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Count of eigenvalues in the generalized eigenvalue problem

Marina Chugunova and Dmitry Pelinovsky[a)]
Department of Mathematics, McMaster University, Hamilton, Ontario L8S 4K1, Canada

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We study isolated and embedded eigenvalues in the generalized eigenvalue problem defined by two self-adjoint operators with a positive essential spectrum and a finite number of isolated eigenvalues. The generalized eigenvalue problem determines the spectral stability of nonlinear waves in infinite-dimensional Hamiltonian systems. The theory is based on Pontryagin's invariant subspace theorem and extends beyond the scope of earlier papers of Pontryagin, Krein, Grillakis, and others. Our main results are (i) the number of unstable and potentially unstable eigenvalues *equals* the number of negative eigenvalues of the self-adjoint operators, (ii) the total number of isolated eigenvalues of the generalized eigenvalue problem is *bounded from above* by the total number of isolated eigenvalues of the self-adjoint operators, and (iii) the quadratic forms defined by the two self-adjoint operators are *strictly positive* on the subspace related to the continuous spectrum of the generalized eigenvalue problem. Applications to the localized solutions of the nonlinear Schrödinger equations are developed from the general theory.
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I. INTRODUCTION

Nonlinear partial differential equations that conserve energy can often be written as infinite-dimensional Hamiltonian systems in the abstract form

$$\frac{du}{dt} = JE'(u(t)), \quad u(t) \in \mathcal{X}, \tag{1.1}$$

where $J:\mathcal{X}\to\mathcal{X}$ is a symplectic operator with the property $J^*=-J$ and $E:\mathcal{X}\to\mathbb{R}$ is a C^2 functional in a Hilbert space $\mathcal{X}$. A critical point $\phi\in\mathcal{X}$ of the Hamiltonian functional E, defined by $E'(\phi)=0$, represents a localized solution of the nonlinear partial differential equation. The spectral stability of a localized solution ϕ is determined by the spectrum of the non-self-adjoint eigenvalue problem

$$JE''(\phi)v = \lambda v, \quad v \in \mathcal{X}, \tag{1.2}$$

which is obtained after a linearization of the Hamiltonian system (1.1). Although the operator $JE''(\phi)$ is non-self-adjoint, it is related to the self-adjoint operator $E''(\phi)$ by multiplication of the symplectic operator J. In many specific examples, such as the nonlinear Schrödinger (NLS) equations, the non-self-adjoint eigenvalue problem (1.2) can be rewritten as the generalized eigenvalue problem

$$Aw = \gamma Kw, \quad w \in \mathcal{X}, \tag{1.3}$$

where A and K are self-adjoint operators and $\gamma=-\lambda^2$. The critical point ϕ is said to be *spectrally*

[a)]Electronic mail: dmpeli@math.mcmaster.ca.

unstable if there exists an eigenvalue γ with either $\gamma < 0$ or $\text{Im}(\gamma) \neq 0$. Otherwise, the critical point is said to be *weakly spectrally stable*. Moreover, it is a minimizer of the Hamiltonian functional $E(\phi)$ if all eigenvalues γ are positive and the quadratic forms $(A\cdot,\cdot)$ and $(K\cdot,\cdot)$, evaluated at the eigenvectors of the generalized eigenvalue problem (1.3), are strictly positive.

The same generalized eigenvalue problem (1.3) arises in the analysis of spectral stability of equilibrium configurations for a Hamiltonian system of finitely many interacting particles.[5] In this context, $\mathcal{X}=\mathbb{R}^n$, while A and K are symmetric matrices in $\mathbb{R}^{n\times n}$ for the potential and kinetic energies, respectively. When the matrix K is positive definite, all eigenvalues γ are real and semisimple (that is, the geometric and algebraic multiplicities coincide). By Sylvester's inertia law,[6] the numbers of positive, zero, and negative eigenvalues of the generalized eigenvalue problem (1.3) *equal* the numbers of positive, zero, and negative eigenvalues of the matrix A. When K is not positive definite, a complete classification of eigenvalues γ in terms of real eigenvalues of A and K has been developed with the use of Pontryagin's invariant subspace theorem, originally proved for a single negative eigenvalue of K (Ref. 29) and then extended to finitely many negative eigenvalues.[8].

Our examples in Sec. VI show that spectral stability of spatially localized solutions in NLS equations leads to the generalized eigenvalue problem (1.3), where A and K^{-1} are self-adjoint differential operators in a constrained L^2 space. There has been recently a rapidly growing sequence of publications about the characterization of unstable eigenvalues of this generalized eigenvalue problem in terms of isolated eigenvalues of the operators A and K^{-1}.[4,13,15,18,23,31] Besides predictions of spectral stability or instability of spatially localized solutions in Hamiltonian systems, analysis of the linear eigenvalue problems is important for the studies of orbital stability[9,10] and asymptotic stability,[3,27,30] the existence of stable manifolds,[19,32] and the construction of self-similar solutions in nonlinear evolution equations.[28]

It is the purpose of this article to develop analysis of the generalized eigenvalue problem (1.3) by using the Pontryagin space decomposition. The theory of Pontryagin spaces was developed by Krein and his students (see books[1,14]) and partly used in the context of spectral stability of spatially localized solutions by MacKay,[20] Grillakis,[12] and Buslaev and Perelman[2] (see also a recent application in Ref. 13). We shall give an elegant geometric proof of Pontryagin's invariant subspace theorem and then apply this theorem to establish our main results.

(i) The number of unstable and potentially unstable eigenvalues of the generalized eigenvalue problem (1.3) *equals* the number of negative eigenvalues of the self-adjoint operators A and K^{-1}.
(ii) The total number of isolated eigenvalues of the generalized eigenvalue problem (1.3) is *bounded from above* by the total number of isolated eigenvalues of the self-adjoint operators A and K^{-1}.
(iii) The quadratic forms defined by the two self-adjoint operators A and K^{-1} are *strictly positive* on the subspace related to the continuous spectrum of the generalized eigenvalue problem (1.3).

The first result is a remake of the main results obtained in Refs. 4, 15, and 23, although the method of proof presented therein is quite different than that given here. The second result gives a new inequality on the number of isolated eigenvalues of the generalized eigenvalue problem (1.3), which can be useful to control the number of neutrally stable eigenvalues in the gap of the continuous spectrum of the linearized operator associated with the stable localized solutions. The third result has a technical significance since it establishes a similarity between Sylvester's inertial law used in Ref. 23 and Pontryagin's space decomposition used here. With this construction, one can bypass the topological theory developed in Ref. 12 and used in Ref. 15.

The structure of the paper is as follows. Formalism of the generalized eigenvalue problem and the main results are described in Sec. II. Pontryagin's invariant subspace theorem is proved in Sec. III. The spectrum of a self-adjoint operator in the Pontryagin space is characterized in Sec. IV. The count of eigenvalues of the generalized eigenvalue problem and the proofs of the main theorems are given in Sec. V. Section VI contains applications of our main results to spatially localized solutions of the NLS equations.

II. FORMALISM AND THE MAIN RESULTS

Let $\mathcal{X}$ be a Hilbert space with the inner product $(\cdot,\cdot)$. Let L_+ and L_- be two real-valued self-adjoint operators with $\mathrm{Dom}(L_\pm)\subseteq\mathcal{X}$. Our two assumptions on operators L_+ and L_- are listed here.

P1 The essential spectrum $\sigma_e(L_\pm)$ includes the absolute continuous part bounded from below by $\omega_\pm>0$ and, possibly, embedded eigenvalues.
P2 The discrete spectrum $\sigma_d(L_\pm)$ in $\mathcal{X}$ includes finitely many isolated eigenvalues of finite multiplicities with $p(L_\pm)$ positive, $z(L_\pm)$ zero, and $n(L_\pm)$ negative eigenvalues. (These indices can be zero and the corresponding subspaces can be empty.)

We consider the linear eigenvalue problem defined by the self-adjoint operators $L_\pm$ in the form

$$L_+u=-\lambda w,\quad L_-w=\lambda u,\quad u,w\in\mathcal{X}, \tag{2.1}$$

where $\lambda\in\mathbb{C}$. By assumption (P1), the kernel of L_- is isolated from the essential spectrum of L_-. Let $\mathcal{P}$ be the orthogonal projection from $\mathcal{X}$ to $\mathcal{H}$, where $\mathcal{H}$ is the constrained Hilbert space

$$\mathcal{H}=\{u\in\mathcal{X}: u\perp\mathrm{Ker}(L_-)\}. \tag{2.2}$$

We are only interested in nonzero eigenvalues of the spectral problem (2.1) because only nonzero eigenvalues λ determined spectral stability or instability of the underlying solution. If $\lambda\neq 0$ and $u\in\mathcal{X}$, then $u\in\mathcal{H}$ so that $u=\mathcal{P}u\in\mathrm{Ran}(L_-)$. As a result, we express w from the second equation of system (2.1)

$$w=\lambda\mathcal{P}L_-^{-1}\mathcal{P}u+w_0,\quad w_0\in\mathrm{Ker}(L_-). \tag{2.3}$$

Substituting w into the first equation of system (2.1) and using the projection operator $\mathcal{P}$, we obtain a closed equation for u,

$$\mathcal{P}L_+\mathcal{P}u=-\lambda^2\mathcal{P}L_-^{-1}\mathcal{P}u,\quad u\in\mathcal{H} \tag{2.4}$$

and a unique expression for w_0,

$$w_0=-\frac{1}{\lambda}(\mathcal{I}-\mathcal{P})L_+\mathcal{P}u, \tag{2.5}$$

where $\lambda\neq 0$ and $(\mathcal{I}-\mathcal{P})$ is the orthogonal projection from $\mathcal{X}$ to $\mathrm{Ker}(L_-)$.

Equation (2.4) shows that the linear eigenvalue problem (2.1) for nonzero λ is equivalent to the generalized eigenvalue problem for nonzero γ,

$$Au=\gamma Ku,\quad u\in\mathcal{H}, \tag{2.6}$$

where $A=\mathcal{P}L_+\mathcal{P}$, $K=\mathcal{P}L_-^{-1}\mathcal{P}$, and $\gamma=-\lambda^2$. The following proposition gives an important result on the equivalence of quadratic forms (u,u) and (Ku,u) for solutions of the generalized eigenvalue problem $Au=\gamma Ku$. The quadratic form (Ku,u) will be used in the construction of the Pontryagin space for the generalized eigenvalue problem (2.6).

Proposition 2.1: The generalized eigenvalue problem $Au=\gamma Ku$ with $(u,u)<\infty$ is equivalent to the generalized eigenvalue problem $Au=\gamma Ku$ with $|(Ku,u)|<\infty$.

