqrt{2(E - V(\gamma^{-1/2}X(\tau)) - X^2(\tau))} \leq \sqrt{2E}, \quad (51)$$

where we have used bounds (39) and (46). By bounds (40) and (49), hence we have

$$\|X\|_{H^1(\hat{\tau}_0, T_0/4)} + \|Z\|_{H^1(\hat{\tau}_0, T_0/4)} \leq C(\varepsilon, \delta, E)\gamma^{1/4}. \quad (52)$$

As a conclusion, estimate (28) follows from estimates (48) and (52). Estimate (29) follows from (48), (50), and (51) for $\gamma$ small enough. $\square$

**Remark 5.** A $T_0$-periodic solution of the perturbed oscillator equation (24) may not exist if the constraint (27) is not met. For instance, if $V \equiv 0$ (this corresponds to the case $a_0 = 0$) and $F(\tau) = \cos(\sqrt{2}\tau)$, the perturbed linear oscillator (24) has no $T_0$-periodic solutions because of the resonance of the linear oscillator and the $T_0$-periodic force for $T_0 = \sqrt{2}\pi$. Since $\Delta_0 = 0$, expansion (35) yields the classical constraint

$$\int_0^{\frac{\pi}{2\sqrt{2}}} F(\tau)\cos(\sqrt{2}\tau)d\tau = 0$$

for the existence of a $\sqrt{2}\pi$-periodic solution under symmetric periodic forcing. The constraint is not satisfied if $F(\tau) = \cos(\sqrt{2}\tau)$.

For the arguments in the proof of Theorem 1 based on the Schauder fixed-point theorem, we will need a continuity of the nonlinear map $F \mapsto Z$ from $L^2_e$ to $H^2_e$.

**Lemma 4.** *Under Assumptions of Lemma 3, for all $\gamma \in (0, \gamma_0)$ and all $F \in B_\delta$, the solution $Z = \mathcal{G}_{\gamma,\epsilon}(F) \in H^2_e$ is continuous with respect to $F \in L^2_e$.*

*Proof.* From Lemma 3, we know that for given $F_1 \in B_\delta$ and $F_2 \in B_\delta$, there exist roots $E^* = E_1$ and $E^* = E_2$ of equation (43) and if $F_1 \to F_2$ in $L^2_e$, then $E_1 \to E_2$. In what follows, we denote by $W_{1,2}$, $Z_0^{(1),(2)}$, and $\tau_{1,2}$ the variables $W_0$, $Z_0$, and $tau_0$ introduced in Lemma 3, in the case when $E^* = E_{1,2}$. Expansion (35) implies that

$$\exists C > 0: \quad |\tau_1 - \tau_2| \leq C\gamma^{1/2}\left(|E_1 - E_2| + \gamma^{\varepsilon}\|F_1 - F_2\|_{L^2(0,T_0)}\right).$$

Let $\hat{\tau}_0 = \min(\tau_1, \tau_2)$. From the solution (32) of the linear oscillator equation (31) and the expansion (34), we obtain

$$\exists C > 0: \quad \|Z_1 - Z_2\|_{H^2(0,\hat{\tau}_0)} \leq C\left(|E_1 - E_2| + \gamma^{\varepsilon+1/2}\|F_1 - F_2\|_{L^2(0,T_0)}\right). \quad (53)$$

Consequently, if $F_1 \to F_2$ in $L^2_e$, then $E_1 \to E_2$ and $Z_1 \to Z_2$ in $H^2(0, \hat{\tau}_0)$.

There remains to complete (53) by a continuity result on the time interval $[\hat{\tau}_0, \tau_*]$, on which we rewrite the differential equation (36) in variables

$$z_{1,2}(t) = \gamma^{-1/2}Z_{1,2}(\tau), \quad f_{1,2}(t) = F_{1,2}(\tau), \quad t = \gamma^{-1/2}(\tau - \hat{\tau}_0), \quad (54)$$

or explicitly,

$$\ddot{z}_{1,2}(t) + 2\gamma z_{1,2}(t) + V'(z_{1,2}(t)) = \gamma^{\varepsilon+1}f_{1,2}(t), \quad t \in [0, \hat{t}_0], \quad (55)$$

where $\hat{t}_0 = \gamma^{-1/2}(\tau_* - \hat{\tau}_0) = O(1)$ thanks to (30) and $\tau_* = T_0/4$.

For definiteness, let us assume that $\tau_1 > \tau_2$ so that $\hat{\tau}_0 = \tau_2$. The initial-value problems for differential equations (55) are started with $(z_1(0), \dot{z}_1(0))$ and $(z_2(0), \dot{z}_2(0))$, where $z_2(0) = a_0$, $\dot{z}_2(0) = \dot{Z}_2(\tau_2) = W_2$, and $z_1(0) = \gamma^{-1/2}Z_1(\tau_2) > a_0$, $\dot{z}_1(0) = \dot{Z}_1(\tau_2)$ are determined by (32).

Since $V'$ is globally Lipschitzian by the assumptions (P1) and (P4), Gronwall's inequality implies the existence of $C > 0$ such that

$$\begin{aligned}
&\|z_1 - z_2\|_{C^1(0,\hat{t}_0)} \\
\leq & C\left(|z_1(0) - z_2(0)| + |\dot{z}_1(0) - \dot{z}_2(0)| + \gamma^{\epsilon+1}\int_0^{\hat{t}_0} |f_1(t) - f_2(t)|dt\right) \\
\leq & C\left(\frac{|Z_1(\tau_2) - Z_2(\tau_2)|}{\gamma^{1/2}} + |\dot{Z}_1(\tau_2) - \dot{Z}_2(\tau_2)| + \gamma^{\epsilon+1}\hat{t}_0^{1/2}\|f_1 - f_2\|_{L^2(0,\hat{t}_0)}\right).
\end{aligned}$$

When $F_1 \to F_2$ in $L^2_e$, the first equation of (33) yields $Z_0^{(1)} \to Z_0^{(2)}$ (since $W_1 \to W_2$ and $\tau_1 \to \tau_2$), which implies $Z_1(\tau_2) \to Z_2(\tau_2)$ and $\dot{Z}_1(\tau_2) \to \dot{Z}_2(\tau_2)$ thanks to expression (32). Consequently we obtain $\|z_1 - z_2\|_{C^1(0,\hat{t}_0)} \to 0$ and $\|Z_1 - Z_2\|_{C^1(\hat{\tau}_0,\tau_*)} \to 0$. Combining this result with (53), we see that if $F_1 \to F_2$ in $L^2_e$ then $Z_1 \to Z_2$ in $H^1_e$.

