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Nonexistence of self-similar blowup for the nonlinear Dirac equations in (1+1) dimensions

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ABSTRACT

We address a general system of nonlinear Dirac equations in (1+1) dimensions and prove nonexistence of self-similar blowup solutions in the space of bounded functions. While this argument does not exclude the possibility of finite-time blowup, it still suggests that self-similar singularities do not develop in the nonlinear Dirac equations in (1+1) dimensions in a finite time. In the particular case of the cubic Dirac equations, we characterize (unbounded) self-similar solutions in the closed analytical form.

1. Introduction

Smooth solutions of many nonlinear dispersive wave equations may blow up in a finite time depending on the power of nonlinearity. The classical example is the nonlinear Schrödinger equation (NLS) with power nonlinearity, where smooth solutions are global only in the subcritical case. For critical and supercritical powers, smooth solutions of the NLS may blow up in a finite time [1,2].

Nonlinear Dirac equations are considered to be the relativistic generalization of the NLS equation, yet they display many new dynamical properties compared to the NLS equation [3]. In particular, smooth solutions to many examples of the nonlinear Dirac equations in (1+1) dimensions escape blowup in a finite time [4–8].

The general system of massless nonlinear Dirac equations in (1+1) dimensions can be written in the form:

$$\begin{cases} i(\partial_t U_1 + \partial_x U_1) = \partial_{\bar{U}_1} W(U_1, U_2, \bar{U}_1, \bar{U}_2), \\ i(\partial_t U_2 - \partial_x U_2) = \partial_{\bar{U}_2} W(U_1, U_2, \bar{U}_1, \bar{U}_2), \end{cases} \tag{1.1}$$

where $(U_1, U_2) : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \times \mathbb{C}$, $\bar{U}$ is a complex conjugate of U, and the nonlinear potential W is assumed to satisfy the following properties:

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(1) Symmetry: $W(U_1, U_2, \bar{U}_1, \bar{U}_2) = W(U_2, U_1, \bar{U}_2, \bar{U}_1)$
(2) Phase invariance: $W(e^{i\theta}U_1, e^{i\theta}U_2, e^{-i\theta}\bar{U}_1, e^{-i\theta}\bar{U}_2) = W(U_1, U_2, \bar{U}_1, \bar{U}_2)$ with $\theta \in \mathbb{R}$.
(3) Homogeneous polynomial in $(U_1, U_2, \bar{U}_1, \bar{U}_2)$.

It was shown in [9] that the nonlinear potential W can be characterized as a homogeneous polynomial in variables $(|U_1|^2 + |U_2|^2)$, $|U_1|^2|U_2|^2$, and $(\bar{U}_2 U_1 + U_2 \bar{U}_1)$. In particular, the most general quartic polynomial for W is represented by

$$W = a_1|U_1|^2|U_2|^2 + a_2(\bar{U}_1 U_2 + \bar{U}_2 U_1)^2 + a_3(|U_1|^4 + |U_2|^4) + a_4(|U_1|^2 + |U_2|^2)(\bar{U}_1 U_2 + \bar{U}_2 U_1),$$

where (a_1, a_2, a_3, a_4) are real constants.

When $W = |U_1|^2|U_2|^2$, the system (1.1) is called Thirring model [10]. The Cauchy problem for the Thirring model was found to be globally well-posed in Sobolev space $H^s(\mathbb{R})$ with $s \in \mathbb{N}$ [4] and in $L^2(\mathbb{R})$ [11–13]. Orbital stability of solitary wave solutions in the massive Thirring model was proven in [14,15].

When $W = (\bar{U}_1 U_2 + \bar{U}_2 U_1)^2$, the system (1.1) is called Gross–Neveu model [16]. The Cauchy problem for the Gross–Neveu model was proven to be globally well-posed in $H^s(\mathbb{R})$ with $s > 1/2$ [8,17] by obtaining bounds on the $L^\infty(\mathbb{R})$ of the solution and in $L^2(\mathbb{R})$ [18,19] by using characteristics. Spectral stability of solitary wave solutions in the massive Gross–Neveu model was studied numerically in the general case [20–22] and analytically in the nonrelativistic limit [23,24].

When $W = |U_1|^4 + 4|U_1|^2|U_2|^2 + |U_2|^4$, the system (1.1) is called the coupled-mode model [5]. The Cauchy problem for the coupled-mode system was found to be well-posed in Sobolev space $H^s(\mathbb{R})$ with $s \in \mathbb{N}$ [5,7] and in $L^2(\mathbb{R})$ [6]. Existence and spectral stability of solitary wave solutions have been analyzed in this model in many details (see [9] and references therein).