Proof: Since K is a bounded invertible self-adjoint operator with $\mathrm{Dom}(K)\equiv\mathcal{H}$, there exists $C>0$ such that

$$\forall u \in \mathcal{H}: \quad |(Ku,u)| \leq C(u,u). \tag{2.7}$$

Therefore, if u is an eigenvector of $Au=\gamma Ku$ and $(u,u)<\infty$, then $|(Ku,u)|<\infty$. On the other hand, A is generally an unbounded noninvertible self-adjoint operator with $\mathrm{Dom}(A)\subseteq\mathcal{H}$. If u is the eigenvector of $Au=\gamma Ku$ with $|(Ku,u)|<\infty$, then $u\in\mathrm{Dom}(A)\subseteq\mathcal{H}$ so that $(u,u)<\infty$. ■

Finitely many isolated eigenvalues of the operators A and K^{-1} in $\mathcal{H}$ are distributed between negative, zero, and positive eigenvalues away from their essential spectra. By the spectral theory of self-adjoint operators, the Hilbert space $\mathcal{H}$ can be equivalently decomposed into two orthogonal sums of subspaces, which are invariant with respect to the operators K and A,

$$\mathcal{H} = \mathcal{H}_K^- \oplus \mathcal{H}_K^+ \oplus \mathcal{H}_K^{\sigma_e(K)}, \tag{2.8}$$

$$\mathcal{H} = \mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+ \oplus \mathcal{H}_A^{\sigma_e(A)}, \tag{2.9}$$

where notation $-$ $(+)$ stands for negative (positive) isolated eigenvalues, 0 for the isolated kernel, and σ_e for the essential spectrum that includes the absolute continuous part and embedded eigenvalues. Since $\mathcal{P}$ is a projection defined by the eigenspace of L_- and $K=\mathcal{P}L_-^{-1}\mathcal{P}$, it is obvious that

$$\dim(\mathcal{H}_K^-) = n(L_-), \quad \dim(\mathcal{H}_K^+) = p(L_-), \quad \sigma_e(K) \subset (0,\omega_-^{-1}]. \tag{2.10}$$

The eigenvalues of A are related to the eigenvalues of L_+ according to the standard variational theory in constrained Hilbert spaces.[10,4] The following proposition summarizes the main result of the variational theory.

Proposition 2.2: Let $\mathrm{Ker}(L_-)=\mathrm{Span}\{v_1,v_2,\ldots,v_n\}\in\mathcal{X}$ *and define the matrix function* $M(\mu)$ *by*

$$\forall \mu \notin \sigma(L_+): \quad M_{ij}(\mu) = ((\mu - L_+)^{-1}v_i, v_j), \quad 1 \leq i,j \leq n. \tag{2.11}$$

Let n_0, z_0, *and* p_0 *be the number of negative, zero, and positive eigenvalues of* $M(\mu)$ as $\mu\uparrow 0$ *and* $N_0=n-n_0-z_0-p_0\geq 0$. *Then,*

$$\dim(\mathcal{H}_A^-) = n(L_+) - p_0 - z_0, \quad \dim(\mathcal{H}_A^0) = z(L_+) + z_0 - N_0. \tag{2.12}$$

Proof: According to the results in Ref. 4, all n eigenvalues of $M(\mu)$ are strictly decreasing functions of μ on the intervals $(-\infty,\omega_+)\backslash\sigma_d(L_+)$. These functions may have infinite jump discontinuities from minus infinity to plus infinity across the points of $\sigma_d(L_+)$ and have a uniform limit to minus zero as $\mu\to-\infty$. The count of jumps of the eigenvalues of $M(\mu)$ gives the count of eigenvalues of the constrained variational problem

$$(\mu - L_+)v = \sum_{j=1}^{n} \nu_j v_j, \quad v \in \mathcal{H}, \quad \mu \in (-\infty,\omega_+), \tag{2.13}$$

where $(\nu_1,\nu_2,\ldots,\nu_n)$ are Lagrange multipliers. Equalities (2.12) are proved in Lemma 3.4 in Ref. 4 for the case $z(L_+)=0$ and in Theorem 2.9 of in Ref. 4 for the case $z(L_+)\neq 0$. ■

Since L_+ is generally noninvertible, some eigenvalues of $M(\mu)$ can be unbounded as $\mu\uparrow 0$ if $\mathrm{Ker}(L_+)\nsubseteq\mathcal{H}$. The numbers n_0, z_0, and p_0 denote bounded eigenvalues of $M(\mu)$ as $\mu\uparrow 0$ so that $N_0=n-n_0-z_0-p_0\geq 0$.

If $\mathcal{H}_A^0$ is trivial [that is, $\dim(\mathcal{H}_A^0)=0$], operator A is invertible in $\mathcal{H}$ and we can proceed with the analysis of the generalized eigenvalue problem (2.6). However, if A is not invertible, we would like to reduce the generalized eigenvalue problem to the one defined by invertible operators.

Let γ_{-1} be the smallest (in absolute value) negative eigenvalue of $K^{-1}A$. Since A has finitely many negative eigenvalues and K has no kernel in $\mathcal{H}$, there exists a small number $\delta\in(0,|\gamma_{-1}|)$ such that operator $A+\delta K$ is continuously invertible in $\mathcal{H}$ and the generalized eigenvalue problem (2.6) is rewritten in the shifted form

$$(A+\delta K)u=(\gamma+\delta)Ku, \quad u \in \mathcal{H}. \tag{2.14}$$

By the spectral theory, an alternative decomposition of the Hilbert space $\mathcal{H}$ exists for $\delta \in (0,|\gamma_{-1}|)$,

$$\mathcal{H}=\mathcal{H}^-_{A+\delta K} \oplus \mathcal{H}^+_{A+\delta K} \oplus \mathcal{H}^{\sigma_e(A+\delta K)}_{A+\delta K}, \tag{2.15}$$

where $\sigma_e(A+\delta K) \geqslant \omega_{A+\delta K}$. If $\omega_+>0$, then $\omega_{A+\delta K}>0$ for sufficiently small $\delta>0$.

Since we shift A to $A+\delta K$ for sufficiently small $\delta>0$, all zero eigenvalues of A become small nonzero eigenvalues of $A+\delta K$. We need to know how many zero eigenvalues of A becomes small negative eigenvalues of $A+\delta K$. The following proposition gives the dimension of $\mathcal{H}^-_{A+\delta K}$.

Proposition 2.3: Assume that $\dim(\mathcal{H}^0_A)=1$ *and let* $\{u_1,\ldots,u_n\}$ *be the Jordan chain of eigenvectors of (2.6) given by*

$$\begin{cases} Au_1=0 \\ Au_2=Ku_1 \\ \ldots \\ Au_n=Ku_{n-1}, \end{cases}$$

such that $(Ku_j,u_1)=0$ *for all* $j\in\{1,2,\ldots,n-1\}$ *and* $(Ku_n,u_1)\neq 0$. *Fix* $\delta\in(0,|\gamma_{-1}|)$, *where* γ_{-1} *is the smallest (in absolute value) negative eigenvalue of (2.6). Then*

- $\dim(\mathcal{H}^-_{A+\delta K})=\dim(\mathcal{H}^-_A)$ *if n is odd and* $(Ku_n,u_1)>0$ *or if n is even and* $(Ku_n,u_1)<0$
- $\dim(\mathcal{H}^-_{A+\delta K})=\dim(\mathcal{H}^-_A)+1$ *if n is odd and* $(Ku_n,u_1)<0$ *or if n is even and* $(Ku_n,u_1)>0$.

Proof: Since we shift a self-adjoint operator A to a self-adjoint operator $A+\delta K$ for a sufficiently small $\delta>0$, the zero eigenvalue of operator A becomes a small real eigenvalue $\mu(\delta)$ of operator $A+\delta K$. By perturbation theory for isolated eigenvalues of self-adjoint operators (see Chap. VII.3 in Ref. 17), eigenvalue $\mu(\delta)$ is a continuous function of δ and

$$\lim_{\delta\to 0^+}\frac{\mu(\delta)}{\delta^n}=(-1)^{n+1}\frac{(Ku_n,u_1)}{(u_1,u_1)}. \tag{2.16}$$

Since $\omega_+>0$, the zero eigenvalue of A is isolated from the continuous spectrum of $K^{-1}A$ so that $(Ku_n,u_1)\neq 0$ by the Fredholm theory for the generalized eigenvalue problem (2.6). The assertion of the proposition follows from the perturbation theory (2.16). Since no eigenvalues of $K^{-1}A$ exists in $(\gamma_{-1},0)$, the eigenvalue $\mu(\delta)$ remains sign-definite for $\delta\in(0,|\gamma_{-1}|)$. ■

Remark 2.4: Assumption $\dim(\mathcal{H}^0_A)=1$ of Proposition 2.3 can be removed by considering the Jordan block decomposition for the zero eigenvalue and by summing contributions from all Jordan blocks.

Remark 2.5: If 0 is a semisimple eigenvalue of $Au=\gamma Ku$, the statement of Proposition 2.3 can be simplified as follows.

Let $\mathrm{Ker}(A)=\mathrm{span}\{u_1,u_2,\ldots,u_n\}$ and $M_K\in\mathbb{R}^{n\times n}$ be the matrix with elements

$$(M_K)_{ij}=(Ku_i,u_j), \quad 1\leqslant i,j\leqslant n.$$

Then for small $\delta>0$

$$\dim(\mathcal{H}^-_{A+\delta K})=\dim(\mathcal{H}^-_A)+\dim(\mathcal{H}^-_{M_K}). \tag{2.17}$$

For the proof, let $\{u_1,u_2,\ldots,u_n\}$ be a basis for $\mathrm{Ker}(A)$, which is orthogonal with respect to $(K\cdot,\cdot)$ (such a basis always exists if 0 is a semisimple eigenvalue of $Au=\gamma Ku$). Then, for the *jth* Jordan block, the result of Proposition 2.3 with $n=1$ shows that $\dim(\mathcal{H}^-_{A+\delta K})=\dim(\mathcal{H}^-_A)+1$ if $(Ku_j,u_j)<0$. The Equality (2.17) holds after summing contributions from all Jordan blocks for this basis. The number of negative eigenvalues of M_K is invariant with respect to the choice of basis in $\mathrm{Ker}(A)$.

We shall now introduce notations for particular eigenvalues of the generalized eigenvalue problem (2.6) and formulate our main results. Let $N_p^-(N_n^-)$, $N_p^0(N_n^0)$, and $N_p^+(N_n^+)$ be, respectively, the numbers of negative, zero, and positive eigenvalues γ of the generalized eigenvalue problem (2.6) with the account of their algebraic multiplicities whose eigenvectors are associated with the non-negative (nonpositive) values of the quadratic form $(K\cdot,\cdot)$. Let $N_{c^+}(N_{c^-})$ be the number of complex eigenvalues in the upper (lower) half-plane $\gamma \in \mathbb{C}$, $\mathrm{Im}(\gamma) > 0$ $[\mathrm{Im}(\gamma) < 0]$. Because A and K are real-valued operators, it is obvious that $N_{c^+} = N_{c^-}$.

Theorem 1: *Let assumptions (P1) and (P2) be satisfied. Eigenvalues of the generalized eigenvalue problem (2.6) satisfy the following two equalities:*

$$N_p^- + N_n^0 + N_n^+ + N_{c^+} = \dim(\mathcal{H}_{A+\delta K}^-), \tag{2.18}$$

$$N_n^- + N_n^0 + N_n^+ + N_{c^+} = \dim(\mathcal{H}_K^-). \tag{2.19}$$

Proof: The theorem is proved in Sec. V. ■

Corollary 2.6: Let $N_{\mathrm{neg}} = \dim(\mathcal{H}_{A+\delta K}^-) + \dim(\mathcal{H}_K^-)$ *be the total negative index of the generalized eigenvalue problem (2.6). Let* $N_{\mathrm{unst}} = N_p^- + N_n^- + 2N_{c^+}$ *be the total number of unstable eigenvalues that includes* $N^- = N_p^- + N_n^-$ *negative eigenvalues* $\gamma < 0$ *and* $N_c = N_{c^+} + N_{c^-}$ *complex eigenvalues with* $\mathrm{Im}(\gamma) \neq 0$. *Then,*

$$N_{\mathrm{neg}} = N_{\mathrm{unst}} + 2N_n^+ + 2N_n^0. \tag{2.20}$$

Proof: The equality (2.20) follows by the sum of (2.18) and (2.19). ■

Theorem 2: *Let assumptions (P1) and (P2) be satisfied. Let* $N_A = \dim(\mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+)$ *be the total number of isolated eigenvalues of A. Let* $N_K = \dim(\mathcal{H}_K^- \oplus \mathcal{H}_K^+)$ *be the total number of isolated eigenvalues of K. Assume that all eigenvalues of the generalized eigenvalue problem (2.6) are isolated from the continuous spectrum. Then, isolated eigenvalues satisfy the following inequality:*

$$N_p^- + N_p^0 + N_p^+ + N_{c^+} \leq N_A + N_K. \tag{2.21}$$

Proof: This theorem is proved in Sec. V. ■

Corollary 2.7: Let $N_{\mathrm{tot}} = N_A + N_K$ *be the total number of isolated eigenvalues of operators A and K. Let* $N_{\mathrm{isol}} = N_p^- + N_n^- + N_p^0 + N_n^0 + N_p^+ + N_n^+ + N_{c^+} + N_{c^-}$ *be the total number of isolated eigenvalues of the generalized eigenvalue problem (2.6). Then,*

$$N_{\mathrm{isol}} \leq N_{\mathrm{tot}} + \dim(\mathcal{H}_K^-). \tag{2.22}$$

Proof: The inequality (2.22) follows by the sum of (2.19) and (2.21). ■

To prove Theorems 1 and 2, we shall prove Pontryagin's invariant subspace theorem and apply this theorem to the count of isolated and embedded eigenvalues for the non-self-adjoint operator $K^{-1}A$.