Now observing that $\Delta = \frac{d^2}{d\tau^2} : H^2_e \to L^2_e$ is invertible and considering equation (24) that defines $\mathcal{G}_{\gamma,\epsilon}(F)$ implicitly, $Z_{1,2} = \mathcal{G}_{\gamma,\epsilon}(F_{1,2})$ satisfy the equality

$$Z_{1,2} = \Delta^{-1}\left[\gamma^{\varepsilon+1/2}F_{1,2} - 2Z_{1,2} - \gamma^{-1/2}V'(\gamma^{-1/2}Z_{1,2})\right].$$

Hence continuity in $H^1_e$ norm implies continuity in $H^2_e$ norm. □

4. **Small-amplitude oscillations on other sites.** In our construction, the large-amplitude breather bifurcating from infinity as $\gamma \to 0$ is localized at a single site $n = 0$, and close to the periodic solution of Lemma 1. The other sites for $n \neq 0$ display the oscillatory motion guided by the oscillation at the central site $n = 0$ and powered by a small amplitude in $\gamma$. By symmetry of the discrete Laplacian, we may assume that

$$x_n = x_{-n}, \quad n \geq 1.$$

Since the amplitudes of oscillations for $n \neq 0$ are small, we shall consider the linearized discrete Klein–Gordon equation (1) at the zero solution to study possible resonances with the oscillatory motion at $n = 0$.

To this end, let us introduce the function spaces

$$\mathbb{X} := L^2_{\text{per}}((0,T_0); l^2(\mathbb{N})), \quad \mathbb{D} := H^2_{\text{per}}((0,T_0); l^2(\mathbb{N})),$$

and denote $\omega_0 = \frac{2\pi}{T_0}$. Let us use assumption (P2) and consider the inhomogeneous linear problem

$$\begin{aligned}
&\gamma U_1''(\tau) + \kappa^2 U_1(\tau) = \gamma(U_2(\tau) - 2U_1(\tau)) + F_1(\tau), \\
&\gamma U_n''(\tau) + \kappa^2 U_n(\tau) = \gamma(U_{n+1}(\tau) + U_{n-1}(\tau) - 2U_n(\tau)) + F_n(\tau), \quad n \geq 2,
\end{aligned} \tag{56}$$

where $\tau = \gamma^{1/2}t$ denotes the rescaled time. We denote the sequence $\{F_n(\tau)\}_{n\in\mathbb{N}} \in \mathbb{X}$ by $\mathbf{F}$ and look for solution $\mathbf{U} = \{U_n(\tau)\}_{n\in\mathbb{N}} \in \mathbb{D}$ of the inhomogeneous linear equations (56). We shall prove the following.

**Lemma 5.** *Assume the non-resonance condition (9) to be satisfied for given values of $\kappa$, $\gamma$ and $\omega_0$, that is,*

$$\kappa^2 + 2\gamma(1 - \cos q) \neq m^2\gamma\omega_0^2, \quad \forall\, m \in \mathbb{Z}, \quad \forall q \in [-\pi,\pi]. \tag{57}$$

*Then, for all $\mathbf{F} \in \mathbb{X}$, equations (56) admit a unique solution $\mathbf{U} \in \mathbb{D}$. Moreover, the solution satisfies*

$$\|\mathbf{U}\|_{\mathbb{X}} \leq C(\gamma,\omega_0,\kappa)\|\mathbf{F}\|_{\mathbb{X}}, \tag{58}$$

*where*

$$C(\gamma,\omega_0,\kappa) = \Big( \inf_{m\in\mathbb{Z},\, q\in[-\pi,\pi]} \big|\kappa^2 - m^2\gamma\omega_0^2 + 2\gamma(1-\cos q)\big| \Big)^{-1} < +\infty.$$

*If in addition* $\mathbf{F} \in H^1_{\mathrm{per}}((0,T_0); l^2(\mathbb{N}))$ *then*

$$\|\mathbf{U}\|_{H^1_{\mathrm{per}}((0,T_0);l^2(\mathbb{N}))} \le C(\gamma,\omega_0,\kappa)\|\mathbf{F}\|_{H^1_{\mathrm{per}}((0,T_0);l^2(\mathbb{N}))}. \tag{59}$$

*Proof.* To solve (56), we expand $\mathbf{F}$ and $\mathbf{U}$ using Fourier series in the time variable $t$ and band-limited Fourier transforms in the discrete spatial coordinate $n$. Defining

$$\hat{f}_m(q) = \frac{1}{\pi T_0}\sum_{n\in\mathbb{N}} \int_0^{T_0} F_n(\tau)\, \sin(nq)\, e^{-im\omega_0\tau}\, d\tau,$$

$$\hat{u}_m(q) = \frac{1}{\pi T_0}\sum_{n\in\mathbb{N}} \int_0^{T_0} U_n(\tau)\, \sin(nq)\, e^{-im\omega_0\tau}\, d\tau,$$

we have thus

$$F_n(\tau) = \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} \hat{f}_m(q)\sin(nq)e^{im\omega_0\tau}\, dq,$$

$$U_n(\tau) = \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} \hat{u}_m(q)\sin(nq)e^{im\omega_0\tau}\, dq.$$

Note that the Fourier transform defines an isometric isomorphism between the Hilbert spaces $\mathbb{X}$ and $\mathcal{X}$ and between $\mathbb{D}$ and $\mathcal{D}$, where

$$\mathcal{X} := \ell^2(\mathbb{Z}; L^2_{\mathrm{per}}(0,2\pi)), \quad \mathcal{D} := \ell^2_2(\mathbb{Z}; L^2_{\mathrm{per}}(0,2\pi)),$$

and the latter denotes the usual Hilbert space consisting of sequences $\{\hat{u}_m\}_{m\in\mathbb{Z}}$ in $L^2_{\mathrm{per}}(0,2\pi)$ for which $\{m^2\hat{u}_m\} \in \ell^2(\mathbb{Z}; L^2_{\mathrm{per}}(0,2\pi))$.

