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The derivative NLS equation: global existence with solitons

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Abstract. We prove the global existence result for the derivative NLS equation in the case when the initial datum includes a finite number of solitons. This is achieved by an application of the Bäcklund transformation that removes a finite number of zeros of the scattering coefficient. By means of this transformation, the Riemann–Hilbert problem for meromorphic functions can be formulated as the one for analytic functions, the solvability of which was obtained recently. A major difficulty in the proof is to show invertibility of the Bäcklund transformation acting on weighted Sobolev spaces.

1. Introduction

We consider the Cauchy problem for the derivative nonlinear Schrödinger (DNLS) equation

$$(1.1)\qquad \left\{ \begin{array}{l} iu_t + u_{xx} + i(|u|^2u)_x = 0, \quad t \in \mathbb{R}, \\ u|_{t=0} = u_0, \end{array} \right.$$

where the subscripts denote partial derivatives and u_0 is defined in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. Here $H^m(\mathbb{R})$ denotes the Sobolev space of distributions with square integrable derivatives up to the order m, $H^{1,1}(\mathbb{R})$ denotes the weighted Sobolev space given by

$$H^{1,1}(\mathbb{R}) = \left\{ u \in L^{2,1}(\mathbb{R}), \quad \partial_x u \in L^{2,1}(\mathbb{R}) \right\},$$

and the weighted space $L^{2,1}(\mathbb{R})$ is equipped with the norm

$$\|u\|_{L^{2,1}} = \left(\int_{\mathbb{R}} \langle x \rangle^2 |u|^2 dx \right)^{1/2}, \quad \langle x \rangle := (1+x^2)^{1/2}.$$

Global well-posedness of the Cauchy problem (1.1) for u_0 in $H^2(\mathbb{R})$ was shown for initial datum with small $H^1(\mathbb{R})$ norm in the pioneer works of Tsutsumi & Fukuda [**25, 26**]. Hayashi [**11**] and Hayashi & Ozawa [**12**] extended the global well-posedness for u_0 in $H^1(\mathbb{R})$ with small $L^2(\mathbb{R})$ norm. The critical $L^2(\mathbb{R})$ norm corresponds to the stationary solitary waves of the DNLS equation. The question

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of whether global solutions for initial datum with large $L^2(\mathbb{R})$ norm exist in the Cauchy problem (1.1) was addressed very recently by using different analytical and numerical methods.

Wu [**27, 28**] combined the mass, momentum and energy conservation with variational arguments and pushed up the upper bound on the $L^2(\mathbb{R})$-norm of the initial datum required for existence of global solutions. By adding a new result on orbital stability of algebraically decaying solitons [**15**], this upper bound is pushed up even higher, but still within the range of the $L^2(\mathbb{R})$ norm of the travelling solitary waves of the DNLS equation.

Orbital stability of one-soliton solutions was shown long ago by Guo & Wu [**10**] and Colin & Ohta [**2**]. More recently, the orbital stability of multi-soliton solutions was obtained in the energy space, under suitable assumptions on the speeds and frequencies of the single solitons [**16**]. Variational characterization of the DNLS solitary waves and further improvements of the global existence near a single solitary wave were developed in [**20**]. Orbital stability of a sum of two solitary waves was obtained from the variational characterization in [**21**] (see also [**7, 24**] for similar analysis of the generalized DNLS equation).

Numerical simulations of the DNLS equation (1.1) indicate no blow-up phenomenon for initial data in $H^1(\mathbb{R})$ with any large $L^2(\mathbb{R})$ norm [**18, 19**]. The same conclusion is confirmed by means of the asymptotic analysis of the self-similar blow-up solutions [**3**].

Since the DNLS equation (1.1) is formally solvable with the inverse scattering transform method [**14**], one can look at other analytical tools to deal with the same question. Lipschitz continuity of the direct and inverse scattering transform in appropriate function spaces was established very recently [**17, 22**] and this result suggests global well-posedness of the Cauchy problem (1.1) without sharp constraints on the $L^2(\mathbb{R})$ norm of the initial datum. The solvability of the inverse scattering transform was achieved by using the pioneer results of Deift & Zhou [**6**] and Zhou [**30**] but extended from the Zakharov–Shabat (ZS) to the Kaup–Newell (KN) spectral problem. Simplifying assumptions were made in [**17, 22**] to exclude eigenvalues and resonances in the KN spectral problem. Excluding resonances is a natural condition to define so-called generic initial data u_0. On the other hand, eigenvalues are usually excluded if the initial datum satisfies the small-norm constraint, and it is not obvious if there exist the initial datum u_0 with a large $L^2(\mathbb{R})$ norm that yield no eigenvalues in the KN spectral problem.

The goal of the present paper is to extend the result from [**22**] to the case of a finite number of eigenvalues in the KN spectral problem. Working with the Bäcklund transformation, similarly to the work [**4, 5**] for the ZS spectral problem, we are able to apply the inverse scattering transform technique to the initial datum with a finite number of solitons. By using the solvability result from [**22**] and the invertibility of the Bäcklund transformation proved here, we are able to extend the global well-posedness result for the Cauchy problem (1.1) to arbitrarily large initial data in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.

The main algebraic tool used in this paper is definitely not new. Imai [**13**] used the multi-fold Bäcklund transformation to obtain multi-solitons and quasi-periodic solutions of the DNLS equation. Steudel [**23**] gave a very nice overview of the construction of the multi-solitons with the same technique. More recent treatments of the Bäcklund transformations for the DNLS equation can be found in further works [**9, 29**]. What makes this present paper new is the way how the Bäcklund transformation can be applied in the rigorous treatment of the inverse scattering transform and the global well-posedness problem.

The DNLS equation appears to be a compatibility condition for C^2 solutions to the KN spectral problem

$$\partial_x \psi = \left[-i\lambda^2 \sigma_3 + \lambda Q(u)\right] \psi \tag{1.2}$$

and the time-evolution problem

$$\partial_t \psi = \left[-2i\lambda^4 \sigma_3 + 2\lambda^3 Q(u) + i\lambda^2 |u|^2 \sigma_3 - \lambda |u|^2 Q(u) + i\lambda \sigma_3 Q(u_x)\right] \psi, \tag{1.3}$$

where $\lambda \in \mathbb{C}$ is the (t,x)-independent spectral parameter, $\psi(t,x)$ is the $\mathbb{C}^2$ vector for the wave function, and $Q(u)$ is the (t,x)-dependent matrix potential given by

$$Q(u) = \begin{bmatrix} 0 & u \\ -\overline{u} & 0 \end{bmatrix}. \tag{1.4}$$

The Pauli matrices that include σ_3 are given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.5}$$

A long but standard computation shows that the compatibility condition $\partial_t \partial_x \psi = \partial_x \partial_t \psi$ for C^2 solutions of system (1.2) and (1.3) is equivalent to the DNLS equation $iu_t + u_{xx} + i(|u|^2 u)_x = 0$ for classical solutions u.

The following theorem presents the main result.

Theorem 1. *For every $u_0 \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ such that the KN spectral problem (1.2) admits no resonances in the sense of Definition 1 and only simple eigenvalues in the sense of Definition 2, there exists a unique global solution $u(t,\cdot) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ of the Cauchy problem* (1.1) *for every $t \in \mathbb{R}$.*

The organization of the paper is as follows. Section 2 contains the review of Jost functions and scattering coefficients from [**22**]. Section 3 presents the Bäcklund transformation for the KN spectral problem in the form suitable for our analysis. Section 4 adds the time evolution for the Bäcklund transformation according to the DNLS equation. Section 5 gives an example of the Bäcklund transformation connecting the one-soliton and zero-soliton solutions. Section 6 completes the proof of Theorem 1. Appendix A lists useful properties of operators used in the definition of the Bäcklund transformation. Appendix B gives a technical result on the regularity of Jost functions for the KN spectral problem.

2. Review of the direct scattering transform

We introduce Jost functions for the KN spectral problem (1.2) under some conditions on the potential u. In this section, we freeze the time variable t and drop it from the argument list of the dependent functions. The following two propositions were proved in the previous work (see Lemma 1, Corollary 2, and Corollary 3 in [**22**]). Here $e_{1,2}$ are standard basis vectors in $\mathbb{R}^2$.

Proposition 1. *Let $u \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\partial_x u \in L^1(\mathbb{R})$. For every $\lambda \in \mathbb{R} \cup i\mathbb{R}$, there exist unique solutions $\varphi_\pm(x;\lambda)e^{-i\lambda^2 x}$ and $\phi_\pm(x;\lambda)e^{i\lambda^2 x}$ to the KN spectral problem* (1.2) *with $\varphi_\pm(\cdot;\lambda) \in L^\infty(\mathbb{R})$ and $\phi_\pm(\cdot;\lambda) \in L^\infty(\mathbb{R})$ such that*

$$\left.\begin{array}{l} \varphi_\pm(x;\lambda) \to e_1, \\ \phi_\pm(x;\lambda) \to e_2, \end{array}\right\} \quad \text{as} \quad x \to \pm\infty. \tag{2.1}$$

Proposition 2. *Under the same assumption on u as in Proposition 1, for every $x \in \mathbb{R}$, the Jost functions $\varphi_-(x;\cdot)$ and $\phi_+(x;\cdot)$ are analytic in the first and third quadrant of the λ plane (where $\mathrm{Im}(\lambda^2) > 0$), whereas the functions $\varphi_+(x;\cdot)$ and $\phi_-(x;\cdot)$ are analytic in the second and fourth quadrant of the λ plane (where $\mathrm{Im}(\lambda^2) < 0$). Furthermore, for every λ with $\mathrm{Im}(\lambda^2) > 0$ and for all u satisfying $\|u\|_{L^1\cap L^\infty} + \|\partial_x u\|_{L^1} \leq M$ there exists a constant C_M which does not depend on u, such that*

$$\|\varphi_-(\cdot;\lambda)\|_{L^\infty} + \|\phi_+(\cdot;\lambda)\|_{L^\infty} \leq C_M. \tag{2.2}$$

The set of Jost functions $[\varphi_-(x;\lambda), \psi_-(x;\lambda)]e^{-i\lambda^2\sigma_3 x}$ at the left infinity is linearly dependent from the set of Jost functions $[\varphi_+(x;\lambda), \psi_+(x;\lambda)]e^{-i\lambda^2\sigma_3 x}$ at the right infinity. Therefore, for every $\lambda \in \mathbb{R} \cup i\mathbb{R}$ there exists the transfer matrix $S(\lambda)$ that connects the two sets as follows:

$$[\varphi_-(x;\lambda), \phi_-(x;\lambda)]e^{-i\lambda^2\sigma_3 x} = [\varphi_+(x;\lambda), \phi_+(x;\lambda)]e^{-i\lambda^2\sigma_3 x} S(\lambda), \tag{2.3}$$

where $x \in \mathbb{R}$ is arbitrary. Thanks to the symmetry relations

$$\phi_\pm(x;\lambda) = \sigma_1\sigma_3\overline{\varphi_\pm(x;\overline{\lambda})}, \tag{2.4}$$

the transfer matrix S has the structure

$$S(\lambda) = \begin{bmatrix} a(\lambda) & -\overline{b(\overline{\lambda})} \\ b(\lambda) & \overline{a(\overline{\lambda})} \end{bmatrix}, \tag{2.5}$$

defined by the two scattering coefficients $a(\lambda)$ and $b(\lambda)$. Since the determinant of the transfer matrix $S(\lambda)$ is equal to unity for every $\lambda \in \mathbb{R}\cup i\mathbb{R}$, we have the following relation between $a(\lambda)$ and $b(\lambda)$:

$$a(\lambda)\overline{a(\overline{\lambda})} + b(\lambda)\overline{b(\overline{\lambda})} = 1, \quad \lambda \in \mathbb{R} \cup i\mathbb{R}. \tag{2.6}$$

Furthermore, scattering coefficients $a(\lambda)$ and $b(\lambda)$ can be written in terms of Jost functions by using the Wronskian determinant $W(\eta,\xi) = \eta_1\xi_2 - \xi_1\eta_2$ defined for $\eta, \xi \in \mathbb{C}^2$:

$$a(\lambda) = W(\varphi_-(x;\lambda)e^{-i\lambda^2 x}, \phi_+(x;\lambda)e^{+i\lambda^2 x}), \tag{2.7a}$$

$$b(\lambda) = W(\varphi_+(x;\lambda)e^{-i\lambda^2 x}, \varphi_-(x;\lambda)e^{-i\lambda^2 x}). \tag{2.7b}$$

The coefficient $b(\lambda)$ is expressed by the Jost functions whose analytic domains in the λ plane are disjoint. As a result, $b(\lambda)$ cannot be continued into the complex plane of λ. On the other hand, $a(\lambda)$ can be continued analytically into the complex plane of λ, according to the following result (see Lemma 4 in [**22**]).