III. PONTRYAGIN'S INVARIANT SUBSPACE THEOREM

We develop here an abstract theory of Pontryagin spaces with sign-indefinite metric, where the main result is Pontryagin's invariant subspace theorem.

Definition 3.1: Let $\mathcal{H}$ *be a Hilbert space equipped with the inner product* $(\cdot,\cdot)$ *and the sesquilinear form* $[\cdot,\cdot]$. (We say that a complex-valued form $[u,v]$ on the product space $\mathcal{H} \times \mathcal{H}$ is a sesquilinear form if it is linear in u for each fixed v and linear with complex conjugate in v for each fixed u.) *The Hilbert space* $\mathcal{H}$ *is called the Pontryagin space (denoted as* Π_κ*) if it can be decomposed into the direct sum, which is orthogonal with respect to* $[\cdot,\cdot]$,

$$\mathcal{H} \coloneqq \Pi_\kappa = \Pi_+ \oplus \Pi_-, \quad \Pi_+ \cap \Pi_- = \varnothing, \tag{3.1}$$

where Π_+ *is a Hilbert space with the inner product* $(\cdot,\cdot) = [\cdot,\cdot]$, Π_- *is a Hilbert space with the inner product* $(\cdot,\cdot) = -[\cdot,\cdot]$, *and* $\kappa = \dim(\Pi_-) < \infty$.

We shall write components of an element x in the Pontryagin space Π_κ as a vector $x=\{x_-,x_+\}$. The direct orthogonal sum (3.1) implies that any nonzero element $x \in \Pi_\kappa$ can be uniquely represented by the sum of two terms,

$$\forall x \in \Pi_\kappa: \quad x = x_+ + x_-, \tag{3.2}$$

so that

$$[x_+,x_-]=0, \quad [x_+,x_+]>0, \quad [x_-,x_-]<0. \tag{3.3}$$

Definition 3.2: We say that Π *is a nonpositive subspace of* Π_κ *if* $[x,x]\leqslant 0\,\forall x \in \Pi$. *We say that* Π *is a maximal nonpositive subspace if any subspace of* Π_κ *of dimension higher than* $\dim(\Pi)$ *is not a nonpositive subspace of* Π_κ. *Similarly, we say that* Π *is a non-negative (neutral) subspace of* Π_κ *if* $[x,x]\geqslant 0([x,x]=0)\,\forall x \in \Pi$.

Lemma 3.3: The dimension of the maximal nonpositive subspace of Π_κ *is* κ.

Proof: By contradiction, we assume that there exists a $(\kappa+1)$-dimensional nonpositive subspace $\tilde{\Pi}$. Let $\{e_1,e_2,\ldots,e_\kappa\}$ be a basis in Π_-. We fix two elements $y_1,y_2 \in \tilde{\Pi}$ with the same projections to $\{e_1,e_2,\ldots,e_\kappa\}$ so that

$$y_1 = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_\kappa e_\kappa + y_{1p},$$

$$y_2 = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_\kappa e_\kappa + y_{2p},$$

where $y_{1p},y_{2p} \in \Pi_+$. It is clear that $y_1-y_2=y_{1p}-y_{2p} \in \Pi_+$ so that $[y_{1p}-y_{2p},y_{1p}-y_{2p}]\geqslant 0$. On the other hand, $y_1-y_2 \in \tilde{\Pi}$ so that $[y_1-y_2,y_1-y_2]\leqslant 0$. Hence, $y_{1p}=y_{2p}$ and $y_1=y_2 \in \Pi$. Therefore, $\tilde{\Pi}$ is still a κ-dimensional nonpositive subspace of Π_κ. ■

Theorem 3: *(Pontryagin) Let T be a self-adjoint bounded operator in* Π_κ *in the sense of* $[T\cdot,\cdot]=[\cdot,T\cdot]$. *There exists a* κ*-dimensional, maximal nonpositive*, T*-invariant subspace of* Π_κ.

There are historically two completely different approaches to the proof of this theorem. A proof based on the theory of analytic functions was given by Pontryagin,[29] while a proof based on angular operators was given by Krein and his students (see books[8,1,14]). Theorem 3 was rediscovered by Grillakis[12] with the use of topology. We describe a geometric proof of Theorem 3 based on Shauder's fixed point theorem. The proof uses the Cayley transformation of a self-adjoint operator in Π_κ to a unitary operator in Π_κ (Lemma 3.4) and the Krein representation of the maximal nonpositive subspace of Π_κ in terms of a graph of the contraction map (Lemma 3.6). While many statements of our analysis are available in literature, details of the proofs are missing. Our presentation gives full details of the proof of Theorem 3 (see Ref. 13 for a similar treatment in the case of compact operators).

Lemma 3.4: Let T be a linear operator in Π_κ *and* $z \in \mathbb{C}$, $\mathrm{Im}(z)>0$ *be a regular point of the operator* T, *such that* $z \in \rho(T)$. *Let U be the Cayley transform of T defined by* $U=(T-\bar{z})(T-z)^{-1}$. *The operators T and U have the same invariant subspaces in* Π_κ.

Proof: Let Π be a finite-dimensional invariant subspace of the operator T in Π_κ. It follows from $z \in \rho(T)$ that $(T-z)\Pi=\Pi$ then $(T-z)^{-1}\Pi=\Pi$ and $(T-\bar{z})(T-z)^{-1}\Pi \subseteq \Pi$, i.e. $U\Pi \subseteq \Pi$. Conversely, let Π be an invariant subspace of the operator U. It follows from $U-I=(z-\bar{z})(T-z)^{-1}$ that $1 \in \rho(U)$ therefore $\Pi=(U-I)\Pi=(T-z)^{-1}\Pi$. From there, $\Pi \subseteq \mathrm{dom}(T)$ and $(T-z)\Pi=\Pi$ so $T\Pi \subseteq \Pi$. ■

Corollary 3.5: If T is a self-adjoint operator in Π_κ, *then U is a unitary operator in* Π_κ.

Proof: We shall prove that $[Ug,Ug]=[g,g]$, where $g \in \mathrm{dom}(U)$, by the explicit computation,

$$[Ug,Ug] = [(T-\bar{z})f,(T-\bar{z})f] = [Tf,Tf] - \bar{z}[f,Tf] - z[Tf,f] + |z|^2[f,f],$$

$$[g,g] = [(T-z)f,(T-z)f] = [Tf,Tf] - \bar{z}[f,Tf] - z[Tf,f] + |z|^2[f,f],$$

where we have introduced $f \in \mathrm{dom}(T)$ such that $f=(T-z)^{-1}g$. ■

Lemma 3.6: A linear subspace $\Pi \subseteq \Pi_\kappa$ *is a* κ*-dimensional nonpositive subspace of* Π_κ *if and* *only if it is a graph of the contraction map* $\mathcal{K}:\Pi_- \to \Pi_+$ *such that* $x=\{x_-,\mathcal{K}x_-\}$, $\forall x \in \Pi$, *and* $\|\mathcal{K}x_-\| \leq \|x_-\|$.

Proof: Let Π be a κ-dimensional nonpositive subspace of Π_κ. We will show that there exist a contraction map $\mathcal{K}:\Pi_- \mapsto \Pi_+$ such that Π is a graph of $\mathcal{K}$. Indeed, the subspace Π is a graph of a linear operator $\mathcal{K}$ if and only if it follows from $\{0,x_+\} \in \Pi$ that $x_+=0$. Since Π is nonpositive with respect to $[\cdot,\cdot]$, then $[x,x]=\|x_+\|^2-\|x_-\|^2 \leq 0$, where $\|\cdot\|$ is a norm in $\mathcal{H}$. As a result, $0 \leq \|x_+\| \leq \|x_-\|$ and if $x_-=0$ then $x_+=0$. Moreover, for any $x_- \in \Pi_-$, it is true that $\|\mathcal{K}x_-\| \leq \|x_-\|$ so that $\mathcal{K}$ is a contraction map. Conversely, let $\mathcal{K}$ be a contraction map $\mathcal{K}:\Pi_- \mapsto \Pi_+$. Then, we have

$$\forall x = \{x_-,\mathcal{K}x_-\} \in \Pi: \quad [x,x] = \|x_+\|^2 - \|x_-\|^2 = \|\mathcal{K}x_-\|^2 - \|x_-\|^2 \leq 0,$$

so that the graph of $\mathcal{K}$ belongs to the nonpositive subspace of Π_κ and $\dim(\Pi)=\dim(\Pi_\kappa)=\kappa$. ■

Extending arguments of Lemma 3.6, one can prove that the subspace Π is strictly negative with respect to $[\cdot,\cdot]$ if and only if it is a graph of the strictly contraction map $\mathcal{K}:\Pi_- \mapsto \Pi_+$ such that $x=\{x_-,\mathcal{K}x_-\}$, $\forall x \in \Pi$, and $\|\mathcal{K}x_-\| < \|x_-\|$.