In the Fourier space, differential equations (56) reduce to the set of algebraic equations

$$[\,\kappa^2 - m^2\gamma\omega_0^2 + 2\gamma(1-\cos q)\,]\,\hat{u}_m(q) = \hat{f}_m(q), \tag{60}$$

where $\hat{\mathbf{f}} \in \mathcal{X}$ and we look for $\hat{\mathbf{u}} \in \mathcal{D}$. Thanks to condition (57), for all $m \in \mathbb{Z}$, equation (60) has a unique solution $\hat{u}_m = \hat{H}_m\,\hat{f}_m \in L^2_{\mathrm{per}}(0,2\pi)$, where

$$\hat{H}_m(q) = [\,\kappa^2 - m^2\gamma\omega_0^2 + 2\gamma(1-\cos q)\,]^{-1}$$

is $2\pi$-periodic and analytic over $\mathbb{R}$. To check that $\hat{\mathbf{u}} \in \mathcal{D}$, we use the estimate

$$\begin{aligned}\sum_{m\in\mathbb{Z}} m^4\,\|\hat{u}_m\|^2_{L^2_{\mathrm{per}}} &\le \sum_{m\in\mathbb{Z}} m^4\,\|\hat{H}_m\|^2_{L^\infty}\,\|\hat{f}_m\|^2_{L^2_{\mathrm{per}}} \\ &\le \Big(\sup_{m\in\mathbb{Z}}\,(m^2\,\|\hat{H}_m\|_{L^\infty})\Big)^2\,\|\hat{\mathbf{f}}\|^2_{\mathcal{X}},\end{aligned} \tag{61}$$

where

$$\|\hat{H}_m\|_{L^\infty} = \frac{1}{\inf_{q\in[-\pi,\pi]}|\kappa^2 - m^2\gamma\omega_0^2 + 2\gamma(1-\cos(q))|}.$$

For $|m| \ge \omega_0^{-1}(4+\gamma^{-1}\kappa^2)^{1/2}$ one has

$$\|\hat{H}_m\|_{L^\infty} = \frac{1}{m^2\gamma\omega_0^2 - \kappa^2 - 4\gamma}$$

and $\sum_{m\in\mathbb{Z}} m^4 \|\hat{u}_m\|^2_{L^2_{\rm per}} < \infty$ according to (61). Consequently, there exists a unique solution $\mathbf{U} \in \mathbb{D}$ of the inhomogeneous system (56) in the form

$$U_n(\tau) = \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{\hat{f}_m(q)\sin(nq)e^{im\omega_0\tau}}{\kappa^2 - m^2\gamma\omega_0^2 + 2\gamma(1-\cos q)} dq. \tag{62}$$

Moreover, we have

$$\|\mathbf{U}\|^2_{\mathbb{X}} = \sum_{m\in\mathbb{Z}} \|\hat{u}_m\|^2_{L^2_{\rm per}} \le \sum_{m\in\mathbb{Z}} \|\hat{H}_m\|^2_{L^\infty} \|\hat{f}_m\|^2_{L^2_{\rm per}} \le \Big(\sup_{m\in\mathbb{Z}} \|\hat{H}_m\|_{L^\infty}\Big)^2 \|\mathbf{F}\|^2_{\mathbb{X}}. \tag{63}$$

This proves estimate (58). Estimate (59) follows by differentiating (56) with respect to $\tau$ and using (58). ☐

Now we come back to the case in which $\omega_0$ depends in fact on $\gamma$ and on the fixed energy $E$ of the excitation at site $n=0$. More precisely, we recall that

$$\omega_0(E,\gamma) = \frac{2\pi}{T(E,\gamma)}\gamma^{-1/2} = \sqrt{2} + O(\gamma^{1/2}) \quad \text{as} \quad \gamma \to 0. \tag{64}$$

Non-resonance conditions (57) are satisfied if $\gamma$ belongs to the disjoint set $C_E = \cup_{m\ge m_0}(\Gamma_m,\gamma_m)$, where $\Gamma_m$ and $\gamma_m$ are roots of equations (10) for $m$ large enough. For each $\gamma \in C_E$, a unique solution of the inhomogeneous system (56) exists in the form (62). However, the norm $\|\mathbf{U}\|_{\mathbb{X}}$ diverges as $\gamma$ approaches the boundary of $C_E$, according to estimate (58). Following the approach introduced in [7], we quantify this divergence when $\gamma$ tends towards 0 but remains in a well-chosen subset $\tilde{C}_{E,\nu}$ of $C_E$ far enough from resonances.