Proposition 3. *Under the same assumption on u as in Proposition 1, the scattering coefficient $a(\lambda)$ can be continued analytically into $\{\lambda \in \mathbb{C} : Im(\lambda^2) > 0\}$ with the limit*

$$a_\infty = \lim_{|\lambda|\to\infty} a(\lambda) = e^{\frac{1}{2i}\|u\|^2_{L^2}}.$$

Similarly, $\overline{a(\overline{\lambda})}$ is continued analytically into $\{\lambda \in \mathbb{C} : Im(\lambda^2) < 0\}$ with the limit

$$\overline{a}_\infty = \lim_{|\lambda|\to\infty} \overline{a(\overline{\lambda})} = e^{-\frac{1}{2i}\|u\|^2_{L^2}}.$$

Since $a_\infty \neq 0$ if $u \in L^2(\mathbb{R})$, the following corollary holds by a theorem of complex analysis on zeros of analytic functions.

Corollary 1. *Under the same assumption on u as in Proposition 1, the scattering coefficient $a(\lambda)$ has at most finite number of zeros in $\{\lambda \in \mathbb{C} : Im(\lambda^2) > 0\}$.*

If a potential u is sufficiently small, then one can easily deduce that $a(\lambda)$ has no zeros in the domain of its analyticity. As is explained in Remark 5 in [**22**], $a(\lambda) \neq 0$ for every $\mathrm{Im}(\lambda^2) \geq 0$ if

$$\|u\|_{L^2}^2 + \sqrt{\|u\|_{L^1}(2\|\partial_x u\|_{L^1} + \|u\|_{L^3}^3)} < 1.$$

However, for sufficiently large u, the spectral coefficient $a(\lambda)$ may have zeros for some $\mathrm{Im}(\lambda^2) \geq 0$. We distinguish two cases, according to the following definitions.

Definition 1. *If $a(\lambda_0) = 0$ for $\lambda_0 \in \mathbb{R} \cup i\mathbb{R}$, we say that λ_0 is a resonance of the spectral problem (1.2).*

Definition 2. *If $a(\lambda_0) = 0$ for $\lambda_0 \in \mathbb{C}_I := \{\mathrm{Re}(\lambda) > 0, \quad \mathrm{Im}(\lambda) > 0\}$, we say that λ_0 is an eigenvalue of the spectral problem (1.2) in $\mathbb{C}_I$. An eigenvalue is called simple if $a'(\lambda_0) \neq 0$.*

Remark 1. *By the symmetry of the KN spectral problem (1.2), if $a(\lambda_0) = 0$ for $\lambda_0 \in \mathbb{C}_I$, then $a(-\lambda_0) = 0$.*

Remark 2. *If $u \in H^{1,1}(\mathbb{R})$, then the assumption of Propositions 1, 2, and 3 are satisfied so that $u \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\partial_x u \in L^1(\mathbb{R})$. To enable the inverse scattering transform, we will work with u in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.*

The main assumption of Theorem 1 excludes resonances but includes simple eigenvalues. Thanks to Corollary 1, the number of eigenvalues is finite under the assumptions in Proposition 1. Therefore, the initial datum u_0 of the Cauchy problem (1.1) in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ may include at most finitely many solitons.

Let Z_N be a subset of $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ such that $a(\lambda)$ has N simple zeros in the first quadrant $\mathbb{C}_I$. Zeros of $a(\lambda)$ are assumed to be simple in order to simplify our presentation. This is not a restricted assumption because the union of $\{Z_N\}_{N\in\mathbb{N}}$ is dense in space $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ thanks to the classical result of Beals & Coifman [**1**]. Indeed, as is known from [**14**] (see also [**22**]), the Kaup-Newell spectral system (1.2) can be transformed to the Zakharov–Shabat spectral system by the transformation

$$\widetilde{\psi}(x) = \left[\begin{array}{cc} e^{\frac{1}{2i}\int_x^\infty |u(y)|^2 dy} & 0 \\ 0 & e^{-\frac{1}{2i}\int_x^\infty |u(y)|^2 dy} \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ -\overline{u}(x) & 2i\lambda \end{array}\right] \psi(x),$$

where $\widetilde{\psi}$ satisfies

$$\partial_x \widetilde{\psi} = \left[-i\lambda^2 \sigma_3 + \widetilde{Q}(u)\right] \widetilde{\psi}, \tag{2.8}$$

with

$$\widetilde{Q}(u) = \frac{1}{2i} \left[\begin{array}{cc} 0 & u e^{-i\int_x^\infty |u(y)|^2 dy} \\ -(2i\overline{u}_x + \overline{u}|u|^2) e^{i\int_x^\infty |u(y)|^2 dy} & 0 \end{array}\right].$$

Eigenvalues of the spectral problems (1.2) and (2.8) coincide and the potential $\widetilde{Q}(u)$ is now defined in $L^1(\mathbb{R})$ under the assumption that $u \in H^{1,1}(\mathbb{R})$. Proposition 2.30 in [**1**] yields the following result.

Proposition 4. *The subset $Z := \bigcup_{N=1}^\infty Z_N$ is dense in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.*

Let $u \in Z_N$ and $a(\lambda)$ vanishes at some $\lambda_j \in \mathbb{C}_I$, $j \in \{1, 2, ..., N\}$. It follows from the definition of $a(\lambda)$ in (2.7a) that the Jost functions $\varphi_-(x; \lambda_j)e^{-i\lambda_j^2 x}$ and $\phi_+(x; \lambda_j)e^{i\lambda_j^2 x}$ are linearly dependent. This implies that there is a norming coefficient $\gamma_j \in \mathbb{C}$ such that

$$\varphi_-(x; \lambda_j)e^{-i\lambda_j^2 x} = \gamma_j \phi_+(x; \lambda_j)e^{i\lambda_j^2 x}, \quad x \in \mathbb{R}. \tag{2.9}$$

Since $\varphi_-(x; \lambda_j)e^{-i\lambda_j^2 x}$ and $\phi_+(x; \lambda_j)e^{i\lambda_j^2 x}$ are uniquely determined by Proposition 1, the norming coefficient γ_j is determined uniquely.

REMARK 3. *Because $\lambda_j \in \mathbb{C}_I$, the Jost functions $\varphi_-(x; \lambda_j)e^{-i\lambda_j^2 x}$ and $\phi_+(x; \lambda_j)e^{i\lambda_j^2 x}$ in (2.9) decay to zero as $|x| \to \infty$ exponentially fast. Hence, they represent an eigenvector of the spectral problem (1.2) for the simple eigenvalue λ_j.*

3. Bäcklund transformation

In order to define the Bäcklund transformation in the simplest form, let us introduce the bilinear form d_λ that acts on $\mathbb{C}^2$ for a fixed $\lambda \in \mathbb{C}$. If $\eta = (\eta_1, \eta_2)^t$ and $\xi = (\xi_1, \xi_1)^t$ are in $\mathbb{C}^2$, then

$$d_\lambda(\eta, \xi) := \lambda \eta_1 \overline{\xi}_1 + \overline{\lambda} \eta_2 \overline{\xi}_2. \tag{3.1}$$

We further introduce

$$G_\lambda(\eta) := \frac{d_{\overline{\lambda}}(\eta, \eta)}{d_\lambda(\eta, \eta)} \quad \text{and} \quad S_\lambda(\eta) := 2i(\lambda^2 - \overline{\lambda}^2)\frac{\eta_1 \overline{\eta}_2}{d_\lambda(\eta, \eta)}. \tag{3.2}$$

Useful algebraic properties of d_λ, G_λ, and S_λ are reviewed in Appendix A.

The Bäcklund transformation can be expressed by using operators G_λ and S_λ. Let us first give an informal definition of the Bäcklund transformation and then make it precise.

Suppose that u is a smooth solution of the DNLS equation and η is a smooth nonzero solution of the KN spectral problem (1.2) associated with the potential u for a fixed $\lambda \in \mathbb{C} \backslash \{0\}$. The Bäcklund transformation $\mathbf{B}_\lambda(\eta)$ is given as

$$\mathbf{B}_\lambda(\eta)u := G_\lambda(\eta)[-G_\lambda(\eta)u + S_\lambda(\eta)]. \tag{3.3}$$

We intend to show that $\mathbf{B}_\lambda(\eta)u$ is another smooth solution of the DNLS equation. Note that

$$G_\lambda(\eta) = -1 \quad \text{and} \quad S_\lambda(\eta) = 0 \quad \text{if} \quad \lambda \in \mathbb{R} \cup i\mathbb{R},$$

which implies $\mathbf{B}_\lambda(\eta)u = -u$ in this case. Therefore, it makes sense to use the Bäcklund transformation (3.3) for a value of λ outside the continuous spectrum, e.g., for $\lambda \in \mathbb{C}_I$.

The transformation (3.3) has been derived by a constructive algorithm in [**29**], where it is called the 2-fold Darboux transformation. It must be noted that, since η depends on u via the KN spectral problem (1.2), the transformation (3.3) is nonlinear in u. The function $\mathbf{B}_\lambda(\eta)u$ depends on variables t and x, whereas the value of λ is fixed. The quantities u, η, as well as $\lambda \in \mathbb{C} \backslash \{0\}$ affect $\mathbf{B}_\lambda(\eta)u$, e.g., depending on η and λ, the transformation can be used to obtain different families of solutions from the same solution u.

Let $u(t, \cdot) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ be a local solution of the Cauchy problem (1.1) defined for $t \in (-T, T)$ for some $T > 0$. Such solutions always exist by the local well-posedness theory [**12**]. Assume that $u(t, \cdot) \in Z_1$ which means that the solution to the DNLS equation contains a single soliton related to a simple eigenvalue $\lambda_1 \in \mathbb{C}_I$

of the KN spectral problem (1.2). By using the Bäcklund transformation (3.3) with $\lambda = \lambda_1$ and η being an eigenvector of the KN spectral problem (1.2) for the same λ_1, we define $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u$ as a function of (t,x). We would like to show that

(i) $u^{(1)} \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$;
(ii) $u^{(1)} \in Z_0$, that is, the new solution does not contain solitons;
(iii) $\mathbf{B}_{\lambda_1}(\eta)$ has the (left) inverse so that $u = [\mathbf{B}_{\lambda_1}(\eta)]^{-1} u^{(1)}$;
(iv) $u^{(1)}(t,\cdot) = \mathbf{B}_{\lambda_1}(\eta(t,\cdot))u(t,\cdot)$ is a solution of the DNLS equation for $t \in (-T,T)$.

Properties (i) and (iii) are shown in Lemma 1. Property (ii) is shown in Lemma 6. Property (iv) is shown in Lemma 9.

In order to obtain the global well-posedness of the Cauchy problem (1.1), we want to extend an existence time T of the solution $u(t,\cdot) \in Z_1$ to an arbitrary large number. Importantly, the global existence of the solution $u^{(1)}(t,\cdot) \in Z_0$ is known from the previous works [**17, 22**].

Let $\mathbf{B}_{\lambda_1}(\eta^{(1)})$ be the inverse of $\mathbf{B}_{\lambda_1}(\eta)$ for some function $\eta^{(1)}$, that is,

$$\mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)} = u.$$

Although the choice of $\eta^{(1)}$ is generally not unique, we will show in Lemmas 2 and 7 that $\eta^{(1)}$ can be fixed as a unique linear combination of Jost functions of the KN spectral problem (1.2) associated with the potential $u^{(1)}$. By analyzing the Bäcklund transformation (3.3), we obtain from Lemma 8 the global estimate in the form

$$\|u(t,\cdot)\|_{H^2\cap H^{1,1}} = \|\mathbf{B}_{\lambda_1}(\eta^{(1)}(t,\cdot))u^{(1)}(t,\cdot)\|_{H^2\cap H^{1,1}} \leq C_M, \tag{3.4}$$

for every $u^{(1)}(t,\cdot)$ satisfying $\|u^{(1)}(t,\cdot)\|_{H^2\cap H^{1,1}} \leq M$, where the constant C_M depends on M but does not depend on $u^{(1)}$. Since $\|u^{(1)}(t,\cdot)\|_{H^2\cap H^{1,1}}$ is finite for all times $t \in \mathbb{R}$ (but may grow as $|t| \to \infty$) by the previous results [**17, 22**], the bound (3.4) yields the proof of Theorem 1 in the case of one soliton. By using recursively the Bäcklund transformation (3.3), the result for any number of solitons follows from the result for one soliton. Thus, the proof of Theorem 1 relies on the proof of the properties (i)–(iv), the unique construction of $\eta^{(1)}$ for the inverse Bäcklund transformation $\mathbf{B}_{\lambda_1}(\eta^{(1)}) = [\mathbf{B}_{\lambda_1}(\eta)]^{-1}$, and the estimate (3.4).

3.1. Transformation of potentials. The following lemma shows that the transformation (3.3) can be defined as an invertible operator from u to $u^{(1)}$ in the same function space $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. Since we only use the KN spectral problem (1.2) here, we drop the time variable t from all function arguments.