Proof of Theorem 3: Let $z \in \mathbb{C}$, $\mathrm{Im}(z) > 0$ be a regular point of the self-adjoint operator T in Π_κ. Let $U=(T-\bar{z})(T-z)^{-1}$ be the Cayley transform of T. By Corollary 3.5, U is a unitary operator in Π_κ. By Lemma 3.4, T and U have the same invariant subspaces in Π_κ. Therefore, the existence of the maximal nonpositive invariant subspace for the self-adjoint operator T can be proved from the existence of such a subspace for the unitary operator U. Let $x=\{x_-,x_+\}$ and

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

be the matrix representation of the operator U with respect to the decomposition (3.1). Let Π denote a κ-dimensional nonpositive subspace in Π_κ. Since U has a trivial kernel in Π_κ and U is unitary in Π_κ such that $[Ux,Ux]=[x,x] \leq 0$, then $\tilde{\Pi}=U\Pi$ is also a κ-dimensional nonpositive subspace of Π_κ. By Lemma 3.6, there exist two contraction mappings $\mathcal{K}$ and $\tilde{\mathcal{K}}$ for subspaces Π and $\tilde{\Pi}$, respectively. Therefore, the assignment $\tilde{\Pi}=U\Pi$ is equivalent to the system,

$$\begin{pmatrix} \tilde{x}_- \\ \tilde{\mathcal{K}}\tilde{x}_- \end{pmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{pmatrix} x_- \\ \mathcal{K}x_- \end{pmatrix} = \begin{pmatrix} (U_{11} + U_{12}\mathcal{K})x_- \\ (U_{21} + U_{22}\mathcal{K})x_- \end{pmatrix},$$

so that

$$U_{21} + U_{22}\mathcal{K} = \tilde{\mathcal{K}}(U_{11} + U_{12}\mathcal{K}). \tag{3.4}$$

We shall prove that the operator $(U_{11}+U_{12}\mathcal{K})$ is invertible. By contradiction, we assume that there exists $x_- \neq 0$ such that $\tilde{x}_-=(U_{11}+U_{12}\mathcal{K})x_-=0$. Since $\tilde{x}_-=0$ implies that $\tilde{x}_+=\tilde{\mathcal{K}}\tilde{x}_-=0$, we obtain that $\{x_-,\mathcal{K}x_-\}$ is an eigenvector in the kernel of U. However, U has a trivial kernel in Π_κ so that $x_-=0$. Let $F(\mathcal{K})$ be an operator-valued function in the form,

$$F(\mathcal{K}) = (U_{21} + U_{22}\mathcal{K})(U_{11} + U_{12}\mathcal{K})^{-1},$$

and rewrite (3.4) in the form $\tilde{\mathcal{K}}=F(\mathcal{K})$, where $F(\mathcal{K})$ is defined for any contraction operator K. By Lemma 3.6, the operator $F(\mathcal{K})$ maps the operator unit ball $\|\mathcal{K}\| \leq 1$ to itself. Since U is a continuous operator and U_{12} is a finite-dimensional operator, then U_{12} is a compact operator. Hence, the operator ball $\|\mathcal{K}\| \leq 1$ is a weakly compact set and the function $F(\mathcal{K})$ is continuous with respect to weak topology. By Schauder's fixed-point principle, there exists a fixed point $\mathcal{K}_0$ such that $F(\mathcal{K}_0)=\mathcal{K}_0$ and $\|\mathcal{K}_0\| \leq 1$. By Lemma 3.6, the graph of $\mathcal{K}_0$ defines the κ-dimensional nonpositive subspace Π, which is invariant with respect to U. By Lemma 3.3, the κ-dimensional nonpositive subspace Π is a maximal nonpositive subspace of Π_κ. ■

IV. SPECTRUM OF A SELF-ADJOINT OPERATOR IN PONTRYAGIN SPACE

We apply here Pontryagin's invariant subspace theorem (Theorem 3) to the product of two bounded invertible self-adjoint operators $T=BC$ in Pontryagin space Π_κ, where $\kappa=\dim(\mathcal{H}_C^-)$. In the context of the shifted generalized eigenvalue problem (2.14), the operator T can be constructed in two equivalent settings. In the first setting, $B=(A+\delta K)^{-1}$ and $C=K$ with $\kappa=\dim(\mathcal{H}_K^-)$, while in the second setting, $B=K$ and $C=(A+\delta K)^{-1}$ with $\kappa=\dim(\mathcal{H}_{A+\delta K}^-)$. With a slight abuse of notations [spectral parameter λ here does not correspond to parameter λ used in the linear eigenvalue problem (2.1)], we shall denote eigenvalues of the operator $T=BC$ by λ. In the context of the shifted generalized eigenvalue problem (2.14), $\lambda=(\gamma+\delta)^{-1}$ in the first setting and $\lambda=(\gamma+\delta)$ in the second setting.

Lemma 4.1: Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot,\cdot)$ and $B,C:\mathcal{H}\to\mathcal{H}$ be bounded invertible self-adjoint operators in $\mathcal{H}$. Define the sesquilinear form

$$[\cdot,\cdot]=(C\cdot,\cdot) \tag{4.1}$$

and extend $\mathcal{H}$ to the Pontryagin space Π_κ, where κ is the finite number of negative eigenvalues of C counted with their algebraic multiplicities. The operator $T=BC$ is self-adjoint in Π_κ and there exists a κ-dimensional maximal nonpositive subspace of Π_κ which is invariant with respect to T.

Proof: It follows from the orthogonal sum decomposition in the Hilbert space $\mathcal{H}$ that the quadratic form $(C\cdot,\cdot)$ is strictly negative on the κ-dimensional subspace $\mathcal{H}_C^-$ and strictly positive on the infinite-dimensional subspace $\mathcal{H}_C^+\oplus\mathcal{H}_C^{\sigma_e(C)}$. By continuity and Gram–Schmidt orthogonalization, the Hilbert space $\mathcal{H}$ is extended to the Pontryagin space Π_κ with respect to the sesquilinear form (4.1). The bounded operator $T=BC$ is self-adjoint in Π_κ since B and C are self-adjoint in $\mathcal{H}$ and

$$[T\cdot,\cdot]=(CBC\cdot,\cdot)=(C\cdot,BC\cdot)=[\cdot,T\cdot].$$

The existence of the κ-dimensional maximal nonpositive T-invariant subspace of Π_κ follows from Pontryagin's invariant subspace theorem (Theorem 3). ∎

We consider now various sign-definite subspaces of Π_κ, which are invariant with respect to the operator $T=BC$. In general, these invariant sign-definite subspaces do not provide a canonical decomposition of Π_κ, unlike the direct orthogonal sum (3.1).

Let $\mathcal{H}_{c^+}(\mathcal{H}_{c^-})$ denote the T-invariant subspace associated with complex eigenvalues λ in the upper (lower) half-plane and $\mathcal{H}_n(\mathcal{H}_p)$ denote the nonpositive (non-negative) T-invariant subspace associated with real eigenvalues λ. Spectrum of T may consist of three disjoint sets: isolated and embedded eigenvalues, continuous spectrum, and residual spectrum (see Definitions 4.2 and 4.3). We will show that the maximal nonpositive T-invariant subspace in Lemma 4.1 does not include the residual and continuous spectra but may include isolated and embedded eigenvalues.

Definition 4.2: We say that λ is a point of the residual spectrum of T if

$$\mathrm{Ker}(T-\lambda I)=\varnothing,\quad \overline{\mathrm{Ran}(T-\lambda I)}\neq\Pi_\kappa$$

and λ is a point of the continuous spectrum of T if

$$\mathrm{Ker}(T-\lambda I)=\varnothing,\quad \mathrm{Ran}(T-\lambda I)\neq\overline{\mathrm{Ran}(T-\lambda I)}=\Pi_\kappa.$$

Definition 4.3: We say that λ is a point of the discrete spectrum of T (an eigenvalue) if

$$\mathrm{Ker}(T-\lambda I)\neq\varnothing.$$

The eigenvalue is said to be semisimple if

$$\mathrm{Ker}(T-\lambda I)=\dim(\cap_{k\in\mathbb{N}}\mathrm{Ker}(T-\lambda I)^k)$$

and multiple if

$$\mathrm{Ker}(T-\lambda I)<\dim(\cap_{k\in\mathbb{N}}\mathrm{Ker}(T-\lambda I)^k).$$

In the latter case, the eigenspace can be represented by the union of the Jordan blocks and the canonical basis for each Jordan block is built by the generalized eigenvectors

$$f_j \in \Pi_\kappa: \quad Tf_j = \lambda_0 f_j + f_{j-1}, \quad j = 1, \ldots, n, \tag{4.2}$$

where $f_0 = 0$. *If* $n = \infty$, *the eigenvalue* λ_0 *is said to have an infinite multiplicity.*

Lemma 4.4: The residual spectrum of T is empty.

Proof: By a contradiction, assume that λ belongs to the residual part of the spectrum of T such that $\mathrm{Ker}(T-\lambda I)=\varnothing$ but $\mathrm{Ran}(T-\lambda I)$ is not dense in Π_κ. Let $g \in \Pi_\kappa$ be orthogonal to $\mathrm{Ran}(T-\lambda I)$ so that

$$\forall f \in \Pi_\kappa: \quad 0 = [(T-\lambda I)f, g] = [f, (T-\bar{\lambda} I)g].$$

Therefore, $(T-\bar{\lambda}I)g=0$, that is, $\bar{\lambda}$ is an eigenvalue of T. Since T is real-valued operator, λ is also an eigenvalue of T and hence it cannot be in the residual part of the spectrum of T. ∎

Lemma 4.5: The continuous spectrum of T is real.

Proof: Let P^+ and P^- be orthogonal projectors to Π^+ and Π^-, respectively, so that $I=P^+ + P^-$. Since $\Pi^{\pm}$ are defined by the quadratic form (4.1), the self-adjoint operator C admits the polar decomposition $C=J|C|$, where $J=P^+-P^-$ and $|C|$ is a positive operator. Since $J^2=I$ and C is self-adjoint, we have $J|C|J=|C|$. As a result, $J|C|^{1/2}J=|C|^{1/2}$ and the operator $T=BC$ is similar to the operator

$$|C|^{1/2}BJ|C|^{1/2} = |C|^{1/2}BJ|C|^{1/2}(J+2P^-) = |C|^{1/2}B|C|^{1/2} + 2|C|^{1/2}BJ|C|^{1/2}P^-.$$

Since P^- is a projection to a finite-dimensional subspace, the operator $|C|^{1/2}BJ|C|^{1/2}$ is a finite-rank perturbation of the self-adjoint operator $|C|^{1/2}B|C|^{1/2}$. By Theorem 18 on p. 22 in, Ref. 7, the continuous part of the self-adjoint operator $|C|^{1/2}B|C|^{1/2}$ is the same as that of $|C|^{1/2}BJ|C|^{1/2}$. By similarity transformation, it is the same as that of T. ∎

Theorem 4: *Let* Π_c *be an invariant subspace associated with the continuous spectrum of* T. *Then,* $[f,f]>0$ *for any nonzero* $f \in \Pi_c$.

Proof: By Lemma 4.1, the operator T has a κ-dimensional maximal nonpositive invariant subspace of Π_κ. Let us denote this subspace by Π. By Lemma 4.4, the spectrum of T is decomposed into disjoint sets of eigenvalues and the continuous spectrum. Since any finite-dimensional invariant subspace of T cannot be a part of Π_c, Π and Π_c intersect trivially. Assume now that there exists a nonzero $f_0 \in \Pi_c$ such that $[f_0, f_0] \leq 0$. Since $f_0 \notin \Pi$, the subspace spanned by f_0 and the basis vectors in Π is a $(\kappa+1)$-dimensional nonpositive subspace of Π_κ. However, by Lemma 3.3, the maximal dimension of any nonpositive subspace of Π_κ is κ. Therefore, $[f_0, f_0]>0$ for any nonzero $f_0 \in \Pi_c$. ∎

V. EIGENVALUES OF THE GENERALIZED EIGENVALUE PROBLEM

We count here isolated and embedded eigenvalues for the product operator $T=BC$. This operator is self-adjoint in the Pontryagin space Π_κ, which is defined by the sesquilinear form (4.1) with $\kappa=\dim(\mathcal{H}_C^-)$. This count is used in the proofs of our main Theorems 1 and 2. We assume that the eigenspaces associated with eigenvalues of T are represented by the union of the Jordan blocks, according to Definition 4.3. Each Jordan block of generalized eigenvectors (4.2) is associated with a single eigenvector of T. We start with an elementary result about the generalization of the Fredholm theory in the Hilbert space $\mathcal{H}$ to that in the Pontryagin space Π_κ.