**Lemma 6.** *Fix $E > V_L$ and $\nu \in (0,1)$. Let $\gamma \in \tilde{C}_{E,\nu} = \cup_{m\ge m_0}(\tilde{\Gamma}_m,\tilde{\gamma}_m) \subset C_E$, where $m_0 \ge 1$ is large enough and $\tilde{\Gamma}_m$, $\tilde{\gamma}_m$ are found from the equations*

$$\frac{\sqrt{\kappa^2 + 4\tilde{\Gamma}_m}}{\sqrt{(m+1)^2 - \nu(m+1)}} = \sqrt{\tilde{\Gamma}_m}\omega_0(E,\tilde{\Gamma}_m), \qquad \frac{\kappa}{\sqrt{m^2+\nu m}} = \sqrt{\tilde{\gamma}_m}\omega_0(E,\tilde{\gamma}_m), \tag{65}$$

*for $m \ge m_0$, and satisfy as $m \to +\infty$*

$$\begin{aligned} \tilde{\Gamma}_m &= \frac{\kappa^2}{2m^2}\Big(1 + \frac{\kappa\lambda(E)}{\pi m} - \frac{2-\nu}{m} + O(m^{-2})\Big), \\ \tilde{\gamma}_m &= \frac{\kappa^2}{2m^2}\Big(1 + \frac{\kappa\lambda(E)}{\pi m} - \frac{\nu}{m} + O(m^{-2})\Big). \end{aligned}$$

*There exist $\gamma_0(\nu) > 0$ and $C_0(\nu) > 0$ such that for any $\gamma \in \tilde{C}_{E,\nu} \cap (0,\gamma_0(\nu))$ and any $\mathbf{F} \in \mathbb{X}$, the solution $\mathbf{U} \in \mathbb{D}$ of the inhomogeneous equation (56) satisfies*

$$\|\mathbf{U}\|_{\mathbb{X}} \le C_0(\nu)\gamma^{-1/2}\|\mathbf{F}\|_{\mathbb{X}}. \tag{66}$$

*Moreover, if in addition $\mathbf{F} \in H^1_{\rm per}((0,T_0);l^2(\mathbb{N}))$*

$$\|\mathbf{U}\|_{H^1_{\rm per}((0,T_0);l^2(\mathbb{N}))} \le C_0(\nu)\gamma^{-1/2}\|\mathbf{F}\|_{H^1_{\rm per}((0,T_0);l^2(\mathbb{N}))}. \tag{67}$$

*Proof.* Equations (65) can be solved for $m$ large enough (say $m \ge m_0(E)$) and small $\tilde{\Gamma}_m, \tilde{\gamma}_m$ by combining expansion (64) and the implicit function arguments. To deduce estimates (66) and (67) from Lemma 5, we need a lower bound for

$$M(n,q) = \kappa^2 - n^2\gamma\omega_0^2 + 2\gamma(1-\cos(q)), \quad n \in \mathbb{Z}, \ \ q \in [-\pi,\pi]. \tag{68}$$

Let us assume $\gamma \in (\tilde{\Gamma}_m,\tilde{\gamma}_m)$ with $m$ large enough. In what follows we will show that $M(m,q) > 0$ and $M(m+1,q) < 0$. For $n \le m-1$ it follows that

$M(n,q) > M(m,q) > 0$, and for $n \geq m+2$ we have also $M(n,q) < M(m+1,q) < 0$, hence the infimum of $|M(n,q)|$ will be reached for $n = m$ or $n = m+1$.

Let us start with the case $n = m$. It follows from (64) that

$$\partial_\gamma(\gamma\omega_0^2(E,\gamma)) = 2 + O(\sqrt{\gamma}) > 0$$

for small $\gamma > 0$. This property implies that the minimum of $M(m,q)$ occurs at $\gamma = \tilde{\gamma}_m$ and $q = 0$. Using the definition of $\tilde{\gamma}_m$ by (65), we obtain

$$M(m,q) \geq \kappa^2 - m^2\tilde{\gamma}_m\omega_0^2(E,\tilde{\gamma}_m) = \frac{\nu\kappa^2}{m+\nu} > 0.$$

Next, for $n = m+1$ one has for $m$ large enough

$$\begin{aligned}\partial_\gamma(4\gamma - (m+1)^2\gamma\omega_0^2(E,\gamma)) &= 4 - 2(m+1)^2 + O(m^2\gamma^{1/2}) \\ &= -2m^2 + O(m) < 0.\end{aligned}$$

Therefore, the maximum of $M(m+1,q)$ occurs at $\gamma = \tilde{\Gamma}_m$ and $q = \pi$, where

$$M(m+1,q) \leq \kappa^2 - (m+1)^2\tilde{\Gamma}_m\omega_0^2(E,\tilde{\Gamma}_m) + 4\tilde{\Gamma}_m.$$

Using the definition of $\tilde{\Gamma}_m$ by (65), we obtain

$$\kappa^2 - (m+1)^2\tilde{\Gamma}_m\omega_0^2(E,\tilde{\Gamma}_m) + 4\tilde{\Gamma}_m = -\frac{\nu(m+1)\kappa^2\omega_0^2(E,\tilde{\Gamma}_m)}{\omega_0^2(E,\tilde{\Gamma}_m)((m+1)^2 - \nu(m+1)) - 4},$$

and consequently

$$M(m+1,q) \leq -\frac{\nu(m+1)\kappa^2\omega_0^2(E,\tilde{\Gamma}_m)}{\omega_0^2(E,\tilde{\Gamma}_m)((m+1)^2 - \nu(m+1)) - 4} < -\frac{\nu\kappa^2}{m+1-\nu} < 0.$$

As a result of the above analysis, we have

$$\inf_{n\in\mathbb{Z},q\in[-\pi,\pi]} |\, M(n,q)\,| \geq \frac{\nu\kappa^2}{m+1}.$$

Since $\gamma \leq \tilde{\gamma}_m \leq \frac{\kappa^2}{m^2}$ for $m$ large enough, we get finally for all $\gamma \in \tilde{C}_{E,\nu}$ small enough

$$\inf_{n\in\mathbb{Z},q\in[-\pi,\pi]} |\, M(n,q)\,| \geq \frac{\nu\kappa}{2}\,\gamma^{1/2}. \tag{69}$$