Finally, when $W = (|U_1|^2 + |U_2|^2)(\bar{U}_1 U_2 + \bar{U}_2 U_1)$, the nonlinear Dirac equation with pseudoscalar potential [25] occurs in the context of photonic crystals with the nonlinear refractive index [26]. As far as we know, it has been an open problem for many years to address global existence or finite time blowup of solutions to the Cauchy problem for this system [7]. This problem is the subject of the present work.

The self-similar blowup has played an important role in the formation of singularities of partial differential equations (see [27] and references therein). In particular, self-similar blowup solutions have been investigated for the nonlinear Schrödinger equation [28,29], the relativistic wave equation [30–32], and for the Navier–Stokes equations [33].

The main goal of this study is to prove nonexistence of self-similar blowup solutions in the space of bounded functions to the nonlinear Dirac equation (1.1) with the nonlinear potential in the form:

$$W = (|U_1|^2 + |U_2|^2)^k (\bar{U}_1 U_2 + \bar{U}_2 U_1)^\ell, \tag{1.2}$$

where k, ℓ are nonnegative integers with $p := k + \ell - 1 \in \mathbb{N}$. Besides the space and time translation invariance, the system of nonlinear Dirac equations (1.1) with (1.2) has the following scaling invariance property: if $[U_1(x,t),\, U_2(x,t)]$ is a solution, then

$$\left[\lambda^{\frac{1}{2p}} U_1(\lambda x, \lambda t),\, \lambda^{\frac{1}{2p}} U_2(\lambda x, \lambda t)\right], \quad \lambda > 0$$

is also a solution of the same system. Thanks to the scaling invariance property and the separation of variables, the self-similar solutions to nonlinear Dirac equations (1.1) with (1.2) are defined in the form:

$$U_1(x,t) = \frac{1}{(1-t)^{\frac{1}{2p}}} U\left(\frac{x}{1-t}\right), \quad U_2(x,t) = \frac{1}{(1-t)^{\frac{1}{2p}}} V\left(\frac{x}{1-t}\right), \tag{1.3}$$

where U and V are functions of $y := x/(1-t)$. A singularity of the self-similar solutions (1.3) is placed at the point $(x,t) = (0,1)$ thanks to the space and time translation symmetries. Thanks to the unit speed of propagation, the variable y can be restricted to the interval $[-1, 1]$.

Existence of bounded solutions $(U,V) \in L^\infty([-1,1])$ implies self-similar blowup of smooth solutions to the Cauchy problem to the nonlinear Dirac equations (1.1) with (1.2) in a finite time because global solutions $(U_1,U_2) \in C(\mathbb{R},H^s(\mathbb{R}))$ with $s>1/2$ belong to $L^\infty(\mathbb{R})$ for every $t \in \mathbb{R}$ thanks to the Sobolev embedding of $H^s(\mathbb{R})$ into $L^\infty(\mathbb{R})$. While nonexistence of bounded self-similar solutions does not exclude the possibility of finite-time blowup completely, it still suggests that self-similar singularities do not develop in the nonlinear Dirac equations in a finite time.

The following theorem presents the main result of this work.

Theorem 1.1. *For every $p \in \mathbb{N}$, there exist no self-similar solutions in the form (1.3) with $(U,V) \in C^1(-1,1) \cap L^\infty([-1,1])$.*

Theorem 1.1 is proven in Section 2 by using the polar decomposition, dynamical system methods, and a continuation argument. The proof is simpler in the case of odd ℓ and more technically involved in the case of even ℓ. Note that the nonexistence of self-similar blowup solutions in Theorem 1.1 is guaranteed by the global well-posedness results in the particular cases: $k \in \mathbb{N}$, $\ell=0$ [5,7] and $k=0$, $\ell=2$ [8,17].

Bounded self-similar solutions in the form (1.3) do not exist because U and V break either before they reach the end points $y=\pm1$ of the interval $[-1,1]$ or at the end points $y=\pm1$. In the general case, we are not able to obtain the precise rate of how U and V diverge before or at $y=\pm1$. However, in the case $k=\ell=1$, which corresponds to the physically relevant model (1.1) with $W=(|U_1|^2+|U_2|^2)(\bar{U}_1U_2+\bar{U}_2U_1)$ derived in [26], we are able to prove the following theorem.

