INVERSE SCATTERING FOR THE MASSIVE THIRRING MODEL

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ABSTRACT. We consider the massive Thirring model in the laboratory coordinates and explain how the inverse scattering transform can be developed with the Riemann–Hilbert approach. The key ingredient of our technique is to transform the corresponding spectral problem to two equivalent forms: one is suitable for the spectral parameter at the origin and the other one is suitable for the spectral parameter at infinity. Global solutions to the massive Thirring model are recovered from the reconstruction formulae at the origin and at infinity.

1. INTRODUCTION

The massive Thirring model (MTM) was derived by Thirring in 1958 [33] in the context of general relativity. It represents a relativistically invariant nonlinear Dirac equation in the space of one dimension. Another relativistically invariant one-dimensional Dirac equation is given by the Gross–Neveu model [12] also known as the massive Soler model [32] when it is written in the space of three dimensions.

It was discovered in 1970s by Mikhailov [24], Kuznetsov and Mikhailov [21], Orfanidis [25], Kaup and Newell [18] that the MTM is integrable with the inverse scattering transform method in the sense that it admits a Lax pair, countably many conserved quantities, the Bäcklund transformation, and other common features of integrable models. We write the MTM system in the laboratory coordinates by using the normalized form:

$$\begin{cases} i(u_t + u_x) + v + |v|^2 u = 0, \\ i(v_t - v_x) + u + |u|^2 v = 0. \end{cases} \tag{1.1}$$

The MTM system (1.1) appears as the compatibility condition in the Lax representation

$$L_t - A_x + [L, A] = 0, \tag{1.2}$$

where the 2×2-matrices L and A are given by

$$L = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{1.3}$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3. \tag{1.4}$$

Other forms of L and A with nonzero trace have also been introduced by Barashenkov and Getmanov [1]. The traceless representation of L and A in (1.3) and (1.4) is more useful for inverse scattering transforms.

Formal inverse scattering results for the linear operators (1.3) and (1.4) can be found in [21]. The first steps towards rigorous developments of the inverse scattering transform for the MTM system (1.1) were made in 1990s by Villarroel [34] and Zhou [38]. In the former work, the treatment of the Riemann–Hilbert problems is sketchy, whereas in the latter work, an abstract method to

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solve Riemann–Hilbert problems with rational spectral dependence is developed with applications to the sine–Gordon equation in the laboratory coordinates. Although the MTM system (1.1) does not appear in the list of examples in [38], one can show that the abstract method of Zhou is also applicable to the MTM system.

The present paper relies on recent progress in the inverse scattering transform method for the derivative NLS equation [27, 29]. The key element of our technique is a transformation of the spectral plane λ for the operator L in (1.3) to the spectral plane $z = \lambda^2$ for a different spectral problem. This transformation can be performed uniformly in the λ plane for the Kaup–Newell spectral problem related to the derivative NLS equation [19]. In the contrast, one needs to consider separately the subsets of the λ plane near the origin and near infinity for the operator L in (1.3) due to its rational dependence on λ. Therefore, two Riemann–Hilbert problems are derived for the MTM system (1.1) with the components (u, v): the one near $\lambda = 0$ recovers u and the other one near $\lambda = \infty$ recovers v.

Let $\dot{L}^{2,m}(\mathbb{R})$ denote the space of square integrable functions with the weight $|x|^m$ for $m \in \mathbb{Z}$ so that $L^{2,m}(\mathbb{R}) \equiv \dot{L}^{2,m}(\mathbb{R}) \cap L^2(\mathbb{R})$. Let $\dot{H}^{n,m}(\mathbb{R})$ denote the Sobolev space of functions, the n-th derivative of which is square integrable with the weight $|x|^m$ for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ so that $H^{n,m}(\mathbb{R}) \equiv \dot{H}^{n,m}(\mathbb{R}) \cap \dot{L}^{2,m}(\mathbb{R}) \cap H^n(\mathbb{R})$ with $H^n(\mathbb{R}) \equiv \dot{H}^n(\mathbb{R}) \cap L^2(\mathbb{R})$. Norms on any of these spaces are introduced according to the standard convention.

The inverse scattering transform for the linear operators (1.3) and (1.4) can be controlled when the potential (u, v) belongs to the function space

$$X_{(u,v)} := H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R}). \tag{1.5}$$

Transformations of the spectral plane employed here allow us to give a sharp requirement on the L^2-based Hilbert spaces, for which the Riemann–Hilbert problem can be solved by using the technique from Deift and Zhou [11, 37]. Note that both the direct and inverse scattering transforms for the NLS equation are solved in function space $H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, which is denoted by the same symbol $H^{1,1}(\mathbb{R})$ in the previous works [11, 37]. Compared to this space, the reflection coefficients (r_+, r_-) introduced in our paper for the linear operators (1.3) and (1.4) belong to the function space

$$X_{(r_+,r_-)} := \dot{H}^1(\mathbb{R}\setminus[-1,1]) \cap \dot{H}^{1,1}([-1,1]) \cap \dot{L}^{2,1}(\mathbb{R}) \cap \dot{L}^{2,-2}(\mathbb{R}). \tag{1.6}$$

In the application of the inverse scattering transform to the derivative NLS equation, alternative methods were recently developed [16, 22] based on a different (gauge) transformation of the Kaup–Newell spectral problem to the spectral problem for the Gerdjikov–Ivanov equation. Both the potentials and the reflection coefficients were controlled in the same function space $H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R})$ [16, 22]. These function spaces are more restrictive compared to the function spaces for the potential and the reflection coefficients used in [27, 29].

Unlike the recent literature on the derivative NLS equation, our interest to the inverse scattering for the MTM system (1.1) is not related to the well-posedness problems. Indeed, the local and global existence of solutions to the Cauchy problem for the MTM system (1.1) in the L^2-based Sobolev spaces $H^m(\mathbb{R})$, $m \in \mathbb{N}$ can be proven with the standard contraction and energy methods, see review of literature in [26]. Low regularity solutions in $L^2(\mathbb{R})$ were already obtained for the MTM system by Selberg and Tesfahun [31], Candy [5], Huh [13, 14, 15], and Zhang [35, 36]. The well-posedness results can be formulated as follows.

Theorem 1. [5, 15] *For every $(u_0, v_0) \in H^m(\mathbb{R})$, $m \in \mathbb{N}$, there exists a unique global solution $(u, v) \in C(\mathbb{R}, H^m(\mathbb{R}))$ such that $(u, v)|_{t=0} = (u_0, v_0)$ and the solution (u, v) depends continuously on the initial data (u_0, v_0). Moreover, for every $(u_0, v_0) \in L^2(\mathbb{R})$, there exists a global solution $(u, v) \in C(\mathbb{R}, L^2(\mathbb{R}))$ such that $(u, v)|_{t=0} = (u_0, v_0)$. The solution (u, v) is unique in a certain subspace of $C(\mathbb{R}, L^2(\mathbb{R}))$ and it depends continuously on the initial data (u_0, v_0).*

The inverse scattering transform and the reconstruction formulas for the global solutions (u, v) to the MTM system (1.1) can be used to solve other interesting analytical problems such as long-range scattering to zero [6], orbital and asymptotic stability of the Dirac solitons [9, 28], and an analytical proof of the soliton resolution conjecture. Similar questions have been recently addressed in the context of the cubic NLS equation [8, 10, 30] and the derivative NLS equation [17, 23].

The goal of our paper is to explain how the inverse scattering transform for the linear operators (1.3) and (1.4) can be developed by using the Riemann–Hilbert problem. For simplicity of presentation, we assume that the initial data to the MTM system (1.1) admit no eigenvalues and resonances in the sense of Definition 1 given in Section 3. Note that eigenvalues can be easily added by using Bäcklund transformation for the MTM system [9], whereas resonances can be removed by perturbations of initial data [3] (see relevant results in [27]). The following theorem represents the main result of this paper.