Lemma 1. *Fix $\lambda_1 \in \mathbb{C}_I$. Given a potential $u \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, define $\eta(x) := \varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x}$, where φ_- is the Jost function for the KN spectral problem* (1.2) *in Propositions 1 and 2. Then, $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u$ belongs to $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. Moreover, the left inverse of $\mathbf{B}_{\lambda_1}(\eta)$ exists.*

Proof. First, we notice that $d_{\lambda_1}(\eta,\eta) = 0$ if and only if $\eta = 0$ because $\mathrm{Re}(\lambda_1) > 0$. However, if $\eta(x_0) = 0$ at a point $x_0 \in \mathbb{R}$, then the system (1.2) suggests $\eta'(x_0) = 0$, which implies $\eta(x) = 0$ for every $x \in \mathbb{R}$. Since $\varphi_-(x;\lambda_1)$ satisfies the nonzero asymptotic limit (2.1) as $x \to -\infty$, then $\eta(x) = \varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x} \neq 0$ and $d_{\lambda_1}(\eta,\eta) \neq 0$ for every finite $x \in \mathbb{R}$.

In order to deal with the limits as $x \to \pm\infty$, we note that $G_\lambda(\eta) = G_\lambda(\varphi_-)$ and $S_\lambda(\eta) = S_\lambda(\varphi_-)$ by properties (A.4) and (A.5) of Appendix A. Therefore, it is sufficient to consider $d_{\lambda_1}(\varphi_-,\varphi_-)$ instead of $d_{\lambda_1}(\eta,\eta)$. If $a(\lambda_1)\neq 0$, we claim that there exists $\varepsilon_0>0$ such that

$$|d_{\lambda_1}(\varphi_-,\varphi_-)|\geq \varepsilon_0, \quad \text{for all } \ x\in\mathbb{R}\,. \tag{3.5}$$

Indeed, since $d_{\lambda_1}(\varphi_-,\varphi_-)\to\lambda_1$ as $x\to-\infty$ and thanks to the arguments above, $d_{\lambda_1}(\varphi_-,\varphi_-)$ may only vanish in the limit $x\to+\infty$. However, it follows from the representation (2.7a) that the limit $\phi_+(x;\lambda_1)\to e_2$ as $x\to+\infty$, and the fact that $\varphi_-(\cdot;\lambda_1)\in L^\infty(\mathbb{R})$ imply that $\varphi_{-,1}(x;\lambda_1)\to a(\lambda_1)$ as $x\to+\infty$ so that $d_{\lambda_1}(\varphi_-,\varphi_-)\nrightarrow 0$ as $x\to+\infty$. Therefore, the claim (3.5) follows.

By using the triangle inequality, the bounds (B.1)–(B.2) of Appendix B, the bound (3.5), and $|G_{\lambda_1}(\eta)|=1$, we obtain

$$\begin{aligned}\|u^{(1)}\|_{L^{2,1}} &\leq \|u\|_{L^{2,1}}+\|S_{\lambda_1}(\varphi_-)\|_{L^{2,1}}\\ &\leq \|u\|_{L^{2,1}}+2\varepsilon_0^{-1}|\lambda_1^2-\overline{\lambda}_1^2|\left\|\varphi_{-,1}(\cdot,\lambda_1)\overline{\varphi_{-,2}(\cdot,\lambda_1)}\right\|_{L^{2,1}}<\infty.\end{aligned} \tag{3.6}$$

The norms $\|\partial_x u^{(1)}\|_{L^{2,1}}$ as well as $\|\partial_x^2 u^{(1)}\|_{L^2}$ are estimated similarly with the bounds (B.1)–(B.2) of Appendix B and the bound (3.5).

If $a(\lambda_1)=0$, the uniform bound (3.5) is no longer valid because $d_{\lambda_1}(\varphi_-,\varphi_-)\to 0$ as $x\to+\infty$. The estimate (3.6) can only be proved on the interval $(-\infty,R)$ with arbitrary $R>0$. In order to extend the estimate (3.6) on the interval (R,∞), we use (2.9) and write $\eta(x)=\varphi_-(x;\lambda_1)e^{-i\lambda_1^2x}=\gamma_1\phi_+(x;\lambda_1)e^{i\lambda_1^2x}$, so that $u^{(1)}=\mathbf{B}_{\lambda_1}(\varphi_-)u$ can be rewritten as $u^{(1)}=\mathbf{B}_{\lambda_1}(\phi_+)u$. Since $d_{\lambda_1}(\phi_+,\phi_+)\to\overline{\lambda}_1$ as $x\to+\infty$, we repeat the same estimates on the interval (R,∞) by using the equivalent representation of $u^{(1)}$.

Next, we show the existence of the left inverse for $\mathbf{B}_{\lambda_1}(\eta)u$. Let η^* be a vector function and define

$$\begin{aligned}u^{(2)}&=\mathbf{B}_{\lambda_1}(\eta^*)\mathbf{B}_{\lambda_1}(\eta)u\\ &=-G_{\lambda_1}(\eta^*)^2[-G_{\lambda_1}(\eta)^2u+G_{\lambda_1}(\eta)S_{\lambda_1}(\eta)]+G_{\lambda_1}(\eta^*)S_{\lambda_1}(\eta^*)\\ &=G_{\lambda_1}(\eta^*)^2G_{\lambda_1}(\eta)^2u+G_{\lambda_1}(\eta^*)[-G_{\lambda_1}(\eta^*)G_{\lambda_1}(\eta)S_{\lambda_1}(\eta)+S_{\lambda_1}(\eta^*)].\end{aligned}$$

$\mathbf{B}_{\lambda_1}(\eta^*)$ is the left inverse of $\mathbf{B}_{\lambda_1}(\eta)u$ if η^* satisfies

$$G_{\lambda_1}(\eta^*)^2G_{\lambda_1}(\eta)^2=1 \tag{3.7}$$

and

$$-G_{\lambda_1}(\eta^*)G_{\lambda_1}(\eta)S_{\lambda_1}(\eta)+S_{\lambda_1}(\eta^*)=0. \tag{3.8}$$

System (3.7) and (3.8) is satisfied either for

$$\overline{G_{\lambda_1}(\eta^*)}=G_{\lambda_1}(\eta),\quad S_{\lambda_1}(\eta^*)=S_{\lambda_1}(\eta) \tag{3.9}$$

or for

$$\overline{G_{\lambda_1}(\eta^*)}=-G_{\lambda_1}(\eta),\quad S_{\lambda_1}(\eta^*)=-S_{\lambda_1}(\eta). \tag{3.10}$$

Let us show that the choice (3.10) is impossible if $\lambda_1\in\mathbb{C}_I$.

Since $\eta(x)=\varphi_-(x;\lambda_1)e^{-i\lambda_1^2x}$, we have $G_{\lambda_1}(\eta)\to\overline{\lambda}_1/\lambda_1$ as $x\to-\infty$. Writing

$$\overline{G_{\lambda_1}(\eta^*)}=\frac{\lambda_1\frac{|\eta_1^*|^2}{|\eta_2^*|^2}+\overline{\lambda}_1}{\overline{\lambda}_1\frac{|\eta_1^*|^2}{|\eta_2^*|^2}+\lambda_1},$$

we realize that $|\eta_1^*|/|\eta_2^*|\nrightarrow 0$ as $x\to-\infty$, as it would contradict to the first equation in (3.10) with $\lambda_1\neq 0$. From the second equation in (3.10), we can see that

$S_{\lambda_1}(\eta^*) \to 0$ as $x \to -\infty$ because $S_{\lambda_1}(\eta) \to 0$ as $x \to -\infty$. Since $|\eta_1^*|/|\eta_2^*| \not\to 0$ as $x \to -\infty$, the limit $S_{\lambda_1}(\eta^*) \to 0$ as $x \to -\infty$ implies that $|\eta_2^*|/|\eta_1^*| \to 0$ as $x \to -\infty$. This implies that $\overline{G_{\lambda_1}(\eta^*)} \to \lambda_1/\overline{\lambda}_1$ as $x \to -\infty$, or in view of the first equation in (3.10), we obtain $\mathrm{Re}(\lambda_1^2) = 0$. Since $\lambda_1 \in \mathbb{C}_I$, then $\arg(\lambda_1) = \pi/4$. Finally writing $\lambda_1 = |\lambda_1| e^{i\pi/4}$ and using the first equation in (3.10) yields

$$|\eta_1^*|^2 |\eta_2|^2 + |\eta_2^*|^2 |\eta_1|^2 = 0,$$

which cannot be satisfied with $\eta^* \neq 0$. This contradiction eliminates possibility of the choice (3.10).

Thus, we only have the choice (3.9) to define η^* and to satisfy system (3.7) and (3.8). Since $\lambda_1 \in \mathbb{C}_I$, the condition $\overline{G_{\lambda_1}(\eta^*)} = G_{\lambda_1}(\eta)$ is equivalently written as

$$|\eta_1|^2 |\eta_1^*|^2 = |\eta_2|^2 |\eta_2^*|^2,$$

so that there exists a positive number k such that

$$|\eta_1^*| = k|\eta_2|, \quad |\eta_2^*| = k|\eta_1|. \tag{3.11}$$

On the other hand, the condition $S_{\lambda_1}(\eta^*) = S_{\lambda_1}(\eta)$ yields

$$\frac{\eta_1 \overline{\eta}_2}{\eta_1^* \overline{\eta}_2^*} = \frac{\lambda_1 |\eta_1|^2 + \overline{\lambda}_1 |\eta_2|^2}{\lambda_1 |\eta_1^*|^2 + \overline{\lambda}_1 |\eta_2^*|^2},$$

which transforms after substitution of (3.11) to

$$k^2 \frac{\eta_1 \overline{\eta}_2}{\eta_1^* \overline{\eta}_2^*} = \frac{\lambda_1 |\eta_1|^2 + \overline{\lambda}_1 |\eta_2|^2}{\lambda_1 |\eta_2|^2 + \overline{\lambda}_1 |\eta_1|^2}, \tag{3.12}$$

where the right-hand side is of modulus one. Combining (3.11) and (3.12), we obtain the most general solution of the system (3.9) in the form

$$\eta_1^* = k_1 \overline{\eta}_2, \quad \eta_2^* = k_2 \overline{\eta}_1, \tag{3.13}$$

where $k_1, k_2 \in \mathbb{C}$ satisfying $|k_1| = |k_2|$. Thus, $\mathbf{B}_{\lambda_1}(\eta^*)$ with η^* given by (3.13) is the left inverse of the transformation $\mathbf{B}_{\lambda_1}(\eta)$. □

The following lemma specifies a unique choice for the function η^* constructed in the proof of Lemma 1 and shows that η^* is a solution of the KN spectral problem (1.2) associated with the new potential $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta) u$ and the same value λ_1.

LEMMA 2. *Under the same assumption as in Lemma 1, let $\eta^{(1)}$ be given by*

$$\eta_1^{(1)} = \frac{\overline{\eta}_2}{d_{\lambda_1}(\eta, \eta)}, \quad \eta_2^{(1)} = \frac{\overline{\eta}_1}{d_{\overline{\lambda}_1}(\eta, \eta)}. \tag{3.14}$$

Then, $\eta^{(1)}$ is the solution of the KN spectral problem (1.2) associated with the potential $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta) u$ and the same value λ_1.