Proposition 5.1: Let λ_0 *be an isolated eigenvalue of* $T=BC$ *associated with a one-dimensional eigenspace* $\mathcal{H}_{\lambda_0}=\mathrm{Span}\{f_0\}$. *Then,* $\lambda_0 \in \mathbb{R}$ *is algebraically simple if and only if* $[f_0, f_0] \neq 0$, *while* $\lambda_0 \notin \mathbb{R}$ *is algebraically simple if and only if* $[f_0, \bar{f}_0] \neq 0$.

Proof: Since B and C are bounded invertible self-adjoint operators in the Hilbert space $\mathcal{H}$, the eigenvalue problem $Tf=\lambda f$ in the Pontryagin space Π_κ is rewritten as the generalized eigenvalue problem $Cf=\lambda B^{-1}f$ in the Hilbert space $\mathcal{H}$. Since λ_0 is an isolated eigenvalue, the Fredholm theory for the generalized eigenvalue problem implies that $\lambda_0 \in \mathbb{R}$ is algebraically simple if and only if $(B^{-1}f_0,f_0)\neq 0$, while $\lambda_0\notin\mathbb{R}$ is algebraically simple if and only if $(B^{-1}f_0,\bar{f}_0)\neq 0$. Since $\lambda_0\neq 0$ (otherwise, C would not be invertible), the condition of the Fredholm theory is equivalent to the condition that $[f_0,f_0]=(Cf_0,f_0)\neq 0$ and $[f_0,\bar{f}_0]=(Cf_0,\bar{f}_0)\neq 0$, respectively. ∎

Lemma 5.2: Let $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ be eigenspaces associated with eigenvalues λ and μ of the operator T in Π_κ and $\lambda\neq\bar{\mu}$. Then $\mathcal{H}_\lambda$ is orthogonal to $\mathcal{H}_\mu$ with respect to $[\cdot,\cdot]$.

Proof: Let $n\geqslant 1$ and $m\geqslant 1$ be dimensions of $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$, respectively. By Definition 4.3, it is clear that

$$f\in\mathcal{H}_\lambda \Leftrightarrow (T-\lambda I)^n f=0, \tag{5.1}$$

$$g\in\mathcal{H}_\mu \Leftrightarrow (T-\mu I)^m g=0. \tag{5.2}$$

We should prove that $[f,g]=0$ by induction for $n+m\geqslant 2$. If $n+m=2$ $(n=m=1)$, then it follows from systems (5.1) and (5.2) that

$$(\lambda-\bar{\mu})[f,g]=0,\quad f\in\mathcal{H}_\lambda,\quad g\in\mathcal{H}_\mu,$$

so that $[f,g]=0$ for $\lambda\neq\bar{\mu}$. Let us assume that subspaces $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are orthogonal for $2\leqslant n+m\leqslant k$ and prove that an extended subspace $\tilde{\mathcal{H}}_\lambda$ with $\tilde{n}=n+1$ remains orthogonal to $\mathcal{H}_\mu$. To do so, we define $\tilde{f}=(T-\lambda I)f$ and verify that

$$f\in\tilde{\mathcal{H}}_\lambda \Leftrightarrow (T-\lambda I)^{\tilde{n}}f=(T-\lambda I)^n\tilde{f}=0.$$

By the inductive assumption, we have $[\tilde{f},g]=0$ so that

$$[(T-\lambda I)f,g]=0. \tag{5.3}$$

Using systems (5.1) and (5.2) and relation (5.3), we obtain

$$(\lambda-\bar{\mu})[f,g]=0\quad f\in\tilde{\mathcal{H}}_\lambda,\quad g\in\mathcal{H}_\mu,$$

so that $[f,g]=0$ for all $f\in\tilde{\mathcal{H}}_\lambda$ and $g\in\mathcal{H}_\mu$. Using the same analysis, one can prove that an extended subspace $\tilde{\mathcal{H}}_\mu$ with $\tilde{m}=m+1$ remains orthogonal to $\mathcal{H}_\lambda$. The assertion of the lemma follows by the induction method. ∎

Lemma 5.3: Let $\mathcal{H}_{\lambda_0}$ be an eigenspace associated with an isolated eigenvalue $\lambda_0\in\mathbb{R}$ of T in Π_κ and $\{f_1,f_2,\ldots f_n\}$ be the Jordan chain of eigenvectors. Let $\mathcal{H}_0=\mathrm{Span}\{f_1,f_2,\ldots,f_k\}\subset\mathcal{H}_{\lambda_0}$, where $k=n/2$ if n is even and $k=(n-1)/2$ if n is odd, and $\tilde{\mathcal{H}}_0=\mathrm{Span}\{f_1,f_2,\ldots,f_k,f_{k+1}\}\subset\mathcal{H}_{\lambda_0}$.

- *If n is even $(n=2k)$, the neutral subspace $\mathcal{H}_0$ is the maximal sign-definite subspace of $\mathcal{H}_{\lambda_0}$.*
- *If n is odd $(n=2k+1)$, the subspace $\tilde{\mathcal{H}}_0$ is the maximal nonnegative subspace of $\mathcal{H}_{\lambda_0}$ if $[f_1,f_n]>0$ and the maximal nonpositive subspace of $\mathcal{H}_{\lambda_0}$ if $[f_1,f_n]<0$, while the neutral subspace $\mathcal{H}_0$ is the maximal nonpositive subspace of $\mathcal{H}_{\lambda_0}$ if $[f_1,f_n]>0$ and the maximal nonnegative subspace of $\mathcal{H}_{\lambda_0}$ if $[f_1,f_n]<0$.*

Proof: Without loss of generality, we will consider the case $\lambda_0=0$ (if $\lambda_0\neq 0$ the same argument is applied to the shifted self-adjoint operator $\tilde{T}=T-\lambda_0 I$). We will show that $[f,f]=0$, $\forall f\in\mathcal{H}_0$. By a decomposition over the basis in $\mathcal{H}_0$, we obtain

$$\forall f=\sum_{i=1}^{k}\alpha_i f_i:\quad [f,f]=\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha_i\bar{\alpha}_j[f_i,f_j]. \tag{5.4}$$

We use that

$$[f_i,f_j] = [Tf_{i+1},Tf_{j+1}] = \cdots = [T^k f_{i+k},T^k f_{j+k}] = [T^{2k} f_{i+k},f_{j+k}]$$

for any $1 \leq i,j \leq k$. In the case of even $n=2k$, we have

$$[f_i,f_j] = [T^n f_{i+k},f_{j+k}] = 0, \quad 1 \leq i,j \leq k.$$

In the case of odd $n=2k+1$, we have

$$[f_i,f_j] = [T^{n+1} f_{i+k+1},f_{j+k+1}] = 0, \quad 1 \leq i,j \leq k.$$

Therefore, $\mathcal{H}_0$ is a neutral subspace of $\mathcal{H}_{\lambda_0}$. To show that it is the maximal neutral subspace of $\mathcal{H}_{\lambda_0}$, let $\mathcal{H}_0' = \text{Span}\{f_1,f_2,\ldots,f_k,f_{k_0}\}$, where $k+1 \leq k_0 \leq n$. Since f_{n+1} does not exist in the Jordan chain (4.2) (otherwise, the algebraic multiplicity is $n+1$) and λ_0 is an isolated eigenvalue, then $[f_1,f_n] \neq 0$ by Proposition 5.1. It follows from the Jordan chain (4.2) that

$$[f_1,f_n] = [T^{m-1} f_m,f_n] = [f_m,T^{m-1} f_n] = [f_m,f_{n-m+1}] \neq 0. \tag{5.5}$$

When $n=2k$, we have $1 \leq n-k_0+1 \leq k$, such that $[f_{k_0},f_{n-k_0+1}] \neq 0$ and the subspace $\mathcal{H}_0'$ is sign-indefinite in the decomposition (5.4). When $n=2k+1$, we have $1 \leq n-k_0+1 \leq k$ for $k_0 \geq k+2$ and $n-k_0+1=k+1$ for $k_0=k+1$. In either case, $[f_{k_0},f_{n-k_0+1}] \neq 0$ and the subspace $\mathcal{H}_0'$ is sign-indefinite in the decomposition (5.4) unless $k_0=k+1$. In the latter case, we have

$$[f_{k+1},f_{k+1}] = [f_1,f_n] \neq 0 \quad \text{and } [f_j,f_{k+1}] = [T^{2k} f_{j+k},f_n] = 0, \quad 1 \leq j \leq k,$$

so that this subspace $\tilde{\mathcal{H}}_0 \equiv \mathcal{H}_0'$ with $k_0=k+1$ is non-negative for $[f_1,f_n]>0$ and nonpositive for $[f_1,f_n]<0$. ■

Remark 5.4: If $\lambda_0 \in \mathbb{R}$ is an embedded eigenvalue of T, the Jordan chain (4.2) can be truncated at f_n even if $[f_1,f_n]=0$. Indeed, the Fredholm theory for the generalized eigenvalue problem in Proposition 5.1 gives a necessary but not a sufficient condition for existence of the solution f_{n+1} in the Jordan chain (4.2) if the eigenvalue λ_0 is embedded into the continuous spectrum. If $[f_1,f_n]=0$ but f_{n+1} does not exist in Π_κ, the neutral subspaces $\mathcal{H}_0$ for $n=2k$ and $\tilde{\mathcal{H}}_0$ for $n=2k+1$ in Lemma 5.3 do not have to be the maximal nonpositive or non-negative subspaces. The construction of a maximal nonpositive subspace for embedded eigenvalues depends on the computations of the projection matrix $[f_i,f_j]$ in the eigenspace $\mathcal{H}_\lambda = \text{Span}\{f_1,\ldots,f_n\}$. If λ_0 is an algebraically simple embedded eigenvalue, then the corresponding eigenspace $\mathcal{H}_{\lambda_0} = \text{Span}\{f_1\}$ is either positive or negative or neutral, depending on the value of $[f_1,f_1]$.

Lemma 5.5: Let $\lambda_0 \in \mathbb{C}$, $\text{Im}(\lambda_0)>0$ be an eigenvalue of T in Π_κ, $\mathcal{H}_{\lambda_0}$ be the corresponding eigenspace, and $\tilde{\mathcal{H}}_{\lambda_0} = \{\mathcal{H}_{\lambda_0},\mathcal{H}_{\bar{\lambda}_0}\} \subset \Pi_\kappa$. Then, the neutral subspace $\mathcal{H}_{\lambda_0}$ is the maximal sign-definite subspace of $\tilde{\mathcal{H}}_{\lambda_0}$ such that $[f,f]=0$, $\forall f \in \mathcal{H}_{\lambda_0}$.

Proof: By Lemma 5.2 with $\lambda=\mu=\lambda_0$, the eigenspace $\mathcal{H}_{\lambda_0}$ is orthogonal to itself with respect to $[\cdot,\cdot]$, such that $\mathcal{H}_{\lambda_0}$ is a neutral subspace of $\tilde{\mathcal{H}}_{\lambda_0}$. It remains to prove that $\mathcal{H}_{\lambda_0}$ is the maximal sign-definite subspace in $\tilde{\mathcal{H}}_\lambda$. Let $\mathcal{H}_{\lambda_0} = \text{Span}\{f_1,f_2,\ldots,f_n\}$, where $\{f_1,f_2,\ldots,f_n\}$ is the Jordan chain of eigenvectors (4.2). Consider a subspace $\tilde{\mathcal{H}}_{\lambda_0}' = \text{Span}\{f_1,f_2,\ldots,f_n,\bar{f}_j\}$ for any $1 \leq j \leq n$ and construct a linear combination of f_{n+1-j} and $\bar{f}_j$,

$$\forall \alpha \in \mathbb{C}: \quad [f_{n+1-j} + \alpha \bar{f}_j, f_{n+1-j} + \alpha \bar{f}_j] = 2\,\text{Re}(\alpha [\bar{f}_j, f_{n+1-j}]). \tag{5.6}$$

By Proposition 5.1, we have $[f_n,\bar{f}_1] \neq 0$ and, by virtue of the chain (5.5), we obtain $[\bar{f}_j,f_{n+1-j}] \neq 0$. As a result, the linear combination $f_{n+1-j} + \alpha \bar{f}_j$ in equality (5.6) is sign-indefinite with respect to $[\cdot,\cdot]$. ■

We shall summarize the count of the dimensions of the maximal nonpositive and non-negative subspaces associated with eigenspaces of T in Π_κ.