Estimates (66) and (67) follow directly from Lemma 5 and estimate (69). □

**Remark 6.** If the set $\tilde{C}_{E,\nu} = \cup_{m\geq m_0}(\tilde{\Gamma}_m,\tilde{\gamma}_m) \subset C_E$ is defined by

$$\frac{\sqrt{\kappa^2 + 4\,\tilde{\Gamma}_m}}{\sqrt{(m+1)^2 - \nu(m+1)^q}} = \sqrt{\tilde{\Gamma}_m}\omega_0(E,\tilde{\Gamma}_m), \qquad \frac{\kappa}{\sqrt{m^2 + \nu m^q}} = \sqrt{\tilde{\gamma}_m}\omega_0(E,\tilde{\gamma}_m),$$

for some $q \in (0,2)$ and $\nu > 0$, then $\tilde{\Gamma}_m$ and $\gamma_m$ satisfy as $m \to +\infty$

$$\begin{aligned}\tilde{\Gamma}_m &= \frac{\kappa^2}{2m^2}\Big(1 + \frac{\kappa\lambda(E)}{\pi m} - \frac{2}{m} + \frac{\nu}{m^{2-q}} + O(m^{-2})\Big), \\ \tilde{\gamma}_m &= \frac{\kappa^2}{2m^2}\Big(1 + \frac{\kappa\lambda(E)}{\pi m} - \frac{\nu}{m^{2-q}} + O(m^{-2})\Big).\end{aligned}$$

From the requirement that $\tilde{\Gamma}_m < \tilde{\gamma}_m$ for large $m \geq m_0$, we can see that either $q \in (0,1)$ and $\nu > 0$ or $q = 1$ and $\nu \in (0,1)$. On the other hand, we have from the proof of Lemma 6 that

$$\inf_{n\in\mathbb{Z},q\in[-\pi,\pi]} |\, M(n,q)\,| \geq C(\nu)\gamma^{(2-q)/2}. \tag{70}$$

For $q \in (0,1)$, the interval $(\tilde{\Gamma}_m, \tilde{\gamma}_m)$ converges faster to the interval $(\Gamma_m, \gamma_m)$ for the price of losing too much power of $\gamma$ in the bound (67) by inverting of (70). Therefore, the estimate of Lemma 6 is sharp in this sense.

5. **Proof of Theorem 1.** Let us represent

$$x_0(t) = \frac{1}{\gamma^{1/2}} X_0(\tau), \quad x_{-n}(t) = x_n(t) = X_n(\tau), \ \ n \geq 1,$$

where $\{X_n\}_{n\in\mathbb{Z}}$ is a new set of unknowns in time $\tau = \gamma^{1/2} t$. From the discrete Klein–Gordon equation (1), we obtain

$$\ddot{X}_0 + 2X_0 + \gamma^{-1/2} V'(\gamma^{-1/2} X_0) = 2\gamma^{1/2} X_1, \tag{71}$$

$$\gamma \ddot{X}_1 + \kappa^2 X_1 + N(X_1) = \gamma(X_2 - 2X_1) + \gamma^{1/2} X_0, \tag{72}$$

$$\gamma \ddot{X}_n + \kappa^2 X_n + N(X_n) = \gamma(X_{n+1} - 2X_n + X_{n-1}), \quad n \geq 2, \tag{73}$$

where $N(X) := V'(X) - \kappa^2 X$.

Let $B_\delta \subset H^1_e$ be a ball of small radius $\delta > 0$ centered at $0 \in H^1_e$. By assumption (P2), $N(X) : B_\delta \to H^1_e$ is a $C^5$ map. Moreover, expansion $V'(x) = \kappa^2 x + O(x^5)$ near $x = 0$ implies the existence of $C > 0$ such that for all $\delta > 0$ small enough we have

$$\forall X \in B_\delta, \quad \|N(X)\|_{H^1_e} \leq C \|X\|^5_{H^1_e}, \tag{74}$$

$$\forall X_1, X_2 \in B_\delta, \quad \|N(X_1) - N(X_2)\|_{H^1_e} \leq C\, \delta^4 \, \|X_1 - X_2\|_{H^1_e}. \tag{75}$$

From system (72) and (73), we can see that oscillations near the zero solution would involve inverting the linearized operator in the inhomogeneous system (56). By estimate (67), we are going to loose $\gamma^{1/2}$, which is the size of the inhomogeneous term $\gamma^{1/2} X_0$. This would prevent us from using the contraction mapping theorem in the neighborhood of the zero solution. To overcome this obstacle, we introduce the near-identity transformation

$$X_1 = Y_1 + \gamma^{1/2} \kappa^{-2} X_0, \quad X_n = Y_n, \ \ n \geq 2$$

and rewrite system (71)–(73) in the equivalent form

$$\ddot{X}_0 + 2X_0 + \gamma^{-1/2} V'(\gamma^{-1/2} X_0) = 2\gamma\kappa^{-2} X_0 + 2\gamma^{1/2} Y_1, \tag{76}$$

$$\gamma \ddot{Y}_1 + \kappa^2 Y_1 - \gamma(Y_2 - 2Y_1) + N(Y_1 + \gamma^{1/2}\kappa^{-2} X_0) = -\gamma^{3/2}\kappa^{-2} \left(\ddot{X}_0 + 2X_0\right), \tag{77}$$

$$\gamma \ddot{Y}_2 + \kappa^2 Y_2 - \gamma(Y_3 - 2Y_2 + Y_1) + N(Y_2) = \gamma^{3/2}\kappa^{-2} X_0, \tag{78}$$

$$\gamma \ddot{Y}_n + \kappa^2 Y_n - \gamma(Y_{n+1} - 2Y_n + Y_{n-1}) + N(Y_n) = 0, \quad n \geq 3. \tag{79}$$