PROOF. We recall that η is a solution of

$$\partial_x \eta = [-i\lambda_1^2 \sigma_3 + \lambda_1 Q(u)]\eta, \tag{3.15}$$

as follows from the KN spectral problem (1.2) for $\lambda = \lambda_1$. By using system (3.15), we obtain

$$\partial_x d_{\lambda_1}(\eta, \eta) = (\lambda_1^2 - \overline{\lambda}_1^2) \left[u \overline{\eta}_1 \eta_2 - i\lambda_1 |\eta_1|^2 + i\overline{\lambda}_1 |\eta_2|^2 \right]. \tag{3.16}$$

By using (3.14), (3.15), and (3.16), we obtain

$$\begin{aligned}
\partial_x \eta_1^{(1)} + i\lambda_1^2 \eta_1^{(1)} &= \frac{1}{d_{\lambda_1}(\eta,\eta)} \left[-\overline{\lambda}_1 u \overline{\eta}_1 + i(\lambda_1^2 - \overline{\lambda}_1^2)\overline{\eta}_2 \right] \\
&\qquad - \frac{(\lambda_1^2 - \overline{\lambda}_1^2)\overline{\eta}_2}{[d_{\lambda_1}(\eta,\eta)]^2} \left[u\overline{\eta}_1 \eta_2 - i\lambda_1 |\eta_1|^2 + i\overline{\lambda}_1 |\eta_2|^2 \right] \\
&= \frac{\lambda_1 \overline{\eta}_1}{[d_{\lambda_1}(\eta,\eta)]^2} \left[-u d_{\overline{\lambda}_1}(\eta,\eta) + 2i(\lambda_1^2 - \overline{\lambda}_1^2)\eta_1 \overline{\eta}_2 \right] \\
&= \lambda_1 u^{(1)} \eta_2^{(1)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\partial_x \eta_2^{(1)} - i\lambda_1^2 \eta_1^{(1)} &= \frac{1}{d_{\overline{\lambda}_1}(\eta,\eta)} \left[\overline{\lambda}_1 \overline{u \eta}_2 - i(\lambda_1^2 - \overline{\lambda}_1^2)\overline{\eta}_1 \right] \\
&\qquad + \frac{(\lambda_1^2 - \overline{\lambda}_1^2)\overline{\eta}_1}{[d_{\overline{\lambda}_1}(\eta,\eta)]^2} \left[\overline{u} \eta_1 \overline{\eta}_2 + i\overline{\lambda}_1 |\eta_1|^2 - i\lambda_1 |\eta_2|^2 \right] \\
&= -\frac{\lambda_1 \overline{\eta}_2}{[d_{\overline{\lambda}_1}(\eta,\eta)]^2} \left[-\overline{u} d_{\lambda_1}(\eta,\eta) + 2i(\lambda_1^2 - \overline{\lambda}_1^2)\overline{\eta_1}\eta_2 \right] \\
&= -\lambda_1 \overline{u}^{(1)} \eta_1^{(1)}.
\end{aligned}$$

Thus, we have proven that $\eta^{(1)}$ satisfies the KN spectral problem (1.2) with the potential $u^{(1)}$ and the same value $\lambda = \lambda_1$. $\square$

In the construction of Lemmas 1 and 2, the Jost function φ_- was used in the choice for η. The following lemma shows that the same potential $u^{(1)}$ can be equivalently obtained from all four Jost functions of the KN spectral problem (1.2) if λ_1 is selected to be a root of the scattering coefficient $a(\lambda)$.

Lemma 3. *Assume that $\lambda_1 \in \mathbb{C}_I$ is chosen so that $a(\lambda_1) = 0$. Given a potential $u \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, it is true that*

$$u^{(1)}(x) = \mathbf{B}_{\lambda_1}(\varphi_-(x;\lambda_1) e^{-i\lambda_1^2 x}) u(x) \tag{3.17a}$$
$$= \mathbf{B}_{\lambda_1}(\phi_+(x;\lambda_1) e^{i\lambda_1^2 x}) u(x) \tag{3.17b}$$
$$= \mathbf{B}_{\overline{\lambda}_1}(\varphi_+(x;\overline{\lambda}_1) e^{-i\overline{\lambda}_1^2 x}) u(x) \tag{3.17c}$$
$$= \mathbf{B}_{\overline{\lambda}_1}(\phi_-(x;\overline{\lambda}_1) e^{i\overline{\lambda}_1^2 x}) u(x), \tag{3.17d}$$

where the four Jost functions to the KN spectral problem (1.2) *are given in Propositions 1 and 2.*

Proof. Representation (3.17a) was defined in Lemma 1. If $a(\lambda_1) = 0$, the representation (3.17b) was also obtained in Lemma 1, thanks to the invariance of G_λ and S_λ under a multiplication by a nonzero complex number a in properties (A.4) and (A.5) of Appendix A and the relation (2.9) between $\varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x}$ and $\phi_+(x;\lambda_1)e^{i\lambda_1^2 x}$. In order to establish (3.17c), we use the symmetry relation (2.4) as well as properties (A.6) and (A.7) of Appendix A and obtain

$$G_{\lambda_1}(\varphi_-(x;\lambda_1)) = G_{\overline{\lambda}_1}(\sigma_1\sigma_3\varphi_-(x;\lambda_1)) = G_{\overline{\lambda}_1}\left(\overline{\phi_-(x;\overline{\lambda}_-)}\right) = G_{\overline{\lambda}_1}(\phi_-(x;\overline{\lambda}_-))$$

and

$$S_{\lambda_1}(\varphi_-(x;\lambda_1)) = -\overline{S_{\lambda_1}(\sigma_1\sigma_3\varphi_-(x;\lambda_1))} = -\overline{S_{\lambda_1}\left(\overline{\phi_-(x;\overline{\lambda}_1)}\right)} = S_{\overline{\lambda}_1}(\phi_-(x;\overline{\lambda}_1)).$$

The transformation formula (3.3) yields (3.17c). Finally, the representation (3.17d) is obtained from the relation between $\varphi_+(x;\overline{\lambda}_1)e^{-i\overline{\lambda}_1^2 x}$ and $\phi_-(x;\overline{\lambda}_1)e^{i\overline{\lambda}_1^2 x}$ in the case $a(\lambda_1)=0$ that corresponds to $\overline{a(\overline{\lambda}_1)}=0$. □

3.2. Transformation of Jost functions. For values of $\lambda \in \mathbb{C}\backslash\{\pm\lambda_1\}$, Jost functions of the KN spectral problem (1.2) associated with the new potential $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u$ can be constructed from the old Jost functions by using the transformation matrix

$$M[\eta,\lambda,\lambda_1] := \frac{\lambda_1}{\overline{\lambda}_1}\frac{1}{\lambda^2-\lambda_1^2}\begin{bmatrix}\lambda^2 G_{\lambda_1}(\eta) - |\lambda_1|^2 & -\frac{\lambda}{2i}S_{\lambda_1}(\eta) \\ \frac{\lambda}{2i}\overline{S_{\lambda_1}(\eta)} & -\lambda^2\overline{G_{\lambda_1}(\eta)} + |\lambda_1|^2\end{bmatrix}. \tag{3.18}$$

The following lemma presents the new Jost functions of the KN spectral problem (1.2) associated with the new potential $u^{(1)}$. Since $u^{(1)} \in H^2(\mathbb{R})\cap H^{1,1}(\mathbb{R})$ by Lemma 1, the new Jost functions exist according to Proposition 1.

Lemma 4. *Under the same assumption as in Lemma 1, let us define for $\lambda \in \mathbb{C}\backslash\{\pm\lambda_1,\pm\overline{\lambda}_1\}$,*

$$\varphi_-^{(1)}(x;\lambda) = M[\varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x},\lambda,\lambda_1]\varphi_-(x;\lambda), \tag{3.19a}$$

$$\varphi_+^{(1)}(x;\lambda) = M[\varphi_+(x;\overline{\lambda}_1)e^{-i\overline{\lambda}_1^2 x},\lambda,\overline{\lambda}_1]\varphi_+(x;\lambda), \tag{3.19b}$$

$$\phi_+^{(1)}(x;\lambda) = -M[\phi_+(x;\lambda_1)e^{i\lambda_1^2 x},\lambda,\lambda_1]\phi_+(x;\lambda), \tag{3.19c}$$

$$\phi_-^{(1)}(x;\lambda) = -M[\phi_-(x;\overline{\lambda}_1)e^{i\overline{\lambda}_1^2 x},\lambda,\overline{\lambda}_1]\phi_-(x;\lambda). \tag{3.19d}$$

Then, $\{\varphi_\pm^{(1)}(x;\lambda)e^{-i\lambda^2 x},\phi_\pm^{(1)}(x;\lambda)e^{i\lambda^2 x}\}$ are Jost functions of the KN spectral problem (1.2) associated with the potential $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u$.

Proof. First, we prove that the transformations (3.19a)–(3.19d) produce solutions of the KN spectral problem associated with the potential $u^{(1)}$. It is sufficient to consider the first Jost function in (3.19a). Therefore we shall verify that

$$\partial_x(\varphi_-^{(1)}(x;\lambda)e^{-i\lambda^2 x}) = \left[-i\lambda^2\sigma_3 + \lambda Q(u^{(1)})\right]\varphi_-^{(1)}(x;\lambda)e^{-i\lambda^2 x}. \tag{3.20}$$

Denoting entries of $M[\varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x},\lambda,\lambda_1]$ by M_{ij} for $1 \le i,j \le 2$ and using the KN spectral problem (1.2) for $\varphi_-(x;\lambda)e^{-i\lambda^2 x}$, we obtain the four differential equations:

$$\partial_x M_{11} - \lambda\overline{u}M_{12} = \lambda u^{(1)}M_{21} \tag{3.21a}$$

$$\partial_x M_{12} + \lambda u M_{11} = \lambda u^{(1)}M_{22} - 2i\lambda^2 M_{12} \tag{3.21b}$$

$$\partial_x M_{21} - \lambda\overline{u}M_{22} = -\lambda\overline{u}^{(1)}M_{11} + 2i\lambda^2 M_{21} \tag{3.21c}$$

$$\partial_x M_{22} + \lambda u M_{21} = -\lambda\overline{u}^{(1)}M_{12} \tag{3.21d}$$

By using equation (3.16), we obtain

$$\partial_x G_{\lambda_1}(\eta) = \frac{\lambda_1^2-\overline{\lambda}_1^2}{[d_{\lambda_1}(\eta,\eta)]^2}\left[2i(\lambda_1^2-\overline{\lambda}_1^2)|\eta_1|^2|\eta_2|^2 - d_{\lambda_1}(\eta,\eta)\overline{u}\eta_1\overline{\eta}_2 - d_{\overline{\lambda}_1}(\eta,\eta)u\overline{\eta}_1\eta_2\right],$$

from which we verify equation (3.21a) as follows:

$$\begin{aligned}\partial_x M_{11} - \lambda \overline{u} M_{12} &= \frac{\lambda_1}{\overline{\lambda}_1} \frac{\lambda^2}{\lambda^2 - \lambda_1^2} \frac{\lambda_1^2 - \overline{\lambda}_1^2}{[d_{\lambda_1}(\eta,\eta)]^2} \left[2i(\lambda_1^2 - \overline{\lambda}_1^2)|\eta_1|^2|\eta_2|^2 - d_{\overline{\lambda}_1}(\eta,\eta) u \overline{\eta}_1 \eta_2 \right] \\ &= \lambda u^{(1)} M_{21}.\end{aligned}$$

The proof of (3.21d) is based on the complex conjugate equation and similar computations.

Equation (3.21b) is equivalent to

$$\partial_x (S_{\lambda_1}(\eta)) = -2i(u + u^{(1)})|\lambda_1|^2.$$

This equality holds by means of the following two explicit computations:

$$\partial_x S_{\lambda_1}(\eta) = \frac{2i(\lambda_1^2 - \overline{\lambda}_1^2)}{[d_{\lambda_1}(\eta,\eta)]^2} \left[-u|\lambda_1|^2(|\eta_1|^4 - |\eta_2|^4) - 2i|\lambda_1|^2 \eta_1 \overline{\eta}_2 d_{\overline{\lambda}_1}(\eta,\eta) \right]$$

and

$$u^{(1)} + u = \frac{u(\lambda_1^2 - \overline{\lambda}_1^2)(|\eta_1|^4 - |\eta_2|^4)}{[d_{\lambda_1}(\eta,\eta)]^2} + \frac{2i(\lambda_1^2 - \overline{\lambda}_1^2)\eta_1 \overline{\eta}_2 d_{\overline{\lambda}_1}(\eta,\eta)}{[d_{\lambda_1}(\eta,\eta)]^2}.$$

Hence, we have proven (3.21b). Equation (3.21c) is obtained from complex conjugate equations and similar computations.

Thus, the function $\varphi_-^{(1)}(x;\lambda)e^{-i\lambda^2 x}$ satisfies equation (3.20), that is, it is a solution of the KN spectral problem (1.2) associated with the potential $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u$. Similar computations are performed for the other functions $\varphi_+^{(1)}(x;\lambda)e^{-i\lambda^2 x}$ and $\phi_\pm^{(1)}(x;\lambda)e^{i\lambda^2 x}$ given by (3.19b)–(3.19d). Since $G_{\lambda_1}(\eta)$ and $S_{\lambda_1}(\eta)$ are bounded in x for all considered choices for η, the functions $\varphi_\pm^{(1)}(x;\lambda)$ and $\phi_\pm^{(1)}(x;\lambda)$ are bounded functions of x for every $\lambda \in \mathbb{C} \setminus \{\pm\lambda_1, \pm\overline{\lambda}_1\}$.

It is left to check the boundary conditions (2.1) in Proposition 1. The boundary conditions (2.1) follow from properties (A.8) and (A.9) in Appendix A:

$$M[e_1, \lambda, \lambda_1]e_1 = e_1, \quad M[e_2, \lambda, \lambda_1]e_2 = -e_2. \tag{3.22}$$

Since $u^{(1)} \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ satisfies the assumption of Proposition 1, the four functions (3.19a)–(3.19d) are the unique Jost functions of the KN spectral problem (1.2) associated with $u^{(1)}$. □

Since the definitions (3.19a)–(3.19d) with the transformation matrix (3.18) are singular as $\lambda \to \{\pm\lambda_1, \pm\overline{\lambda}_1\}$, we show that the singularity is removable, so that the definitions (3.19a)–(3.19d) can be extended in the domains of analyticity of the Jost functions $\varphi_\pm(x;\lambda)$ and $\phi_\pm(x;\lambda)$ according to Proposition 2.