Theorem 1.2. *For $k=\ell=1$, there exists a unique local self-similar solution in the form (1.3) with $|U(0)|=|V(0)|$ that extends to $y\to1$ and satisfies the following asymptotic behavior*

$$U(y) \sim (1-y)^{\frac{1}{4}}, \quad V(y) \sim (1-y)^{-\frac{1}{4}} \quad \text{as} \quad y \to 1. \tag{1.4}$$

However, this solution does not extend to $y\to-1$ in the sense that there exists $y_0 \in (-1,0)$ such that $\lim_{y\to y_0} U(y)$ and $\lim_{y\to y_0} V(y)$ diverge. All other local solutions with $|U(0)|=|V(0)|$ extend neither to $y\to1$ nor to $y\to-1$.

Theorem 1.2 is proven in Section 3, where the system of differential equations for U and V with the initial condition $|U(0)|=|V(0)|$ is integrated in a closed form. Although we are not able to integrate the system of differential equations for U and V with $|U(0)| \neq |V(0)|$, the same method used in the proof of Theorem 1.2 suggests that more general solutions with $|U(0)| \neq |V(0)|$ do not extend simultaneously to $y\to1$ and $y\to-1$ (see Remark 3.1). Therefore, bounded self-similar solutions in the form (1.3) do not exist in Theorem 1.1 because they blow up before reaching $y=\pm1$ at least for $k=\ell=1$.

2. Proof of Theorem 1.1

Substituting (1.3) into (1.1) with W in (1.2) yields the following system of differential equations for U and V:

$$\begin{cases} i\left[(y+1)U' + \frac{1}{2p}U\right] = FV + GU, \\ i\left[(y-1)V' + \frac{1}{2p}V\right] = FU + GV, \end{cases} \tag{2.1}$$

where the prime denotes derivative in y, $p=k+\ell-1$, and

$$F = \ell(|U|^2+|V|^2)^k(U\bar{V}+\bar{U}V)^{\ell-1}, \qquad G = k(|U|^2+|V|^2)^{k-1}(U\bar{V}+\bar{U}V)^{\ell}.$$

We are studying existence of bounded solutions to the system (2.1) on the interval $[-1,1]$ including the limits $y\to\pm1$, hence we require $(U,V) \in L^\infty([-1,1])$. By the ODE theory, bounded solutions to the system (2.1) belong to $(U,V) \in C^1(-1,1)$.

By inspecting the integrating factors for the left-hand side of the system (2.1), we introduce the new variables:

$$u(y) := (1+y)^{1/2p} U(y), \qquad v(y) := (1-y)^{1/2p} V(y). \tag{2.2}$$

New variables allow us to rewrite the system (2.1) in the equivalent form:

$$\begin{cases} i(1+y)^{1-\frac{1}{2p}} u' = Fv(1-y)^{-\frac{1}{2p}} + Gu(1+y)^{-\frac{1}{2p}}, \\ -i(1-y)^{1-\frac{1}{2p}} v' = Fu(1+y)^{-\frac{1}{2p}} + Gv(1-y)^{-\frac{1}{2p}}, \end{cases} \tag{2.3}$$

where

$$F = \ell \left(\frac{|u|^2}{(1+y)^{\frac{1}{p}}} + \frac{|v|^2}{(1-y)^{\frac{1}{p}}} \right)^k \frac{(u\bar{v} + \bar{u}v)^{\ell-1}}{(1-y^2)^{\frac{\ell-1}{2p}}}$$

and

$$G = k \left(\frac{|u|^2}{(1+y)^{\frac{1}{p}}} + \frac{|v|^2}{(1-y)^{\frac{1}{p}}} \right)^{k-1} \frac{(u\bar{v} + \bar{u}v)^{\ell}}{(1-y^2)^{\frac{\ell}{2p}}}.$$

If $(U,V) \in C^1(-1,1) \cap L^\infty([-1,1])$, then $(u,v) \in C^1(-1,1) \cap L^\infty([-1,1])$ with $u(-1) = 0$ and $v(1) = 0$.