Extracting $\ddot{X}_0 + 2X_0$ from (76), we can rewrite (77) in the equivalent form

$$\begin{aligned} &\gamma \ddot{Y}_1 + \kappa^2 Y_1 - \gamma(Y_2 - 2Y_1) + N(Y_1 + \gamma^{1/2}\kappa^{-2} X_0) \\ &\qquad = -2\gamma^2\kappa^{-2} Y_1 - 2\gamma^{5/2}\kappa^{-4} X_0 + \gamma\kappa^{-2} V'(\gamma^{-1/2} X_0). \end{aligned} \tag{80}$$

We shall solve the above system in two steps, using the contraction mapping theorem to solve (78)-(80) at fixed $X_0$, and then Schauder's fixed point theorem to solve (76). In the latter case, we shall consider equation (76) similar to equation (24) with $X_0 \in H^1_e$ being close to the solution $X \in H^1_e$ of Lemma 1 rescaled by (16). The source term depends on $Y_1 \in B_\delta \subset H^1_e$ and $X_0$, and it will be proved that $\delta = O(\gamma^\varepsilon)$ is small as $\gamma \to 0$.

Let us now describe our functional setting in more detail. Given $\mu \in (0, \frac{1}{2})$ and $\gamma > 0$ small enough, we define

$$D_{\mu,\gamma} = \left\{ X_0 \in H^1_e \cap C^1_e : \; \|X_0\|_{C^1} \leq 3\sqrt{E}, \; X_0(\tau) \geq a_0 \gamma^{1/2}, 0 \leq \tau \leq \frac{T_0}{4} - \gamma^{1/2-\mu} \right\}. \tag{81}$$

When $\gamma$ is small enough, Corollary 1 and Lemma 2 imply that $X \in D_{\mu,\gamma}$. Moreover, $D_{\mu,\gamma}$ defines a closed, bounded and convex subset of $C^1_{\rm per}(0,T_0)$. Repeating the same arguments as in the proof of Corollary 2, we obtain

$$\exists C > 0 : \quad \forall X_0 \in D_{\mu,\gamma} : \quad \|V'(\gamma^{-1/2} X_0)\|_{H^1_e} \leq C \gamma^{-(1+2\mu)/4}. \tag{82}$$

For sufficiently small $\gamma$ in the set $\tilde{C}_{\omega_0,\nu}$ for fixed $\nu \in (0,1)$, we can rewrite system (78), (79), and (80) in the form

$$\mathbf{Y} + \mathcal{L}^{-1} \mathbf{N}(\mathbf{Y}, X_0) = \mathcal{L}^{-1} \mathbf{F}(Y_1, X_0),$$

where $\mathcal{L}^{-1}$ is the Green operator of Lemma 6 solving the linear inhomogeneous problem (56),

$$\mathbf{N}(\mathbf{Y}, X_0) : H^1_e((0,T_0); l^2(\mathbb{N})) \times D_{\mu,\gamma} \to H^1_e((0,T_0); l^2(\mathbb{N}))$$

is the nonlinear operator at the left side of (78)-(80), and

$$\mathbf{F}(Y_1, X_0) : B_\delta \times D_{\mu,\gamma} \to H^1_e((0,T_0); l^2(\mathbb{N}))$$

is the right side of (78)-(80). To use the estimates of Lemma 6, we assume $\gamma$ small enough in $\tilde{C}_{\omega_0,\nu}$. Using (67) and (82), we obtain that

$$\exists M > 0 : \;\; \forall Y_1 \in B_1, \;\; \forall X_0 \in D_{\mu,\gamma} : \;\; \|\mathcal{L}^{-1} \mathbf{F}(Y_1, X_0)\|_{H^1_e((0,T_0); l^2(\mathbb{N}))} \leq \frac{M}{2} \gamma^\epsilon, \tag{83}$$

where $\epsilon = \frac{1}{4} - \frac{\mu}{2}$. Now let us denote by $\mathbb{B}_\delta$ the ball of radius $\delta = M\,\gamma^\epsilon$ centered at 0 in $H^1_e((0,T_0); l^2(\mathbb{N}))$. Using (67) and (74), we obtain

$$\exists C > 0 : \; \forall \mathbf{Y} \in \mathbb{B}_\delta, \forall X_0 \in D_{\mu,\gamma} : \|\mathcal{L}^{-1} \mathbf{N}(\mathbf{Y}, X_0)\|_{H^1_e((0,T_0); l^2(\mathbb{N}))} \leq C \gamma^{5\epsilon - 1/2}. \tag{84}$$

Let us further assume $\mu \in \left(0, \frac{1}{4}\right)$, which implies $\epsilon \in \left(\frac{1}{8}, \frac{1}{4}\right)$. From (83)-(84) and the triangle inequality, it follows that the map $\mathcal{L}^{-1}(\mathbf{N} - \mathbf{F})(\cdot, X_0)$ maps $\mathbb{B}_\delta$ into itself for $\gamma$ small enough in $\tilde{C}_{\omega_0,\nu}$ and for all $X_0 \in D_{\mu,\gamma}$. Moreover, thanks to bound (75) and Lemma 6, for all sufficiently small $\gamma$ in $\tilde{C}_{\omega_0,\nu}$ and for all $X_0 \in D_{\mu,\gamma}$, the map $\mathcal{L}^{-1}(\mathbf{N} - \mathbf{F})(\cdot, X_0)$ is a contraction in $\mathbb{B}_\delta$, with Lipschitz constant $O(\gamma^{4\epsilon - \frac{1}{2}})$. By the contraction mapping theorem (and using the fact that $\mathbf{N}$ and $\mathbf{F}$ are in addition locally Lipschitzian with respect to $X_0 \in H^1_e$), there exists a unique continuous map

$$D_{\mu,\gamma} \ni X_0 \mapsto \mathbf{Y} \in H^1_e((0,T_0); l^2(\mathbb{N})) \tag{85}$$

such that $\{Y_n\}_{n \geq 1}$ solves (78)-(80) and satisfies the bound

$$\exists M > 0 : \quad \forall X_0 \in D_{\mu,\gamma} : \quad \|\mathbf{Y}\|_{H^1_e((0,T_0); l^2(\mathbb{N}))} \leq M \gamma^\epsilon. \tag{86}$$