Lemma 5. *Let $\varphi_\pm^{(1)}(x;\lambda)$ and $\phi_\pm^{(1)}(x;\lambda)$ be defined by (3.19a)–(3.19d). Then, $\lambda = \pm\lambda_1$ and $\lambda = \pm\overline{\lambda}_1$ are removable singularities in the corresponding domains of analyticity of $\varphi_\pm^{(1)}(x;\lambda)$ and $\phi_\pm^{(1)}(x;\lambda)$.*

Proof. It is sufficient again to consider the first Jost function $\varphi_-^{(1)}(x;\lambda)$ represented by (3.19a). By using the notations $\varphi_- = (\varphi_{-,1}, \varphi_{-,2})^t$ and $\varphi_-^{(1)} = (\varphi_{-,1}^{(1)}, \varphi_{-,2}^{(1)})^t$

for the 2-vectors and dropping the dependence on x, we obtain for $\lambda \in \mathbb{C}_I \cup \mathbb{C}_{III} \setminus \{\pm\lambda_1\}$

$$\begin{aligned}
\varphi^{(1)}_{-,1}(\lambda) &= \frac{\lambda_1}{\overline{\lambda}_1}\left\{ \frac{\left(\lambda^2 d_{\overline{\lambda}_1}(\varphi_-,\varphi_-) - |\lambda_1|^2 d_{\lambda_1}(\varphi_-,\varphi_-)\right)\varphi_{-,1}(\lambda)}{(\lambda^2-\lambda_1^2)\, d_{\lambda_1}(\varphi_-,\varphi_-)} \right.\\
&\qquad\qquad \left. - \frac{\lambda(\lambda_1^2-\overline{\lambda}_1^2)\varphi_{-,1}(\lambda_1)\overline{\varphi_{-,2}(\lambda_1)}\varphi_{-,2}(\lambda)}{(\lambda^2-\lambda_1^2)\, d_{\lambda_1}(\varphi_-,\varphi_-)}\right\}\\
&= \frac{\lambda_1}{\overline{\lambda}_1}\frac{(\lambda^2-\lambda_1^2)\overline{\lambda}_1|\varphi_{-,1}(\lambda_1)|^2\varphi_{-,1}(\lambda) + F(\lambda)}{(\lambda^2-\lambda_1^2)\, d_{\lambda_1}(\varphi_-,\varphi_-)},
\end{aligned}$$

where

$$F(\lambda) := (\lambda^2-\overline{\lambda}_1^2)\lambda_1|\varphi_{-,2}(\lambda_1)|^2\varphi_{-,1}(\lambda) - \lambda(\lambda_1^2-\overline{\lambda}_1^2)\varphi_{-,1}(\lambda_1)\overline{\varphi_{-,2}(\lambda_1)}\varphi_{-,2}(\lambda).$$

Since $\varphi_{-,1}(\lambda)$ is even in λ and $\varphi_{-,2}(\lambda)$ is odd in λ [**22**], we obviously have $F(\lambda_1) = F(-\lambda_1) = 0$. Furthermore, F is analytic in $\mathbb{C}_I \cup \mathbb{C}_{III}$ by Proposition 2, hence $F(\lambda) = (\lambda^2-\lambda_1^2)\widetilde{F}(\lambda)$, where $\widetilde{F}$ is analytic in $\mathbb{C}_I \cup \mathbb{C}_{III}$. Thus, we obtain

$$\varphi^{(1)}_{-,1}(\lambda) = \frac{\lambda_1}{\overline{\lambda}_1}\frac{\overline{\lambda}_1|\varphi_{-,1}(\lambda_1)|^2\varphi_{-,1}(\lambda) + \widetilde{F}(\lambda)}{d_{\lambda_1}(\varphi_-,\varphi_-)},$$

so that $\pm\lambda_1$ are removable singularities of $\varphi^{(1)}_{-,1}(\lambda)$. Similar calculations show that $\pm\lambda_1$ are also removable singularities of $\varphi^{(1)}_{-,2}(\lambda)$. □

3.3. Transformation of scattering coefficients. We next transform the scattering coefficients $a(\lambda)$ and $b(\lambda)$ given by (2.7a)–(2.7b) and show that the new potential $u^{(1)}$ belongs to $Z_0 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ if the old potential u belongs to $Z_1 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ and the value $\lambda_1 \in \mathbb{C}_I$ is chosen to be the root of $a(\lambda)$ in $\mathbb{C}_I$. The following lemma gives the corresponding result.

Lemma 6. *Let $u \in Z_1$ and $\lambda_1 \in \mathbb{C}_I$ such that $a(\lambda_1) = 0$. Let $\eta(x) = \varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x}$, where φ_- is the Jost function of the KN spectral problem* (1.2) *given in Propositions 1 and 2. Then, $u^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u$ belongs to Z_0.*

Proof. In order to show that $u^{(1)} \in Z_0$, we show that if the only simple zero $a(\lambda) = W(\varphi_-(\cdot;\lambda), \phi_+(\cdot;\lambda))$ in $\mathbb{C}_I$ is located at $\lambda = \lambda_1$, then $a^{(1)}(\lambda) := W(\varphi^{(1)}_-(\cdot;\lambda), \phi^{(1)}_+(\cdot;\lambda))$ has no zero in $\mathbb{C}_I$, where $\varphi^{(1)}_-$ and $\phi^{(1)}_+$ are given by (3.19a) and (3.19c) in Lemma 4. This follows from the direct computation as follows:

$$\begin{aligned}
&(3.23a) & a^{(1)}(\lambda) &= W(\varphi^{(1)}_-(x;\lambda), \phi^{(1)}_+(x;\lambda))\\
&(3.23b) & &= W\Big(M[\varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x}, \lambda, \lambda_1]\varphi_-(x;\lambda),\\
& & &\qquad -M[\phi_+(x;\lambda_1)e^{i\lambda_1^2 x}, \lambda, \lambda_1]\phi_+(x;\lambda)\Big)\\
&(3.23c) & &= W\Big(M[\varphi_-(x;\lambda_1), \lambda, \lambda_1]\varphi_-(x;\lambda),\\
& & &\qquad -M[\varphi_-(x;\lambda_1), \lambda, \lambda_1]\phi_+(x;\lambda)\Big)\\
&(3.23d) & &= -a(\lambda)\,\det\left(M[\varphi_-(x;\lambda_1), \lambda, \lambda_1]\right)\\
&(3.23e) & &= -a(\lambda)\det\left(M[e_1, \lambda, \lambda_1]\right)\\
&(3.23f) & &= a(\lambda)\frac{\lambda_1^2}{\overline{\lambda}_1^2}\frac{\lambda^2-\overline{\lambda}_1^2}{\lambda^2-\lambda_1^2},
\end{aligned}$$

where we have used (3.19a) and (3.19c) to get (3.23b), (2.9), (A.4), and (A.5) to get (3.23c), (2.7a) to get (3.23d), the limit $x \to -\infty$ to get (3.23e), and (A.5), (A.8) and (A.9) to get (3.23f). Since λ_1 is the only simple zero of $a(\lambda)$ in $\mathbb{C}_I$, then $a^{(1)}(\lambda)$ has no zeros for λ in $\mathbb{C}_I$. □

REMARK 4. *For completeness, we also give transformation of $b(\lambda)$ to $b^{(1)}(\lambda)$ as follows:*

$$\begin{aligned}
b^{(1)}(\lambda) &= W(e^{-2i\lambda^2 x}\varphi_+^{(1)}(x;\lambda), \varphi_-^{(1)}(x;\lambda)) \\
&= W(e^{-2i\lambda^2 x}\varphi_+^{(1)}(x;\lambda), M[\varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x}, \lambda, \lambda_1]\varphi_-(x;\lambda)) \\
&= W\Big(e^{-2i\lambda^2 x}\varphi_+^{(1)}(x;\lambda), \\
&\qquad M[\phi_+(x;\lambda_1)e^{i\lambda_1^2 x}, \lambda, \lambda_1][a(\lambda)\varphi_+(x;\lambda) + e^{i2\lambda^2 x} b(\lambda)\phi_+(x;\lambda)]\Big) \\
&= b(\lambda)W(e_1, M[e_2, \lambda, \lambda_1]e_2) \\
&= -b(\lambda),
\end{aligned}$$

where the term with $a(\lambda)$ vanishes in the limit $x \to +\infty$ because $W(e_1, e_1) = 0$ and we have used the following limits as $x \to +\infty$

$$M[\phi_+(x;\lambda_1), \lambda, \lambda_1]\varphi_+(x;\lambda) \to M[e_2, \lambda, \lambda_1]e_1 = \frac{\lambda_1^2(\lambda^2 - \overline{\lambda}_1^2)}{\overline{\lambda}_1^2(\lambda^2 - \lambda_1^2)} e_1$$

and $\varphi_+^{(1)}(x;\lambda) \to e_1$.

By Lemmas 1, 2, and 6, we have shown the existence of a sequence of invertible Bäcklund transformations $Z_1 \to Z_0 \to Z_1$ given by

$$u \to \mathbf{B}_{\lambda_1}(\eta) \to u^{(1)} \to \mathbf{B}_{\lambda_1}(\eta^{(1)}) \to u.$$

Next, we express $\eta^{(1)}$ in Lemma 2 in terms of the new Jost functions $\varphi_\pm^{(1)}$ and $\phi_\pm^{(1)}$ associated with $u^{(1)}$ in Lemma 4.

LEMMA 7. *Fix $\lambda_1 \in \mathbb{C}_I$ such that $a(\lambda_1) = 0$ and $a'(\lambda_1) \neq 0$. Let η and $\eta^{(1)}$ be given as in Lemmas 1 and 2. Then, $\eta^{(1)}$ is decomposed as*

$$\eta^{(1)}(x) = \frac{1}{\gamma_1 \overline{\lambda}_1 a^{(1)}(\lambda_1)} e^{-i\lambda_1^2 x}\varphi_-^{(1)}(x;\lambda_1) + \frac{1}{\overline{\lambda}_1 a^{(1)}(\lambda_1)} e^{i\lambda_1^2 x}\phi_+^{(1)}(x;\lambda_1), \tag{3.24}$$

where the new Jost functions $\varphi_-^{(1)}$ and $\phi_+^{(1)}$ are constructed in Lemmas 4 and 5, $\gamma_1 \neq 0$ is the norming constant in (2.9), and $a^{(1)}(\lambda_1) \neq 0$ as in Lemma 6.

PROOF. We use notations $\eta^{(1)} = (\eta_1^{(1)}, \eta_2^{(1)})^t$ and $\varphi_- = (\varphi_{-,1}, \varphi_{-,2})^t$ for the 2-vectors. Components of $\eta^{(1)}$ given by (3.14) are rewritten explicitly by

$$\eta_1^{(1)}(x) = \frac{e^{i\lambda_1^2 x}\overline{\varphi_{-,2}(x;\lambda_1)}}{d_{\lambda_1}(\varphi_-(x;\lambda_1), \varphi_-(x;\lambda_1))}$$

and

$$\eta_2^{(1)}(x) = \frac{e^{i\lambda_1^2 x}\overline{\varphi_{-,1}(x;\lambda_1)}}{d_{\overline{\lambda}_1}(\varphi_-(x;\lambda_1), \varphi_-(x;\lambda_1))}.$$

Since $\lim_{x\to-\infty} \varphi_-(x;\lambda_1) = e_1$, we have

$$\lim_{x\to-\infty} e^{-i\lambda_1^2 x}\eta^{(1)}(x) = \frac{1}{d_{\overline{\lambda}_1}(e_1, e_1)} e_2 = \frac{1}{\overline{\lambda}_1} e_2. \tag{3.25}$$

By using the relation (2.9) with the norming coefficient γ_1, components of $\eta^{(1)}$ can be rewritten in the equivalent form:

$$\eta_1^{(1)}(x) = \frac{e^{-i\lambda_1^2 x}\overline{\phi_{+,2}(x;\lambda_1)}}{\gamma_1 d_{\lambda_1}(\phi_+(x;\lambda_1),\phi_+(x;\lambda_1))}$$

and

$$\eta_2^{(1)}(x) = \frac{e^{-i\lambda_1^2 x}\overline{\phi_{+,1}(x;\lambda_1)}}{\gamma_1 d_{\overline{\lambda}_1}(\phi_+(x;\lambda_1),\phi_+(x;\lambda_1))}.$$

Since $\lim_{x\to+\infty}\phi_+(x;\lambda_1) = e_2$, we have

$$\lim_{x\to+\infty} e^{i\lambda_1^2 x}\eta^{(1)}(x) = \frac{1}{\gamma_1 d_{\lambda_1}(e_2,e_2)} e_1 = \frac{1}{\gamma_1\overline{\lambda}_1} e_1. \tag{3.26}$$

By Lemma 2, $\eta^{(1)}$ is a solution of the KN spectral problem (1.2) associated with the new potential $u^{(1)}$ for $\lambda = \lambda_1$. By Lemmas 4 and 5, the two new Jost functions $\varphi_-^{(1)}(x;\lambda)$ and $\phi_+^{(1)}(x;\lambda)$ are analytic at λ_1. Any solution of the second-order system is a linear combination of the two linearly independent solutions, so that we have

$$\eta^{(1)}(x) = c_1\varphi_-^{(1)}(x;\lambda_1)e^{-i\lambda_1^2 x} + c_2\phi_+^{(1)}(x;\lambda_1)e^{i\lambda_1^2 x},$$

where c_1, c_2 are some numerical coefficients. Thanks to the boundary conditions (2.1) and the representation (2.7a), we obtain the boundary conditions

$$\lim_{x\to-\infty} e^{-i\lambda_1^2 x}\eta^{(1)}(x) = c_2 a^{(1)}(\lambda_1)e_2, \quad \lim_{x\to+\infty} e^{i\lambda_1^2 x}\eta^{(1)}(x) = c_1 a^{(1)}(\lambda_1)e_1, \tag{3.27}$$

where we have recalled that $\lambda_1 \in \mathbb{C}_I$. Since $a^{(1)}(\lambda_1) \neq 0$ by Lemma 6, c_1 and c_2 are found uniquely from (3.25), (3.26), and (3.27) to yield the decomposition (3.24). □

Remark 5. *Instead of the decomposition (3.24), we can write*

$$\eta^{(1)}(x) := \varphi_-^{(1)}(x;\lambda_1)e^{-i\lambda_1^2 x} + \gamma_1\,\phi_+^{(1)}(x;\lambda_1)e^{i\lambda_1^2 x} \tag{3.28}$$

because the Bäcklund transformation (3.3) is invariant if $\eta^{(1)}$ is multiplied by a nonzero constant.