Theorem 5: *Let $N_n(\lambda_0)$ $(N_p(\lambda_0))$ denote the dimension of the maximal nonpositive (non-*

negative) subspace of Π_κ *corresponding to the eigenspace* $\mathcal{H}_{\lambda_0}$ *for an eigenvalue* λ_0. *If* $\lambda_0 \in \mathbb{R}$ *is isolated from the continuous spectrum, then*

$$\dim(\mathcal{H}_{\lambda_0}) = N_p(\lambda_0) + N_n(\lambda_0) \tag{5.7}$$

and, for each Jordan block of generalized eigenvectors, we have

(i) *if* $n=2k$, *then* $N_p(\lambda_0)=N_n(\lambda_0)=k$.
(ii) *If* $n=2k+1$ *and* $[f_1,f_n]>0$, *then* $N_p(\lambda_0)=k+1$ *and* $N_n(\lambda_0)=k$.
(iii) *If* $n=2k+1$ *and* $[f_1,f_n]<0$, *then* $N_p(\lambda_0)=k$ *and* $N_n(\lambda_0)=k+1$.

If $\lambda_0 \in \mathbb{R}$ *is a simple embedded eigenvalue, then*

(i) *if* $[f_1,f_1]>0$, *then* $N_p(\lambda_0)=1$, $N_n(\lambda_0)=0$.
(ii) *If* $[f_1,f_1]<0$, *then* $N_p(\lambda_0)=0$, $N_n(\lambda_0)=1$.
(iii) *If* $[f_1,f_1]=0$, *then* $N_p(\lambda_0)=N_n(\lambda_0)=1$.

If $\lambda_0 \notin \mathbb{R}$, *then* $\dim(\mathcal{H}_{\lambda_0})=N_p(\lambda_0)=N_n(\lambda_0)$.

Proof: The assertion follows from Lemma 5.3, Remark 5.4, and Lemma 5.5. ■

Remark 5.6: Note that the intersection of the maximal nonpositive and non-negative subspaces of $\mathcal{H}_{\lambda_0}$ can be nonempty even for an isolated eigenvalue $\lambda_0 \in \mathbb{C}$. For an embedded eigenvalue $\lambda_0 \in \mathbb{R}$, equality (5.7) does not hold in case (iii) as

$$1 = \dim(\mathcal{H}_{\lambda_0}) < N_p(\lambda_0) + N_n(\lambda_0) = 2.$$

If $\lambda_0 \in \mathbb{R}$ is a multiple embedded eigenvalue, computations of the projection matrix $[f_i,f_j]$ is needed in order to find the dimensions $N_p(\lambda_0)$ and $N_n(\lambda_0)$.

We can now prove Theorems 1 and 2.

Proof of Theorem 1: We use the shifted generalized eigenvalue problem (2.14) for sufficiently small $\delta>0$ and consider the bounded operator $T=(A+\delta K)^{-1}K$, that is, $B=(A+\delta K)^{-1}$ and $C=K$. By Lemma 4.1, the operator T is self-adjoint with respect to $[\cdot,\cdot]=(K\cdot,\cdot)$ and it has a κ-dimensional maximal nonpositive invariant subspace, where $\kappa=\dim(\mathcal{H}_K^-)$. Counting all eigenvalues of the shifted generalized eigenvalue problem (2.14) using Theorem 5, we obtain equality (2.19) from Lemma 4.1.

Now, let $B=K$ and $C=(A+\delta K)^{-1}$ and consider the bounded operator $\tilde{T}=K(A+\delta K)^{-1}$ which is self-adjoint with respect to $[\cdot,\cdot]=((A+\delta K)^{-1}\cdot,\cdot)$. The self-adjoint operator $(A+\delta K)^{-1}$ defines the indefinite metric in the Pontryagin space $\tilde{\Pi}_{\tilde{\kappa}}$, where $\tilde{\kappa}=\dim(\mathcal{H}_{A+\delta K}^-)$. For a simple eigenvalue γ_0 of the shifted eigenvalue problem (2.14), we have

$$\forall f,g \in \mathcal{H}_{\gamma_0}: \quad ((A+\delta K)f,g) = (\gamma_0+\delta)(Kf,g).$$

If $\gamma_0 \geq 0$ or $\mathrm{Im}(\gamma_0) \neq 0$, the maximal nonpositive eigenspace of $\tilde{T}$ in $\tilde{\Pi}_{\tilde{\kappa}}$ associated with γ_0 coincides with the maximal nonpositive eigenspace of T in Π_κ. If $\gamma_0<0$, the maximal nonpositive eigenspace of $\tilde{T}$ in $\tilde{\Pi}_{\tilde{\kappa}}$ coincides with the maximal non-negative eigenspace of T in Π_κ. The same assertion holds in the case of a multiple eigenvalue γ_0. Therefore, the dimension of the maximal nonpositive eigenspace of $\tilde{T}$ in $\tilde{\Pi}_{\tilde{\kappa}}$ is $N_p^-+N_n^0+N_n^++N_{c^+}$ and equality (2.18) follows by Lemma 4.1. ■

Proof of Theorem 2: Let us introduce T and Π_κ according to the choice $B=(A+\delta K)^{-1}$ and $C=K$. Let Π be a non-negative invariant subspace in Π_κ, which is spanned by eigenvectors of the generalized eigenvalue problem (2.6) for N_p^- negative eigenvalues $\gamma<0$, N_p^0 zero eigenvalues $\gamma=0$, N_p^+ positive isolated eigenvalues $\gamma>0$, and N_{c^+} complex eigenvalues with $\mathrm{Im}(\gamma)>0$. Let us assume that $N_p^-+N_p^0+N_p^++N_{c^+}>N_A+N_K$ and derive a contradiction.

By Gram–Schmidt orthogonalization with respect to the inner product in the Hilbert space $\mathcal{H}$, if $N_p^-+N_p^0+N_p^++N_{c^+}>N_A+N_K$, then there exist a vector $h\in\Pi$ such that $(h,f)=0$ and $(h,g)=0$ for any $f\in\mathcal{H}_A^-\oplus\mathcal{H}_A^0\oplus\mathcal{H}_A^+$ and $g\in\mathcal{H}_K^-\oplus\mathcal{H}_K^+$. Therefore, $h\in\mathcal{H}_A^{\sigma_e(A)}\cap\mathcal{H}_K^{\sigma_e(K)}$ such that

$$(Ah,h) \geq \omega_+(h,h), \quad (Kh,h) \leq \omega_-^{-1}(h,h),$$

and

$$(Ah,h) \geq \omega_+\omega_-(Kh,h).$$

On the other hand, since $h \in \Pi$, then it can be represented by $h=\sum_{i=1}^{N_p^-+N_p^0+N_p^++N_{c^+}} \alpha_i h_i$, where $(h_1,h_2,\ldots,h_{N_p^-+N_p^0+N_p^++N_{c^+}})$ is a basis in Π associated with the eigenspaces of the generalized eigenvalue problem (2.6). By Lemmas 5.2, 5.3, and 5.5, we obtain

$$\begin{aligned}(Ah,h) &= \sum_{i,j} \alpha_i\bar{\alpha}_j(Ah_i,h_j) = \sum_{\gamma_i=\gamma_j<0} \alpha_i\bar{\alpha}_j(Ah_i,h_j) + \sum_{\gamma_i=\gamma_j=0} \alpha_i\bar{\alpha}_j(Ah_i,h_j) + \sum_{\gamma_i=\gamma_j>0} \alpha_i\bar{\alpha}_j(Ah_i,h_j) \\ &= \sum_{\gamma_j<0} |\alpha_j|^2(Ah_j,h_j) + \sum_{\gamma_j=0} |\alpha_j|^2(Ah_j,h_j) + \sum_{\gamma_j>0} |\alpha_j|^2(Ah_j,h_j) = \sum_{\gamma_j<0} \gamma_j|\alpha_j|^2(Kh_j,h_j) \\ &\quad + \sum_{\gamma_j>0} \gamma_j|\alpha_j|^2(Kh_j,h_j) < \omega_+\omega_- \sum_{\gamma_j>0} |\alpha_j|^2(Kh_j,h_j),\end{aligned}$$

where in the last inequality we have used the facts that $(Kh_j,h_j) \geq 0$ for any eigenvector $h_j \in \Pi$ and that $\gamma_j < \omega_+\omega_-$ for any isolated eigenvalue γ_j. On the other hand,

$$\begin{aligned}(Kh,h) &= \sum_{i,j} \alpha_i\bar{\alpha}_j(Kh_i,h_j) = \sum_{\gamma_j<0} |\alpha_j|^2(Kh_j,h_j) + \sum_{\gamma_j=0} |\alpha_j|^2(Kh_j,h_j) + \sum_{\gamma_j>0} |\alpha_j|^2(Kh_j,h_j) \\ &\geq \sum_{\gamma_j>0} |\alpha_j|^2(Kh_j,h_j).\end{aligned}$$

Therefore, $(Ah,h) < \omega_+\omega_-(Kh,h)$, which is a contradiction. As a result, inequality $N_p^-+N_p^0+N_p^+ +N_{c^+} \leq N_A+N_K$ is proved. ■

Remark 5.7: Isolated eigenvalues of infinite multiplicities are excluded by the counts of Theorems 1 and 2. Embedded eigenvalues of infinite multiplicity are possible but they may only correspond to finitely many Jordan blocks of finite length, according to Theorem 1.

VI. APPLICATIONS OF MAIN RESULTS

We describe here two applications of our main results related to recent studies of stability of the localized solutions in the NLS equations.

A. Solitons of the NLS equation

Consider a NLS equation in the form

$$i\psi_t = -\Delta\psi + F(|\psi|^2)\psi, \quad \Delta = \partial^2_{x_1x_1} + \ldots + \partial^2_{x_dx_d}, \tag{6.1}$$

where $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ and $\psi \in \mathbb{C}$. For a suitable nonlinear function $F(|\psi|^2)$, where F is C^∞ and $F(0)=0$, the NLS equation (6.1) possesses a solitary wave solution $\psi=\phi(x)e^{i\omega t}$, where $\omega>0$ and $\phi: \mathbb{R}^d \to \mathbb{R}$ is an exponentially decaying C^∞ function. See Ref. 21 for the existence and uniqueness of positive radial solutions of the stationary NLS equation

$$-\Delta\phi + \omega\phi + F(\phi^2)\phi = 0. \tag{6.2}$$

Linearization of the NLS equation (6.1) with the ansatz,

$$\psi = (\phi(x) + [u(x) + iw(x)]e^{\lambda t} + [\bar{u}(x) + i\bar{w}(x)]e^{\bar{\lambda} t})e^{i\omega t}, \tag{6.3}$$

where $\lambda \in \mathbb{C}$ and $(u,w) \in \mathbb{C}^2$, results in the linear eigenvalue problem (2.1), where $L_\pm$ are Schrödinger operators given by

$$L_+ = -\Delta + \omega + F(\phi^2) + 2\phi^2F'(\phi^2), \tag{6.4}$$

$$L_- = -\Delta + \omega + F(\phi^2). \tag{6.5}$$

We note that $L_\pm$ are unbounded operators and $\sigma_e(L_\pm) \geq \omega_\pm$ with $\omega_+ = \omega_- = \omega > 0$. The kernel of L_- includes at least one eigenvector $\phi(x)$ and the kernel of L_+ includes at least d eigenvectors $\partial_{x_j}\phi(x)$, $j=1,\ldots,d$. The Hilbert space is defined as $\mathcal{X}=L^2(\mathbb{R}^d,\mathbb{C})$ and the main assumptions (P1) and (P2) are satisfied due to the exponential decay of the functions $F(\phi^2)$ and $\phi^2 F'(\phi^2)$. Theorems 1 and 2 give precise count of eigenvalues of the stability problem (2.1) provided that the numbers $\dim(\mathcal{H}_K^-)$, $\dim(\mathcal{H}_{A+\delta K}^-)$, N_K, and N_A can be computed from the count of isolated eigenvalues of $A=\mathcal{P}L_+\mathcal{P}$ and $K=\mathcal{P}L_-^{-1}\mathcal{P}$, where $\mathcal{P}$ is the orthogonal projection to the complement of $\mathrm{Ker}(L_-)$. We illustrate these computations with two examples.