Theorem 2. *For every $(u_0, v_0) \in X_{(u,v)}$ admitting no eigenvalues or resonances in the sense of Definition 1, there is a direct scattering transform with the spectral data (r_+, r_-) defined in $X_{(r_+,r_-)}$. The unique solution $(u, v) \in C(\mathbb{R}, X_{(u,v)})$ to the MTM system (1.1) can be uniquely recovered by means of the inverse scattering transform for every $t \in \mathbb{R}$.*

The paper is organized as follows. Section 2 describes Jost functions obtained after two transformations of the differential operator L given by (1.3). Section 3 is used to set up scattering coefficients (r_+, r_-) and to introduce the scattering relations between the Jost functions. Section 4 explains how the Riemann–Hilbert problems can be solved and how the potentials (u, v) can be recovered in the time evolution of the MTM system (1.1). Section 5 concludes the paper with a review of open questions.

2. Jost functions

The linear operator L in (1.3) can be rewritten in the form:

$$L = Q(\lambda; u, v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3,$$

where

$$Q(\lambda; u, v) = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix}.$$

Here we freeze the time variable t and drop it from the list of arguments. Assuming fast decay of (u, v) to $(0, 0)$ as $|x| \to \infty$, solutions to the spectral problem

$$\psi_x = L\psi \tag{2.1}$$

can be defined by the following asymptotic behavior:

$$\psi_1^{(-)}(x;\lambda) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ix(\lambda^2-\lambda^{-2})/4}, \qquad \psi_2^{(-)}(x;\lambda) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ix(\lambda^2-\lambda^{-2})/4} \qquad \text{as } x \to -\infty$$

and

$$\psi_1^{(+)}(x;\lambda) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ix(\lambda^2-\lambda^{-2})/4}, \qquad \psi_2^{(+)}(x;\lambda) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ix(\lambda^2-\lambda^{-2})/4} \qquad \text{as } x \to +\infty.$$

The *normalized Jost functions*

$$\varphi_\pm(x;\lambda) = \psi_1^{(\pm)}(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}, \qquad \phi_\pm(x;\lambda) = \psi_2^{(\pm)}(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} \tag{2.2}$$

satisfy the constant boundary conditions at infinity:

$$\lim_{x\to\pm\infty} \varphi_\pm(x;\lambda) = e_1 \quad \text{and} \quad \lim_{x\to\pm\infty} \phi_\pm(x;\lambda) = e_2, \tag{2.3}$$

where $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$. The normalized Jost functions are solutions to the following Volterra integral equations:

$$\varphi_\pm(x;\lambda) = e_1 + \int_{\pm\infty}^x \left(\begin{array}{cc} 1 & 0 \\ 0 & e^{-\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} \end{array}\right) Q(\lambda;u(y),v(y))\, \varphi_\pm(y;\lambda)dy, \tag{2.4a}$$

$$\phi_\pm(x;\lambda) = e_2 + \int_{\pm\infty}^x \left(\begin{array}{cc} e^{\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} & 0 \\ 0 & 1 \end{array}\right) Q(\lambda;u(y),v(y))\, \phi_\pm(y;\lambda)dy. \tag{2.4b}$$

A standard assumption in analyzing Volterra integral equations is $Q(\lambda;u(\cdot),v(\cdot)) \in L^1(\mathbb{R})$ for fixed $\lambda \neq 0$ which is equivalent to $(u,v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by the definition of Q. In this case, for every $\lambda \in (\mathbb{R} \cup i\mathbb{R}) \setminus \{0\}$, Volterra integral equations (2.4) admit unique solutions $\varphi_\pm(\cdot;\lambda)$ and $\phi_\pm(\cdot;\lambda)$ in the space $L^\infty(\mathbb{R})$. However, even if $(u,v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the L^1-norm of $Q(\lambda;u(\cdot),v(\cdot))$ is not controlled uniformly in λ as $\lambda \to 0$ and $|\lambda| \to \infty$. This causes difficulties in studying the behaviour of $\varphi_\pm(\cdot;\lambda)$ and $\phi_\pm(\cdot;\lambda)$ as $\lambda \to 0$ and $|\lambda| \to \infty$ and thus we need to transform the spectral problem (2.1) to two equivalent forms. These two transformations generalize the exact transformation of the Kaup–Newell spectral problem to the Zakharov–Shabat spectral problem, see [19, 29].

2.1. **Transformation of the Jost functions for small λ.** Assume $u \in L^\infty(\mathbb{R})$, $\lambda \neq 0$, and define the transformation matrix by

$$T(u;\lambda) := \left(\begin{array}{cc} 1 & 0 \\ u & \lambda^{-1} \end{array}\right). \tag{2.5}$$

Let ψ be a solution of the spectral problem (2.1) and define $\Psi := T\psi$. Straightforward computations show that Ψ satisfies the equivalent linear equation

$$\Psi_x = \mathcal{L}\Psi, \tag{2.6}$$

with new linear operator

$$\mathcal{L} = Q_1(u,v) + \lambda^2 Q_2(u,v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{2.7}$$

where

$$Q_1(u,v) = \left(\begin{array}{cc} -\frac{i}{4}(|u|^2+|v|^2) & \frac{i}{2}\overline{u} \\ u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v & \frac{i}{4}(|u|^2+|v|^2) \end{array}\right), \quad Q_2(u,v) = \frac{i}{2}\left(\begin{array}{cc} u\overline{v} & -\overline{v} \\ u + u^2\overline{v} & -u\overline{v} \end{array}\right).$$

Let us define $z := \lambda^2$ and introduce the partition $\mathbb{C} = B_0 \cup \mathbb{S}^1 \cup B_\infty$ with

$$B_0 := \{z \in \mathbb{C}: \quad |z| < 1\}, \quad \mathbb{S}^1 := \{z \in \mathbb{C}: \quad |z| = 1\}, \quad B_\infty := \{z \in \mathbb{C}: \quad |z| > 1\}. \tag{2.8}$$

The second term in (2.7) is bounded if $z \in B_0$. The normalized Jost functions associated to the spectral problem (2.6) denoted by $\{m_\pm, n_\pm\}$ can be obtained from the original Jost functions $\{\varphi_\pm, \psi_\pm\}$ by the transformation formulas:

$$m_\pm(x;z) = T(u(x);\lambda)\varphi_\pm(x;\lambda), \qquad n_\pm(x;z) = \lambda\, T(u(x);\lambda)\phi_\pm(x;\lambda), \tag{2.9}$$

subject to the constant boundary conditions at infinity:

$$\lim_{x\to\pm\infty} m_\pm(x;\lambda) = e_1 \quad \text{and} \quad \lim_{x\to\pm\infty} n_\pm(x;\lambda) = e_2. \tag{2.10}$$

The transformed Jost functions are solutions to the following Volterra integral equations:

$$m_\pm(x;z) = e_1 + \int_{\pm\infty}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(z-z^{-1})(x-y)} \end{pmatrix} \left[Q_1(u(y),v(y)) + zQ_2(u(y),v(y))\right] m_\pm(y;z)dy, \tag{2.11a}$$

$$n_\pm(x;z) = e_2 + \int_{\pm\infty}^{x} \begin{pmatrix} e^{\frac{i}{2}(z-z^{-1})(x-y)} & 0 \\ 0 & 1 \end{pmatrix} \left[Q_1(u(y),v(y)) + zQ_2(u(y),v(y))\right] n_\pm(y;z)dy. \tag{2.11b}$$

Compared to [29], we have an additional term $\frac{i}{2}z(x-y)$ in the argument of the oscillatory kernel and the additional term $zQ_2(u,v)$ under the integration sign. However, both additional terms are bounded in B_0 where $|z|<1$. Therefore, the same analysis as in the proof of Lemmas 1 and 2 in [29] yields the following.

