# Solitons in nonintegrable systems

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We introduce the concept of this special focus issue on solitons in nonintegrable systems. A brief overview of some recent developments is provided, and the various contributions are described. The topics covered in this focus issue are the modulation of solitons, bores, and shocks, the dynamical evolution of solitary waves, and existence and stability of solitary waves and embedded solitons. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2047867]

**Solitons first arose as special solutions of integrable equations such as the Korteweg–de Vries and nonlinear Schrödinger equations. These soliton solutions may exist in nonintegrable modifications of the basic governing equations, needed to model real physical systems. This focus issue brings together some recent theoretical developments that deal with solitary waves in nonintegrable physical systems. In particular, analytical and numerical methods are developed in the cases when the governing equation may not be close to being integrable. The physical applications of these analytical and numerical studies cover fluid flows, solid state physics and nonlinear optics.**

Solitons as solutions of integrable equations such as the Korteweg–de Vries or nonlinear Schrodinger equation are widely known, appreciated and utilized; there is indeed a vast literature dealing with the amazing properties of integrable systems. However, the history of solitons has always been associated with real physical applications, and in this context the governing equations may not typically be integrable, and further they may be close, or not close, to being integrable. Such situations arise, for instance, in fluid flows, solid state physics, and in nonlinear optics. Some of the issues involved are the occurrence of variable-coefficient nonlinear evolution equations, leading to slowly varying solitary waves with trailing shelves and radiation; forced nonlinear systems, where time evolution of undular bores is studied with the Whitham modulation theory; stability of solitary waves, both in the context of integrable equations and in the wider context within which these integrable equations reside as asymptotic reductions; embedded solitons, which correspond to nonlinear waves coexisting with the linear continuous-wave radiation; and strongly nonlinear solitary waves. In this special focus issue we have brought together a sample of some of the current research work dealing with these and related topics. Note that, appropriately for this focus issue, we will use the terminology "soliton" in its wider sense of a solitary wave, in particular, including situations when the underlying physical system is not necessarily conservative.

The contributions to this special focus issue can be divided into three related groups. The first group is devoted to analytical solutions of the modulation problem that arises in the time evolution of solitons, bores, and shocks. Analytical solutions are often complemented by numerical computations of the governing nonlinear evolution equations. The main applications of these problems are in fluid dynamics, particularly for surface and internal waves, plasma physics, and solid state physics.

The second group of articles focuses on the dynamical properties of propagating solitary waves that are modeled by reduced nonlinear evolution equations. Temporal instabilities, shape distortions and generation of nonlinear and dispersive waves are detected by means of full-scale numerical simulations. The main applications of these soliton propagation problems are in fluid dynamics and nonlinear optics.

The third group of articles is devoted to fundamental problems of soliton theory: the existence of special solutions of partial differential equations, and their spectral stability in the underlying time evolution. The techniques of dynamical systems, partial differential equations, and spectral theory are used here for a mathematical description of regular and generalized solitary waves and embedded solitons.

We shall next describe individual contributions to the three groups.

**• Modulation theory of solitons, bores, and shocks**

The focus issue opens with the paper by El, Grimshaw, and Kamchatnov,[1] which is devoted to weakly dissipative shallow-water undular bores. Undular bores are nonlinear wave structures, which are generated in the breaking profiles of large-scale nonlinear waves propagating in dispersive media. The Whitham modulation theory is developed in this work, by using an integrable version of the bidirectional Boussinesq equations, but modified by a small Burgers viscous term.

The topic of Whitham modulation equations is continued in the paper by El,[2] where a shock is considered in a hyperbolic quasilinear system modified by weak dispersion. The Whitham system derived from an integrable nonlinear evolution equation can be solved exactly. The author considers a nonintegrable evolution problem and develops an adequate description of a dispersive shock transition between two different constant states, bypassing full integration of the modulation equations. In particular, the results are applied to a model for nonlinear ion-acoustic waves in a collisionless plasma.

The role of conserved quantities, collective coordinates, and geometric optics solutions is exploited in the paper by Jorge, Cruz-Pacheco, Mier-y-Teran-Romero, and Smyth,[3] which describes the modulation of lump solutions in the context of electromigration in nanocircuits. A coupled system of the Zakharov–Kuznetsov equation for surface waves and an elliptic equation for the electrostatic potential is considered. The system has lump solitons in two spatial dimensions that shed radiation during their time evolution.

Another version of modulation theory is based on the averaging methods applied to nonlinear evolution equations with time-periodic coefficients. This topic is addressed in the paper by Zharnitsky and Pelinovsky[4] in the context of the Feshbach resonance for Bose–Einstein condensates. The methods of Hamiltonian averaging theory, asymptotic multiscale expansions and perturbation series are exploited to study the averaged dynamics of localized pulses in the nonlinear Schrödinger equation under nonlinearity management.

- **The dynamics of solitary waves**

Forced solitary waves and fronts are modeled in the paper by Binder, Vanden-Broeck, and Dias[5] in the context of steady nonlinear two-dimensional potential free surface flows. The authors develop an efficient numerical method based on the boundary integral equation and apply it to the flow of a liquid in an open channel with various types of bottom configurations.

