

GROWTH OF PERTURBATIONS TO THE PEAKED PERIODIC WAVES IN THE CAMASSA–HOLM EQUATION*

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Abstract. Peaked periodic waves in the Camassa–Holm equation are revisited. Linearized evolution equations are derived for perturbations to the peaked periodic waves, and linearized instability is proven in both H^1 and $W^{1,\infty}$ norms. Dynamics of perturbations in H^1 is related to the existence of two conserved quantities and is bounded in the full nonlinear system due to these conserved quantities. On the other hand, perturbations to the peaked periodic wave grow in the $W^{1,\infty}$ norm and may blow up in a finite time in the nonlinear evolution of the Camassa–Holm equation.

Key words. peaked periodic waves, Camassa–Holm equation, characteristics, stability

AMS subject classifications. 35B35, 35B10, 35Q53, 35P30, 37K50, 37K55

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1. Introduction. We address the Camassa–Holm (CH) equation given by

$$(1.1) \quad u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

in the setting of $\mathbb{T} := [-\pi, \pi]$ subject to the periodic boundary conditions at $\pm\pi$. Following the original derivation in [5, 6], the CH equation arises in hydrodynamical applications as a model for propagation of unidirectional shallow water waves [10, 17]. A generalized version of this equation also models propagation of nonlinear waves inside a cylindrical hyperelastic rod with a small diameter [13]. The CH equation can be interpreted geometrically in terms of geodesic flows on the diffeomorphism group [27, 34]. Generalizations of the CH equation with multipeakons were constructed in [1, 2].

Let $\varphi \in H^1_{\text{per}}(\mathbb{T})$ be the Green function satisfying

$$(1.2) \quad (1 - \partial_x^2)\varphi = 2\delta_0, \quad x \in \mathbb{T},$$

with δ_0 being Dirac delta distribution centered at $x = 0$. The CH equation (1.1) can be rewritten in the convolution form

$$(1.3) \quad u_t + uu_x + \frac{1}{2}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right) = 0,$$

where $(f * g)(x) := \int_{\mathbb{T}} f(x-y)g(y)dy$ denotes the convolution operator and φ' denotes piecewise continuous derivative of φ in x . The Green function φ can be expressed explicitly in the form

$$(1.4) \quad \varphi(x) = \frac{\cosh(\pi - |x|)}{\sinh(\pi)}, \quad x \in \mathbb{T},$$

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which shows that $\varphi \in H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ is a piecewise C^1 function with the maximum at

$$(1.5) \quad M := \varphi(0) = \coth(\pi)$$

and the minima at

$$(1.6) \quad m := \varphi(\pm\pi) = \operatorname{csch}(\pi).$$

The central peak of φ is located at $x = 0$ with $\varphi'(0^\pm) = \mp 1$, from which φ is monotonically decreasing towards the turning points at $x = \pm\pi$, where $\varphi'(\pm\pi) = 0$. The graph of φ versus x is shown on Figure 1.

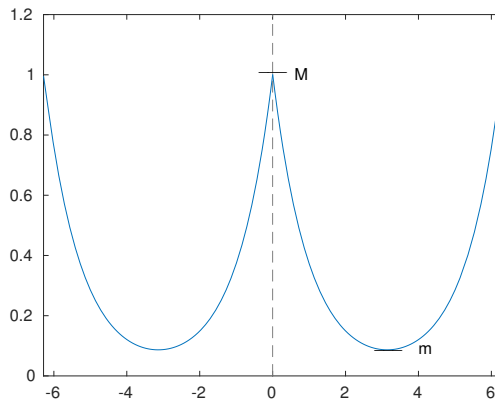


FIG. 1. The graph of φ on $[-2\pi, 2\pi]$ with M and m given by (1.5) and (1.6).

The Green function φ determines also the traveling periodic wave solution $u(t, x) = \varphi(x - ct)$ to the CH equation (1.3) rewritten in the weak form, where c is the wave speed. The wave speed is uniquely determined by $c = M$, which gives the unique solution

$$(1.7) \quad u(t, x) = \varphi(x - Mt).$$

By using the elementary scaling transformation, this unique solution generates a family of traveling periodic waves of the form

$$(1.8) \quad u(t, x) = \gamma\varphi(x - \gamma Mt), \quad \gamma \in \mathbb{R}.$$

Since the traveling periodic waves (1.8) have a peaked profile φ , we call them *peaked periodic waves*. Stability of such peaked periodic waves in $H^1_{\text{per}}(\mathbb{T})$ was studied by Lenells in [28, 29] after similar results on stability of peaked solitary waves in $H^1(\mathbb{R})$ in [11, 12]. Note that $c = 1$ was incorrectly typed in [28, 29], which did not affect the corresponding stability arguments.

The initial-value problem for the periodic CH equation (1.3) is locally well-posed in the space $H^3_{\text{per}}(\mathbb{T})$ [8, 9], $H^s_{\text{per}}(\mathbb{T})$ with $s > \frac{3}{2}$ [14, 24], $C^1_{\text{per}}(\mathbb{T})$ [35], and $H^1_{\text{per}}(\mathbb{T}) \cap \text{Lip}(\mathbb{T})$ [15], where $\text{Lip}(\mathbb{T})$ stands for Lipschitz continuous functions. Local well-posedness includes the existence, uniqueness, and continuous dependence of solutions from the initial data. Peaked periodic waves and their perturbations can be considered in the space of lower regularity $H^1_{\text{per}}(\mathbb{T}) \cap \text{Lip}(\mathbb{T})$.

Cusped periodic waves (given by bounded functions with unbounded derivatives on each side of the peaks) exist in $H_{\text{per}}^1(\mathbb{T})$. Cusped waves were used in [4, 25] to show that local solutions in $H_{\text{per}}^1(\mathbb{T})$ to the initial-value problem for the periodic CH equation (1.3) are not uniformly continuous with respect to the initial data. The same norm inflation occurs in $H_{\text{per}}^s(\mathbb{T})$ for every $s \leq \frac{3}{2}$ [22]. The initial-value problem for cusped periodic waves and their perturbations in $H_{\text{per}}^1(\mathbb{T})$ is ill-posed due to the lack of continuous dependence on the initial data.

Two energy quantities are well-defined for solutions to the CH equation (1.3) in $H_{\text{per}}^1(\mathbb{T})$:

$$(1.9) \quad E(u) = \int_{\mathbb{T}} (u^2 + u_x^2) dx, \quad F(u) = \int_{\mathbb{T}} u(u^2 + u_x^2) dx.$$

These quantities are constant continuously in time before the first time instance, for which the $W^{1,\infty}$ norm of the solution blows up.

By considering perturbations to the peaked periodic wave in $H_{\text{per}}^1(\mathbb{T})$ and using conservation of the mean value as well as conservation of the two energy quantities (1.9), it was proven in [28, 29] that the peaked periodic wave $u(t, x) = \varphi(x - ct)$ is orbitally stable in the following sense.

THEOREM 1 (see [28, 29]). *For every $\varepsilon > 0$, there is a $\nu > 0$ such that if $u \in C([0, T], H_{\text{per}}^1(\mathbb{T}))$ is a solution to the CH equation (1.1) with the initial data u_0 satisfying*

$$(1.10) \quad \|u_0 - \varphi\|_{H^1(\mathbb{T})} < \nu,$$

then

$$(1.11) \quad \|u(t, \cdot) - \varphi(\cdot - \xi(t))\|_{H^1(\mathbb{T})} < \varepsilon, \quad t \in [0, T],$$

where $\xi(t) \in \mathbb{T}$ is a point where the function $u(t, \cdot)$ attains its maximum on \mathbb{T} and $T > 0$ is either finite or infinite.

The main purpose of this work is to clarify the meaning of the stability result of Theorem 1. We will prove that the peaked periodic waves are strongly unstable with respect to perturbations in $H_{\text{per}}^1(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$. In this regard, Theorem 1 just ensures that the H^1 norm of the solution does not drift far from the H^1 norm of a translated peaked periodic wave, when the $W^{1,\infty}$ norm grows and even blows up in a finite time.

The following theorem represents the main result of this work.

THEOREM 2. *For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H_{\text{per}}^1(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ satisfying*

$$(1.12) \quad \|u_0 - \varphi\|_{H^1(\mathbb{T})} + \|u_0' - \varphi'\|_{L^\infty(\mathbb{T})} < \delta,$$

such that the local solution $u \in C([0, T], H_{\text{per}}^1(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ to the CH equation (1.3) with the initial data u_0 and $T > t_0$ satisfies

$$(1.13) \quad \|u_x(t_0, \cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty(\mathbb{T})} \geq 1,$$

where $\xi(t) \in \mathbb{T}$ is a point of peak of the function $u(t, \cdot)$ on \mathbb{T} . Moreover, there exist u_0 such that the maximal existence time T is finite.

The proof of Theorem 2 is based on several recent developments. A similar theorem for peaked solitary waves of the CH equation on an infinite line was proven in [38], to which the present work has many common points. Analogous study was performed for peaked solitary waves in a different model (the Novikov equation) in [7], where the local well-posedness result of the initial-value problem in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ was not previously available.

In the context of the peaked periodic waves, linear instability of peaked periodic waves was obtained for a different model (the reduced Ostrovsky equation) in [19]. More recently, spectral instability for perturbations in $L^2(\mathbb{T})$ was obtained in [20] for the reduced Ostrovsky equation with either quadratic or cubic nonlinearities. Neither local well-posedness nor the nonlinear instability was considered for the reduced Ostrovsky equation in [19, 20].

Let us explain the organization of the paper.

In section 2, we define weak solutions to the CH equation (1.3) in $H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$. For solutions with a single peak at $\xi(t)$ on \mathbb{T} , we prove that the single peak propagates with the local characteristic speed so that

$$(1.14) \quad \frac{d\xi}{dt} = u(t, \xi(t)).$$

This allows us to define the precise form of the evolution equations for perturbations to the peaked periodic wave (see Appendix A). It follows from (1.14) that the speed c of the traveling peaked periodic wave $u(t, x) = \varphi(x - ct)$ is uniquely determined as $c = \varphi(0) = M$. This result is well known: all weak traveling wave solutions to the CH equation were classified in [30], and the general family of peaked periodic waves is given in (8.5) of [30] (see also (C.4) in Appendix C).