Lemma 1. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u_x \in L^1(\mathbb{R})$. For every $z \in (-1,1)$, there exist unique solutions $m_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ and $n_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ satisfying the integral equations (2.11). For every $x \in \mathbb{R}$, $m_\pm(x,\cdot)$ and $n_\mp(x,\cdot)$ are continued analytically in $\mathbb{C}^\pm \cap B_0$. There exist a positive constant C such that*

$$\|m_\pm(\cdot;z)\|_{L^\infty} + \|n_\mp(\cdot;z)\|_{L^\infty} \le C, \quad z \in \mathbb{C}^\pm \cap B_0. \tag{2.12}$$

Lemma 2. *Under the conditions of Lemma 1, for every $x \in \mathbb{R}$ the normalized Jost functions $m_\pm$ and $n_\pm$ satisfy the following limits as $\mathrm{Im}(z) \to 0$ along a contour in the domains of their analyticity:*

$$\lim_{z\to 0} \frac{m_\pm(x;z)}{m_\pm^\infty(x)} = e_1, \quad \lim_{z\to 0} \frac{n_\pm(x;z)}{n_\pm^\infty(x)} = e_2, \tag{2.13}$$

where

$$m_\pm^\infty(x) = e^{-\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}, \quad n_\pm^\infty(x) = e^{\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}.$$

If in addition $u \in C^1(\mathbb{R})$, then

$$\lim_{z\to 0} \frac{1}{z}\left[\frac{m_\pm(x;z)}{m_\pm^\infty(x)} - e_1\right] = \begin{pmatrix} -\int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right]dy \\ 2iu_x + u|v|^2 + v \end{pmatrix}, \tag{2.14a}$$

$$\lim_{z\to 0} \frac{1}{z}\left[\frac{n_\pm(x;z)}{n_\pm^\infty(x)} - e_2\right] = \begin{pmatrix} \overline{u} \\ \int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right]dy \end{pmatrix}. \tag{2.14b}$$

Remark 1. *By Sobolev's embedding of $H^1(\mathbb{R})$ into the space of continuous, bounded, and decaying at infinity functions, if $u \in H^1(\mathbb{R})$, then $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u(x) \to 0$ as $|x| \to \infty$. By the embedding of $L^{2,1}(\mathbb{R})$ into $L^1(\mathbb{R})$, if $u \in H^{1,1}(\mathbb{R})$, then $u \in L^1(\mathbb{R})$ and $u_x \in L^1(\mathbb{R})$. Thus, requirements of Lemma 1 are satisfied if $(u,v) \in H^{1,1}(\mathbb{R})$. The additional requirement $u \in C^1(\mathbb{R})$ of Lemma 2 is satisfied if $u \in H^2(\mathbb{R})$. Hence, $X_{(u,v)}$ in (1.5) is an optimal L^2-based Sobolev space for direct scattering of the MTM system (1.1).*

Remark 2. *Notations $(m_\pm, n_\pm)$ for the Jost functions used here are different from notations $(m_\pm, n_\pm)$ used in [29], where an additional transformation was used to generate $n_\pm$ (denoted by $p_\pm$ in [29]). This additional transformation is not necessary for our further work.*

2.2. **Transformation of the Jost functions for large λ.** Assume $v \in L^\infty(\mathbb{R})$ and define the transformation matrix by

$$\widehat{T}(v;\lambda) := \begin{pmatrix} 1 & 0 \\ v & \lambda \end{pmatrix}. \tag{2.15}$$

Let ψ be a solution of the spectral problem (2.1) and define $\widehat{\Psi} := \widehat{T}\psi$. Straightforward computations show that $\widehat{\Psi}$ satisfies the equivalent linear equation

$$\widehat{\Psi}_x = \widehat{\mathcal{L}}\widehat{\Psi}, \tag{2.16}$$

with new linear operator

$$\widehat{\mathcal{L}} = \widehat{Q}_1(u,v) + \frac{1}{\lambda^2}\widehat{Q}_2(u,v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{2.17}$$

where

$$\widehat{Q}_1(u,v) = \left(\begin{array}{cc} \frac{i}{4}(|u|^2+|v|^2) & -\frac{i}{2}\overline{v} \\ v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u & -\frac{i}{4}(|u|^2+|v|^2) \end{array}\right), \quad \widehat{Q}_2(u,v) = -\frac{i}{2}\left(\begin{array}{cc} \overline{u}v & -\overline{u} \\ v + \overline{u}v^2 & -\overline{u}v \end{array}\right).$$

We introduce the same variable $z := \lambda^2$ and note that the second term in (2.17) is now bounded for $z \in B_\infty$. The normalized Jost functions associated to the spectral problem (2.6) denoted by $\{\widehat{m}_\pm, \widehat{n}_\pm\}$ can be obtained from the original Jost functions $\{\varphi_\pm, \psi_\pm\}$ by the transformation formulas:

$$\widehat{m}_\pm(x;z) = \widehat{T}(v(x);\lambda)\varphi_\pm(x;\lambda), \qquad \widehat{n}_\pm(x;z) = \lambda^{-1}\widehat{T}(v(x);\lambda)\phi_\pm(x;\lambda), \tag{2.18}$$

subject to the constant boundary conditions at infinity:

$$\lim_{x\to\pm\infty} \widehat{m}_\pm(x;\lambda) = e_1 \quad \text{and} \quad \lim_{x\to\pm\infty} \widehat{n}_\pm(x;\lambda) = e_2. \tag{2.19}$$

The transformed Jost functions are solutions to the following Volterra integral equations:

(2.20a) $\widehat{m}_\pm(x;z) =$
$$e_1 + \int_{\pm\infty}^x \left(\begin{array}{cc} 1 & 0 \\ 0 & e^{-\frac{i}{2}(z-z^{-1})(x-y)} \end{array}\right)\left[\widehat{Q}_1(u(y),v(y)) + z^{-1}\widehat{Q}_2(u(y),v(y))\right]\widehat{m}_\pm(y;z)dy,$$

(2.20b) $\widehat{n}_\pm(x;z) =$
$$e_2 + \int_{\pm\infty}^x \left(\begin{array}{cc} e^{\frac{i}{2}(z-z^{-1})(x-y)} & 0 \\ 0 & 1 \end{array}\right)\left[\widehat{Q}_1(u(y),v(y)) + z^{-1}\widehat{Q}_2(u(y),v(y))\right]\widehat{n}_\pm(y;z)dy.$$

Again, we have an additional term $\frac{i}{2}z^{-1}(x-y)$ in the argument of the oscillatory kernel and the additional term $z^{-1}\widehat{Q}_2(u,v)$ under the integration sign. However, both additional terms are bounded in B_∞ where $|z| > 1$. The following two lemmas contain results analogous to Lemmas 1 and 2.

Lemma 3. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v_x \in L^1(\mathbb{R})$. For every $z \in \mathbb{R}\backslash[-1,1]$, there exist unique solutions $\widehat{m}_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ and $\widehat{n}_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ satisfying the integral equations (2.20). For every $x \in \mathbb{R}$, $\widehat{m}_\pm(x,\cdot)$ and $\widehat{n}_\mp(x,\cdot)$ are continued analytically in $\mathbb{C}^\pm \cap B_\infty$. There exist a positive constant C such that*

$$\|\widehat{m}_\pm(\cdot;z)\|_{L^\infty} + \|\widehat{n}_\mp(\cdot;z)\|_{L^\infty} \le C, \quad z \in \mathbb{C}^\pm \cap B_\infty. \tag{2.21}$$

Lemma 4. *Under the conditions of Lemma 3, for every $x \in \mathbb{R}$ the normalized Jost functions $\widehat{m}_\pm$ and $\widehat{n}_\pm$ satisfy the following limits as $\mathrm{Im}(z) \to \infty$ along a contour in the domains of their analyticity:*

$$\lim_{|z|\to\infty} \frac{\widehat{m}_\pm(x;z)}{\widehat{m}_\pm^\infty(x)} = e_1, \quad \lim_{|z|\to\infty} \frac{\widehat{n}_\pm(x;z)}{\widehat{n}_\pm^\infty(x)} = e_2, \tag{2.22}$$

where

$$\widehat{m}_\pm^\infty(x) = e^{\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}, \quad \widehat{n}_\pm^\infty(x) = e^{-\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}.$$