Let us use the polar decomposition for complex-valued amplitudes:

$$u = |u| e^{i\alpha}, \quad v = |v| e^{i\beta}, \tag{2.4}$$

where all functions depend on y and the phases (α, β) are defined uniquely on the torus $\mathbb{T} := [-\pi, \pi]$ closed with the periodic boundary conditions. Because $(u,v) \in C^1(-1,1)$,

$$\frac{d|u|}{dy}, \quad \frac{d|v|}{dy}, \quad |u|\frac{d\alpha}{dy}, \quad |v|\frac{d\beta}{dy}$$

are all bounded and piecewise continuous on the interval $(-1,1)$. Therefore, substituting the polar decomposition (2.4) into (2.3) and separating the real and imaginary parts yield the following system of differential equations for amplitudes and phases:

$$\begin{cases} \dfrac{d|u|}{dy} = \dfrac{F|v|\sin(\beta-\alpha)}{(1+y)^{1-\frac{1}{2p}}(1-y)^{\frac{1}{2p}}}, \\ \dfrac{d|v|}{dy} = \dfrac{F|u|\sin(\beta-\alpha)}{(1+y)^{\frac{1}{2p}}(1-y)^{1-\frac{1}{2p}}}, \end{cases} \tag{2.5}$$

and

$$\begin{cases} -|u|\dfrac{d\alpha}{dy} = \dfrac{F|v|\cos(\beta-\alpha)}{(1+y)^{1-\frac{1}{2p}}(1-y)^{\frac{1}{2p}}} + \dfrac{G|u|}{(1+y)}, \\ |v|\dfrac{d\beta}{dy} = \dfrac{F|u|\cos(\beta-\alpha)}{(1+y)^{\frac{1}{2p}}(1-y)^{1-\frac{1}{2p}}} + \dfrac{G|v|}{(1-y)}, \end{cases} \tag{2.6}$$

where

$$F = \ell \left(\frac{|u|^2}{(1+y)^{\frac{1}{p}}} + \frac{|v|^2}{(1-y)^{\frac{1}{p}}} \right)^k \frac{[2|u||v|\cos(\beta-\alpha)]^{\ell-1}}{(1-y^2)^{\frac{\ell-1}{2p}}}$$

and

$$G = k \left(\frac{|u|^2}{(1+y)^{\frac{1}{p}}} + \frac{|v|^2}{(1-y)^{\frac{1}{p}}} \right)^{k-1} \frac{[2|u||v|\cos(\beta-\alpha)]^{\ell}}{(1-y^2)^{\frac{\ell}{2p}}}.$$

The vector field of the system (2.5) and (2.6) is piecewise continuous on $(-1,1)$. We shall now proceed differently depending whether ℓ is zero, odd, or even.

2.1. The case of $\ell = 0$

In this case, $F = 0$ and the system (2.5) implies that $|u(y)|$ and $|v(y)|$ are constant in y. Therefore, it is impossible to satisfy $u(-1) = 0$ and $v(1) = 0$ except for the trivial (zero) solution.

2.2. The case of odd ℓ

Since ℓ is odd, we have $F \geq 0$ and $G\cos(\beta-\alpha) \geq 0$. Combining the two equations in the system (2.6) yields

$$\begin{aligned} |u||v|\frac{d}{dy}\sin(\beta-\alpha) = F\cos^2(\beta-\alpha)\left(\frac{|v|^2}{(1+y)^{1-\frac{1}{2p}}(1-y)^{\frac{1}{2p}}} + \frac{|u|^2}{(1+y)^{\frac{1}{2p}}(1-y)^{1-\frac{1}{2p}}}\right) \\ + \frac{2G|u||v|\cos(\beta-\alpha)}{1-y^2} \geq 0. \end{aligned} \tag{2.7}$$

From here, we obtain a contradiction against the existence of solutions $(u,v) \in C^1(-1,1) \cap L^\infty([-1,1])$ satisfying $u(-1) = 0$ and $v(1) = 0$.

Indeed, if $u(-1) = 0$, then $\frac{d}{dy}|u| \geq 0$ at least near $y = -1$. The first equation of the system (2.5) with odd ℓ implies $\sin(\beta-\alpha) \geq 0$ at least near $y = -1$. Thanks to monotonicity (2.7), we have $\sin(\beta-\alpha) \geq 0$ for every $y \in (-1,1)$. The second equation of the system (2.5) with odd ℓ implies then that $\frac{d}{dy}|v| \geq 0$ for every $y \in (-1,1)$. Hence $|v(y)| \geq |v(-1)|$ for every $y \in (-1,1)$ and it is impossible to satisfy $v(1) = 0$ except for the trivial (zero) solution.