Lemma 8. *Under the same conditions as in Lemma 7, for every $u^{(1)} \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ satisfying $\|u^{(1)}\|_{H^2\cap H^{1,1}} \leq M$ for some $M > 0$, the transformation*

$$\mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)} \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$$

satisfies

$$\|\mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)}\|_{H^2\cap H^{1,1}} \leq C_M, \tag{3.29}$$

where the constant C_M does not depend on $u^{(1)}$.

Proof. By the representation (2.7a), we have

$$\begin{aligned} |a^{(1)}(\lambda_1)| &= \Big| \left(\varphi_{-,1}^{(1)}(x;\lambda_1)e^{-i\lambda_1^2 x} + \gamma_1\,\phi_{+,1}^{(1)}(x;\lambda_1)e^{i\lambda_1^2 x}\right)\phi_{+,2}^{(1)}(x;\lambda_1)e^{i\lambda_1^2 x} \\ &\qquad - \left(\varphi_{-,2}^{(1)}(x;\lambda_1)e^{-i\lambda_1^2 x} + \gamma_1\,\phi_{+,2}^{(1)}(x;\lambda_1)e^{i\lambda_1^2 x}\right)\phi_{+,1}^{(1)}(x;\lambda_1)e^{i\lambda_1^2 x}\Big| \\ &\leq \|\phi_+^{(1)}(\cdot;\lambda_1)\|_{L^\infty}\left(|e^{i\lambda_1^2 x}\eta_1^{(1)}(x)| + |e^{i\lambda_1^2 x}\eta_2^{(1)}(x)|\right). \end{aligned}$$

Since $a^{(1)}(\lambda_1) \neq 0$ by Lemma 6 and $|d_\lambda(\eta,\eta)| \geq |\mathrm{Re}(\lambda_1)|(|\eta_1|^2 + |\eta_2|^2)$, it follows from (2.2) that there is a constant $C_M > 0$ independently of $u^{(1)}$ such that

$$\frac{1}{|d_{\lambda_1}(e^{i\lambda_1^2 x}\eta^{(1)}(x), e^{i\lambda_1^2 x}\eta^{(1)}(x))|} \leq C_M \quad \text{for all } x \in \mathbb{R}.$$

By using the same argument, we also obtain

$$|a^{(1)}(\lambda_1)| \leq |\gamma_1|^{-1} \|\varphi_-^{(1)}(\cdot;\lambda_1)\|_{L^\infty} \left(|e^{-i\lambda_1^2 x}\eta_1^{(1)}(x)| + |e^{-i\lambda_1^2 x}\eta_2^{(1)}(x)|\right),$$

such that

$$\frac{1}{|d_{\lambda_1}(e^{-i\lambda_1^2 x}\eta^{(1)}(x), e^{-i\lambda_1^2 x}\eta^{(1)}(x))|} \leq C_M \quad \text{for all } x \in \mathbb{R}.$$

As a consequence, by using the bound

$$\begin{aligned}
\left|\frac{\eta_1^{(1)}\overline{\eta}_2^{(1)}}{d_{\lambda_1}(\eta^{(1)},\eta^{(1)})}\right| \leq\ & \frac{|\varphi_{-,1}^{(1)}(x;\lambda_1)\overline{\varphi_{-,2}^{(1)}(x;\lambda_1)}|}{|d_{\lambda_1}(e^{i\lambda_1^2 x}\eta^{(1)}, e^{i\lambda_1^2 x}\eta^{(1)})|} + |\gamma_1|^2 \frac{|\phi_{-,1}^{(1)}(x;\lambda_1)\overline{\phi_{-,2}^{(1)}(x;\lambda_1)}|}{|d_{\lambda_1}(e^{-i\lambda_1^2 x}\eta^{(1)}, e^{-i\lambda_1^2 x}\eta^{(1)})|} \\
& + |\gamma_1| \frac{|\varphi_{-,1}^{(1)}(x;\lambda_1)\overline{\phi_{-,2}^{(1)}(x;\lambda_1)}| + |\phi_{-,1}^{(1)}(x;\lambda_1)\overline{\varphi_{-,2}^{(1)}(x;\lambda_1)}|}{|d_{\lambda_1}(\eta^{(1)},\eta^{(1)})|},
\end{aligned}$$

and the bounds (B.1)–(B.2) of Appendix B, we obtain

$$\|S_{\lambda_1}(\eta^{(1)})u^{(1)}\|_{L^{2,1}} \leq C_M.$$

Similar to the proof of Lemma 1, this implies by the triangle inequality that

$$\|\mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)}\|_{L^{2,1}} \leq C_M.$$

The norms $\|\partial_x(\mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)})\|_{L^{2,1}}$ and $\|\partial_x^2(\mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)})\|_{L^2}$ can be estimated similarly with the use of estimates (B.1)–(B.2) of Appendix B, so that the proof of the bound (3.29) is complete. □

4. Time evolution of the Bäcklund transformation

Here we will prove property (iv) claimed in Section 3. In other words, extending the Jost function $\varphi_-(t,x;\lambda)$ to be time-dependent according to the linear system (1.2) and (1.3), we will prove the following lemma, which is a time-dependent analogue of Lemma 1.

LEMMA 9. *Fix $\lambda_1 \in \mathbb{C}_I$. Given a local solution $u(t,\cdot) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, $t \in (-T,T)$ to the Cauchy problem (1.1) for some $T > 0$, define*

$$\eta(t,x) := \varphi_-(t,x;\lambda_1)e^{-i(\lambda_1^2 x + 2\lambda_1^4 t)},$$

where φ_- is the Jost function of the linear system (1.2) *and* (1.3)*. Then, $u^{(1)}(t,\cdot) = \mathbf{B}_{\lambda_1}(\eta(t,\cdot))u(t,\cdot)$ belongs to $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ for every $t \in [0,T)$ and satisfies the Cauchy problem (1.1) for $u^{(1)}(0,\cdot) = \mathbf{B}_{\lambda_1}(\eta(0,\cdot))u(0,\cdot)$.*

One way to prove Lemma 9 is to show that the time-dependent versions of the transformations (3.19a)–(3.19d) satisfy the time evolution equation (1.3) associated with the potential $u^{(1)}(t,\cdot) = \mathbf{B}_{\lambda_1}(\eta(t,\cdot))u(t,\cdot)$. By compatibility of the linear system (1.2) and (1.3) as well as smoothness of the new Jost functions and the new potential $u^{(1)}$, it then follows that $u^{(1)}(t,x)$ is a new solution of the DNLS equation $iu_t + u_{xx} + i(|u|^2u)_x = 0$.

However, the proof of the above claim is straightforward but enormously lengthy. Therefore we will avoid the technical proof and instead make use of the inverse scattering transform for the soliton-free solutions to the Cauchy problem (1.1), which was developed in the recent works [**17, 22**]. We explain the idea for the case of one soliton and then extend the argument to the case of finitely many solitons.

Let $u(t,\cdot) \in Z_1 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ be a local solution of the Cauchy problem (1.1) on $(-T,T)$ for some $T>0$. For every fixed time $t \in (-T,T)$ we find a new potential of the KN spectral problem (1.2) by means of the Bäcklund transformation $u^{(1)}(t,\cdot) = \mathbf{B}_{\lambda_1}(\eta(t,\cdot))u(t,\cdot)$. If $\lambda_1 \in \mathbb{C}_I$ is taken such that $a(\lambda_1)=0$, then $u^{(1)}(t,\cdot) \in Z_0$. On the other hand, let $\widetilde{u}(t,\cdot) \in Z_0$ be a solution to the Cauchy problem (1.1) starting with the initial condition $\widetilde{u}(0,\cdot) = u^{(1)}(0,\cdot) \in Z_0$. Since assumptions of [**22**, Theorem 1.1] are satisfied, the solution $\widetilde{u}(t,\cdot) \in Z_0$ exists for every $t \in \mathbb{R}$, in particular, for $t \in (-T,T)$. The following diagram illustrates the scheme.

$$\begin{tikzcd}
u(0,\cdot) \in Z_1 \arrow[r] \arrow[d, "\text{DNLS}"'] & u^{(1)}(0,\cdot) \in Z_0 \arrow[dr, "\text{DNLS}"] & \\
u(t,\cdot) \in Z_1 \arrow[r] & u^{(1)}(t,\cdot) \in Z_0 \arrow[r, leftrightarrow, "?"] & \widetilde{u}(t,\cdot) \in Z_0, \qquad t \in (-T,T).
\end{tikzcd}$$

Thus, the proof of Lemma 9 in the case of one soliton will rely on the proof that $\widetilde{u}(t,\cdot) = u^{(1)}(t,\cdot)$ for every $t \in (-T,T)$. To show this, we will first prove that the two functions have the same scattering data.

LEMMA 10. *For every $t \in (-T,T)$, the potentials $\widetilde{u}(t,\cdot)$ and $u^{(1)}(t,\cdot)$ produce the same scattering data.*

PROOF. We know that both functions $\widetilde{u}(t,\cdot)$ and $u^{(1)}(t,\cdot)$ remain in Z_0 for every $t \in (-T,T)$. Hence the scattering data consist only of the reflection coefficient which is introduced in [**22**]. For the potential $u(t,\cdot) \in Z_1$ with $t \in (-T,T)$, we have $r(t,\lambda) = b(t,\lambda)/a(t,\lambda)$ for $\lambda \in \mathbb{R} \cup i\,\mathbb{R}$. Let us denote by $r^{(1)}(t,\lambda) = b^{(1)}(t,\lambda)/a^{(1)}(t,\lambda)$ the reflection coefficient of $u^{(1)}(t,\cdot) \in Z_0$ for $t \in (-T,T)$. Lemma 6 and Remark 4 tell us how the old and the new reflection coefficient are connected:

$$r^{(1)}(t,\lambda) = -r(t,\lambda)\frac{\lambda_1^2}{\overline{\lambda}_1^2}\frac{\lambda^2-\overline{\lambda}_1^2}{\lambda^2-\lambda_1^2}, \quad \lambda \in \mathbb{R} \cup i\mathbb{R}, \quad t \in (-T,T). \tag{4.1}$$

When $u(t,x)$ is a solution to the DNLS equation in Z_1, the time evolution of the reflection coefficient $r(t,\lambda)$ is given by

$$r(t,\lambda) = r(0,\lambda)e^{4i\lambda^4 t}, \quad t \in (-T,T), \tag{4.2}$$

which implies by virtue of (4.1) that

$$r^{(1)}(t,\lambda) = r^{(1)}(0,\lambda)e^{4i\lambda^4 t}, \quad t \in (-T,T). \tag{4.3}$$

Note that equation (4.2) coincides with Eq. (5.2) in [**22**], where it was derived for u in Z_0. The proof for u in Z_1 is the same.