If in addition $v \in C^1(\mathbb{R})$, then

$$\lim_{|z|\to\infty} z\left[\frac{\widehat{m}_\pm(x;z)}{\widehat{m}_\mp^\infty(x)} - e_1\right] = \left(\begin{array}{c} -\int_{\pm\infty}^x \left[\overline{v}(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u) + \frac{i}{2}\overline{u}v\right] dy \\ -2iv_x + |u|^2 v + u \end{array}\right), \tag{2.23a}$$

$$\lim_{|z|\to\infty} z\left[\frac{\widehat{n}_\pm(x;z)}{\widehat{n}_\pm^\infty(x)} - e_2\right] = \left(\begin{array}{c} \overline{v} \\ \int_{\pm\infty}^x \left[\overline{v}(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u) + \frac{i}{2}\overline{u}v\right] dy \end{array}\right). \tag{2.23b}$$

2.3. **Continuation of the transformed Jost functions across $\mathbb{S}^1$.** In Lemmas 1 and 3 we showed the existence of the transformed Jost functions

$$\{m_\pm(\cdot;z), n_\pm(\cdot;z)\}, \quad z \in B_0, \qquad \text{and} \qquad \{\widehat{m}_\pm(\cdot;z), \widehat{n}_\pm(\cdot;z)\}, \quad z \in B_\infty,$$

respectively, where the partition (2.8) is used. Because both sets of the transformed Jost functions are connected to the set $\{\varphi_\pm, \phi_\pm\}$ of the original Jost functions by the transformation formulas (2.9) and (2.18), respectively, we find the following connection formulas for every $z \in \mathbb{S}^1$:

$$m_\pm(x;z) = \left(\begin{array}{cc} 1 & 0 \\ u(x) - z^{-1}v(x) & z^{-1} \end{array}\right) \widehat{m}_\pm(x;z), \tag{2.24a}$$

$$n_\pm(x;z) = \left(\begin{array}{cc} z & 0 \\ u(x)z - v(x) & 1 \end{array}\right) \widehat{n}_\pm(x;z), \tag{2.24b}$$

or in the opposite direction,

$$\widehat{m}_\pm(x;z) = \left(\begin{array}{cc} 1 & 0 \\ v(x) - zu(x) & z \end{array}\right) m_\pm(x;z), \tag{2.25a}$$

$$\widehat{n}_\pm(x;z) = \left(\begin{array}{cc} z^{-1} & 0 \\ v(x)z^{-1} - u(x) & 1 \end{array}\right) n_\pm(x;z). \tag{2.25b}$$

By Lemmas 3 and 4, the right-hand sides of (2.24a) and (2.24b) yield analytic continuations of $m_\pm(x;\cdot)$ and $n_\mp(x;\cdot)$ in $\mathbb{C}^\pm \cap B_\infty$ respectively with the following limits as $\mathrm{Im}(z) \to \infty$ along a contour in the domains of their analyticity:

$$\lim_{|z|\to\infty} \frac{m_\pm(x;z)}{\widehat{m}_\mp^\infty(x)} = e_1 + u(x)e_2, \quad \lim_{|z|\to\infty} \frac{n_\pm(x;z)}{\widehat{n}_\pm^\infty(x)} = \bar{v}(x)e_1 + (1 + u(x)\bar{v}(x))e_2. \tag{2.26}$$

Analogously, by Lemmas 1 and 2, the right-hand sides of (2.25a) and (2.25b) yield analytic continuations of $\widehat{m}_\pm(x;\cdot)$ and $\widehat{n}_\mp(x;\cdot)$ in $\mathbb{C}^\pm \cap B_0$ respectively with the following limits as $\mathrm{Im}(z) \to 0$ along a contour in the domains of their analyticity:

$$\lim_{z\to 0} \frac{\widehat{m}_\pm(x;z)}{m_\mp^\infty(x)} = e_1 + v(x)e_2, \quad \lim_{z\to 0} \frac{\widehat{n}_\pm(x;z)}{n_\pm^\infty(x)} = \bar{u}(x)e_1 + (1 + \bar{u}(x)v(x))e_2. \tag{2.27}$$

By Lemmas 1, 2, 3, 4, and the continuation formulas (2.24), (2.25), we obtain the following result.

Lemma 5. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $(u_x, v_x) \in L^1(\mathbb{R})$.. For every $x \in \mathbb{R}$ the Jost functions defined by the integral equations (2.11) and (2.20) can be continued such that $m_\pm(x;\cdot)$, $n_\mp(x;\cdot)$, $\widehat{m}_\pm(x;\cdot)$, and $\widehat{n}_\mp(x;\cdot)$ are analytic in $\mathbb{C}^\pm$ and continuous in $\mathbb{C}^\pm \cup \mathbb{R}$ with bounded limits as $z \to 0$ and $|z| \to \infty$ given by (2.13), (2.22), (2.26), (2.27).*

3. Scattering coefficients

In order to define the scattering coefficients between the transformed Jost functions $\{m_\pm, n_\pm\}$ and $\{\widehat{m}_\pm, \widehat{n}_\pm\}$, we go back to the original Jost functions $\{\varphi_\pm, \phi_\pm\}$. For every $\lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}$, we define the standard form of the scattering relation by

$$\left(\begin{array}{c} \varphi_-(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} \\ \phi_-(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4} \end{array}\right) = \left(\begin{array}{cc} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{array}\right) \left(\begin{array}{c} \varphi_+(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} \\ \phi_+(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4} \end{array}\right). \tag{3.1}$$

Since the operator L in (1.3) admits the symmetry

$$\overline{\phi_\pm(x;\lambda)} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \varphi_\pm\left(x;\overline{\lambda}\right),$$

we obtain

$$\gamma(\lambda) = -\overline{\beta(\overline{\lambda})}, \quad \delta(\lambda) = \overline{\alpha(\overline{\lambda})}, \quad \lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}. \tag{3.2}$$

Since the matrix operator L in (1.3) has zero trace, the Wronskian determinant W of any two solutions to the spectral problem (2.1) for any $\lambda \in \mathbb{C}$ is independent of x. By computing the Wronskian determinants of the solutions $\{\varphi_-, \phi_+\}$ and $\{\varphi_+, \varphi_-\}$ as $x \to +\infty$ and using the scattering relation (3.1) and the asymptotic behavior of the Jost functions $\{\varphi_\pm, \psi_\pm\}$, we obtain

$$\left\{ \begin{array}{l} \alpha(\lambda) = W\left(\varphi_-(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4}, \phi_+(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}\right), \\ \beta(\lambda) = W\left(\varphi_+(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4}, \varphi_-(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4}\right). \end{array} \right. \tag{3.3}$$

It follows from the asymptotic behavior of $\{\varphi_-, \phi_-\}$ as $x \to -\infty$ that $W(\varphi_-, \phi_-) = 1$. Substituting (3.1) and using the the asymptotic behavior of $\{\varphi_+, \phi_+\}$ as $x \to +\infty$ yield the following constraint on the scattering data:

$$\alpha(\lambda)\delta(\lambda) - \beta(\lambda)\gamma(\lambda) = 1, \quad \lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}. \tag{3.4}$$

In view of the constraints (3.2), the constraint (3.4) can be written as

$$\alpha(\lambda)\overline{\alpha(\overline{\lambda})} + \beta(\lambda)\overline{\beta(\overline{\lambda})} = 1, \quad \lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}. \tag{3.5}$$

By using the transformation formulas (2.9) we can rewrite the scattering relation (3.1) in terms of the transformed Jost functions $\{m_\pm, n_\pm\}$. In particular, we apply $T(u;\lambda)$ to the first equation in (3.1) and $\lambda T(u;\lambda)$ to the second equation in (3.1), so that we obtain for $z \in \mathbb{R}\backslash\{0\}$,

$$\begin{pmatrix} m_-(x;z)e^{ix(z-z^{-1})/4} \\ n_-(x;z)e^{-ix(z-z^{-1})/4} \end{pmatrix} = \begin{pmatrix} a(z) & b_+(z) \\ -\overline{b_-(z)} & \overline{a(z)} \end{pmatrix} \begin{pmatrix} m_+(x;z)e^{ix(z-z^{-1})/4} \\ n_+(x;z)e^{-ix(z-z^{-1})/4} \end{pmatrix}, \tag{3.6}$$

where we have recalled $z = \lambda^2$ and defined the scattering coefficients:

$$a(z) := \alpha(\lambda), \quad b_+(z) := \lambda^{-1}\beta(\lambda), \quad b_-(z) := \lambda\beta(\lambda), \quad z \in \mathbb{R}\backslash\{0\}. \tag{3.7}$$