In section 3, we study the linearized evolution equations for perturbations to the peaked periodic wave. Similarly to [38] and [7], we are able to simplify the linearized evolution equation in a compact form, which requires no convolution integrals and which can be solved explicitly by using the method of characteristics (see Appendix B). Linear instability of the peaked periodic wave is proven both in H^1 and $W^{1,\infty}$. Compared to the previous work [38], we add a new result, where we show that the linearized instability in H^1 is related to the conservation of the two energy quantities (1.9).

In section 4, we consider nonlinear dynamics of peaked perturbations to the peaked periodic wave. Although the H^1 norm of the perturbation does not grow in the nonlinear evolution due to the same two conserved quantities (1.9), as is shown in [28, 29], we prove that the $W^{1,\infty}$ norm of the perturbation can grow. Moreover, we prove that the $W^{1,\infty}$ norm of the perturbation can blow up in a finite time as follows:

$$(1.15) \quad u_x(t, x) \rightarrow -\infty \quad \text{at some } x \in \mathbb{T} \quad \text{as } t \rightarrow T^-.$$

The wave breakdown criterion (1.15) is natural for the inviscid Burgers equation

$$(1.16) \quad u_t + uu_x = 0,$$

which contributes to the local part of the CH equation (1.3). The precise blow-up rate was derived for the strong solutions in $H^3_{\text{per}}(\mathbb{T})$ by using the method of characteristics [9]. Compared to the previous work [38], we do not take for granted the existence of local solutions in $H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ and prove local well-posedness for piecewise C^1 perturbations with a single peak on \mathbb{T} by using the method of characteristics. We

also show that the wave breakdown (1.15) does occur as a result of the nonlinear instability of Theorem 2.

Similar to the case of peaked solitary waves in the CH equation [38] and contrary to the case of peaked solitary waves in the Novikov equation [7], we confirm that the passage from the linear to the nonlinear theory is false in H^1 , as was expected in [11]. Although the linearized instability in H^1 is replaced by the nonlinear stability result of Theorem 1 in H^1 , the linearized instability in $W^{1,\infty}$ persists as the nonlinear instability result of Theorem 2. This nonlinear instability arises due to the negative slopes on the right side of the peak, which grow unboundedly from below.

Finally, we mention two open problems for the CH equation (1.1).

Peaked periodic waves (1.8) appear to be at the border between two families of smooth and cusped periodic waves (see [30] and Appendix C). Existence of smooth periodic waves was studied in [21], whereas cusped periodic waves were analyzed in [4, 25]. It is naturally to expect that the smooth periodic waves are orbitally stable in $H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$, similarly to the case of the reduced Ostrovsky equation considered in [18]. Orbital stability of the smooth periodic waves in $H^1_{\text{per}}(\mathbb{T})$ was obtained in [31] with the inverse scattering methods for initial data in $H^3_{\text{per}}(\mathbb{T})$. On the other hand, it is difficult to consider stability of cusped periodic waves due to the lack of continuity with respect to initial data and the norm inflation in $H^1_{\text{per}}(\mathbb{T})$.

Another interesting problem is to study how the nonlinear instability of peaked waves complicates the interaction dynamics of multi-peaked solutions. Asymptotic stability of multi-peaked solitary waves is proven in [36] (without antipeakons) and in [37] (in the presence of antipeakons). Collisions of peakons and antipeakons lead to jumps of the energy quantities (1.9) [3]; see also relevant results in [16, 23, 26, 32, 33]. Perturbations to multi-peakons are expected to grow in the $W^{1,\infty}$ norm similarly to perturbations to single peakons considered in [38] and in this work.

2. Peaked periodic waves as weak solutions. Let us rewrite the initial-value problem for the CH equation (1.3) in the form

$$(2.1) \quad \begin{cases} u_t + uu_x + Q[u] = 0, & t > 0, \\ u|_{t=0} = u_0, \end{cases}$$

where

$$(2.2) \quad Q[u](x) := \frac{1}{2} \int_{\mathbb{T}} \varphi'(x-y)q[u](y)dy, \quad q[u] := u^2 + \frac{1}{2}u_x^2, \quad x \in \mathbb{T},$$

and the time dependence of $Q[u]$ and $q[u]$ is dropped for convenience. The following lemma describes properties of $Q[u]$ depending on the class of functions for u .

LEMMA 2.1. *If $u \in H^1_{\text{per}}(\mathbb{T})$, then $Q[u] \in C^0_{\text{per}}(\mathbb{T})$. If, in addition, $u \in W^{1,\infty}(\mathbb{T})$, then $Q[u]$ is Lipschitz on \mathbb{T} .*

Proof. The integration in (2.2) can be split as a sum of two terms:

$$Q[u](x) = \frac{m}{2} \left[\int_{-\pi}^x \sinh(x-y-\pi)q[u](y)dy - \int_x^{\pi} \sinh(y-x-\pi)q[u](y)dy \right],$$

where m is given by (1.6). Since $q[u]$ is absolutely integrable if $u \in H^1_{\text{per}}(\mathbb{T})$, each integral is continuous on \mathbb{T} . If $u \in H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$, then $q[u]$ is also bounded, so that each integral is Lipschitz on \mathbb{T} . \square

We say that $u \in C([0, T], H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ is a weak solution to the initial-value problem (2.1) for some maximal existence time $T > 0$ if

$$(2.3) \quad \int_0^T \int_{\mathbb{T}} \left(u\psi_t + \frac{1}{2}u^2\psi_x - Q[u]\psi \right) dxdt + \int_{\mathbb{T}} u_0(x)\psi(0, x)dx = 0$$

is satisfied for every test function $\psi \in C^1([0, T] \times \mathbb{T})$ such that $\psi(T, \cdot) = 0$.

We consider the class of peaked periodic wave solutions with a single peak on \mathbb{T} placed at the point $x = \xi(t)$ for every $t \in [0, T]$. Hence we introduce the following notation:

$$(2.4) \quad C^1_{\xi} := \{u \in H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}) : u_x \in C(\mathbb{T} \setminus \{\xi\})\}.$$

The following lemma shows that the single peak moves with its local characteristic speed as in (1.14).

LEMMA 2.2. *Assume that $u \in C([0, T], H^1_{\text{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ is a weak solution to the CH equation in the form (2.3) and there exists $\xi(t) \in \mathbb{T}$ for $t \in [0, T]$ such that $u(t, \cdot) \in C^1_{\xi(t)}$ for $t \in [0, T]$. Then $\xi \in C^1(0, T)$ satisfies*

$$(2.5) \quad \frac{d\xi}{dt} = u(t, \xi(t)), \quad t \in (0, T).$$

Proof. Integrating (2.3) by parts for $x < \xi(t)$ and $x > \xi(t)$ on \mathbb{T} and using the fact that $u(t, \cdot) \in C^0_{\text{per}}(\mathbb{T})$ and $u(t, \cdot) \in C^1_{\xi(t)}$ for $t \in [0, T]$, we obtain the following equations piecewise outside the peak’s location:

$$(2.6) \quad u_t(t, x) + u(t, x)u_x(t, x) + Q[u](t, x) = 0, \quad \pm[x - \xi(t)] > 0, \quad t \in (0, T).$$

By Lemma 2.1, $Q[u]$ is a continuous function of x on \mathbb{T} for $t \in [0, T]$, and hence it follows from (2.6) that

$$(2.7) \quad [u_t]_{\pm}^+ + u(t, \xi(t))[u_x]_{\pm}^+ = 0, \quad t \in (0, T),$$

where

$$[v]_{\pm}^+ := \lim_{x \rightarrow \xi(t)^+} v(t, x) - \lim_{x \rightarrow \xi(t)^-} v(t, x)$$

is the jump of v across the peak location at $x = \xi(t)$. On the other hand, since $u(t, \cdot) \in C^1_{\xi(t)}$ for $t \in [0, T]$, we differentiate $u(t, \xi(t))$ continuously on both sides from $x = \xi(t)$ and define

$$(2.8) \quad \dot{u}^{\pm}(t) := \lim_{x \rightarrow \xi(t)^{\pm}} \left[u_t + \frac{d\xi}{dt}u_x \right], \quad t \in (0, T).$$

Since $u(t, \xi(t))$ is continuous for $t \in [0, T]$, we have $\dot{u}^+(t) = \dot{u}^-(t)$ almost everywhere for $t \in (0, T)$. Therefore, it follows from (2.8) that

$$(2.9) \quad [u_t]_{\pm}^+ + \frac{d\xi}{dt}[u_x]_{\pm}^+ = 0 \quad \text{a.e. } t \in (0, T).$$

Since $[u_x]_{\pm}^+ \neq 0$ if $u \notin C^1_{\text{per}}(\mathbb{T})$, then it follows from (2.7) and (2.9) that $\xi(t)$ satisfies (2.5) almost everywhere for $t \in (0, T)$. Since $u \in C([0, T] \times \mathbb{T})$ due to Sobolev embedding of $H^1_{\text{per}}(\mathbb{T})$ into $C^0_{\text{per}}(\mathbb{T})$, then $u(t, \xi(t)) \in C^0(0, T)$, so that (2.5) is satisfied everywhere for $t \in (0, T)$ and $\xi \in C^1(0, T)$. \square

As a corollary, it follows from Lemma 2.2 that if $u(t, x) = \varphi(x - ct)$ is the traveling peaked periodic wave, then $c = \varphi(0) = M$ is uniquely defined with M given by (1.5). The following lemma proves that $c = M$ by explicit computation.

LEMMA 2.3. *The traveling peaked periodic wave $u(t, x) = \varphi(x - ct)$ satisfies the stationary equation*

$$(2.10) \quad -c\varphi + \frac{1}{2}\varphi^2 + \frac{3}{4}\varphi * \varphi^2 = d, \quad x \in \mathbb{T},$$

with $c = M$ and $d = m^2$, where the nonlocal equation is piecewise C^1 on both sides from the peak at $x = 0$.