2.3. The case of even ℓ

Since ℓ is even, we have $G \geq 0$ and $F\cos(\beta-\alpha) \geq 0$. Combining the two equations in the system (2.6) yields

$$\begin{aligned} |u||v|\frac{d}{dy}(\beta-\alpha) = F\cos(\beta-\alpha)\left(\frac{|v|^2}{(1+y)^{1-\frac{1}{2p}}(1-y)^{\frac{1}{2p}}} + \frac{|u|^2}{(1+y)^{\frac{1}{2p}}(1-y)^{1-\frac{1}{2p}}}\right) \\ + \frac{2G|u||v|}{1-y^2} \geq 0. \end{aligned} \tag{2.8}$$

As $\ell \geq 2$, we have $F = G = 0$ if $\cos(\beta-\alpha) = 0$, hence $\beta-\alpha = \pm\frac{\pi}{2}$ are invariant lines, which cannot be crossed for finite $y \in (-1,1)$. From here, we obtain a contradiction against the existence of solutions $(u,v) \in C^1(-1,1) \cap L^\infty([-1,1])$ satisfying $u(-1) = 0$ and $v(1) = 0$.

Indeed, if $u(-1) = 0$, then $\frac{d}{dy}|u| \geq 0$ at least near $y = -1$. The first equation of the system (2.5) with even ℓ implies $\sin(\beta-\alpha)\cos(\beta-\alpha) \geq 0$ at least near $y = -1$. Thanks to monotonicity (2.8) and invariance of $\beta-\alpha = \pm\frac{\pi}{2}$, we have

$$\text{either} \quad 0 \leq \beta-\alpha \leq \frac{\pi}{2} \quad \text{or} \quad \pi \leq \beta-\alpha \leq \frac{3\pi}{2},$$

for every $y \in (-1,1)$, which means that $\sin(\beta-\alpha)\cos(\beta-\alpha) \geq 0$ for every $y \in (-1,1)$. The second equation of the system (2.5) with even ℓ implies then that $\frac{d}{dy}|v| \geq 0$ for every $y \in (-1,1)$. Hence $|v(y)| \geq |v(-1)|$ for every $y \in (-1,1)$ and it is impossible to satisfy $v(1) = 0$ except for the trivial (zero) solution.

3. Proof of Theorem 1.2

Here we investigate the case of $k = \ell = 1$ in the system (2.5) and (2.6). The system is rewritten explicitly as follows:

$$\begin{cases} \dfrac{d|u|}{dy} = \dfrac{|v|\sin(\beta-\alpha)}{\sqrt{1-y^2}}\left[\dfrac{|u|^2}{1+y}+\dfrac{|v|^2}{1-y}\right], \\ \dfrac{d|v|}{dy} = \dfrac{|u|\sin(\beta-\alpha)}{\sqrt{1-y^2}}\left[\dfrac{|u|^2}{1+y}+\dfrac{|v|^2}{1-y}\right], \end{cases} \tag{3.1}$$

and

$$\begin{cases} -|u|\dfrac{d\alpha}{dy} = \dfrac{|v|\cos(\beta-\alpha)}{\sqrt{1-y^2}}\left[\dfrac{3|u|^2}{1+y}+\dfrac{|v|^2}{1-y}\right], \\ |v|\dfrac{d\beta}{dy} = \dfrac{|u|\cos(\beta-\alpha)}{\sqrt{1-y^2}}\left[\dfrac{|u|^2}{1+y}+\dfrac{3|v|^2}{1-y}\right]. \end{cases} \tag{3.2}$$

We are looking for local solutions $(u,v) \in C^1(-1,1)$ satisfying the constraint $|u(0)| = |v(0)|$ on the initial condition. The system (3.1) yields the first-order invariant

$$|u(y)|^2 = |v(y)|^2 + C, \tag{3.3}$$

where C is constant. It follows from the constraint $|u(0)| = |v(0)|$ that $C = 0$, hence $|u(y)| = |v(y)|$ for every $y \in [-1,1]$. With this reduction, the system (3.1) and (3.2) reduces to a simpler form:

$$\begin{cases} \dfrac{d|v|}{dy} = \dfrac{2|v|^3\sin(\beta-\alpha)}{\sqrt{(1-y^2)^3}}, \\ \dfrac{d(\beta-\alpha)}{dy} = \dfrac{8|v|^2\cos(\beta-\alpha)}{\sqrt{(1-y^2)^3}}. \end{cases} \tag{3.4}$$