For the reflection coefficient $\widetilde{r}$ of the potential $\widetilde{u}$ we know $r^{(1)}(0,\lambda) = \widetilde{r}(0,\lambda)$ since $u^{(1)}(0,\cdot) = \widetilde{u}(0,\cdot)$. By using the time evolution of the reflection coefficient from [**22**] and the expression (4.3), we obtain

$$\widetilde{r}(t,\lambda) = \widetilde{r}(0,\lambda)e^{4i\lambda^4 t} = r^{(1)}(0,\lambda)e^{4i\lambda^4 t} = r^{(1)}(t,\lambda), \quad t \in (-T,T). \tag{4.4}$$

The assertion of the lemma is proved. $\square$

Corollary 2. *The potential* $u^{(1)}(t,\cdot) = \mathbf{B}_{\lambda_1}(\eta(t,\cdot))u(t,\cdot)$ *is a new solution of the DNLS equation for* $t \in (-T,T)$.

Proof. In [**17, 22**], existence and Lipschitz continuity of the mapping

$$L^{2,1}(\mathbb{R} \cup i\mathbb{R}) \supset X \ni r \mapsto u \in Z_0 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$$

was established by means of the solvability of the associated Riemann–Hilbert problem (see [**17, 22**] for details on X). Therefore, the mapping is bijective and $\widetilde{u}(t,\cdot) = u^{(1)}(t,\cdot)$ for every $t \in (-T,T)$ follows from Lemma 10. Since $\widetilde{u}$ is a solution of the DNLS equation, so does $u^{(1)}$. □

The proof of Lemma 9 in the case of finitely many solitons relies on the iterative use of the argument above. For a given $u \in Z_k$, $k \in \mathbb{N}$, we remove the distinct eigenvalues $\{\lambda_1, ...\lambda_k\}$ in $\mathbb{C}_I$ by iterating the Bäcklund transformation k times. We set $u^{(0)} = u$ and

$$u^{(l)} = \mathbf{B}_{\lambda_l}(\eta^{(l-1)})u^{(l-1)}, \quad (1 \le l \le k),$$

such that eventually $u^{(k)} \in Z_0$. The arguments of Lemma 10 and Corollary 2 apply to the last potential $u^{(k)}$. As a result, we know that the k-fold iteration of the Bäcklund transformation of a solution $u(t,\cdot) \in Z_k$ of the Cauchy problem (1.1) for $t \in [0,T)$ produces a new solution $u^{(k)}(t,\cdot) \in Z_0$ of the Cauchy problem (1.1). Thus, the following diagram commutes.

$$\begin{array}{ccccccc}
u(0,\cdot) \in Z_k & \longrightarrow & u^{(1)}(0,\cdot) \in Z_{k-1} & \longrightarrow & \cdots & \longrightarrow & u^{(k)}(0,\cdot) \in Z_0 \\
\text{DNLS}\downarrow & & & & & & \downarrow\text{DNLS} \\
u(t,\cdot) \in Z_k & \longrightarrow & u^{(1)}(t,\cdot) \in Z_{k-1} & \longrightarrow & \cdots & \longrightarrow & u^{(k)}(t,\cdot) \in Z_0
\end{array}$$

Remark 6. *We do not prove here that every step in the chain* $u \to u^{(1)} \to \cdots \to u^{(k)}$ *yields a solution of the DNLS equation. Although this claim is likely to be true, the proof would require the inverse scattering theory of* [**17, 22**] *to be extended to the cases of eigenvalues.*

5. An example of the Bäcklund transformation

Let us give an example of the explicit Bäcklund transformation that connects the zero and one-soliton solutions of the DNLS equation. In order to find the one-soliton solution in the explicit form, we assume that we start with a potential $u_{\lambda_1,\gamma_1} \in Z_1$ with eigenvalue $\lambda_1 \in \mathbb{C}_I$ and norming constant $\gamma_1 \neq 0$, for which the Bäcklund transformation (3.3) in Lemma 1 yields exactly the zero solution:

$$u^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u_{\lambda_1,\gamma_1} = 0. \tag{5.1}$$

For the zero solution, we know that the Jost functions of the linear system (1.2) and (1.3) are given by

$$\varphi_{\pm}^{(1)}(t,x;\lambda) = e^{-i(\lambda^2 x + 2\lambda^4 t)}e_1, \quad \phi_{\pm}^{(1)}(t,x;\lambda) = e^{i(\lambda^2 x + 2\lambda^4 t)}e_2.$$

Hence, $a^{(1)}(\lambda) = 1$ and $b^{(1)}(\lambda) = 0$. Now we set

$$\eta^{(1)}(t,x) = \frac{1}{\gamma_1\bar{\lambda}_1}e^{-i(\lambda^2 x + 2\lambda^4 t)}e_1 + \frac{1}{\bar{\lambda}_1}e^{i(\lambda^2 x + 2\lambda^4 t)}e_2. \tag{5.2}$$

By Lemma 7, the potential u_{λ_1,γ_1}, which we started with, can be recovered by means of the inverse Bäcklund transformation

$$u_{\lambda_1,\gamma_1} = \mathbf{B}_{\lambda_1}(\eta^{(1)})0. \tag{5.3}$$

Explicit calculations with (5.2) and (5.3) yield the explicit expression

$$\begin{aligned} u_{\lambda_1,\gamma_1}(t,x) = 2i(\lambda_1^2 - \overline{\lambda}_1^2)\frac{\overline{\gamma}_1}{|\gamma_1|}\frac{e^{-2i(\lambda_1^2 x + 2\lambda_1^4 t)}}{|e^{-i(\lambda_1^2 x+2\lambda_1^4 t)}|^2}\times \\ \frac{\overline{\lambda}_1|\gamma_1|^{-1}|e^{-i(\lambda_1^2 x+2\lambda_1^4 t)}|^2 + \lambda_1|\gamma_1||e^{i(\lambda_1^2 x+2\lambda_1^4 t)}|^2}{(\lambda_1|\gamma_1|^{-1}|e^{-i(\lambda_1^2 x+2\lambda_1^4 t)}|^2 + \overline{\lambda}_1|\gamma_1||e^{i(\lambda_1^2 x+2\lambda_1^4 t)}|^2)^2}, \end{aligned} \tag{5.4}$$

which coincides with the one-soliton of the DNLS equation in the literature (see e.g. [**14**]).

Remark 7. *It is less straightforward to find the explicit expressions for the Jost functions of the linear system (1.2) and (1.3) with the one-soliton potential* u_{λ_1,γ_1} *because the expressions (3.19a)–(3.19d) can only be used in one way from* $\{\varphi_\pm, \phi_\pm\}$ *to* $\{\varphi_\pm^{(1)}, \phi_\pm^{(1)}\}$*, which is hard to invert.*

Remark 8. *For sake of completeness, we can rewrite the one-soliton solution (5.4) in physical notations. By defining*

$$\omega = 4|\lambda_1|^4, \quad v = -4\mathrm{Re}(\lambda_1^2)$$

and

$$x_0 = \frac{2\ln(|\gamma_1|)}{\sqrt{4\omega - v^2}}, \quad \delta = \arg(\gamma_1) + \pi + 3\arctan\left(\frac{\mathrm{Im}(\lambda_1)}{\mathrm{Re}(\lambda_1)}\right)$$

with the obvious constraint $4\omega - v^2 > 0$*,* u_{λ_1,γ_1} *is rewritten in the form used in* [**2**]*:*

$$u_{\lambda_1,\gamma_1}(t,x) = \phi_{\omega,v}(x - vt - x_0)e^{-i\delta + i\omega t + i\frac{v}{2}(x-vt) - \frac{3}{4}i\int_\infty^{x-vt-x_0}|\phi_{\omega,v}(y)|^2 dy}, \tag{5.5}$$

where

$$\phi_{\omega,v}(x) = \left[\frac{2\sqrt{\omega}\cosh(\sqrt{4\omega - v^2}x) - v}{2(4\omega - v^2)}\right]^{-1/2}.$$

By the computations in Lemma 6 and Remark 4, we obtain

$$a(\lambda) = \frac{\overline{\lambda}_1^2}{\lambda_1^2}\frac{\lambda^2 - \lambda_1^2}{\lambda^2 - \overline{\lambda}_1^2}, \quad b(\lambda) = 0, \tag{5.6}$$

for the one-soliton potential u_{λ_1,γ_1}. We also find

$$\begin{aligned} \|u_{\lambda_1,\gamma_1}\|_{L^2}^2 &= 2\sqrt{4\omega - v^2}\int_{-\infty}^{\infty}\frac{dz}{2\sqrt{\omega}\cosh(z) - v} && (5.7a) \\ &= 8\arctan\left(\frac{2\sqrt{\omega}+v}{2\sqrt{\omega}-v}\right) && (5.7b) \\ &= 8\arctan\left(\frac{\mathrm{Im}(\lambda_1)}{\mathrm{Re}(\lambda_1)}\right) && (5.7c) \\ &= 8\arg(\lambda_1), && (5.7d) \end{aligned}$$

where we have used an explicit integral formula from [**8**, Section 2.451] in order to obtain (5.7b). The equality between (5.7c) and (5.7d) holds because of $\lambda_1 \in \mathbb{C}_I$. The computation (5.7) confirms the asymptotic limit in Proposition 3:

$$a_\infty = \lim_{|\lambda|\to\infty} a(\lambda) = \frac{\overline{\lambda}_1^2}{\lambda_1^2} = e^{-4i\arg(\lambda_1)} = e^{\frac{1}{2i}\|u_{\lambda_1,\gamma_1}\|_{L^2}^2}.$$

By using the representation (3.14) in Lemma 2, the explicit formula (5.2), as well as the relation $d_{\lambda_1}(\eta,\eta) = [d_{\lambda_1}(\eta^{(1)},\eta^{(1)})]^{-1}$, we can also find the function $\eta = (\eta_1,\eta_2)^t$ in the transformation (5.1):

$$\eta_1(t,x) = \frac{\overline{\lambda}_1 e^{-i(\overline{\lambda}_1^2 x + 2\overline{\lambda}_1^4 t)}}{\lambda_1|e^{-i(\lambda_1^2 x+2\lambda_1^4 t)}|^2 + \overline{\lambda}_1|e^{i(\lambda_1^2 x+2\lambda_1^4 t)}|^2} \tag{5.8}$$

and

$$\eta_2(t,x) = \frac{\overline{\lambda}_1 e^{i(\overline{\lambda}_1^2 x + 2\overline{\lambda}_1^4 t)}}{\overline{\lambda}_1|e^{-i(\lambda_1^2 x+2\lambda_1^4 t)}|^2 + \lambda_1|e^{i(\lambda_1^2 x+2\lambda_1^4 t)}|^2}, \tag{5.9}$$

where $\gamma_1 = 1$ is set for convenience. Since

$$d_{\lambda_1}(\eta,\eta) = \frac{|\lambda_1|^2}{\lambda_1|e^{-i(\lambda_1^2 x+2\lambda_1^4 t)}|^2 + \overline{\lambda}_1|e^{i(\lambda_1^2 x+2\lambda_1^4 t)}|^2}$$

satisfies the constraint

$$-d_{\overline{\lambda}_1}(\eta,\eta)u_{\lambda_1,\gamma_1} + 2i(\lambda_1^2 - \overline{\lambda}_1^2)\eta_1\overline{\eta}_2 = 0,$$

we confirm the transformation (5.1) by using (5.4), (5.8), and (5.9).

6. Proof of Theorem 1

Let $u_0 \in Z_1 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ and $\lambda_1 \in \mathbb{C}_I$ be the only root of $a(\lambda)$ in $\mathbb{C}_I$. By Lemma 6, if $\eta(x) = \varphi_-(x;\lambda_1)e^{-i\lambda_1^2 x}$, where φ_- is the Jost function of the KN spectral problem (1.2) associated with u_0, then $u_0^{(1)} = \mathbf{B}_{\lambda_1}(\eta)u_0$ belongs to $Z_0 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. Also, by Lemmas 1, 2, and Lemma 7, the mapping is invertible with $u_0 = \mathbf{B}_{\lambda_1}(\eta^{(1)})u_0^{(1)}$, where $\eta^{(1)}$ is expressed from the new Jost functions $\varphi_-^{(1)}$ and $\phi_+^{(1)}$ by the decomposition formula (3.24).

Let $T > 0$ be a maximal existence time for the solution $u(t,\cdot) \in Z_1$, $t \in (-T,T)$ to the Cauchy problem (1.1) with the initial data $u_0 \in Z_1$ and eigenvalue λ_1. For every fixed $t \in (-T,T)$, the solution $u(t,\cdot) \in Z_1$ admits the Jost functions $\{\varphi_\pm(t,x;\lambda), \phi_\pm(t,x;\lambda)\}$. For every $t \in (-T,T)$, define $u^{(1)}$ by the Bäcklund transformation

$$u^{(1)} := \mathbf{B}_{\lambda_1}(\eta)u, \quad \eta(t,x) := \varphi_-(t,x;\lambda_1)e^{-i(\lambda_1^2 x + 2\lambda_1^4 t)},$$

where we have used the boundary conditions (2.1) in the definition of $\varphi_-(t,x;\lambda_1)$ for every $t \in (-T,T)$.