Fluid dynamics applications are continued in the paper by Nakoulima, Zahibo, Pelinovsky, Talipova, and Kurkin,[6] who consider solitary waves propagating above a piecewise-constant periodic bottom topography in shallow water. Both linear and weakly nonlinear models of long-wave scattering by the periodic irregularities of the sea bottom are simulated. As a result of the scattering and fission processes, the solitary wave amplitude attenuates exponentially during the propagation dynamics.

The next two papers of this group are devoted to applications in nonlinear optics. The paper by Zafrany, Malomed, and Merhasin[7] addresses gap solitons in a coupled system with separated dispersion and nonlinearity. Starting with a model problem in the context of photonic-crystal fibers, the authors develop a careful numerical search for gap soliton solutions, and describe different scenarios of their propagation dynamics.

The paper by Porsezian and Senthilnathan[8] deals with optical transmission using a short optical pulse in fiber Bragg gratings. While neglecting the fiber dispersion, the coupled-mode system with Kerr cubic nonlinearities is simulated numerically with the aim of describing modulation instability of a continuous wave, and the generation of gap solitons. The correspondence between stable propagation of ultrashort pulses through Bragg gratings, and modulation instability phenomena, is established in this paper.

Two papers in this group are of a review character. The generation, propagation, and shear-induced instability of internal solitary waves are reviewed in the paper by Grue.[9] Numerical simulations are described for an interfacial model of fully nonlinear and fully dispersive waves generated due to tidal flow over bottom topography, and compared with some experimental observations.

The paper by Ostrovsky and Stepanyants[10] is a review of laboratory experiments concerning internal solitons (motivated by their common occurrence in the coastal oceans). The experimental results obtained over the last few decades are briefly described and compared with the existing theoretical models of solitons in both integrable and nonintegrable systems, the latter including the effects of dissipation, rotation, and strong nonlinearity.

- **Existence and stability of solitary waves and embedded solitons**

The articles in this last group exploit mathematical methods in the analysis of solitons. These themes are opened with the paper by Barrandon and Iooss,[11] which describes the spatial dynamical formulation of surface and interfacial water waves, in the context of a fluid of infinite depth. In the spatial formulation, an ill-posed evolution problem is considered, where the horizontal space variable plays the role of "time." The very difficult case of an infinitely deep layer is considered, when the linearized operator possesses an essential spectrum filling the whole real axis and bifurcations may occur when imaginary eigenvalues meet at the origin and disappear in the essential spectrum. Both regular (algebraically decaying) and generalized (oscillatory-tailed) solitary waves are constructed without the use of center manifold reductions.

Another type of bifurcation analysis is considered by Bridges,[12] when the criticality of a constrained variational principle for Stokes waves determines the bifurcation of dark solitary waves. Dark solitary waves appear in shallow-water wave hydrodynamics under the interactions of dispersive wave packets and mean-flow currents. Although the dark solitary waves could be steady within the model equations, such as the Hasimoto–Ono and Benny–Roskes equations, they are intrinsically unsteady within the original water-wave equations.

Linear eigenvalue problems for the spectral stability of envelope solitons are reviewed in the paper by Kapitula and Kevrekidis.[13] The authors consider the Bose–Einstein condensates trapped by a magnetic parabolic potential and an optical lattice potential. As well as the standard approach using the linear eigenvalues, they applied a powerful method for the bifurcations of nonlinear solutions, that is, the method of Lyapunov–Schmidt reductions. The exact count of real, complex, and imaginary eigenvalues of negative Krein signatures in the spectral stability problem is performed both analytically and numerically.

Spectral stability is also addressed in the paper by Pelinovsky and Yang[14] in the context of embedded solitons of a generalized third-order derivative nonlinear Schrödinger equation. Embedded solitons are fully localized modes that reside inside the continuous spectrum of a nonlinear-wave system. Despite their intrinsic instability, embedded solitons of this generalized third-order NLS model are found to be stable both analytically and numerically. This stability is explained by the bifurcation of a stable resonance pole in the linearized problem, that corresponds to a nondecaying eigenvector responsible for radiative decay of perturbations of embedded solitons.

The issue closes with the paper by Malomed, Wagenknecht, Champneys, and Pearce,[15] who study embedded solitons in a coupled three-wave model with quadratic nonlinearities. Accumulations of embedded solitons near the fourfold zero eigenvalue bifurcation are traced in the normal-form analysis with WKB approximations and numerical continuation software. The cascades of homoclinic orbits for embedded solitons arise often in various models when the radiative tails from two embedded solitons are canceled due to a two-soliton interaction.

The study of solitons in nonintegrable systems is evolving at a very rapid pace. The articles in this special focus issue present a flavor of the current publications in this field. We hope that this issue will provide a valuable reference tool as well as an introduction for young scientists wishing to get involved with the exciting new topics. Exciting new developments in applied mathematics, physical sciences, and engineering applications can be expected in the many problems involving solitary waves and nonintegrable evolution equations.

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