Example 1: Let $\phi(x)$ be the positive radial solution of the stationary NLS equation (6.2). By the spectral theory, $\mathrm{Ker}(L_-)=\mathrm{Span}\{\phi\}$, the subspace $\mathcal{H}_K^-$ is empty, and

$$\mathrm{Ker}(L_+) = \mathrm{Span}\{\partial_{x_1}\phi(x), \ldots, \partial_{x_d}\phi(x)\} \perp \mathrm{Ker}(L_-).$$

- It follows by equality (2.19) that $N_n^- = N_n^0 = N_n^+ = N_{c^+} = 0$. Therefore, the spectrum of the generalized eigenvalue problem (2.6) is real-valued and all eigenvalues γ are semisimple.
- Since $\mathrm{Ker}(L_+) \perp \mathrm{Ker}(L_-)$ and $\mathcal{H}_K^-$ is empty, eigenvectors of $\mathrm{Ker}(L_+)$ are in the positive subspace of K, so that $N_p^0 = z(L_+) = d$. By Proposition 2.3, these eigenvalues become positive eigenvalues of $A+\delta K$ for any $\delta > 0$ so that $\dim(\mathcal{H}_{A+\delta K}^-) = \dim(\mathcal{H}_A^-)$.
- It follows by equality (2.18) that $N_p^- = \dim(\mathcal{H}_{A+\delta K}^-) = \dim(\mathcal{H}_A^-)$. By Proposition 2.2, we have $\dim(\mathcal{H}_A^-) = n(L_+) - p_0 - z_0$, where p_0 and z_0 are the number of positive and zero values of a scalar function $M_0 = -(L_+^{-1}\phi, \phi)$. Since $L_+\partial_\omega \phi(x) = -\phi(x)$, we have

$$M_0 = \frac{1}{2}\frac{d}{d\omega}\|\phi\|_{L^2}^2$$

.

- It follows by inequality (2.21) that $N_p^- + N_p^0 + N_p^+ \leq \dim(\mathcal{H}_A^-) + \dim(\mathcal{H}_A^0) + \dim(\mathcal{H}_A^+) + \dim(\mathcal{H}_K^+)$. By Proposition 2.2 and the previous counts, we obtain $N_p^+ \leq p(L_+) + p(L_-) + p_0$.

Remark 6.1: If $n(L_+) = n \in \mathbb{N}$ and $(d/d\omega)\|\phi\|_{L^2}^2 > 0$, the count above gives $N_p^- = n(L_+) - 1$, which coincides with Theorem 2.1 in Ref. 11 (the case $n=1$ is known as the stability theorem in Ref. 9). If $n(L_+)=1$, $p(L_+)=p(L_-)=0$ and $(d/d\omega)\|\phi\|_{L^2}^2 < 0$, the count above gives $N_p^- = 1$, $N_p^0 = d$, and $N_p^+ = 0$, which is proved, with a direct variational method, in Proposition 2.1.2 (Ref. 28) and Proposition 9.2 (Ref. 19) for $d=1$ and in Lemma 1.8 (Ref. 32) for $d=3$, in the context of the supercritical power NLS equation with $F=|\psi|^{2q}$ and $q > 2/d$.

Remark 6.2: The stability of vector solitons in the coupled NLS equations, which generalize the scalar NLS equation (6.1), is defined by the same linear eigenvalue problem (2.1), where $L_\pm$ are matrix Schrödinger operators. The general results for nonground state solutions are obtained in Refs. 15 and 23 for $d=1$ and in Ref. 4 for $d=3$. Multiple and embedded eigenvalues were either excluded from analysis by an assumption[23,4] or were treated implicitly.[15] The present work generalizes these results with a precise count of multiple and embedded eigenvalues.

Example 2: Let the cubic NLS equation (6.1) with $F=|\psi|^2$ be discretized so that $\Delta \equiv \epsilon \Delta_{\mathrm{disc}}$, where Δ_{disc} is the second-order discrete Laplacian and ϵ is a small parameter. We note that Δ_{disc} is a bounded operator and $\sigma_c(-\Delta_{\mathrm{disc}}) \in [0,4d]$. The Hilbert space is defined as $\mathcal{X}=l^2(\mathbb{Z}^d,\mathbb{C})$. By the Lyapunov–Schmidt reduction method, the solution $\psi = \phi e^{i\omega t}$ with $\omega > 0$ and $\phi \in l^2(\mathbb{Z}^d)$ bifurcates from the limiting solution with N nonzero lattice nodes at $\epsilon=0$. It is proved in Ref. 24 for $d=1$ and in Ref. 25 for $d=2$ that $(d/d\omega)\|\phi\|_{l^2}^2 > 0$, $\mathrm{Ker}(L_+)=\varnothing$, and $\mathrm{Ker}(L_-)=\mathrm{Span}\{\phi\}$ for sufficiently small $\epsilon \neq 0$. It follows by equalities (2.18) and (2.19) that

$$N_p^- + N_n^+ + N_{c^+} = n(L_+) - 1,$$

$$N_n^- + N_n^+ + N_{c^+} = n(L_-),$$

where it is found in Refs. 24 and 25 that $n(L_+)=N$ and $n(L_-)\leqslant N-1$. Lyapunov–Schmidt reductions give, however, more precise information than the general count above since Corollary 3.5 in Ref. 24 for $d=1$ predicts that $N_n^+=n(L_-)$, $N_n^-=N_{c^+}=0$, and $N_p^-=N-1-n(L_-)$. [This precise count is valid only when small positive eigenvalues of L_- are simple. It is shown in Ref. 25 for $d=2$ that the case of multiple small positive eigenvalues of L_- leads to splitting of real eigenvalues N_p^- of the generalized eigenvalue problem (2.6) to complex eigenvalues N_{c^+} beyond the leading-order Lyapunov–Schmidt reduction.] Similarly, it follows by inequality (2.21) and the above count that

$$N_p^+ \leqslant 2n(L_-) + \dim(\mathcal{H}_A^+) + \dim(\mathcal{H}_K^+).$$

If the solution ϕ is a ground state, then $N=1$ and $n(L_-)=0$. In this case, the above inequality shows for a small $\epsilon>0$ that the number of edge bifurcations from the continuous spectrum of $K^{-1}A$ (given by N_p^+) is bounded from above by the number of edge bifurcations from the essential spectrum of A [given by $\dim(\mathcal{H}_A^+)$] and the numbers of edge bifurcations from the essential spectrum of K^{-1} [given by $\dim(\mathcal{H}_K^+)$]. The bound above becomes less useful if $N>1$ and $n(L_-)\neq 0$.

Remark 6.3: The Lyapunov–Schmidt reduction method was also used for continuous coupled NLS equations with and without external potentials. See Refs. 16 and 26 for various results on the count of unstable eigenvalues in parameter continuations of the NLS equations.

B. Vortices of the NLS equation

Consider the two-dimensional NLS equation (6.1) in polar coordinates (r,θ)

$$i\psi_t = -\Delta\psi + F(|\psi|^2)\psi, \quad \Delta = \partial_{rr}^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}^2, \tag{6.6}$$

where $r>0$ and $\theta\in[0,2\pi]$. Assume that the NLS equation (6.6) possesses a charge-m vortex solution $\psi=\phi(r)e^{im\theta+i\omega t}$, where $\omega>0$, $m\in\mathbb{N}$, and $\phi:\mathbb{R}_+\to\mathbb{R}$ is an exponentially decaying C^∞ function with $\phi(0)=0$. See Ref. 22 for existence results of charge-m vortices in the cubic-quintic NLS equation with $F=-|\psi|^2+|\psi|^4$. Linearization of the NLS equation (6.6) with the ansatz,

$$\psi = (\phi(r)e^{im\theta} + \varphi_+(r,\theta)e^{\lambda t} + \bar{\varphi}_-(r,\theta)e^{\bar{\lambda}t})e^{i\omega t}, \tag{6.7}$$

where $\lambda\in\mathbb{C}$ and $(\varphi_+,\varphi_-)\in\mathbb{C}^2$, results in the stability problem,

$$\sigma_3 H\boldsymbol{\varphi} = i\lambda\boldsymbol{\varphi}, \tag{6.8}$$

where $\boldsymbol{\varphi}=(\varphi_+,\varphi_-)^T$, $\sigma_3=\mathrm{diag}(1,-1)$, and

$$H = \begin{pmatrix} -\Delta + \omega + F(\phi^2) + \phi^2 F'(\phi^2) & \phi^2 F'(\phi^2)e^{2im\theta} \\ \phi^2 F'(\phi^2)e^{-2im\theta} & -\Delta + \omega + F(\phi^2) + \phi^2 F'(\phi^2) \end{pmatrix}.$$

Expand $\boldsymbol{\varphi}(r,\theta)$ in the Fourier series

$$\boldsymbol{\varphi} = \sum_{n\in\mathbb{Z}} \boldsymbol{\varphi}^{(n)}(r)e^{in\theta}$$

and reduce the problem to a sequence of spectral problems for ordinary differential equations,

$$\sigma_3 H_n \boldsymbol{\varphi}_n = i\lambda\boldsymbol{\varphi}_n, \quad n\in\mathbb{Z}, \tag{6.9}$$

where $\boldsymbol{\varphi}_n=(\varphi_+^{(n+m)},\varphi_-^{(n-m)})^T$, and

$$H_n = \begin{pmatrix} -\partial_{rr}^2 - \frac{1}{r}\partial_r + \frac{(n+m)^2}{r^2} + \omega + F(\phi^2) + \phi^2 F'(\phi^2) & \phi^2 F'(\phi^2) \\ \phi^2 F'(\phi^2) & -\partial_{rr}^2 - \frac{1}{r}\partial_r + \frac{(n-m)^2}{r^2} + \omega + F(\phi^2) + \phi^2 F'(\phi^2) \end{pmatrix}.$$