Proof. We shall use the relation $(\varphi')^2 = \varphi^2 - m^2$ for $x \in \mathbb{T} \setminus \{0\}$, which follows from (1.4). Substituting $u(t, x) = \varphi(x - ct)$ into (1.3) and integrating in x yields the nonlocal equation (2.10) where d is an integration constant. In order to verify the validity of this equation and the explicit values of c and d , we shall consider $x \in (0, 2\pi)$, for which one can use the expression $\varphi(x) = m \cosh(\pi - x)$ without the modulus sign. By using $\varphi * 1 = 2$ and continuing with explicit evaluation of integrals, we derive

$$\begin{aligned} \varphi * \varphi^2 &= \frac{m^2}{4} [\varphi * e^{2\pi-2x} + 2\varphi * 1 + \varphi * e^{-2\pi+2x}] \\ &= \frac{m^2}{3} [3 + 4 \cosh(\pi) \cosh(\pi - x) - \cosh(2\pi - 2x)], \end{aligned}$$

so that

$$\frac{3}{4}\varphi * \varphi^2 + \frac{1}{2}\varphi^2 = M\varphi + m^2,$$

which coincides with (2.10) for $c = M$ and $d = m^2$. \square

By using Lemmas 2.2 and 2.3, we shall now derive the evolution equations for perturbations near the peaked periodic wave. We are looking for a weak solution $u \in C([0, T], H_{\text{per}}^1(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}))$ to the CH equation in the form (2.3), for which there exists $\xi(t) = ct + a(t) \in \mathbb{T}$ for $t \in [0, T]$ such that $u(t, \cdot) \in C_{\xi(t)}^1$ for $t \in [0, T]$. We present the solution in the form

$$(2.11) \quad u(t, x) = \varphi(x - ct - a(t)) + v(t, x - ct - a(t)), \quad t \in [0, T], \quad x \in \mathbb{T},$$

where $c = M$, $a(t)$ is the deviation of the peak position from its unperturbed position moving with the speed c , and $v(t, x)$ is the perturbation to the peaked periodic wave φ . By Lemma 2.2, $a \in C^1(0, T)$ satisfies the equation

$$(2.12) \quad \frac{da}{dt} = v(t, 0), \quad t \in (0, T).$$

Substituting (2.11) and (2.12) into the initial-value problem (2.1) yields the following problem for the peaked perturbation v :

$$(2.13) \quad \begin{cases} v_t = (c - \varphi)v_x + (v|_{x=0} - v)\varphi' + (v|_{x=0} - v)v_x \\ \quad - \varphi' * (\varphi v + \frac{1}{2}\varphi'v_x) - Q[v], & t \in (0, T), \\ v|_{t=0} = v_0, \end{cases}$$

where we have used the stationary equation (2.10) piecewise on both sides from the peak and replaced $x - ct - a(t)$ by x due to the translational invariance of the system (2.1) with the convolution integral (2.2).

3. Linearized evolution. Here we study the linearized equation of motion arising in the truncation of the nonlinear equation in system (2.13) at the linear terms in v :

$$(3.1) \quad v_t = (c - \varphi)v_x + (v|_{x=0} - v)\varphi' - \varphi' * \left(\varphi v + \frac{1}{2}\varphi'v_x \right).$$

We first simplify the linearized equation (3.1) by using the following elementary result.

LEMMA 3.1. *Assume that $v \in H^1_{\text{per}}(\mathbb{T})$. Then it is true for every $x \in \mathbb{T}$ that*

$$\begin{aligned} & [v(0) - v(x)]\varphi'(x) - (\varphi' * \varphi v)(x) - \frac{1}{2}(\varphi' * \varphi'v_x)(x) \\ &= \varphi(x) \int_0^x v(y)dy - \frac{1}{2}m^2 \sinh(x) \int_{-\pi}^\pi v(y)dy. \end{aligned}$$

Proof. Since integrals of absolutely integrable functions are continuous, the map

$$x \mapsto \varphi' * \left(\varphi v + \frac{1}{2}\varphi'v_x \right)$$

is continuous for every $x \in \mathbb{T}$. Now, $H^1_{\text{per}}(\mathbb{T})$ is continuously embedded into the space of continuous and periodic functions on \mathbb{T} , and hence $v \in C^0_{\text{per}}(\mathbb{T})$. Integrating by parts yields the following explicit expression for every $x \in \mathbb{T}$:

$$\begin{aligned} (\varphi' * \varphi'v_x)(x) &= (\varphi'' * \varphi'v)(x) - (\varphi' * \varphi''v)(x) \\ &= (\varphi * \varphi'v)(x) - 2\varphi'(x)v(x) - (\varphi' * \varphi v)(x) + 2\varphi'(x)v(0), \end{aligned}$$

which yields

$$[v(0) - v(x)]\varphi'(x) - (\varphi' * \varphi v)(x) - \frac{1}{2}(\varphi' * \varphi'v_x)(x) = -\frac{1}{2}(\varphi' * \varphi v)(x) - \frac{1}{2}(\varphi * \varphi'v)(x).$$

Furthermore, we obtain for $x \in (0, \pi]$

$$\begin{aligned} & -\frac{1}{2}(\varphi * \varphi'v)(x) - \frac{1}{2}(\varphi' * \varphi v)(x) \\ &= -\frac{m^2}{2} \left[\int_{-\pi}^0 \sinh(x)v(y)dy - \int_0^x \sinh(2\pi - x)v(y)dy + \int_x^\pi \sinh(x)v(y)dy \right] \\ &= \varphi(x) \int_0^x v(y)dy - \frac{m^2}{2} \sinh(x) \int_{-\pi}^\pi v(y)dy, \end{aligned}$$

which completes the proof of the equality for $x \in (0, \pi]$. For $x \in [-\pi, 0)$, the computations are similar:

$$\begin{aligned} & -\frac{1}{2}(\varphi * \varphi'v)(x) - \frac{1}{2}(\varphi' * \varphi v)(x) \\ &= -\frac{m^2}{2} \left[\int_{-\pi}^x \sinh(x)v(y)dy + \int_x^0 \sinh(2\pi + x)v(y)dy + \int_0^\pi \sinh(x)v(y)dy \right] \\ &= \varphi(x) \int_0^x v(y)dy - \frac{m^2}{2} \sinh(x) \int_{-\pi}^\pi v(y)dy. \end{aligned}$$

The zero value at $x = 0$ is recovered by taking the one-sided limits $x \rightarrow 0^\pm$ in the previous two expressions. □

By Lemma 3.1, we can rewrite the initial-value problem for the linear equation (3.1) in the equivalent form

$$(3.2) \quad \begin{cases} v_t = (c - \varphi)v_x + \varphi w - \pi m^2 \bar{v} \sinh(x), & t > 0, \\ v|_{t=0} = v_0, \end{cases}$$

where we have introduced

$$(3.3) \quad w(t, x) := \int_0^x v(t, y) dy, \quad \bar{v}(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t, y) dy = \frac{w(t, \pi) - w(t, -\pi)}{2\pi}.$$

Let us consider the linearized initial-value problem (3.2) in the space C_0^1 defined by (2.4) with $\xi \equiv 0$. The following lemma shows that the values of $v(t, 0)$ and $\bar{v}(t)$ are independent of t .

LEMMA 3.2. *Assume that there exists a solution $v \in C(\mathbb{R}^+, C_0^1)$ to the initial-value problem (3.2). Then $v(t, 0) = v_0(0)$ and $\bar{v}(t) = \bar{v}_0$ for every $t \in \mathbb{R}^+$.*

Proof. If $v \in C(\mathbb{R}^+, C_0^1)$, then $w \in C(\mathbb{R}^+, C^1(\mathbb{T}))$ so that $w(t, 0) = 0$ follows from (3.3). Hence, it follows from (3.2) that

$$\lim_{x \rightarrow 0^\pm} v_t(t, x) = 0, \quad t \in \mathbb{R}^+,$$

due to $\varphi(0) = c$ and $v(t, \cdot) \in C_0^1$ for every $t \in \mathbb{R}^+$. Hence, $v(t, 0) = v_0(0)$ for every $t \in \mathbb{R}^+$.

Integrating the evolution equation for the solution $v \in C(\mathbb{R}^+, C_0^1)$ in x on \mathbb{T} and using integration by parts piecewise on $[-\pi, 0]$ and $[0, \pi]$, we obtain for every $t \in \mathbb{R}^+$

$$\begin{aligned} \frac{d}{dt} \bar{v}(t) &= \frac{1}{2\pi} \int_{\mathbb{T}} (\varphi(0) - \varphi(x)) v_x(t, x) dx + \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(x) w(t, x) dx \\ &= \frac{1}{2\pi} \left[(\varphi(0) - \varphi(x)) v(t, x) + \varphi'(x) \int_0^x v(t, y) dy \right] \Big|_{x=0^+}^{x=\pi} + \Big|_{x=-\pi}^{x=0^-} \\ &= 0, \end{aligned}$$

due to $\varphi(\pi) = \varphi(-\pi)$, $\varphi'(\pm\pi) = 0$, and $v(t, \cdot) \in C_{\text{per}}^0(\mathbb{T})$ for every $t \in \mathbb{R}^+$. Hence, $\bar{v}(t) = \bar{v}_0$ for every $t \in \mathbb{R}^+$. \square

The evolution problem (3.2) can be solved explicitly by using the method of characteristics piecewise for $x \in \mathbb{T}$ on both sides of the peak at $x = 0$. The family of characteristic curves $x = X(t, s)$ satisfies the initial-value problem

$$(3.4) \quad \begin{cases} \frac{dX}{dt} = \varphi(X) - \varphi(0), \\ X|_{t=0} = s, \end{cases}$$

for every $s \in \mathbb{T}$.

Along each characteristic curve $x = X(t, s)$ satisfying (3.4), function $W(t, s) := w(t, X(t, s))$ satisfies the initial-value problem

$$(3.5) \quad \begin{cases} \frac{dW}{dt} = \varphi'(X(t, s))W + \pi m^2 \bar{v} [1 - \cosh(X(t, s))], \\ W|_{t=0} = w_0(s), \end{cases}$$

where $w_0(x) := \int_0^x v_0(y) dy$. The initial-value problem (3.5) is obtained by linearizing the nonlinear equation (A.2) in Appendix A and posing it at the characteristic curve

$x = X(t, s)$. It follows from (3.2) and (3.4) that $V(t, s) := v(t, X(t, s))$ satisfies the initial-value problem

$$(3.6) \quad \begin{cases} \frac{dV}{dt} = \varphi(X(t, s))W(t, s) - \pi m^2 \bar{v} \sinh(X(t, s)), \\ V|_{t=0} = v_0(s). \end{cases}$$

The following lemma gives the solution of the initial-value problem (3.2) in class $C^1(\mathbb{R}^+, C_0^1)$ from solutions to the initial-value problems (3.4), (3.5), and (3.6).