Since $m_\pm(x;z)$ and $n_\pm(x;z)$ depend on $z = \lambda^2$, we deduce that α is even in λ and β is odd in λ for $\lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}$. The latter condition yields $\overline{\lambda}\beta(\overline{\lambda}) = \lambda\beta(\lambda)$, which has been used already in the expression (3.7) for $b_-(z)$. Thanks to the relation (3.5), we have the following constraints

$$\left\{ \begin{array}{ll} |\alpha(\lambda)|^2 + |\beta(\lambda)|^2 = 1, & \lambda \in \mathbb{R}\backslash\{0\}, \\ |\alpha(\lambda)|^2 - |\beta(\lambda)|^2 = 1, & \lambda \in i\mathbb{R}\backslash\{0\}. \end{array} \right. \tag{3.8}$$

Since the matrix operator $\mathcal{L}$ in (2.7) has zero trace, the Wronskian determinant W of any two solutions to the spectral problem (2.6) is also independent of x. As a result, by computing the Wronskian determinant as $x \to +\infty$ and using the asymptotic behavior of the Jost functions $\{m_\pm, n_\pm\}$, we obtain from the scattering relation (3.6) for $z \in \mathbb{R}\backslash\{0\}$:

$$\left\{ \begin{array}{l} a(z) = W\left(m_-(x;z)e^{ix(z-z^{-1})/4}, n_+(x;z)e^{-ix(z-z^{-1})/4}\right), \\ b_+(z) = W\left(m_+(x;z)e^{ix(z-z^{-1})/4}, m_-(x;z)e^{ix(z-z^{-1})/4}\right), \\ \overline{b_-(z)} = W\left(n_+(x;z)e^{-ix(z-z^{-1})/4}, n_-(x;z)e^{-ix(z-z^{-1})/4}\right), \end{array} \right. \tag{3.9}$$

in accordance with the representation (3.3).

Analogously, by using the transformation formulas (2.18) we can rewrite the scattering relation (3.1) in terms of the transformed Jost functions $\{\widehat{m}_\pm, \widehat{n}_\pm\}$. In particular, we apply $\widehat{T}(u;\lambda)$ to the first equation in (3.1) and $\lambda^{-1}\widehat{T}(u;\lambda)$ to the second equation in (3.1), so that we obtain for $z \in \mathbb{R}\backslash\{0\}$,

$$\begin{pmatrix} \widehat{m}_-(x;z)e^{ix(z-z^{-1})/4} \\ \widehat{n}_-(x;z)e^{-ix(z-z^{-1})/4} \end{pmatrix} = \begin{pmatrix} \widehat{a}(z) & \widehat{b}_+(z) \\ -\overline{\widehat{b}_-(z)} & \overline{\widehat{a}(z)} \end{pmatrix} \begin{pmatrix} \widehat{m}_+(x;z)e^{ix(z-z^{-1})/4} \\ \widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4} \end{pmatrix}, \tag{3.10}$$

where we have recalled $z = \lambda^2$ and defined the scattering coefficients

$$\widehat{a}(z) := \alpha(\lambda), \quad \widehat{b}_+(z) := \lambda\beta(\lambda), \quad \widehat{b}_-(z) := \lambda^{-1}\beta(\lambda), \quad z \in \mathbb{R}\backslash\{0\}. \tag{3.11}$$

Since the matrix operator $\widehat{\mathcal{L}}$ in (2.17) has zero trace, we obtain from the scattering relation (3.10) for $z \in \mathbb{R}\backslash\{0\}$:

$$\left\{ \begin{array}{l} \widehat{a}(z) = W\left(\widehat{m}_-(x;z)e^{ix(z-z^{-1})/4}, \widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4}\right), \\ \widehat{b}_+(z) = W\left(\widehat{m}_+(x;z)e^{ix(z-z^{-1})/4}, \widehat{m}_-(x;z)e^{ix(z-z^{-1})/4}\right), \\ \overline{\widehat{b}_-(z)} = W\left(\widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4}, \widehat{n}_-(x;z)e^{-ix(z-z^{-1})/4}\right), \end{array} \right. \tag{3.12}$$

in accordance with the representation (3.3).

It follows from (3.7) and (3.11) that the two sets of scattering data are actually related by

$$\widehat{a}(z) = a(z), \quad \widehat{b}_+(z) = b_-(z), \quad \widehat{b}_-(z) = b_+(z), \quad z \in \mathbb{R}\backslash\{0\}. \tag{3.13}$$

These relations are in agreement with the continuation formulas (2.24) and (2.25). By using the representations (3.9) and (3.12), as well as Lemma 2, 4, and 5, we obtain the following.

Lemma 6. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $(u_x, v_x) \in L^1(\mathbb{R})$. Then, $a = \widehat{a}$ is continued analytically into $\mathbb{C}^-$ with the following limits in $\mathbb{C}^-$:*

$$\lim_{z\to 0} a(z) = e^{-\frac{i}{4}\int_{\mathbb{R}}(|u|^2+|v|^2)dy} =: a_0 \tag{3.14}$$

and

$$\lim_{|z|\to\infty} a(z) = e^{\frac{i}{4}\int_{\mathbb{R}}(|u|^2+|v|^2)dy} =: a_\infty. \tag{3.15}$$

On the other hand, $b_\pm = \widehat{b}_\pm$ are not continued analytically beyond the real line and satisfy the following limits on $\mathbb{R}$:

$$\lim_{z\to 0} b_\pm(z) = \lim_{|z|\to\infty} b_\pm(z) = 0. \tag{3.16}$$

To simplify the inverse scattering transform, we consider the case of no eigenvalues or resonances in the spectral problem (2.1) in the sense of the following definition.

Definition 1. *We say that the potential (u,v) admits an eigenvalue at $z_0 \in \mathbb{C}^-$ if $a(z_0) = 0$ and a resonance at $z_0 \in \mathbb{R}$ if $a(z_0) = 0$.*

By taking the limit $x \to +\infty$ in the Volterra integral equations (2.11a) and (2.20a) for m_- and $\widehat{m}_-$ respectively and comparing it with the first equations in the scattering relations (3.6) and (3.10), we obtain the following equivalent representations for $a = \widehat{a}$:

$$a(z) = 1 - \frac{i}{4}\int_{\mathbb{R}}\left[(|u|^2+|v|^2)m_-^{(1)} - 2\bar{u}m_-^{(2)} - 2z\bar{v}(um_-^{(1)} - m_-^{(2)})\right]dx, \quad z \in B_0 \cap \mathbb{C}^-, \tag{3.17a}$$

$$a(z) = 1 + \frac{i}{4}\int_{\mathbb{R}}\left[(|u|^2+|v|^2)\widehat{m}_-^{(1)} - 2\bar{v}\widehat{m}_-^{(2)} - 2z^{-1}\bar{u}(v\widehat{m}_-^{(1)} - \widehat{m}_-^{(2)})\right]dx, z \in B_\infty \cap \mathbb{C}^-, \tag{3.17b}$$

where the superscripts denote components of the Jost functions. If $(u,v) \in H^{1,1}(\mathbb{R})$ are defined in the ball of radius δ for some $\delta \in (0,1)$, then constants C in (2.12) and (2.21) are independent of δ.

Then, it follows from (3.17) that if δ is sufficiently small, then the integrals can be made as small as needed for every $z \in \mathbb{C}^- \cup \mathbb{R}$. This implies the following.