We can now substitute $Y_1$ from solutions of system (78)-(80) to equation (76). Applying Lemma 3, we rewrite equation (76) in the form

$$X_0 = \mathcal{F}_{\gamma,\mu}(X_0), \tag{87}$$

where $\mathcal{F}_{\gamma,\mu} : D_{\mu,\gamma} \to C^1_e$ is defined by

$$\mathcal{F}_{\gamma,\mu}(X_0) = \mathcal{G}_{\gamma,\epsilon}(\gamma^{1/2-\epsilon} 2\kappa^{-2} X_0 + 2 Y_1(X_0) \gamma^{-\epsilon}),$$

$\mathcal{G}_{\gamma,\epsilon}$ is the nonlinear Green operator of Lemma 3 solving equation (24), and $Y_1(X_0)$ is defined from the map (85). By Lemma 4 and the continuity of the map (85), the map $\mathcal{F}_{\gamma,\mu}$ is continuous. Moreover, thanks to the estimates of Lemma 3 and the fact that $\mu > 0$, $\mathcal{F}_{\gamma,\mu}$ maps $D_{\mu,\gamma}$ into itself when $\gamma$ is small enough. Observing that $\Delta = \frac{d^2}{d\tau^2} : H_e^3 \to H_e^1$ is invertible and considering equation (24) that defines $\mathcal{G}_{\gamma,\epsilon}(F)$ implicitly, we have the equality

$$\mathcal{G}_{\gamma,\epsilon}(F) = \Delta^{-1} \left[ \gamma^{\varepsilon+1/2} F - 2\mathcal{G}_{\gamma,\epsilon}(F) - \gamma^{-1/2} V'(\gamma^{-1/2}\mathcal{G}_{\gamma,\epsilon}(F)) \right]. \tag{88}$$

Since the embedding of $H_e^3$ into $C_e^1$ is compact, it follows that $\mathcal{G}_{\gamma,\epsilon} : B_\delta \subset H_e^1 \to C_e^1$ is compact, hence $\mathcal{F}_{\gamma,\mu} : D_{\mu,\gamma} \to C_e^1$ is compact. Consequently, by the Schauder fixed-point theorem, there exists a solution $X_0 \in D_{\mu,\gamma}$ of equation (87) for sufficiently small $\gamma > 0$. Moreover, Lemma 3 ensures the existence of $\theta > 0$ such that

$$X_0(\tau) \geq a_0\sqrt{\gamma} \quad \text{for} \quad 0 \leq \tau \leq \frac{T_0}{4} - \theta\gamma^{1/2}.$$

Repeating the same estimates as above with $\mu = 0$, we obtain

$$\exists M > 0 : \quad \forall X_0 \in D_{0,\gamma} : \quad \|\mathbf{Y}\|_{H_e^1((0,T_0);l^2(\mathbb{N}))} \leq M\gamma^{1/4}. \tag{89}$$

Fixing now $\epsilon = \frac{1}{4}$ in Lemma 3, estimate (28) yields finally

$$\exists C > 0 : \quad \|X_0 - X\|_{H_e^1} \leq C\gamma^{1/4}. \tag{90}$$

Combining all transformations above with bounds (89) and (90) as well as using embedding of $H_e^1$ into $L_e^\infty$ and of $l^2(\mathbb{N})$ into $l^\infty(\mathbb{N})$, we obtain bound (13).

**Remark 7.** If assumption (P2) is relaxed with the expansion $V'(x) = \kappa^2 x + O(x^3)$ near $x = 0$, then the map $\mathcal{L}^{-1}\mathbf{N}(\mathbf{Y}, X_0)$ is a contraction operator with respect to $\mathbf{Y}$ in a ball of radius $\delta = \gamma^\varepsilon$ for any $\varepsilon > \frac{1}{4}$. In this case, the inhomogeneous term $\mathcal{L}^{-1}\mathbf{F}(Y_1, X_0)$ with the bound (83) is critical with $\varepsilon = \frac{1}{4}$ and prevent us to close the arguments of the contraction mapping theorem.

**Remark 8.** The Lipschitz continuity of the map (85) can also be established but the Lipschitz constant may have a bad behavior as $\gamma \to 0$ because of the factor $\gamma^{-1/2}$ in the last term $V(\gamma^{-1/2}X_0)$ of equation (80). This obstacle prevents us from the use of the contraction mapping theorem for equation (87).

6. **Exponential decay on $\mathbb{Z}$.** Our last result is to show that the large-amplitude discrete breather constructed in Theorem 1 decays exponentially in $n$ on $\mathbb{Z}$. The arguments repeat those of reference [7], to which we shall refer for some standard steps of the proof.

**Lemma 7.** *Let $\mathbf{x}(t) \in l^2(\mathbb{Z}, L^\infty_{\text{per}}(0,T))$ be the solution in Theorem 1. There exists a constant $D_0 > 0$ such that*

$$\sup_{t\in[0,T]} |x_n(t)| \leq (D_0\gamma)^{(2n-1)/4}, \quad n \in \mathbb{N}, \tag{91}$$

*for all sufficiently small $\gamma \in \tilde{C}_{E,\nu}$.*

*Proof.* The operator $\mathcal{A}_\gamma = \gamma\frac{d^2}{d\tau^2} + \kappa^2 : H_e^3 \to H_e^1$ is unbounded, closed and self-adjoint. Its spectrum consists of simple eigenvalues at $\{\kappa^2 - \gamma k^2\omega_0^2\}_{k\in\mathbb{Z}}$, hence

$$\|\mathcal{A}_\gamma^{-1}\|_{\mathcal{L}(H_e^1)} = \frac{1}{\inf_{k\in\mathbb{Z}} |\kappa^2 - \gamma k^2\omega_0^2|} = \frac{1}{\inf_{k\in\mathbb{Z}} |M(k,0)|},$$

where $M(k,q)$ is defined in (68). Using estimate (69) we get consequently

$$\|\mathcal{A}_\gamma^{-1}\|_{\mathcal{L}(H_e^1)} = O(\gamma^{-1/2}) \tag{92}$$

when $\gamma \to 0$ in $\tilde{C}_{E,\nu}$.