LEMMA 3.3. *For every $v_0 \in C_0^1$, there exists the unique solution $v \in C^1(\mathbb{R}^+, C_0^1)$ to the initial-value problem (3.2).*

Proof. By the existence and uniqueness theory for differential equations, there exists the unique solution $X(\cdot, s) \in C^1(\mathbb{R}^+)$ to (3.4) for every $s \in \mathbb{T}$ due to the Lipschitz continuity of φ on \mathbb{T} . The peak’s location at $X = 0$ is a critical point which remains invariant under the time flow, and hence $X(t, 0) = 0$. The ends of \mathbb{T} at $x = \pm\pi$ are not invariant under the time flow. However, the adjacent peak’s location at $X = 2\pi$ is also a critical point invariant under the time flow with $X(t, 2\pi) = 2\pi$, and hence the interval $[0, 2\pi]$ is invariant under the time flow and it makes sense to consider the initial-value problem (3.4) on the interval $[0, 2\pi]$.

By the continuous dependence of solutions on the initial data, the map $(0, 2\pi) \ni s \mapsto X(t, s) \in (0, 2\pi)$ is C^1 for every $t \in \mathbb{R}^+$. Moreover, the transformation is invertible because

$$(3.7) \quad \frac{\partial X}{\partial s} = e^{\int_0^t \varphi'(X(t', s)) dt'} > 0, \quad t \in \mathbb{R}^+, \quad s \in (0, 2\pi).$$

By substituting the solution $X(t, s)$ to (3.4) into (3.5) and solving the linear inhomogeneous equation for W , we obtain the unique solution $W(\cdot, s) \in C^1(\mathbb{R}^+)$ to (3.5) for every fixed $s \in [0, 2\pi]$. Moreover, since $v_0 \in C_{\text{per}}^0(\mathbb{T})$, the map $(0, 2\pi) \ni s \mapsto W(t, s) \in \mathbb{R}$ is C^1 for every $t \in \mathbb{R}^+$. The right-hand side of (3.5) implies $W(t, 0) = 0 = w_0(0)$ and $W(t, 2\pi) = 2\pi \bar{v} = w_0(2\pi)$, where the last equality is due to invariance of the mean value of a periodic function in $C_{\text{per}}^0(\mathbb{T})$ with respect to any starting point on \mathbb{T} .

Similarly, by substituting the solutions $X(t, s)$ and $W(t, s)$ to (3.4) and (3.5) into (3.6) and integrating it for V , we obtain the unique solution $V(\cdot, s) \in C^1(\mathbb{R}^+)$ to (3.6) for every fixed $s \in [0, 2\pi]$. Moreover, since $v_0 \in C_0^1$, the map $(0, 2\pi) \ni s \mapsto V(t, s) \in \mathbb{R}$ is C^1 for every $t \in \mathbb{R}^+$. The right-hand side of (3.6) implies $V(t, 0) = v_0(0)$ and $V(t, 2\pi) = v_0(2\pi) = v_0(0)$. Since $v_0 \in C_0^1$, then $V(t, \cdot) \in C_0^1$ for every $t \in \mathbb{R}^+$.

Finally, the change of coordinates $\mathbb{R}^+ \times (0, 2\pi) \ni (t, s) \rightarrow (t, X) \in \mathbb{R}^+ \times (0, 2\pi)$ is a diffeomorphism due to (3.7). Since \mathbb{T} is compact, the solution $v(t, \cdot) = V(t, s = X^{-1}(t, \cdot))$ belongs to C_0^1 for every $t \in \mathbb{R}^+$. \square

Although the proof of Lemma 3.3 does not rely on construction of the exact solutions to the initial-value problems (3.4), (3.5), and (3.6), it is easy to obtain exact solutions (see Appendix B). By analyzing the exact solution (B.3), we show that $v(t, \cdot)$ remains bounded in the L^∞ norm, as in the following lemma.

LEMMA 3.4. *Assume that $v_0 \in C_0^1$ in the initial-value problem (3.2). Then there exists $C_0 > 0$ such that*

$$(3.8) \quad \|v(t, \cdot)\|_{L^\infty(\mathbb{T})} \leq C_0, \quad t \in \mathbb{R}^+.$$

Proof. By Lemma 3.3, the unique solution to the initial-value problem (3.2) with $v_0 \in C_0^1$ satisfies $v(t, \cdot) \in C_0^1$ for every $t \in \mathbb{R}^+$. By Sobolev's embedding, $v(t, \cdot) \in L^\infty(\mathbb{T})$ for every $t \in \mathbb{R}^+$. It remains to obtain the bound (3.8) uniformly in $t \in \mathbb{R}^+$.

It suffices to consider the exact solution (B.3) from Appendix B for $s \in [0, 2\pi]$ and $t \in \mathbb{R}^+$ since

$$\|v(t, \cdot)\|_{L^\infty(\mathbb{T})} = \max_{s \in [0, 2\pi]} |V(t, s)|.$$

Since $v_0, w_0 \in L^\infty(\mathbb{T})$, we only need to estimate $Y(t, s)$ in (B.3). By using elementary transformations, we rewrite $Y(t, s)$ in the form

$$Y(t, s) = \frac{(1 - e^{-t})[\sinh(\pi - \frac{s}{2}) \cosh(\pi - \frac{s}{2}) + e^{-t} \sinh(\frac{s}{2}) \cosh(\frac{s}{2})]}{[\sinh(\pi - \frac{s}{2}) + e^{-t} \sinh(\frac{s}{2})]^2 + 2(\cosh \pi - 1)e^{-t} \sinh(\frac{s}{2}) \sinh(\pi - \frac{s}{2})}.$$

For $s \in [0, \pi]$, we have

$$0 \leq Y(t, s) \leq \frac{\sinh(\pi - \frac{s}{2}) \cosh(\pi - \frac{s}{2}) + \sinh(\frac{s}{2}) \cosh(\frac{s}{2})}{\sinh^2(\pi - \frac{s}{2})},$$

which is bounded on $[0, \pi]$ uniformly in $t \in \mathbb{R}^+$. For $s \in [\pi, 2\pi]$, it follows that

$$\begin{aligned} 0 \leq Y(t, s) &\leq \frac{\cosh(\frac{s}{2})[\sinh(\pi - \frac{s}{2}) + e^{-t} \sinh(\frac{s}{2})] - \sinh(\pi - \frac{s}{2})[\cosh(\frac{s}{2}) - \cosh(\pi - \frac{s}{2})]}{[\sinh(\pi - \frac{s}{2}) + e^{-t} \sinh(\frac{s}{2})]^2} \\ &\leq \frac{\cosh(\frac{s}{2})}{\sinh(\pi - \frac{s}{2}) + e^{-t} \sinh(\frac{s}{2})} =: Z(t, s). \end{aligned}$$

Since

$$\frac{\partial Z}{\partial s} = \frac{\cosh(\pi) - e^{-t}}{2[\sinh(\pi - \frac{s}{2}) + e^{-t} \sinh(\frac{s}{2})]^2} > 0,$$

the map $[\pi, 2\pi] \ni s \mapsto Z(t, s) \in \mathbb{R}^+$ is monotonically increasing for $t \in \mathbb{R}^+$ so that

$$\max_{s \in [\pi, 2\pi]} Z(t, s) = Z(t, 2\pi) = Me^t,$$

which is exponentially growing in $t \in \mathbb{R}^+$. We also have in (B.3)

$$\begin{aligned} (3.9) \quad &w_0(s) - \pi m^2 \bar{v} (\cosh s - 1)(1 - e^{-t}) \\ &= \int_{2\pi}^s v_0(s') ds' + \pi m^2 \bar{v} (\cosh 2\pi - \cosh s) + \pi m^2 \bar{v} e^{-t} (\cosh s - 1). \end{aligned}$$

The third term in (3.9) multiplied by $Y(t, s)$ is bounded on $[\pi, 2\pi]$ by

$$\pi m^2 |\bar{v}| e^{-t} (\cosh s - 1) Y(t, s) \leq 2\pi |\bar{v}| e^{-t} Z(t, 2\pi) \leq 2\pi M |\bar{v}|.$$

The second term in (3.9) multiplied by $Y(t, s)$ is bounded on $[\pi, 2\pi]$ by

$$\begin{aligned} \pi m^2 |\bar{v}| (\cosh 2\pi - \cosh s) Y(t, s) &\leq 2\pi m^2 |\bar{v}| \sinh\left(\pi + \frac{s}{2}\right) \sinh\left(\pi - \frac{s}{2}\right) Z(t, s) \\ &\leq 4\pi M |\bar{v}| \frac{\sinh(\pi - \frac{s}{2}) \cosh(\frac{s}{2})}{\sinh(\pi - \frac{s}{2}) + e^{-t} \sinh(\frac{s}{2})} \\ &\leq 4\pi M |\bar{v}| \cosh(\pi). \end{aligned}$$

Finally, since $v_0 \in L^\infty(\mathbb{T})$, it follows for every $s \in [\pi, 2\pi]$ that

$$\left| \int_{2\pi}^s v_0(s') ds' \right| \leq 2\|v_0\|_{L^\infty(\mathbb{T})} \left(\pi - \frac{s}{2} \right) \leq 2\|v_0\|_{L^\infty(\mathbb{T})} \sinh \left(\pi - \frac{s}{2} \right).$$

Hence, the first term in (3.9) multiplied by $Y(t, s)$ is bounded by the same estimate as the second term is. All estimates together yield the bound (3.8). \square

The result of Lemma 3.4 suggests the linear stability of peaked perturbations in the L^∞ norm. The following lemma shows that the perturbations grow in the $W^{1,\infty}$ norm, and this growth is generic at the right side of the peak.

LEMMA 3.5. *Assume that $v_0 \in C_0^1$ in the initial-value problem (3.2). Then it follows that*

$$(3.10) \quad \lim_{x \rightarrow 0^+} v_x(t, x) = e^t v_0'(0^+) + (Mv_0(0) - \pi m^2 \bar{v})(e^t - 1)$$

and

$$(3.11) \quad \lim_{x \rightarrow 0^-} v_x(t, x) = e^{-t} v_0'(0^-) + (Mv_0(0) - \pi m^2 \bar{v})(1 - e^{-t}).$$

Consequently, if $v_0 \in C_{\text{per}}^1(\mathbb{T})$ with either $v_0(0) \neq \frac{m^2}{2M}\bar{v}$ or $v_0'(0) \neq 0$ or both, then $v(t, \cdot) \notin C_{\text{per}}^1(\mathbb{T})$ for almost every $t \in \mathbb{R}^+$.