When $n=0$, the stability problem (6.9) transforms into the linear eigenvalue problem (2.1), where $L_\pm$ is given by (6.4) and (6.5) with $\Delta=\partial_{rr}^2+(1/r)\partial_r-m^2/r^2$ and (u,w) are given by $u=\varphi_+^{(m)}+\varphi_-^{(-m)}$ and $w=-i(\varphi_+^{(m)}-\varphi_-^{(-m)})$. When $n\in\mathbb{N}$, the stability problem (6.9) transforms into the linear eigenvalue problem (2.1) with $L_+=H_n$ and $L_-=\sigma_3 H_n \sigma_3$, where

$$L_+ = L_- + 2\phi^2 F'(\phi^2)\sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and (u,w) are given by $u=\boldsymbol{\varphi}_n$ and $w=-i\sigma_3\boldsymbol{\varphi}_n$. When $-n\in\mathbb{N}$, the stability problem (6.9) admits a transformation with $H_{-n}=\sigma_1 H_n \sigma_1$ and $\sigma_3\sigma_1=-\sigma_1\sigma_3$ into the stability problem with $n\in\mathbb{N}$. Let us introduce the weighted inner product for functions on $r\geqslant 0$,

$$(f,g)_r = \int_0^\infty f(r)g(r)rdr.$$

In all cases $n=0$, $n\in\mathbb{N}$ and $-n\in\mathbb{N}$, $L_\pm$ are unbounded self-adjoint differential operators and $\sigma_e(L_\pm)=[\omega_\pm,\infty)$ with $\omega_+=\omega_-=\omega>0$. The kernel of the linearized operators includes at least three eigenvectors,

$$n = \pm 1: \quad \boldsymbol{\phi}_{\pm 1} = \phi'(r)\mathbf{1} \mp \frac{m}{r}\phi(r)\sigma_3\mathbf{1}, \quad n=0: \quad \boldsymbol{\phi}_0 = \phi(r)\sigma_3\mathbf{1},$$

where $\mathbf{1}=(1,1)^T$. The Hilbert space is defined as $\mathcal{X}=L_r^2(\mathbb{R}_+,\mathbb{C})$ for $n=0$ and $\mathcal{X}=L_r^2(\mathbb{R}_+,\mathbb{C}^2)$ for $\pm n\in\mathbb{N}$. In all cases, the main assumptions (P1) and (P2) are satisfied due to exponential decay of the functions $F(\phi^2)$ and $\phi^2 F'(\phi^2)$.

The case $n=0$ is similar to the case of solitons in Sec. VI A. We shall hence consider adjustments in the count of eigenvalues in the case $\pm n\in\mathbb{N}$, when the stability problem (6.9) is rewritten in the form,

$$\begin{cases} \sigma_3 H_n \boldsymbol{\varphi}_n = i\lambda\boldsymbol{\varphi}_n \\ \sigma_3 H_{-n}\boldsymbol{\varphi}_{-n} = i\lambda\boldsymbol{\varphi}_{-n} \end{cases} \quad n\in\mathbb{N}. \tag{6.10}$$

Let $L_+=\mathrm{diag}(H_n,H_{-n})$ and $L_-=\mathrm{diag}(\sigma_3 H_n\sigma_3,\sigma_3 H_{-n}\sigma_3)$.

Lemma 6.4: Let λ *be an eigenvalue of the stability problem (6.10) with the eigenvector* $(\boldsymbol{\varphi}_n,\mathbf{0})$. *Then there exists another eigenvalue* $-\lambda$ *with the linearly independent eigenvector* $(\mathbf{0},\sigma_1\boldsymbol{\varphi}_n)$. *If* $\mathrm{Re}(\lambda)>0$, *there exist two more eigenvalues* $\bar{\lambda}$, $-\bar{\lambda}$ *with the linearly independent eigenvectors* $(\mathbf{0},\sigma_1\bar{\boldsymbol{\varphi}}_n)$, $(\bar{\boldsymbol{\varphi}}_n,\mathbf{0})$.

Proof: We note that $\sigma_1\sigma_3=-\sigma_3\sigma_1$ and $\sigma_1^2=\sigma_3^2=\sigma_0$, where $\sigma_0=\mathrm{diag}(1,1)$. Therefore, each eigenvalue λ of H_n with the eigenvector $\boldsymbol{\varphi}_n$ generates eigenvalue $-\lambda$ of H_{-n} with the eigenvector $\boldsymbol{\varphi}_{-n}=\sigma_1\boldsymbol{\varphi}_n$. When $\mathrm{Re}(\lambda)\neq 0$, each eigenvalue λ of H_n generates also eigenvalue $-\bar{\lambda}$ of H_n with the eigenvector $\bar{\boldsymbol{\varphi}}_n$ and eigenvalue $\bar{\lambda}$ of H_{-n} with the eigenvector $\boldsymbol{\varphi}_{-n}=\sigma_1\bar{\boldsymbol{\varphi}}_n$. ■

Theorem 6: Let N_{real} *be the number of real eigenvalues in the stability problem (6.10) with* $\mathrm{Re}(\lambda)>0$, N_{comp} *be the number of complex eigenvalues with* $\mathrm{Re}(\lambda)>0$ *and* $\mathrm{Im}(\lambda)>0$, N_{imag}^- *be the number of purely imaginary eigenvalues with* $\mathrm{Im}(\lambda)>0$ *and* $(\boldsymbol{\varphi}_n,H_n\boldsymbol{\varphi}_n)\leqslant 0$, *and* N_{zero}^- *be the algebraic multiplicity of the zero eigenvalue of* $\sigma_3 H_n\boldsymbol{\varphi}_n=i\lambda\boldsymbol{\varphi}_n$ *with* $(\boldsymbol{\varphi}_n,H_n\boldsymbol{\varphi}_n)\leqslant 0$. *Then,* N_{real} *is even and*

$$\tfrac{1}{2}N_{\mathrm{real}} + N_{\mathrm{comp}} = n(H_n) - N_{\mathrm{zero}}^- - N_{\mathrm{imag}}^-. \tag{6.11}$$

Proof: By Lemma 6.4, the multiplicity of N_{real} is even in the stability problem (6.10) and a pair of real eigenvalues of $\sigma_3 H_n \boldsymbol{\varphi}_n = i\lambda \boldsymbol{\varphi}_n$ corresponds to two linearly independent eigenvectors $\boldsymbol{\varphi}_n$ and $\bar{\boldsymbol{\varphi}}_n$. Because $(H_n\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_n)$ is real-valued and hence zero for $\lambda \in \mathbb{R}$, we have

$$(H_n(\boldsymbol{\varphi}_n \pm \bar{\boldsymbol{\varphi}}_n),(\boldsymbol{\varphi}_n \pm \bar{\boldsymbol{\varphi}}_n)) = \pm 2\,\mathrm{Re}(H_n\boldsymbol{\varphi}_n,\bar{\boldsymbol{\varphi}}_n).$$

By counting multiplicities of the real negative and complex eigenvalues of the generalized eigenvalue problem (2.6) associated with the stability problem (6.10), we have $N_n^- = N_p^- = N_{\text{real}}$ and $N_{c^+} = 2N_{\text{comp}}$. By Lemma 6.4, a pair of purely imaginary and zero eigenvalues of the stability problem (6.10) corresponds to two linearly independent eigenvectors $(\boldsymbol{\varphi}_n, \mathbf{0})$ and $(\mathbf{0}, \boldsymbol{\varphi}_{-n})$, where $\boldsymbol{\varphi}_{-n} = \sigma_1 \boldsymbol{\varphi}_n$ and $(H_{-n}\boldsymbol{\varphi}_{-n}, \boldsymbol{\varphi}_{-n}) = (H_n\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_n)$. By counting multiplicities of the real positive and zero eigenvalues of the generalized eigenvalue problem (2.6) associated with the stability problem (6.10), we have $N_n^0 = 2N_{\text{zero}}^-$ and $N_n^+ = 2N_{\text{imag}}^-$. Since the spectra of H_n, $\sigma_1 H_n \sigma_1$, and $\sigma_3 H_n \sigma_3$ coincide, we have $n(L_-) = 2n(H_n)$. As a result, equality (6.11) follows by equality (2.19) of Theorem 1. ■

Corollary 6.5: Let $A = \mathcal{P}L_+\mathcal{P}$ and $K = \mathcal{P}L_-^{-1}\mathcal{P}$, where $\mathcal{P}$ is an orthogonal projection to the complement of $\mathrm{Ker}(L_-) = \mathrm{Span}\{v_1, \ldots, v_n\}$. *The number of small negative eigenvalues of $A + \delta K$ for sufficiently small $\delta > 0$ equals the number of non-negative eigenvalues of $M(\mu)$ as $\mu \uparrow 0$, where* $M_{ij}(\mu) = ((\mu - L_+)^{-1} v_i, v_j)$.

Proof: The same count (6.11) follows by equality (2.18) of Theorem 1 if and only if $\dim(\mathcal{H}^-_{A+\delta K}) = \dim(\mathcal{H}^-_K) = n(L_-)$. Since the zero eigenvalue of A is isolated from the essential spectrum and $n(L_+) = n(L_-)$, the number of small negative eigenvalues of $A + \delta K$ for sufficiently small $\delta \neq 0$ must be equal to

$$\dim(\mathcal{H}^-_{A+\delta K}) - \dim(\mathcal{H}^-_A) = n(L_+) - \dim(\mathcal{H}^-_A) = p_0 + z_0,$$

where we have used equality (2.12) of Proposition 2.2. ■

Example 3: Let $\phi(r)$ be the fundamental charge-m vortex solution such that $\phi(r) > 0$ for $r > 0$ and $\phi(0) = 0$. By spectral theory, $\mathrm{Ker}(H_0) = \mathrm{Span}\{\boldsymbol{\phi}_0\}$ and the analysis for $n = 0$ becomes similar to Example 1. In the case $n \in \mathbb{N}$, let us assume that $\mathrm{Ker}(H_1) = \mathrm{Span}\{\boldsymbol{\phi}_1\}$ and $\mathrm{Ker}(H_n) = \varnothing$ for $n \geqslant 2$.

- By direct computation, we obtain $(\sigma_3 H_1 \sigma_3)^{-1}\boldsymbol{\phi}_1 = -\frac{1}{2} r\phi(r)\mathbf{1}$ and

$$((\sigma_3 H_1 \sigma_3)^{-1}\boldsymbol{\phi}_1, \boldsymbol{\phi}_1) = \int_0^\infty r\phi^2(r)dr > 0.$$

  Since $(\sigma_3\boldsymbol{\phi}_1, \boldsymbol{\phi}_1) = 0$ and $\mathrm{Ker}(\sigma_3 H_1 \sigma_3) = \{\sigma_3\boldsymbol{\phi}_1\}$, then $\boldsymbol{\phi}_1 \perp \mathrm{Ker}(\sigma_3 H_1 \sigma_3)$. By Propositions 2.2 and 2.3, we have $N_n^0 = 0$ for $n = 1$ ($N_n^0 = 0$ holds also for $n \geqslant 2$) so that $p_0 = z_0 = 0$ for all $n \in \mathbb{N}$. Corollary 6.5 is hence confirmed.
- By Theorem 6, we have

$$N_{\text{real}} + 2N_{\text{comp}} = 2n(H_n) - 2N_{\text{imag}}^-, \tag{6.12}$$

  where N_{imag}^- gives the total number of eigenvalues in the stability problem (6.10) with $\mathrm{Re}(\lambda) = 0$, $\mathrm{Im}(\lambda) > 0$, and $(H_n\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_n) < 0$, while $N_{\text{zero}}^- = N_n^0 = 0$.

Remark 6.6: Stability of vortices was considered numerically in Ref. 22, where Lemma 6.4 was also obtained. The closure relation (6.12) was also discussed in Ref. 15 in a more general context. Vortices in the discretized scalar NLS equation were considered with the Lyapunov–Schmidt reduction method in Ref. 25. Although the reduced eigenvalue problems were found in a more complicated form compared to the reduced eigenvalue problem for solitons, equality (6.12) was confirmed for all vortex configurations considered in Ref. 25.

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