Now we rewrite system (79) as

$$Y_n = \mathcal{A}_\gamma^{-1} \left[ \gamma(Y_{n+1} - 2Y_n + Y_{n-1}) - N(Y_n) \right], \quad n \geq 3. \tag{93}$$

By estimate (89) we have $\|Y_n\|_{H_e^1} = O(\gamma^{1/4})$ uniformly in $n \in \mathbb{N}$, which in conjunction with (74) yields $\|N(Y_n)\|_{H_e^1} \leq C\,\gamma\,\|Y_n\|_{H_e^1}$. Using this estimate and the bound (92) in (93), one finds $M > 0$ such that for all $\gamma \in \tilde{C}_{E,\nu}$ small enough and for all $n \geq 3$

$$\|\, Y_n \,\|_{H_e^1} \leq M\,\gamma^{1/2}\,(\,\|\, Y_{n+1} \,\|_{H_e^1} + \|\, Y_{n-1} \,\|_{H_e^1}\,). \tag{94}$$

A simple application of the discrete maximum principle yields (see [7], Lemma 3.3)

$$\|Y_n\|_{H_e^1} \leq (2M\gamma^{1/2})^{n-2}\,\|\, Y_2 \,\|_{H_e^1}, \quad n \geq 3. \tag{95}$$

Using equation (78), estimates (92) and (95) with $n = 3$, the fact that $\|Y_1\|_{H_e^1} = O(\gamma^{1/4})$ and $\|X_0\|_{H_e^1} = O(1)$ (direct consequence of (90) and Corollary 1), we get

$$\|Y_2\|_{H_e^1} = O(\gamma^{3/4}). \tag{96}$$

Then one completes the proof by putting estimates (95) and (96) together and using the continuous embedding of $H_e^1$ in $L_e^\infty$. □

## REFERENCES


[1] J. F. R. Archilla, J. Cuevas, B. Sánchez-Rey and A. Alvarez, *Demonstration of the stability or instability of multibreathers at low coupling*, Physica D, **180** (2003), 235–255.

[2] S. Aubry, *Breathers in nonlinear lattices: Existence, linear stability and quantization*, Lattice Dynamics (Paris, 1995), Physica D, **103** (1997), 201–250.

[3] S. Aubry, G. Kopidakis and V. Kadelburg, *Variational proof for hard discrete breathers in some classes of Hamiltonian dynamical systems*, Discrete and Continuous Dynamical Systems B, **1** (2001), 271–298.

[4] D. Bambusi, *Exponential stability of breathers in Hamiltonian networks of weakly coupled oscillators*, Nonlinearity, **9** (1996), 433–457.

[5] J. Fura and S. Rybicki, *Periodic solutions of second order Hamiltonian systems bifurcating from infinity*, Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire, **24** (2007), 471–490.

[6] G. James, *Centre manifold reduction for quasilinear discrete systems*, J. Nonlinear Sci., **13** (2003), 27–63.

[7] G. James, A. Levitt and C. Ferreira, *Continuation of discrete breathers from infinity in a nonlinear model for DNA breathing*, Applicable Analysis, **89** (2010), 1447–1465.

[8] G. James, B. Sánchez-Rey and J. Cuevas, *Breathers in inhomogeneous nonlinear lattices: An analysis via center manifold reduction*, Rev. Math. Phys., **21** (2009), 1–59.

[9] V. Koukouloyannis and P. Kevrekidis, *On the stability of multibreathers in Klein-Gordon chains*, Nonlinearity, **22** (2009), 2269–2285.

[10] R. S. MacKay and S. Aubry, *Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators*, Nonlinearity, **7** (1994), 1623–1643.

[11] R. S. MacKay and J.-A. Sepulchre, *Stability of discrete breathers*, Localization in Nonlinear Lattices (Dresden, 1997), Physica D, **119** (1998), 148–162.

[12] J. L. Marín and S. Aubry, *Finite size effects on instabilities of discrete breathers*, Physica D, **119** (1998), 163–174.

[13] A. Mielke and C. Patz, *Dispersive stability of infinite-dimensional Hamiltonian systems on lattices*, Applicable Analysis, **89** (2010), 1493–1512.

[14] A. Pankov, "Travelling Waves and Periodic Oscillations in Fermi-Pasta-Ulam Lattices," Imperial College Press, London, 2005.

[15] M. Peyrard, S. Cuesta-López and G. James, *Modelling DNA at the mesoscale: A challenge for nonlinear science?*, Nonlinearity, **21** (2008), T91–T100.

[16] M. Peyrard, S. Cuesta-López and G. James, *Nonlinear analysis of the dynamics of DNA breathing*, J. Biol. Phys., **35** (2009), 73–89.

[17] M. Peyrard, *Nonlinear dynamics and statistical physics of DNA*, Nonlinearity, **17** (2004), R1-R40.

[18] J.-A. Sepulchre and R. S. MacKay, *Localized oscillations in conservative or dissipative networks of weakly coupled autonomous oscillators*, Nonlinearity, **10** (1997), 679–713.

[19] D. Treschev, *Travelling waves in FPU lattices*, Discrete and Continuous Dynamical Systems A, **11** (2004), 867–880.

[20] G. Weber, *Sharp DNA denaturation due to solvent interaction*, Europhys. Lett., **73** (2006), 806.

Received November 2010; revised September 2011.

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