Let us introduce the independent variable $\tau : [-1,1] \mapsto \mathbb{R}$ by

$$\tau(y) := \int_0^y \frac{dy}{\sqrt{(1-y^2)^3}}. \tag{3.5}$$

Then, $\tau(y) \to \pm\infty$ as $y \to \pm 1$. Let us also rewrite the system (3.4) in dependent variables

$$\xi := |v|, \quad \eta := \sin(\beta-\alpha). \tag{3.6}$$

Then, the system (3.4) can be written as the autonomous planar dynamical system:

$$\begin{cases} \dot{\xi} = 2\xi^3\eta, \\ \dot{\eta} = 8\xi^2(1-\eta^2), \end{cases} \tag{3.7}$$

where the dot denotes derivative with respect to τ. The line segment $\Sigma_0 := \{\xi = 0,\ \eta \in [-1,1]\}$ consists of the degenerate critical points, whereas $\Sigma_\pm := \{\xi \in \mathbb{R},\ \eta = \pm 1\}$ are invariant lines with the one-dimensional flow in ξ given by $\dot{\xi} = \pm 2\xi^3$.

The system (3.7) is integrable with the first invariant $E(\xi,\eta) := \xi^8(1-\eta^2)$, where the values of E are constant and $E \geq 0$ since $\eta \in [-1,1]$. The value $E = 0$ is not isolated since Σ_0 intersects $\Sigma_\pm$. The flow on $\Sigma_\pm$ is given by $\dot{\xi} = \pm 2\xi^3$. For both signs, $\xi(\tau)$ does not exist for every $\tau \in \mathbb{R}$ since it blows up in a finite τ either before $\tau \to -\infty$ or before $\tau \to +\infty$. For the minus sign, the solution satisfies $\lim_{\tau\to+\infty}\xi(\tau) = 0$ (that is, $|u(y)| \to 0$ as $y \to 1$) and moreover

$$\xi(\tau) \sim \tau^{-1/2} \quad \Rightarrow \quad |u(y)| \sim (1-y)^{1/4}.$$

This provides the asymptotic scaling (1.4) in variables U and V thanks to the transformation (2.2).

For every $E > 0$, the level curve $E(\xi, \eta) = E > 0$ is unbounded in ξ and does not intersect Σ_0 or $\Sigma_\pm$. It follows from the second equation in the system (3.7) that $\dot{\eta} > 0$, hence the map $\tau \to \eta$ is strictly increasing along the flow with $\eta \in (-1, 1)$. It follows from the first equation of the system (3.7) with $\eta \geq \eta_0 > 0$ that $\dot{\xi} \geq 2\eta_0 \xi^3$. Since the sub-solution of $\dot{\xi}_- = 2\eta_0 \xi_-^3$ with $\xi_-(0) > 0$ blows up in a finite time, the comparison principle implies that $\xi(\tau) \geq \xi_-(\tau)$ for all $\tau > 0$ so that the map $\tau \to \xi$ blows up before $\tau \to +\infty$ in the positive flow in τ. Similarly, it follows from the first equation of the system (3.7) with $\eta \leq -\eta_0 < 0$ that $\dot{\xi} \leq -2\eta_0 \xi^3$. By the comparison principle, the map $\tau \to \xi$ blows up before $\tau \to -\infty$ in the negative flow in τ. Therefore, no other solutions bounded near $y = 1$ exist.

Remark 3.1. For general initial conditions with $|u(0)| \neq |v(0)|$, we have $C \neq 0$ in the local invariant (3.3). For solutions bounded near $y = 1$, we have $v(1) = 0$ and $C = |u(1)|^2 > 0$. For solutions bounded near $y = -1$, we have $u(-1) = 0$ and $C = -|v(-1)|^2 < 0$. In the former case $C > 0$, the system of differential equations takes the form

$$\begin{cases} \dfrac{d|v|}{dy} = \dfrac{\sqrt{C + |v|^2} \sin(\beta - \alpha)}{\sqrt{(1-y^2)^3}} \left[2|v|^2 + C(1-y) \right], \\ \dfrac{d(\beta - \alpha)}{dy} = \dfrac{\cos(\beta - \alpha)}{\sqrt{(1-y^2)^3} |v| \sqrt{C + |v|^2}} \left[8|v|^2 + 2C(4-y)|v|^2 + C^2(1-y) \right], \end{cases} \tag{3.8}$$

so that the same definitions for τ, ξ and η as in (3.5) and (3.6) can be employed. The same monotonicity argument for the map $\tau \to \eta$ and the same comparison principle for the map $\tau \to \xi$ can be employed to show that the solutions bounded near $y = 1$ blow up at a finite $y_0 \in (-1, 0)$ before the other end $y = -1$. However, if $C \neq 0$, solutions extending to $y \to 1$ do not satisfy the same asymptotic behavior as in (1.4).

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