By construction (see Corollary 2), $u^{(1)}(t,\cdot) \in Z_0$, $t \in (-T,T)$ is a solution of the Cauchy problem (1.1) with the initial data $u_0^{(1)} \in Z_0$. By existence and uniqueness theory [**17, 22**], the solution $u^{(1)}(t,\cdot) \in Z_0$ is uniquely continued for every $t \in \mathbb{R}$. Let $\{\varphi_\pm^{(1)}(t,x;\lambda), \phi_\pm^{(1)}(t,x;\lambda)\}$ be the Jost functions for $u^{(1)}(t,x)$. For every $t \in (-T,T)$, we have $u = \mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)}$ with

$$\eta^{(1)}(t,x) = \frac{e^{-i(\lambda_1^2 x + 2\lambda_1^4 t)}}{\gamma_1\overline{\lambda}_1 a^{(1)}(\lambda_1)}\varphi_-^{(1)}(t,x;\lambda_1) + \frac{e^{i(\lambda_1^2 x + 2\lambda_1^4 t)}}{\overline{\lambda}_1 a^{(1)}(\lambda_1)}\phi_+^{(1)}(t,x;\lambda_1),$$

where $a^{(1)}(\lambda_1) \neq 0$ thanks to Lemma 6.

On the other hand, since $u^{(1)}(t,\cdot) \in Z_0$ exists for every $t \in \mathbb{R}$, the associated Jost functions $\{\varphi_\pm^{(1)}(t,x;\lambda), \phi_\pm^{(1)}(t,x;\lambda)\}$ exist for every $t \in \mathbb{R}$ so that we can define

$$\tilde{u} = \mathbf{B}_{\lambda_1}(\eta^{(1)})u^{(1)} \quad t \in \mathbb{R}.$$

Since $u(t,\cdot) = \tilde{u}(t,\cdot) \in Z_1$ for every $t \in (-T,T)$ by uniqueness, the extended function $\tilde{u}$ is an unique extension of the solution u to the same Cauchy problem (1.1) that exists globally in time thanks to the bound (3.4) proven in Lemma 8. Indeed, by [**17, 22**] we have $\|u^{(1)}(t,\cdot)\|_{H^2\cap H^{1,1}} \leq M_T$ for every $t \in (-T,T)$, where $T > 0$ is arbitrary and M_T depends on T. Next, by bound (3.4) we have $\|u(t,\cdot)\|_{H^2\cap H^{1,1}} \leq C_{M_T}$ for every $t \in (-T,T)$. Thus, the solution can not blow up in a finite time and hence there exists a unique global solution $u(t,\cdot) \in Z_1$, $t \in \mathbb{R}$ to the Cauchy problem (1.1) for every $u_0 \in Z_1 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.

By iterating the Bäcklund transformation k times and by the same argument as above, we obtain the global existence of $u(t,\cdot) \in Z_k \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, $t \in \mathbb{R}$ from the global existence of $u^{(k)}(t,\cdot) \in Z_0 \subset H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, $t \in \mathbb{R}$. This completes the proof of Theorem 1.

Appendix A. Useful properties of d_λ, S_λ, and G_λ

Recall the definition (3.1) for the bilinear form d_λ acting on $\mathbb{C}^2$ for a fixed $\lambda \in \mathbb{C}$. One can easily verify the useful algebraic properties of d_λ for every $\eta \in \mathbb{C}^2$ and $a, b \in \mathbb{C}$:

$$d_\lambda(e_1,e_1) = \lambda, \quad d_\lambda(e_2,e_2) = \overline{\lambda}, \tag{A.1}$$

$$\overline{d_\lambda(\eta,\eta)} = d_{\overline{\lambda}}(\eta,\eta), \quad d_\lambda(a\eta, b\eta) = a\overline{b}d_\lambda(\eta,\eta), \tag{A.2}$$

$$d_\lambda(\sigma_3\eta,\sigma_3\eta) = d_\lambda(\eta,\eta), \quad d_\lambda(\sigma_1\eta,\sigma_1\eta) = d_{\overline{\lambda}}(\eta,\eta). \tag{A.3}$$

where $e_1 = (1,0)^t$ and $e_2 = (0,1)^t$ are basis vectors in $\mathbb{C}^2$, whereas σ_1 and σ_3 are Pauli matrices given by (1.5).

Next, we recall the definition (3.2) of the operators G_λ and S_λ acting on $\mathbb{C}^2$ for a fixed $\lambda \in \mathbb{C}$. From (A.1) and (A.2), G_λ and S_λ satisfy for every $\eta \in \mathbb{C}^2$ and nonzero $a \in \mathbb{C}$:

$$G_\lambda(e_1) = \frac{\overline{\lambda}}{\lambda}, \quad G_\lambda(e_2) = \frac{\lambda}{\overline{\lambda}}, \quad \overline{G_\lambda(\eta)} = G_{\overline{\lambda}}(\eta), \quad G_\lambda(a\eta) = G_\lambda(\eta) \tag{A.4}$$

and

$$S_\lambda(e_1) = S(e_2) = 0, \quad S_\lambda(a\eta) = S_\lambda(\eta). \tag{A.5}$$

From (A.3), we also have

$$G_\lambda(\sigma_3\eta) = G_\lambda(\eta), \quad G_\lambda(\sigma_1\eta) = G_{\overline{\lambda}}(\eta) \tag{A.6}$$

and

$$-S_\lambda(\sigma_3\eta) = S_\lambda(\eta), \quad S_\lambda(\sigma_1\eta) = \overline{S_\lambda(\eta)}. \tag{A.7}$$

By using (A.4), one can verify the following properties for every $\lambda \neq \pm\lambda_1$:

$$\frac{\lambda^2 \frac{\lambda_1}{\overline{\lambda}_1} G_{\lambda_1}(e_1) - \lambda_1^2}{\lambda^2 - \lambda_1^2} = 1, \quad \frac{\lambda^2 \frac{\lambda_1}{\overline{\lambda}_1} G_{\lambda_1}(e_2) - \lambda_1^2}{\lambda^2 - \lambda_1^2} = \frac{\lambda_1^2}{\overline{\lambda}_1^2}\frac{\lambda^2 - \overline{\lambda}_1^2}{\lambda^2 - \lambda_1^2} \tag{A.8}$$

and

$$\frac{\lambda^2 \frac{\lambda_1}{\overline{\lambda}_1} G_{\overline{\lambda}_1}(e_2) - \lambda_1^2}{\lambda^2 - \lambda_1^2} = 1, \quad \frac{\lambda^2 \frac{\lambda_1}{\overline{\lambda}_1} G_{\overline{\lambda}_1}(e_1) - \lambda_1^2}{\lambda^2 - \lambda_1^2} = \frac{\lambda_1^2}{\overline{\lambda}_1^2} \frac{\lambda^2 - \overline{\lambda}_1^2}{\lambda^2 - \lambda_1^2}. \tag{A.9}$$

Appendix B. On regularity of Jost functions

Recall that if $u \in H^{1,1}(\mathbb{R})$, then $u \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\partial_x u \in L^1(\mathbb{R})$, so that the assumptions of Propositions 1 and 2 are satisfied. In what follows, we establish more regularity results for Jost functions compared to what was established previously in [**22**].

Lemma B. *For every $u \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ satisfying $\|u\|_{H^2(\mathbb{R})\cap H^{1,1}(\mathbb{R})} \leq M$ for some $M > 0$, let $\varphi_\pm(x;\lambda)e^{-i\lambda^2 x}$ and $\phi_\pm(x;\lambda)e^{+i\lambda^2 x}$ be Jost functions of the KN spectral problem* (1.2) *given in Propositions 1 and 2. Fix $\lambda_1 \in \mathbb{C}$ satisfying $Im(\lambda_1^2) > 0$ and denote $\varphi_- := \varphi_-(\cdot;\lambda_1) = (\varphi_{-,1}, \varphi_{-,2})^t$ and $\phi_+ := \phi_+(\cdot;\lambda_1) = (\phi_{+,1}, \phi_{+,2})^t$. Then,*

$$\|\langle x\rangle \varphi_{-,2}\|_{L^2(\mathbb{R})} + \|\langle x\rangle \partial_x \varphi_-\|_{L^2(\mathbb{R})} + \|\langle x\rangle \partial_x^2 \varphi_-\|_{L^2(\mathbb{R})} + \|\partial_x^3 \varphi_-\|_{L^2(\mathbb{R})} \leq C_M, \tag{B.1}$$

and

$$\|\langle x\rangle \phi_{+,1}\|_{L^2(\mathbb{R})} + \|\langle x\rangle \partial_x \phi_+\|_{L^2(\mathbb{R})} + \|\langle x\rangle \partial_x^2 \phi_+\|_{L^2(\mathbb{R})} + \|\partial_x^3 \phi_+\|_{L^2(\mathbb{R})} \leq C_M, \tag{B.2}$$

where the constant C_M does not depend on u.

Proof. We will prove the statement for φ_- since the proof for ϕ_+ is similar. From Proposition 1, we know that φ_- belongs to $L^\infty(\mathbb{R})$. Let us first show that the second component $\varphi_{-,2}$ is square integrable. Compared with Lemma 1 in [**22**], where the existence of Jost functions is proved uniformly in λ, the assertion of this proposition is easier to prove for just one $\lambda = \lambda_1$. We can work with the integral equation for φ_-:

$$\varphi_- = e_1 + K\varphi_-,$$

where the operator K is given as

$$K\varphi_- = \lambda_1 \int_{-\infty}^{x} \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\lambda_1^2(x-y)} \end{bmatrix} Q(u(y)) \varphi_-(y;\lambda) dy.$$

This integral operator can be bounded as follows,

$$\begin{bmatrix} \|(K\varphi_-)_1\|_{L^\infty(-\infty,x_0)} \\ \|(K\varphi_-)_2\|_{L^2(-\infty,x_0)} \end{bmatrix} \leq |\lambda_1| \|u\|_{L^2(-\infty,x_0)} \begin{bmatrix} 0 & 1 \\ \frac{1}{2\mathrm{Im}(\lambda_1^2)} & 0 \end{bmatrix} \begin{bmatrix} \|\varphi_{-,1}\|_{L^\infty(-\infty,x_0)} \\ \|\varphi_{-,2}\|_{L^2(-\infty,x_0)} \end{bmatrix}.$$

Thus, we deduce that for every fixed λ_1 satisfying $\mathrm{Im}(\lambda_1^2) > 0$, there exists $x_0 \in \mathbb{R}$ such that K is a contraction on $L^\infty(-\infty,x_0) \times L^2(-\infty,x_0)$. Since $u \in L^2(\mathbb{R})$, we can divide $\mathbb{R}$ into finitely many subintervals such that K is a contraction as shown above within each subinterval. By patching solutions together, we obtain that $\varphi_{-,2}$ belongs to $L^2(\mathbb{R})$ and satisfies

$$\|\varphi_{-,2}\|_{L^2(\mathbb{R})} \leq C_M \|u\|_{L^2(\mathbb{R})} \tag{B.3}$$

where C_M does not depend on u.

Next, it follows directly from the Kaup–Newell system (1.2) that

$$\partial_x \varphi_{-,1} = \lambda_1 u \varphi_{-,2} \Longrightarrow \partial_x \varphi_{-,1} \in L^2(\mathbb{R})$$

and

$$\partial_x \varphi_{-,2} = -\lambda_1 \overline{u} \varphi_{-,1} + 2i\lambda_1^2 \varphi_{-,2} \Longrightarrow \partial_x \varphi_{-,2} \in L^2(\mathbb{R}).$$

Differentiating (1.2) once and twice, we also obtain $\partial_x^2 \varphi_-, \partial_x^3 \varphi_- \in L^2(\mathbb{R})$.

In order to show $x\varphi_{-,2} \in L^2(\mathbb{R})$, we write

$$\partial_x (x\varphi_{-,2}) = \varphi_{-,2} + x\partial_x \varphi_{-,2}$$

and use the second component of the Kaup–Newell system (1.2) to get

$$\partial_x (x\varphi_{-,2}) = 2i\lambda_1^2 x\varphi_{-,2} + \varphi_{-,2} - \lambda x \overline{u} \varphi_{-,2}$$

which yields the integral equation

$$x\varphi_{-,2}(x) = \int_{-\infty}^{x} e^{2i\lambda_1^2 (x-y)} \varphi_{-,2}(y) dy - \lambda_1 \int_{-\infty}^{x} e^{2i\lambda_1^2 (x-y)} y \overline{u}(y) \varphi_{-,2}(y) dy.$$

Since the right hand side is bounded in $L^2(\mathbb{R})$, we have $x\varphi_{-,2}$ in $L^2(\mathbb{R})$. Then, it follows from system (1.2) and its derivative that $x\partial_x \varphi_-, x\partial_x^2 \varphi_- \in L^2(\mathbb{R})$. Combining all estimates together, we obtain bounds (B.1) for φ_-. □

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