Lemma 7. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $(u_x, v_x) \in L^1(\mathbb{R})$ be sufficiently small. Then (u,v) does not admit eigenvalues or resonances in the sense of Definition 1.*

Remark 3. *The result of Lemma 7 was first obtained in Theorem 6.1 in* [26]*. No transformation of the spectral problem (2.1) was employed in* [26]*. Transformations similar to those we are using here were employed later in* [29] *in the context of the derivative NLS equation.*

Remark 4. *The result of Lemma 7 is useful for the study of long-range scattering from small initial data. Eigenvalues can always be included by using Bäcklund transformation for the MTM system* [9, 27]*. Resonances are structurally unstable and can be removed by perturbations of initial data* [3, 27]*.*

4. Riemann–Hilbert problems

We will derive two Riemann–Hilbert problems. The first problem is formulated for the transformed Jost functions $\{m_\pm, n_\pm\}$, whereas the second problem is formulated for the transformed Jost functions $\{\widehat{m}_\pm, \widehat{n}_\pm\}$. Thanks to the asymptotic representations (2.14) and (2.23), the first problem is useful for reconstruction of the component u as $z \to 0$, whereas the second problem is useful for reconstruction of the component v as $|z| \to \infty$, both components satisfy the MTM system (1.1). This pioneering idea has first appeared on a formal level in [34]. The following assumption is used to simplify solutions to the Riemann–Hilbert problems.

Assumption 1. *Assume that the scattering coefficient a admits no zeros in $\mathbb{C}^- \cup \mathbb{R}$.*

Assumption 1 corresponds to the initial data (u_0, v_0) which admit no eigenvalues or resonances in the sense of Definition 1. By Lemma 7, the assumption is satisfied if the $H^{1,1}(\mathbb{R})$ norm on the initial data is sufficiently small. Since a is continued analytically into $\mathbb{C}^-$ by Lemma 6 with nonzero limits (3.14) and (3.15), zeros of a lie in a compact set. Therefore, if a admits no zeros in $\mathbb{C}^- \cup \mathbb{R}$ by Assumption 1, then there is $A > 0$ such that $|a(z)| \geq A$ for every $z \in \mathbb{R}$.

4.1. **Riemann-Hilbert problem for the potential u.**

The asymptotic limit (2.26) presents a challenge to use $\{m_\pm, n_\pm\}$ for reconstruction of (u,v) as $|z| \to \infty$. On the other hand, the reconstruction formula for (u,v) in terms of $\{m_\pm, n_\pm\}$ is available from the asymptotic limit (2.14) as $z \to 0$. In order to avoid this complication, we use the inversion transformation $\omega = 1/z$, which maps 0 to ∞ and vice versa. The analyticity regions swap under the inversion transformation so that $\{m_-, n_+\}$ become analytic in $\mathbb{C}^+$ for ω and $\{m_+, n_-\}$ become analytic in $\mathbb{C}^-$ for ω.

Let us define matrices $P_\pm(x;\omega) \in \mathbb{C}^{2\times 2}$ for every $x \in \mathbb{R}$ and $\omega \in \mathbb{R}$ by

$$P_+(x;\omega) := \left[\frac{m_-(x;\omega^{-1})}{a(\omega^{-1})}, n_+(x;\omega^{-1})\right], \quad P_-(x;\omega) := \left[m_+(x;\omega^{-1}), \frac{n_-(x;\omega^{-1})}{\overline{a(\omega^{-1})}}\right], \tag{4.1}$$

and two reflection coefficients

$$r_\pm(\omega) = \frac{b_\pm(\omega^{-1})}{a(\omega^{-1})}, \quad \omega \in \mathbb{R}, \tag{4.2}$$

The scattering relation (3.6) can be rewritten as the following jump condition for the Riemann–Hilbert problem:

$$P_+(x;\omega) = P_-(x;\omega) \left[\begin{array}{cc} 1 + r_+(\omega)\overline{r_-(\omega)} & \overline{r_-(\omega)}e^{-\frac{i}{2}(\omega-\omega^{-1})x} \\ r_+(\omega)e^{\frac{i}{2}(\omega-\omega^{-1})x} & 1 \end{array}\right]$$

If the scattering coefficient a satisfies Assumption 1, then $P_\pm(x;\cdot)$ for every $x \in \mathbb{R}$ are continued analytically in $\mathbb{C}^\pm$ by Lemmas 5 and 6. We denote these continuations by the same letters. Asymptotic limits (2.13) and (3.14) yield the following behavior of $P_\pm(x;\omega)$ for large $|\omega|$ in the domains of their analyticity:

$$P_\pm(x;\omega) \to \left[\begin{array}{cc} m_+^\infty(x) & 0 \\ 0 & n_+^\infty(x) \end{array}\right] =: P^\infty(x) \quad \text{as } |\omega| \to \infty.$$

Since we prefer to work with x-independent boundary conditions, we normalize the boundary conditions by defining

$$M_\pm(x;\omega) := [P^\infty(x)]^{-1}\, P_\pm(x;\omega), \quad \omega \in \mathbb{C}^\pm . \tag{4.3}$$

The following Riemann-Hilbert problem is formulated for the function $M(x;\cdot)$.

Riemann-Hilbert problem 1. *For each $x \in \mathbb{R}$, find a 2×2-matrix valued function $M(x;\cdot)$ such that*

(1) *$M(x;\cdot)$ is piecewise analytic in $\mathbb{C}\setminus\mathbb{R}$ with continuous boundary values*
$$M_\pm(x;\omega) = \lim_{\varepsilon\downarrow 0} M(x;\omega \pm i\varepsilon), \quad z \in \mathbb{R}.$$
(2) *$M(x;\omega) \to I$ as $|\omega| \to \infty$.*
(3) *The boundary values $M_\pm(x;\cdot)$ on $\mathbb{R}$ satisfy the jump relation*
$$M_+(x;\omega) - M_-(x;\omega) = M_-(x;\omega)R(x;\omega), \quad \omega \in \mathbb{R},$$
where
$$R(x;\omega) := \left[\begin{array}{cc} r_+(\omega)\overline{r_-(\omega)} & \overline{r_-(\omega)}e^{-\frac{i}{2}(\omega-\omega^{-1})x} \\ r_+(\omega)e^{\frac{i}{2}(\omega-\omega^{-1})x} & 0 \end{array}\right].$$

It follows from the asymptotic limits (2.14) and the normalization (4.3) that the components (u,v) of the MTM system (1.1) are related to the solution of the Riemann–Hilbert problem 1 by using the following reconstruction formulas:

$$\left[2iu'(x) + u(x)|v(x)|^2 + v(x)\right] e^{\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy} = \lim_{|\omega|\to\infty} \omega[M(x;\omega)]_{21} \tag{4.4}$$

and

$$\overline{u}(x)e^{-\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy} = \lim_{|\omega|\to\infty} \omega[M(x;\omega)]_{12}, \tag{4.5}$$

where the subscript denotes the element of the 2×2 matrix M.

Remark 5. *The gauge factors in (4.4)–(4.5) appear because of the normalization (4.3) and the asymptotic limits (2.14). A different approach was utilized in* [16, 22] *to avoid these gauge factors. The inverse scattering transform was developed to a different spectral problem, which was obtained from the Kaup–Newell spectral problem after a gauge transformation.*

4.2. **Riemann-Hilbert problem for the potential** v. Let us define matrices $\widehat{P}_\pm(x;z) \in \mathbb{C}^{2\times 2}$ for every $x \in \mathbb{R}$ and $z \in \mathbb{R}$ by

$$\widehat{P}_+(x;z) := \left[\widehat{m}_+(x;z), \frac{\widehat{n}_-(x;z)}{\overline{\widehat{a}(z)}}\right], \quad \widehat{P}_-(x;z) := \left[\frac{\widehat{m}_-(x;z)}{\widehat{a}(z)}, \widehat{n}_+(x;z)\right], \tag{4.6}$$

and two reflection coefficients by

$$\widehat{r}_\pm(z) = \frac{\widehat{b}_\pm(z)}{\widehat{a}(z)} = \frac{b_\mp(z)}{a(z)}, \quad z \in \mathbb{R}, \tag{4.7}$$

where the relations (3.13) have been used. The scattering relation (3.10) can be rewritten as the following jump condition for the Riemann–Hilbert problem:

$$\widehat{P}_+(x;z) = \widehat{P}_-(x;z) \begin{bmatrix} 1 & -\overline{\widehat{r}_-(z)} e^{\frac{i}{2}(z-z^{-1})x} \\ -\widehat{r}_+(z) e^{-\frac{i}{2}(z-z^{-1})x} & 1 + \widehat{r}_+(z)\overline{\widehat{r}_-(z)} \end{bmatrix}$$

If the scattering coefficient a satisfies Assumption 1, then $\widehat{P}_\pm(x;\cdot)$ for every $x \in \mathbb{R}$ are continued analytically in $\mathbb{C}^\pm$ by Lemmas 5 and 6. We denote these continuations by the same letters. Asymptotic limits (2.22) and (3.15) yield the following behavior of $\widehat{P}(x;z)$ for large $|z|$ in the domains of their analyticity:

$$\widehat{P}_\pm(x;z) \to \begin{bmatrix} \widehat{m}_+^\infty(x) & 0 \\ 0 & \widehat{n}_+^\infty(x) \end{bmatrix} =: \widehat{P}^\infty(x), \quad \text{as } |z| \to \infty.$$

In order to normalize the boundary conditions, we define

$$\widehat{M}_\pm(x;z) := \left[\widehat{P}^\infty(x)\right]^{-1} \widehat{P}_\pm(x;z), \quad z \in \mathbb{C}^\pm\,. \tag{4.8}$$

The following Riemann-Hilbert problem is formulated for the function $\widehat{M}(x;\cdot)$.