Proof. Along each characteristic curve $x = X(t, s)$ satisfying (3.4), function $U(t, s) := v_x(t, X(t, s))$ satisfies the initial-value problem

$$(3.12) \quad \begin{cases} \frac{dU}{dt} = \varphi'(X(t, s)) [W(t, s) - U(t, s)] + \varphi(X(t, s))V(t, s) \\ \quad - \pi m^2 \bar{v} \cosh(X(t, s)), \\ U|_{t=0} = v_0'(s). \end{cases}$$

The initial-value problem (3.12) is obtained by linearizing the nonlinear equation (A.3) in Appendix A and posing it at the characteristic curve $x = X(t, s)$. Coefficients of the differential equation for U are all $C^1(0, 2\pi)$. The limits $s \rightarrow 0^+$ and $s \rightarrow 2\pi^-$ yield the differential equations

$$(3.13) \quad \frac{d}{dt}U(t, 0^+) = U(t, 0^+) + Mv_0(0) - \pi m^2 \bar{v}$$

and

$$(3.14) \quad \frac{d}{dt}U(t, 2\pi^-) = -U(t, 2\pi^-) + Mv_0(0) - \pi m^2 \bar{v},$$

where we have used $V(t, 0) = V(t, 2\pi) = v_0(0)$, $w_0(2\pi) = 2\pi\bar{v}$, and

$$w_0(2\pi) - \pi m^2 \bar{v} \cosh(2\pi) = -\pi m^2 \bar{v}.$$

Since $U(t, 0^-) = U(t, 2\pi^-)$ due to periodic continuation of all coefficients of the differential equation (3.12) and its solution as $C^1(-2\pi, 0)$, we obtain (3.10) and (3.11) by solving the linear differential equations (3.13) and (3.14) with $U(0, 0^+) = v_0'(0^+)$ and $U(0, 2\pi^-) = U(0, 0^-) = v_0'(0^-)$. \square

The exponential growth of the $W^{1,\infty}$ norm in Lemma 3.5 due to the exact solution (3.10) at $x = 0^+$ is equivalent to the linear instability of the peaked perturbations in $W^{1,\infty}$. The following lemma shows that the peaked perturbations are also linearly unstable in H^1 .

LEMMA 3.6. Assume that $v_0 \in C_0^1$ in the initial-value problem (3.2). Then it follows that

$$(3.15) \quad \|v(t, \cdot)\|_{H^1(\mathbb{T})}^2 = C_+ e^t + C_0 + C_- e^{-t}$$

for some uniquely defined constants C_+, C_0, C_- .

Proof. Since $v(t, \cdot) \in C_0^1$ for every $t \in \mathbb{R}^+$ is the unique solution to the initial-value problem (3.2) by Lemma 3.3, the H^1 norm is independent on the choice of the fundamental interval. Hence we proceed with a priori energy estimates on $[0, 2\pi]$ instead of \mathbb{T} . We write the evolution equation for v and v_x explicitly:

$$(3.16) \quad v_t = (c - \varphi)v_x + \varphi w - \pi m^2 \bar{v} \sinh(x)$$

and

$$(3.17) \quad v_{tx} = (c - \varphi)v_{xx} + \varphi'(w - v_x) + \varphi v - \pi m^2 \bar{v} \cosh(x).$$

By multiplying (3.16) by v and (3.17) by v_x , integrating on $[0, 2\pi]$, and using integration by parts for $v(t, \cdot) \in C_0^1$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v(t, \cdot)\|_{L^2(0, 2\pi)}^2 &= \int_0^{2\pi} \left[(c - \varphi)vv_x + \varphi v \int_0^x v dy - \pi m^2 \bar{v} \sinh(x)v \right] dx \\ &= \int_0^{2\pi} \left[\frac{1}{2} \varphi' v^2 + \varphi v \int_0^x v dy - \pi m^2 \bar{v} \sinh(x)v \right] dx \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_x(t, \cdot)\|_{L^2(0, 2\pi)}^2 &= \int_0^{2\pi} \left[(c - \varphi)v_x v_{xx} + \varphi' v_x \int_0^x v dy - \varphi' v_x^2 + \varphi v v_x - \pi m^2 \bar{v} \cosh(x)v_x \right] dx \\ &= \int_0^{2\pi} \left[-\frac{1}{2} \varphi' v_x^2 - \frac{3}{2} \varphi' v^2 - \varphi v \int_0^x v dy + \pi m^2 \bar{v} \sinh(x)v \right] dx, \end{aligned}$$

where we have used $c = \varphi(0) = \varphi(2\pi)$, $\varphi'(2\pi^-) = 1$, and $v(t, 0) = v(t, 2\pi)$. Adding these two lines together yields

$$(3.18) \quad \frac{d}{dt} E(v) = -2 \int_0^{2\pi} \varphi' \left[v^2 + \frac{1}{2} v_x^2 \right] dx,$$

where $E(v) := \|v(t, \cdot)\|_{L^2(0, 2\pi)}^2 + \|v_x(t, \cdot)\|_{L^2(0, 2\pi)}^2$ in agreement with (1.9).

Next, we multiply (3.16) by φv and (3.17) by φv_x , integrate on $[0, 2\pi]$, and use integration by parts for $v(t, \cdot) \in C_0^1$. After straightforward computations, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^{2\pi} \varphi v^2 dx = \int_0^{2\pi} \left[-\frac{1}{2} (c - 2\varphi) \varphi' v^2 + \varphi^2 v \int_0^x v dy - \pi m^2 \bar{v} \varphi \sinh(x)v \right] dx$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^{2\pi} \varphi v_x^2 dx &= \int_0^{2\pi} \left[-\frac{1}{2} c \varphi' v_x^2 - 2\varphi \varphi' v^2 - (\varphi^2 + (\varphi')^2) v \int_0^x v dy \right] dx \\ &\quad + \pi m^2 \bar{v} \int_0^{2\pi} [\varphi \sinh(x) + \varphi' \cosh(x)] v dx, \end{aligned}$$

where we can use $(\varphi')^2 = \varphi^2 - m^2$ and $\varphi'(x) \cosh(x) - \varphi(x) \sinh(x) = -1$. Adding a linear combination of these two lines multiplied by 2 and 1, respectively, yields

$$(3.19) \quad \frac{d}{dt} \int_0^{2\pi} \varphi \left[v^2 + \frac{1}{2} v_x^2 \right] dx = -c \int_0^{2\pi} \varphi' \left[v^2 + \frac{1}{2} v_x^2 \right] dx,$$

where we have used that

$$\int_0^{2\pi} \left[v \int_0^x v dy - \frac{1}{2} v \int_0^{2\pi} v dy \right] dx = 0.$$

Let us define

$$(3.20) \quad P(t) := \int_0^{2\pi} \varphi \left[v^2 + \frac{1}{2} v_x^2 \right] dx, \quad S(t) := \int_0^{2\pi} \varphi' \left[v^2 + \frac{1}{2} v_x^2 \right] dx.$$

It follows from (3.18) and (3.19) that

$$(3.21) \quad cE(v) = 2P(t) + C_1,$$

where C_1 is arbitrary constant.

Finally, we multiply (3.16) by $\varphi'v$ and (3.17) by $\varphi'v_x$, integrate on $[0, 2\pi]$, and use integration by parts for $v(t, \cdot) \in C_0^1$. After straightforward computations, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^{2\pi} \varphi' v^2 dx = \int_0^{2\pi} \left[-\frac{1}{2} (c\varphi - \varphi^2 - (\varphi')^2) v^2 + \varphi \varphi' v \int_0^x v dy - \pi m^2 \bar{v} \varphi' \sinh(x) v \right] dx$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^{2\pi} \varphi' v_x^2 dx &= \int_0^{2\pi} \left[-\frac{1}{2} (c\varphi - \varphi^2 + (\varphi')^2) v_x^2 - \frac{1}{2} (\varphi^2 + 3(\varphi')^2) v^2 - 2\varphi \varphi' v \int_0^x v dy \right] dx \\ &\quad + \pi m^2 \bar{v} \int_0^{2\pi} [\varphi' \sinh(x) + \varphi \cosh(x)] v dx + M(v|_{x=0})^2 - 2\pi m^2 \bar{v} v|_{x=0}, \end{aligned}$$

where we can use $\varphi(x) \cosh(x) - \varphi'(x) \sinh(x) = M$. Adding a linear combination of these two lines multiplied by 2 and 1, respectively, yields

$$(3.22) \quad \frac{d}{dt} S(t) = -cP(t) + \frac{1}{2} m^2 E(v) + C_2,$$

where C_2 is a constant defined by

$$C_2 := \pi m^2 M \bar{v} + M(v|_{x=0})^2 - 2\pi m^2 \bar{v} v|_{x=0},$$

since $v|_{x=0}$ and \bar{v} are independent of t . Equations (3.19) and (3.22) with conservation (3.21) yield the system of differential equations

$$(3.23) \quad P'(t) = -MS(t), \quad S'(t) = -M^{-1}P(t) + C_3,$$

where $C_3 := \frac{m^2 C_1}{2M} + C_2$, and we have used that $m^2 - M^2 = -1$. Thus, $S''(t) = S(t)$, so that the most general solution of the system (3.23) is given by

$$(3.24) \quad P(t) = -MS_+ e^t + MS_- e^{-t} + MC_3, \quad S(t) = S_+ e^t + S_- e^{-t},$$

where S_+ and S_- are arbitrary constants. Substituting (3.24) into (3.21) yields (3.15) for some constants C_+ , C_0 , and C_- . □

We end this section with two remarks. First we show that the conserved quantity (3.21) for the linearized evolution is obtained from the conserved quantities (1.9) after substituting the solution in the form (2.11) and expanding in powers of the perturbation. By using (1.2) and integrating by parts, we obtain

$$E(u) = E(\varphi) + 2 \int_{\mathbb{T}} (\varphi v + \varphi' v_x) dx + E(v) = E(\varphi) + 4v|_{x=0} + E(v)$$

and

$$\begin{aligned} F(u) &= F(\varphi) + 2 \int_{\mathbb{T}} \varphi(\varphi v + \varphi' v_x) dx + \int_{\mathbb{T}} v[\varphi^2 + (\varphi')^2] dx + 2 \int_{\mathbb{T}} v(\varphi v + \varphi' v_x) dx \\ &\quad + \int_{\mathbb{T}} \varphi[v^2 + (v_x)^2] dx + F(v) \\ &= F(\varphi) + 4M(v|_{x=0}) + 2\pi m^2 \bar{v} + 2P(t) + 2(v|_{x=0})^2 + F(v), \end{aligned}$$

where $P(t)$ is defined by (3.20). By eliminating $v|_{x=0}$ from the first equality and substituting it into the second equality, we obtain

$$(3.25) \quad 2P(t) - ME(v) - \frac{1}{4}[E(u) - E(\varphi)]E(v) + \frac{1}{8}[E(v)]^2 + F(v) = C,$$

where

$$C := F(u) - F(\varphi) - 2\pi m^2 \bar{v} - M[E(u) - E(\varphi)] - \frac{1}{8}[E(u) - E(\varphi)]^2$$

is constant in the time evolution due to the energy conservation. Neglecting the cubic and quartic terms of v in (3.25) recovers the conserved quantity (3.21) of the linearized evolution since $c = M$.

Finally, we illustrate the exact solutions to the initial-value problems (3.4), (3.5), and (3.6) obtained in Appendix B. The solution $v = v(t, x)$ is available in the parametric form of (B.1) and (B.3) due to the method of characteristics with parameter s on $[0, 2\pi]$. The solution is extended periodically to $[-2\pi, 0]$. Figure 2 shows the plots of $v(t, x)$ versus x on $[-2\pi, 2\pi]$ for different values of t for two initial conditions: $v_0(x) = \sin(x)$ (top) and $v_0(x) = \cos(x)$ (bottom), in both cases, $\bar{v} = 0$. These panels give clear illustrations that $v(t, \cdot)$ remains bounded in the L^∞ norm (Lemma 3.4) and that the slope of the perturbation grows on the right side of the peak (Lemma 3.5).

4. Nonlinear evolution. Here we analyze the initial-value problem (2.13) with $v_0 \in C_0^1$ and prove Theorem 2. The challenge is that the local well-posedness in C_0^1 has not been established in the periodic domain \mathbb{T} . Nevertheless, we will get the local well-posedness result by using the method of characteristics and the ODE theory.

By simplifying the linear terms with Lemma 3.1, we rewrite the initial-value problem (2.13) in the equivalent form

$$(4.1) \quad \begin{cases} v_t = (c - \varphi)v_x + \varphi w - \pi m^2 \bar{v} \sinh(x) + (v|_{x=0} - v)v_x - Q[v], \\ v|_{t=0} = v_0, \end{cases}$$

where w and \bar{v} are defined by the same expressions as in (3.3) and

$$(4.2) \quad Q[v](x) := \frac{1}{2} \int_{\mathbb{T}} \varphi'(x - y)q[v](y)dy, \quad q[v] := v^2 + \frac{1}{2}v_x^2, \quad x \in \mathbb{T}.$$

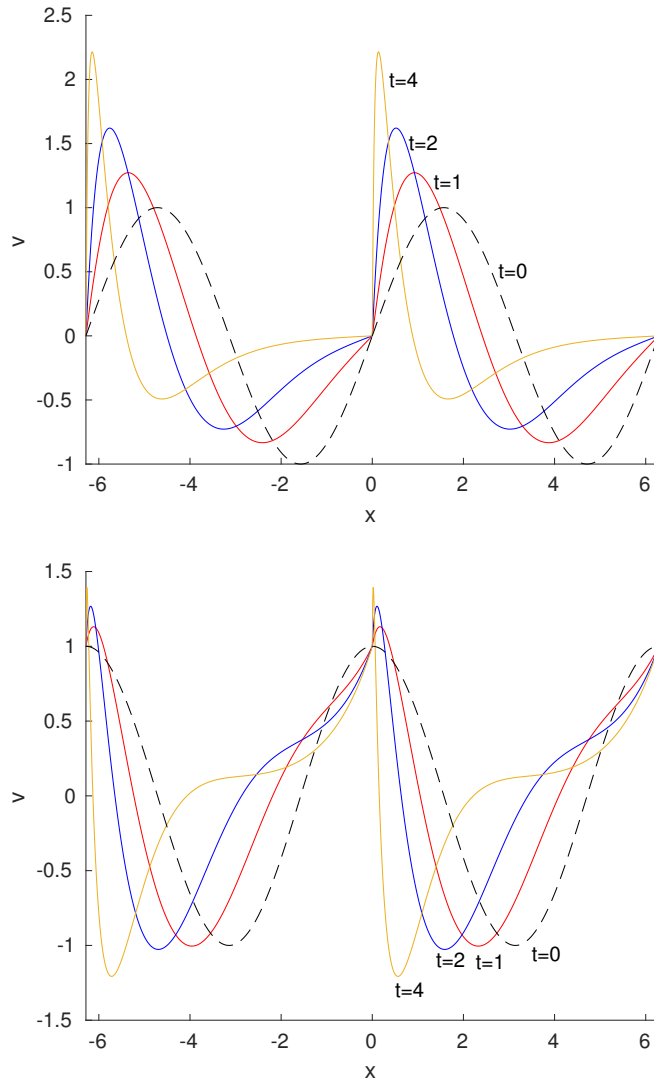


FIG. 2. The plots of $v(t, x)$ versus x on $[-2\pi, 2\pi]$ for different values of t in the case $v_0(x) = \sin(x)$ (top) and $v_0(x) = \cos(x)$ (bottom).

As is derived in Appendix A, w satisfies the initial-value problem

$$(4.3) \quad \begin{cases} w_t = (c - \varphi)w_x + \varphi'w - \pi m^2 \bar{v}[\cosh(x) - 1] - \frac{1}{2}(v|_{x=0} - v)^2 \\ \quad - P[v] + P[v]|_{x=0}, \\ w|_{t=0} = w_0, \end{cases}$$

where $w_0(x) := \int_0^x v_0(y)dy$ and

$$(4.4) \quad P[v](x) := \frac{1}{2} \int_{\mathbb{T}} \varphi(x - y)q[v](y)dy, \quad q[v] := v^2 + \frac{1}{2}v_x^2, \quad x \in \mathbb{T}.$$

Similarly to Lemma 3.2, we prove that \bar{v} is independent of t . At the same time, $v|_{x=0}$ does depend on t in the nonlinear evolution.

LEMMA 4.1. *Assume that there exists a solution $v \in C([0, T], C_0^1)$ to the initial-value problem (4.1). Then $\bar{v}(t) = \bar{v}_0$ for every $t \in [0, T]$.*

Proof. We integrate the evolution equation in system (4.1) for the solution $v \in C([0, T], C_0^1)$ in x on \mathbb{T} and use cancelation of the linear terms in v as in the proof of Lemma 3.2. Then it is true that $\frac{d}{dt}\bar{v}(t) = 0$ if and only if

$$(4.5) \quad \int_{\mathbb{T}} Q[v](t, x) dx = \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi'(x-y) q[v](t, y) dy dx = 0,$$

where both φ' and $q[v]$ are absolutely integrable. Interchanging the integrations by Fubini's theorem and integrating φ' piecewise on both sides of the peak yields for every $t \in [0, T]$

$$\begin{aligned} \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi'(x-y) q[v](t, y) dy dx &= \int_{\mathbb{T}} q[v](t, y) \left(\int_{\mathbb{T}} \varphi'(x-y) dx \right) dy \\ &= \int_{\mathbb{T}} q[v](t, y) (\varphi(\pi-y) - \varphi(-\pi-y)) dy \\ &= 0, \end{aligned}$$

due to periodicity of $\varphi \in C_{\text{per}}^0(\mathbb{T})$. Hence, $\bar{v}(t) = \bar{v}_0$ for every $t \in [0, T]$. \square

We can now develop local well-posedness theory of the initial-value problem (4.1) by means of the method of characteristics. The family of characteristic curves $x = X(t, s)$ satisfies the initial-value problem

$$(4.6) \quad \begin{cases} \frac{dX}{dt} = \varphi(X) - \varphi(0) + v(t, X) - v(t, 0), \\ X|_{t=0} = s, \end{cases}$$

for every $s \in \mathbb{T}$. Assuming that $v(t, \cdot) \in C_0^1$ for every $t \in [0, T]$, we can differentiate (4.6) piecewise in $s \in \mathbb{T} \setminus \{0\}$ and obtain

$$(4.7) \quad \begin{cases} \frac{d}{dt} \frac{\partial X}{\partial s} = [\varphi'(X) + v_x(t, X)] \frac{\partial X}{\partial s}, \\ \frac{\partial X}{\partial s} |_{t=0} = 1, \end{cases}$$

with the exact solution

$$(4.8) \quad \frac{\partial X}{\partial s}(t, s) = \exp \left(\int_0^t [\varphi'(X(t', s)) + v_x(t', X(t', s))] dt' \right).$$

The peak's locations at $X(t, 0) = 0$ and $X(t, 2\pi) = 2\pi$ are invariant in the time evolution if $v(t, \cdot) \in C_0^1$ for every $t \in [0, T]$.

Along each characteristic curve $x = X(t, s)$ satisfying (4.6), functions $V(t, s) := v(t, X(t, s))$, $W(t, s) = w(t, X(t, s))$, and $U(t, s) = v_x(t, X(t, s))$ satisfy the following initial-value problems:

$$(4.9) \quad \begin{cases} \frac{dV}{dt} = \varphi(X)W - \pi m^2 \bar{v} \sinh(X) - Q[v](X), \\ V|_{t=0} = v_0(s), \end{cases}$$

$$(4.10) \quad \begin{cases} \frac{dW}{dt} = \varphi'(X)W - \pi m^2 \bar{v} [\cosh(X) - 1] + \frac{1}{2} [V^2 - (v|_{X=0})^2] \\ \quad - P[v](X) + P[v](0), \\ W|_{t=0} = w_0(s), \end{cases}$$

and

$$(4.11) \quad \begin{cases} \frac{dU}{dt} = \varphi'(X)[W - U] + \varphi(X)V - \pi m^2 \bar{v} \cosh(X) - \frac{1}{2}U^2 + V^2 - P[v](X), \\ U|_{t=0} = v'_0(s). \end{cases}$$

Derivation of (4.9), (4.10), and (4.11) follows from (A.1), (A.2), and (A.3) in Appendix A after using (4.6). The following lemma transfers the local well-posedness theory for differential equations to the initial-value problem (4.1).