Riemann-Hilbert problem 2. *For each $x \in \mathbb{R}$, find a 2×2-matrix valued function $\widehat{M}(x;\cdot)$ such that*

(1) *$\widehat{M}(x;\cdot)$ is piecewise analytic in $\mathbb{C}\setminus\mathbb{R}$ with continuous boundary values*

$$\widehat{M}_\pm(x;z) = \lim_{\varepsilon\downarrow 0} \widehat{M}(x;z \pm i\varepsilon), \quad z \in \mathbb{R}.$$

(2) *$\widehat{M}(x;z) \to I$ as $|z| \to \infty$.*

(3) *The boundary values $\widehat{M}_\pm(x;\cdot)$ on $\mathbb{R}$ satisfy the jump relation*

$$\widehat{M}_+(x;z) - \widehat{M}_-(x;z) = \widehat{M}_-(x;z)\widehat{R}(x;z),$$

where

$$\widehat{R}(x;z) := \begin{bmatrix} 0 & -\overline{\widehat{r}_-(z)} e^{\frac{i}{2}(z-z^{-1})x} \\ -\widehat{r}_+(z) e^{-\frac{i}{2}(z-z^{-1})x} & \widehat{r}_+(z)\overline{\widehat{r}_-(z)} \end{bmatrix}.$$

It follows from the asymptotic limit (2.23) and the normalization (4.8) that the components (u,v) of the MTM system (1.1) can be recovered from the solution of the Riemann–Hilbert problem 2 by using the following reconstruction formulas:

$$\left[-2iv'(x) + |u(x)|^2 v(x) + u(x)\right] e^{-\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy} = \lim_{|z|\to\infty} z \left[\widehat{M}(x;z)\right]_{21} \tag{4.9}$$

and

$$\overline{v}(x) e^{\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy} = \lim_{|z|\to\infty} z \left[\widehat{M}(x;z)\right]_{12}, \tag{4.10}$$

where the subscript denotes the element of the 2×2 matrix M.

Let us now outline the reconstruction procedure for (u,v) as a solution of the MTM system (1.1) in the inverse scattering transform. If the right-hand sides of (4.5) and (4.10) are controlled in the space $H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$, then $(\tilde{u},\tilde{v})\in H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$, where

$$\tilde{u}(x)=u(x)e^{\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy},\quad \tilde{v}(x)=v(x)e^{-\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy}.$$

Since $|\tilde{u}(x)|=|u(x)|$ and $|\tilde{v}(x)|=|v(x)|$, the gauge factors can be immediately inverted, and since $H^1(\mathbb{R})$ is continuously embedded into $L^p(\mathbb{R})$ for any $p\geq 2$, we then have $(u,v)\in H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$. If the right-hand sides of (4.4) and (4.9) are also controlled in $H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$, then similar arguments give $(u',v')\in H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$, that is, $(u,v)\in H^2(\mathbb{R})\cap H^{1,1}(\mathbb{R})$, in agreement with the function space used for direct scattering transform.

Remark 6. *It follows from the limit (3.16) that $R(x;0)=\widehat{R}(x;0)=0$ implying $M_+(x;0)=M_-(x;0)$ and $\widehat{M}_+(x;0)=\widehat{M}_-(x;0)$. More precisely, using (2.26), (2.27), (3.14), (3.15), and $\omega=z^{-1}$ we can derive*

$$M(x;0)=\begin{bmatrix} m_+^\infty(x) & 0\\ 0 & n_+^\infty(x)\end{bmatrix}^{-1}\begin{bmatrix} 1 & \overline{v}(x)\\ u(x) & 1+u(x)\overline{v}(x)\end{bmatrix}\begin{bmatrix} \widehat{m}_+^\infty(x) & 0\\ 0 & \widehat{n}_+^\infty(x)\end{bmatrix}$$

and

$$\widehat{M}(x;0)=\begin{bmatrix} \widehat{m}_+^\infty(x) & 0\\ 0 & \widehat{n}_+^\infty(x)\end{bmatrix}^{-1}\begin{bmatrix} 1 & \overline{u}(x)\\ v(x) & 1+\overline{u}(x)v(x)\end{bmatrix}\begin{bmatrix} m_+^\infty(x) & 0\\ 0 & n_+^\infty(x)\end{bmatrix}.$$

In particular, the following holds:

$$[M(x;0)]_{11}=\frac{\widehat{m}_+^\infty(x)}{m_+^\infty(x)}=e^{-\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy},\qquad [\widehat{M}(x;0)]_{11}=\frac{m_+^\infty(x)}{\widehat{m}_+^\infty(x)}=e^{\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy}.$$

In these formulas, we regain the same exponential factors as those in the reconstruction formulas (4.5) and (4.10). Hence, by substitution we obtain the following two decoupled reconstruction formulas:

$$u(x)=[M(x;0)]_{11}\overline{\lim_{|\omega|\to\infty}\omega[M(x;\omega)]_{12}},\qquad v(x)=[\widehat{M}(x;0)]_{11}\overline{\lim_{|z|\to\infty}z[\widehat{M}(x;z)]_{12}}. \tag{4.11}$$

Whereas equations (4.4), (4.5), (4.9) and (4.10) are suitable for studying the inverse map of the scattering transformation in the sense of Theorem 2, the equivalent formulas (4.11) are useful in the analysis of the asymptotic behavior of $u(x)$ and $v(x)$ as $|x|\to\infty$.

4.3. **Estimates on the reflection coefficients.** In order to be able to solve the Riemann–Hilbert problems 1 and 2, we need to derive estimates on the reflection coefficients $r_\pm$ and $\widehat{r}_\pm$ defined by (4.2) and (4.7). We start with the Jost functions. In order to exclude ambiguity in notations, we write $m_\pm(x;z)\in H^1_z(\mathbb{R})$ for the same purpose as $m_\pm(x;\cdot)\in H^1(\mathbb{R})$.

Thanks to the Fourier theory reviewed in Proposition 1 in [29], the Volterra integral equations (2.11) and (2.20) with the oscillation factors $e^{\frac{i}{2}(\omega^{-1}-\omega)}$ and $e^{\frac{i}{2}(z-z^{-1})x}$ are estimated respectively in the limits $|\omega|\to\infty$ and $|z|\to\infty$, where $\omega:=z^{-1}$, similarly to what was done in the proof of Lemma 3 in [29]. As a result, we obtain the following.