LEMMA 4.2. *For every $v_0 \in C^1_0$, there exists the maximal existence time $T > 0$ (finite or infinite) and the unique solution $v \in C^1([0, T], C^1_0)$ to the initial-value problem (4.1) that depends continuously on $v_0 \in C^1_0$.*

Proof. If $v \in C^1_0$, then $Q[v] \in C^0_{\text{per}}(\mathbb{T}) \cap \text{Lip}(\mathbb{T})$ by Lemma 2.1. It follows from Lemma 4.1 due to (4.5) that if $v \in C^1_0$, then $P[v] \in C^1_{\text{per}}(\mathbb{T})$. Therefore, the nonlocal parts of the initial-value problems (4.9), (4.10), and (4.11) are well-defined and can be considered for $X \in [0, 2\pi]$ that corresponds to $s \in [0, 2\pi]$.

For $s \in [0, 2\pi]$, we rewrite the evolution equations in (4.6), (4.9), (4.10), and (4.11) as the dynamical system

$$\frac{d}{dt} \begin{bmatrix} X \\ V \\ W \\ U \end{bmatrix} = \begin{bmatrix} \varphi(X) - \varphi(0) + V - V|_{s=0} \\ \varphi(X)W - \pi m^2 \bar{v} \sinh(X) - Q[v](X) \\ \varphi'(X)W - \pi m^2 \bar{v} [\cosh(X) - 1] + \frac{1}{2}[V^2 - (V|_{s=0})^2] - P[v](X) + P[v]|_{s=0} \\ \varphi'(X)[W - U] + \varphi(X)V - \pi m^2 \bar{v} \cosh(X) - \frac{1}{2}U^2 + V^2 - P[v](X) \end{bmatrix}$$

subject to the initial condition

$$\begin{bmatrix} X \\ V \\ W \\ U \end{bmatrix} \Big|_{t=0} = \begin{bmatrix} s \\ v_0(s) \\ w_0(s) \\ v'_0(s) \end{bmatrix}$$

and the boundary conditions

$$\begin{cases} X(t, 0) = 0, & X(t, 2\pi) = 2\pi, \\ V(t, 0) = V|_{s=0}, & V(t, 2\pi) = V|_{s=0}, \\ W(t, 0) = 0, & W(t, 2\pi) = 2\pi \bar{v}, \end{cases}$$

where $V|_{s=0}$ satisfies

$$\frac{d}{dt} V \Big|_{s=0} = -Q[v](0).$$

Since the vector field of the dynamical system is C^1 in (X, V, W, U) on $[0, 2\pi] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, there exists a unique local solution $X(\cdot, s), V(\cdot, s), W(\cdot, s), U(\cdot, s) \in C^1([0, T])$ to the initial-value problem for some maximal existence time $T > 0$. The solution depends continuously on the initial data for every $s \in [0, 2\pi]$. Moreover, since the initial data is $C^1(0, 2\pi)$, then $X(t, \cdot), V(t, \cdot), W(t, \cdot), U(t, \cdot) \in C^1(0, 2\pi)$ for every $t \in [0, T]$. The invertibility of the transformation $[0, 2\pi] \ni s \mapsto X(t, s) \in [0, 2\pi]$ is guaranteed along the solution by

$$(4.12) \quad \frac{\partial X}{\partial s}(t, s) = \exp \left(\int_0^t [\varphi'(X(t', s)) + U(t', s)] dt' \right) > 0,$$

which follows from (4.8). Since 0 and 2π are equilibrium points of (4.6), we have $X(t, 0) = 0$ and $X(t, 2\pi) = 2\pi$. Boundary conditions are preserved along the solution

due to consistency of (4.10) and (4.11) with the main equation (4.9). Due to the boundary conditions, the solution $V(t, \cdot) \in C^1(0, 2\pi)$ is extended to $V(t, \cdot) \in C_0^1$ on \mathbb{T} . Due to invertibility of the transformation $[0, 2\pi] \ni s \mapsto X(t, s) \in [0, 2\pi]$, we have $v(t, \cdot) \in C_0^1$ for $t \in [0, T)$ and, moreover, $v \in C^1([0, T), C_0^1)$. \square

The proof of Theorem 2 relies on the study of the evolution of $v_x \in C^1([0, T), C^0(\mathbb{T} \setminus \{0\}))$ on the right side of the peak. By Lemma 4.2, we are allowed to define the one-sided limits $U^\pm(t) := \lim_{s \rightarrow 0^\pm} U(t, s)$ for $t \in [0, T)$, where $U(t, 0^-) = U(t, 2\pi^-)$. The functions $U^\pm \in C^1(0, T)$ satisfy the following for $t \in (0, T)$:

$$(4.13) \quad \frac{dU^\pm}{dt} = \pm U^\pm + MV|_{s=0} - \pi m^2 \bar{v} - \frac{1}{2}(U^\pm)^2 + (V|_{s=0})^2 - P[v](0),$$

which follow from taking the limits $s \rightarrow 0^\pm$ in (4.11). We will now prove Theorem 2.

Due to the decomposition (2.11), we can rewrite the initial bound (1.12) in the form

$$(4.14) \quad \|v_0\|_{H^1(\mathbb{T})} + \|v'_0\|_{L^\infty(\mathbb{T})} < \delta,$$

where $v_0 \in C_0^1 \subset H_{\text{per}}^1(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ and $\delta > 0$ is an arbitrarily small parameter. We first show that there exist $t_0 \in (0, T)$ and $v_0 \in C_0^1$ such that the unique local solution $v \in C^1([0, T), C_0^1)$ to the initial-value problem (4.1) constructed by Lemma 4.2 satisfies

$$(4.15) \quad \|v_x(t_0, \cdot)\|_{L^\infty(\mathbb{T})} \geq 1.$$

It follows from $2P[v](0) = \int_{\mathbb{T}} \varphi(-y)q[v](y)dy > 0$ that (4.13) for the upper sign can be estimated by

$$(4.16) \quad \frac{dU^+}{dt} \leq U^+ + MV|_{s=0} - \pi m^2 \bar{v} + (V|_{s=0})^2.$$

By Theorem 1, for every small $\varepsilon > 0$, there exists $\nu(\varepsilon) > 0$ such that if $\|v_0\|_{H^1(\mathbb{T})} < \nu(\varepsilon)$, then $\|v(t, \cdot)\|_{H^1(\mathbb{T})} < \varepsilon$ for every $t \in [0, T)$. By Sobolev's embedding, there is a positive constant C such that

$$(4.17) \quad \|v(t, \cdot)\|_{L^\infty(\mathbb{T})} \leq C\|v(t, \cdot)\|_{H^1} < C\varepsilon.$$

By using (4.17), we can simplify (4.16) for every $\varepsilon \in (0, 1]$ to the form

$$(4.18) \quad \frac{dU^+}{dt} \leq U^+ + MC\varepsilon + \pi m^2 C\varepsilon + C^2\varepsilon^2 \leq U^+ + C_1\varepsilon$$

for some ε -independent constant $C_1 > 0$. Let us assume that the initial data $v_0 \in C_0^1$ satisfies

$$(4.19) \quad \lim_{x \rightarrow 0^+} v'_0(x) = -\|v'_0\|_{L^\infty(\mathbb{T})} = -2C_1\varepsilon.$$

The initial bound (4.14) is consistent with (4.19) if for every small $\delta > 0$ the small value of ε satisfies

$$\nu(\varepsilon) + 2C_1\varepsilon < \delta,$$

which just specifies small ε in terms of small δ . By integrating (4.18) and using (4.19), we obtain

$$U^+(t) \leq e^t [U^+(0) + C_1\varepsilon] = -C_1\varepsilon e^t.$$

Hence, for every small $\varepsilon > 0$ there exists a sufficiently large $t_1 = -\log(C_1\varepsilon)$ such that $U^+(t_1) \leq -1$. If $t_1 < T$, the bound (4.15) is true with some $t_0 \in (0, t_1]$ since

$$(4.20) \quad \|v_x(t, \cdot)\|_{L^\infty(\mathbb{T})} = \|U(t, \cdot)\|_{L^\infty(0, 2\pi)} \geq |U^+(t)|, \quad t \in [0, T).$$

Assume now $t_1 \geq T$, where T is the finite maximal existence time in Lemma 4.2 for the solution

$$X(\cdot, s), V(\cdot, s), W(\cdot, s), U(\cdot, s) \in C^1([0, T]), \quad s \in [0, 2\pi].$$

By (4.12), we have $X \in C^1([0, T], C^1(0, 2\pi))$ if and only if $U \in C^1([0, T], C^0(0, 2\pi))$. By the bound (4.17), we have $W, V \in C^1([0, T], C^0(0, 2\pi))$ with bounded limits of $\|W(t, \cdot)\|_{L^\infty(0, 2\pi)}$ and $\|V(t, \cdot)\|_{L^\infty(0, 2\pi)}$ as $t \rightarrow T^-$. If $T < \infty$, we have

$$(4.21) \quad \|U(t, \cdot)\|_{L^\infty(0, 2\pi)} \rightarrow \infty \quad \text{as } t \rightarrow T^-,$$

so that there exists $t_0 \in (0, T)$ such that $\|U(t_0, \cdot)\|_{L^\infty(0, 2\pi)} \geq 1$ and the bound (4.15) is true due to (4.20).

Finally, we show that there exists $v_0 \in C_0^1$ such that the maximal existence time T is finite. Due to the bound (4.17), we have for every $\varepsilon \in (0, 1)$

$$|MV|_{s=0} - \pi m^2 \bar{v} + (V|_{s=0})^2 - P[v](0) \leq MC\varepsilon + \pi m^2 C\varepsilon + C^2\varepsilon^2 + \frac{1}{2}M\varepsilon^2 \leq C_2\varepsilon$$

for some ε -independent constant $C_2 > 0$. Let $U_0(\varepsilon)$ be the negative root of the quadratic equation

$$U - \frac{1}{2}U^2 + C_2\varepsilon = 0.$$

It is clear that $U_0(\varepsilon) \in (-C_2\varepsilon, 0)$. Assume that the initial data $v_0 \in C_0^1$ satisfies

$$(4.22) \quad \lim_{x \rightarrow 0^+} v_0'(x) = -\|v_0'\|_{L^\infty(\mathbb{T})} = 2U_0(\varepsilon).$$

The initial bound (4.14) is consistent with (4.22) if for every small $\delta > 0$ the small value of ε satisfies

$$\nu(\varepsilon) + 2|U_0(\varepsilon)| < \delta,$$

which again specifies small ε in terms of small δ . If $U^+(0) = 2U_0(\varepsilon)$, then differential equation (4.13) for U^+ implies that $\frac{d}{dt}U^+ < 0$, and hence $U^+(t) < U^+(0)$ for small positive t and the map $t \mapsto U(t)$ is monotonically decreasing. Let \bar{U} be the supersolution which satisfies

$$\frac{d\bar{U}}{dt} = \bar{U} - \frac{1}{2}\bar{U}^2 + C_2\varepsilon,$$

with $\bar{U}(0) = U^+(0) = 2U_0(\varepsilon)$. It follows by the comparison theory for differential equations that $U^+(t) \leq \bar{U}(t)$ for every $t > 0$ for which $U^+(t)$ exists. Since there exists a finite $\bar{T} > 0$ such that $\bar{U}(t) \rightarrow -\infty$ as $t \rightarrow \bar{T}^-$, then there exists $T \in (0, \bar{T}]$ such that $U^+(t) \rightarrow -\infty$ as $t \rightarrow T^-$, which implies that the blow-up criterion (4.21) is satisfied for finite $T > 0$.

Both parts of Theorem 2 have been proven.

Appendix A. Derivation of the evolution equations for w and v_x . We consider the evolution equation for v given by

$$(A.1) \quad v_t = (c - \varphi)v_x + \varphi w - \pi m^2 \bar{v} \sinh(x) + (v|_{x=0} - v)v_x - Q[v],$$

where

$$Q[v](x) := \frac{1}{2} \int_{\mathbb{T}} \varphi'(x-y) \left[v^2 + \frac{1}{2}(v_y)^2 \right] (y) dy.$$

Substituting $v = w_x$ into (A.1) yields

$$w_{tx} = (c - \varphi)w_{xx} + \varphi w - \pi m^2 \bar{v} \sinh(x) + (v|_{x=0} - w_x)w_{xx} - Q[v].$$

By integrating this evolution equation into x and picking the constant of integration from the boundary conditions $w(t, 0) = 0$, we obtain

$$(A.2) \quad w_t = (c - \varphi)w_x + \varphi' w - \pi m^2 \bar{v} [\cosh(x) - 1] - \frac{1}{2}(v|_{x=0} - v)^2 - P[v] + P[v]|_{x=0},$$

where

$$P[v](x) := \frac{1}{2} \int_{\mathbb{T}} \varphi(x-y) \left[v^2 + \frac{1}{2}(v_y)^2 \right] (y) dy.$$

Differentiating (A.1) in x yields

$$(A.3) \quad \begin{aligned} v_{tx} &= (c - \varphi)v_{xx} + \varphi'(w - v_x) + \varphi v - \pi m^2 \bar{v} \cosh(x) + (v|_{x=0} - v)v_{xx} \\ &+ v^2 - \frac{1}{2}(v_x)^2 - P[v], \end{aligned}$$

where we have used the relation

$$\frac{1}{2} \int_{\mathbb{T}} \varphi''(x-y) \left[v^2 + \frac{1}{2}(v_y)^2 \right] dy = \frac{1}{2} \int_{\mathbb{T}} \varphi(x-y) \left[v^2 + \frac{1}{2}(v_y)^2 \right] dy - v^2 - \frac{1}{2}v_x^2.$$

Appendix B. Explicit solutions to the linearized equations. Separation of variables in (3.4) gives

$$mt = \int_s^X \frac{dx}{\cosh(\pi - x) - \cosh(\pi)} = m \log \frac{(e^{2\pi} - e^X)(1 - e^s)}{(1 - e^X)(e^{2\pi} - e^s)},$$

from which we derive

$$(B.1) \quad X(t, s) = \log \frac{(e^{2\pi} - e^s) + e^{2\pi-t}(e^s - 1)}{(e^{2\pi} - e^s) + e^{-t}(e^s - 1)}, \quad t \in \mathbb{R}^+, \quad s \in [0, 2\pi],$$

such that

$$\lim_{s \rightarrow 0^+} X(t, s) = 0 \quad \text{and} \quad \lim_{s \rightarrow 2\pi^-} X(t, s) = 2\pi.$$

After lengthy manipulations, we can find that

$$\frac{\partial X}{\partial s}(t, s) = \frac{(e^{2\pi} - 1)^2 e^{s-t}}{[(e^{2\pi} - e^s) + e^{-t}(e^s - 1)][(e^{2\pi} - e^s) + e^{2\pi-t}(e^s - 1)]}, \quad t \in \mathbb{R}^+, \quad s \in [0, 2\pi],$$

such that

$$\lim_{s \rightarrow 0^+} \frac{\partial X}{\partial s}(t, s) = e^{-t} \quad \text{and} \quad \lim_{s \rightarrow 2\pi^-} \frac{\partial X}{\partial s}(t, s) = e^t.$$

Integrating (3.5) with an integrating factor yields

$$W(t, s) = \frac{\partial X}{\partial s} \left[w_0(s) + \pi m^2 \bar{v} \int_0^t \frac{1 - \cosh X(t', s)}{\frac{\partial X}{\partial s}(t', s)} dt' \right].$$

By using (B.1), we derive simple expression

$$(B.2) \quad W(t, s) = \frac{\partial X}{\partial s} [w_0(s) - \pi m^2 \bar{v}(\cosh s - 1)(1 - e^{-t})], \quad t \in \mathbb{R}^+, \quad s \in [0, 2\pi],$$

such that

$$\lim_{s \rightarrow 0^+} W(t, s) = 0 \quad \text{and} \quad \lim_{s \rightarrow 2\pi^-} W(t, s) = w_0(2\pi) = 2\pi \bar{v}.$$

Instead of integrating (3.6), we can use the chain rule

$$\frac{\partial W}{\partial s}(t, s) = V(t, s) \frac{\partial X}{\partial s}(t, s),$$

from which we derive

$$(B.3) \quad \begin{aligned} V(t, s) &= v_0(s) - \pi m^2 \bar{v} \sinh s(1 - e^{-t}) \\ &\quad + [w_0(s) - \pi m^2 \bar{v}(\cosh s - 1)(1 - e^{-t})] Y(t, s) \end{aligned}$$

with

$$\begin{aligned} Y(t, s) &:= \frac{\partial}{\partial s} \log \frac{\partial X}{\partial s}(t, s) \\ &= \frac{(1 - e^{-t})[\sinh(2\pi - s) + e^{-t} \sinh(s)]}{\cosh(2\pi - s) - 1 + [\cosh(2\pi) + 1 - \cosh(2\pi - s) - \cosh(s)]e^{-t} + (\cosh s - 1)e^{-2t}}, \end{aligned}$$

such that

$$\lim_{s \rightarrow 0^+} V(t, s) = v_0(0) \quad \text{and} \quad \lim_{s \rightarrow 2\pi^-} V(t, s) = v_0(2\pi) = v_0(0).$$

Appendix C. Smooth, peaked, and cusped periodic waves. Traveling wave solutions to the CH equation (1.1) are of the form $u(t, x) = \phi(x - ct)$, where c is the wave speed and $\phi(x) : \mathbb{T} \mapsto \mathbb{R}$ satisfies the third-order differential equation

$$(C.1) \quad (c - \phi)(\phi''' - \phi') + 2\phi\phi' - 2\phi'\phi'' = 0.$$

Multiplying (C.1) by $(c - \phi)$ allows us to integrate it once:

$$(C.2) \quad \frac{d}{dx} [(c - \phi)^2(\phi'' - \phi)] = 0 \quad \Rightarrow \quad (c - \phi)^2(\phi'' - \phi) = a,$$

where $a \in \mathbb{R}$ is the integration constant. The level $\phi = c$ is singular, and in what follows we consider $c > 0$. For every $\phi \neq c$, the second-order differential equation (C.2) can be integrated once:

$$(C.3) \quad \phi'' - \phi - \frac{a}{(c - \phi)^2} = 0 \quad \Rightarrow \quad (\phi')^2 - \phi^2 - \frac{2a}{c - \phi} = b,$$

where $b \in \mathbb{R}$ is another integration constant.

Let $W(\phi) := -\phi^2 - 2a/(c - \phi)$. If $a = 0$, there is only one critical point of W at $\phi = 0$ so that $(0, 0)$ is a saddle point of the dynamical system on the phase plane (ϕ, ϕ') . Peaked periodic waves are constructed from orbits of the linear equation $\phi'' - \phi = 0$ intersecting the singularity line $\phi = c$ over the length of 2π . If

$$M_\phi := \max_{x \in \mathbb{T}} \phi(x), \quad m_\phi := \min_{x \in \mathbb{T}} \phi(x),$$

then $b = -m_\phi^2$ and $c = M_\phi$. The exact 2π -periodic solution to $\phi'' - \phi = 0$ is given by

$$(C.4) \quad \phi(x) = m_\phi \cosh(\pi - |x|), \quad M_\phi = m_\phi \cosh(\pi).$$

The uniquely selected peaked wave (1.7) corresponds to $\phi = \varphi$ with $M_\phi = M$ and $m_\phi = m$ given by (1.5) and (1.6). The one-parameter family (C.4) corresponds to the one-parameter family of peaked 2π -periodic waves (1.8); this family is given by $\phi = \gamma\varphi$, with $\gamma := \frac{m_\phi}{m}$ being a free parameter and $c = M_\phi = \gamma M$ in agreement between (1.8) and (C.4).

If $a > 0$, there is only one critical point of W at some $\phi_0 < 0$ so that $(\phi_0, 0)$ is a saddle point on the phase plane (ϕ, ϕ') . The periodic waves are constructed from orbits squeezed between the stable and unstable curves from $(\phi_0, 0)$ and the singularity line $\phi = c$. These orbits give the cusped periodic waves with the following local behavior near the singularity:

$$\phi(x) \sim c - \alpha x^{2/3} \quad \text{as } x \rightarrow 0,$$

where $\alpha > 0$ is related to parameter $a > 0$.

If $a < 0$, there are three critical points of W given by $\{\phi_1, \phi_2, \phi_3\}$ ordered as $\phi_1 < \phi_2 < c < \phi_3$ so that $(\phi_1, 0)$ and $(\phi_3, 0)$ are saddle points, whereas $(\phi_2, 0)$ is a center point. Smooth periodic waves are located inside the homoclinic loop connected to $(\phi_1, 0)$ which surrounds $(\phi_2, 0)$. Cusped waves also exist for $\phi > c$ but not for $\phi < c$.

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