Lemma 8. *Let $(u,v)\in H^{1,1}(\mathbb{R})$. Then for every $x\in\mathbb{R}^\pm$, we have*

$$m_\pm(x;\omega^{-1})-m_\pm^\infty(x)e_1\in H^1_\omega(\mathbb{R}\backslash[-1,1]),\quad n_\pm(x;\omega^{-1})-n_\pm^\infty(x)e_2\in H^1_\omega(\mathbb{R}\backslash[-1,1]). \tag{4.12}$$

and

$$\widehat{m}_\pm(x;z)-\widehat{m}_\pm^\infty(x)e_1\in H^1_z(\mathbb{R}\backslash[-1,1]),\quad \widehat{n}_\pm(x;z)-\widehat{n}_\pm^\infty(x)e_2\in H^1_z(\mathbb{R}\backslash[-1,1]). \tag{4.13}$$

If in addition $(u,v) \in H^2(\mathbb{R})$, *then*

$$\omega \left[\frac{m_\pm(x;\omega^{-1})}{m_\pm^\infty(x)} - e_1 \right] - \begin{pmatrix} -\int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right] dy \\ 2iu_x + u|v|^2 + v \end{pmatrix} \in L^2_\omega(\mathbb{R}\backslash[-1,1]), \tag{4.14a}$$

$$\omega \left[\frac{n_\pm(x;\omega^{-1})}{n_\pm^\infty(x)} - e_2 \right] - \begin{pmatrix} \overline{u} \\ \int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right] dy \end{pmatrix} \in L^2_\omega(\mathbb{R}\backslash[-1,1]). \tag{4.14b}$$

and

$$z \left[\frac{\widehat{m}_\pm(x;z)}{\widehat{m}_\pm^\infty(x)} - e_1 \right] - \begin{pmatrix} -\int_{\pm\infty}^x \left[\overline{v}(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u) + \frac{i}{2}\overline{u}v\right] dy \\ -2iv_x + |u|^2 v + u \end{pmatrix} \in L^2_z(\mathbb{R}\backslash[-1,1]), \tag{4.15a}$$

$$z \left[\frac{\widehat{n}_\pm(x;z)}{\widehat{n}_\pm^\infty(x)} - e_2 \right] - \begin{pmatrix} \overline{v} \\ \int_{\pm\infty}^x \left[\overline{v}(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u) + \frac{i}{2}\overline{u}v\right] dy \end{pmatrix} \in L^2_z(\mathbb{R}\backslash[-1,1]). \tag{4.15b}$$

The following lemma transfers the estimates of Lemma 8 to the scattering coefficients a and $b_\pm$ by using the same analysis as in the proof of Lemma 4 in [29].

Lemma 9. *Let* $(u,v) \in H^{1,1}(\mathbb{R})$. *Then,*

$$a(\omega^{-1}) - a_0, \;\; b_+(\omega^{-1}), \;\; b_-(\omega^{-1}) \in H^1_\omega(\mathbb{R}\backslash[-1,1]), \tag{4.16}$$

and

$$a(z) - a_\infty, \;\; b_+(z), \;\; b_-(z) \in H^1_z(\mathbb{R}\backslash[-1,1]). \tag{4.17}$$

If in addition $(u,v) \in H^2(\mathbb{R})$, *then*

$$b_+(\omega^{-1}), \;\; b_-(\omega^{-1}) \in L^{2,1}_\omega(\mathbb{R}\backslash[-1,1]), \tag{4.18}$$

and

$$b_+(z), \;\; b_-(z) \in L^{2,1}_z(\mathbb{R}\backslash[-1,1]). \tag{4.19}$$

The following lemma transfers the estimates of Lemma 9 to the reflection coefficients $r_\pm$ and $\widehat{r}_\pm$. We give an elementary proof of this result since it is based on new computations compared to [29].

Lemma 10. *Assume* $(u,v) \in X_{(u,v)}$, *where* $X_{(u,v)}$ *is given by (1.5), and* a *satisfies Assumption 1. Then* $(r_+, r_-) \in X_{(r_+,r_-)}$, *where* $X_{(r_+,r_-)}$ *is given by (1.6).*

Proof. Under the conditions of the lemma, it follows from Lemma 9 and from the definitions (4.2) and (4.7) that

$$r_\pm(\omega) \in \dot{H}^1_\omega(\mathbb{R}\backslash[-1,1]) \cap \dot{L}^{2,1}_\omega(\mathbb{R}\backslash[-1,1])$$

and

$$\hat{r}_\pm(\omega) \in \dot{H}^1_z(\mathbb{R}\backslash[-1,1]) \cap \dot{L}^{2,1}_z(\mathbb{R}\backslash[-1,1]).$$

It also follows from (4.2) and (4.7) that $r_\pm(\omega) = \widehat{r}_\mp(\omega^{-1})$.

If $f(x) \in \dot{L}^{2,1}_x(1,\infty)$ and $\tilde{f}(y) := f(y^{-1})$, then $\tilde{f}(y) \in \dot{L}^{2,-2}_y(0,1)$, which follows by the chain rule:

$$\int_1^\infty x^2 |f(x)|^2 dx = \int_0^1 y^{-4} |\tilde{f}(y)|^2 dy.$$

Since $\dot{L}^{2,1}(1,\infty)$ is continuously embedded into $\dot{L}^{2,-2}(1,\infty)$ and $\dot{L}^{2,-2}(0,1)$ is continuously embedded into $\dot{L}^{2,1}(0,1)$, we verify that $r_\pm(z) \in \dot{L}^{2,1}_z(\mathbb{R}) \cap \dot{L}^{2,-2}_z(\mathbb{R})$ and $\hat{r}_\pm(\omega) \in \dot{L}^{2,1}_\omega(\mathbb{R}) \cap \dot{L}^{2,-2}_\omega(\mathbb{R})$.

Finally, if $f(x) \in \dot{H}^1_x(1,\infty)$ and $\tilde{f}(y) := f(y^{-1})$, then $\tilde{f}(y) \in \dot{H}^{1,1}_y(0,1)$, which follows by the chain rule $f'(x) = -x^{-2}\tilde{f}'(x^{-1})$ and

$$\int_1^\infty |f'(x)|^2 dx = \int_0^1 y^2 |\tilde{f}'(y)|^2 dy.$$

Combing all requirements together, we obtain the space $X_{(r_+,r_-)}$ both for (r_+,r_-) in z and for $(\hat{r}_+,\hat{r}_-)$ in ω, where $X_{(r_+,r_-)}$ is given by (1.6). □

Remark 7. *It follows from the relations (3.7) and (3.11) that* $r_+(\omega)=\omega r_-(\omega)$ *and* $\widehat{r}_+(z)=z\widehat{r}_-(z)$. *Then, it follows from Lemma 10 and the chain rule that*

$$\text{if } r_+,\hat{r}_+\in \dot{H}^1(\mathbb{R}\backslash[-1,1])\cap \dot{L}^{2,1}(\mathbb{R}), \text{ then } r_-,\hat{r}_-\in \dot{H}^{1,1}(\mathbb{R}\backslash[-1,1])\cap \dot{L}^{2,2}(\mathbb{R})$$

and

$$\text{if } r_-,\hat{r}_-\in \dot{H}^{1,1}([-1,1])\cap \dot{L}^{2,-2}(\mathbb{R}) \text{ then } r_+,\hat{r}_+\in \dot{H}^{1}([-1,1])\cap \dot{L}^{2,-3}(\mathbb{R}).$$

Therefore, we have $r_+,\hat{r}_+\in \dot{H}^1(\mathbb{R})\cap\dot{L}^{2,1}(\mathbb{R})\cap\dot{L}^{2,-3}(\mathbb{R})$ *and* $r_-,\hat{r}_-\in \dot{H}^{1,1}(\mathbb{R})\cap\dot{L}^{2,2}(\mathbb{R})\cap\dot{L}^{2,-2}(\mathbb{R})$.

Remark 8. *It may appear strange for the first glance that the direct and inverse scattering transforms for the MTM system (1.1) connect potentials* $(u,v)\in X_{(u,v)}$ *and reflection coefficients* $(r_+,r_-)\in X_{(r_+,r_-)}$ *in different spaces, whereas the Fourier transform provides an isomorphism in the space* $H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$. *However, the appearance of* $X_{(u,v)}$ *spaces for the potential* (u,v) *is not surprising due to the transformation of the linear operator* L *to the equivalent forms (2.7) and (2.17). The condition* $(u,v)\in X_{(u,v)}$ *ensures that* $(Q_{1,2},\hat{Q}_{1,2})\in H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$*, hence, the direct and inverse scattering transform for the MTM system (1.1) provides a transformation between* $(Q_{1,2},\hat{Q}_{1,2})\in H^1(\mathbb{R})\cap L^{2,1}(\mathbb{R})$ *and* $(r_+,r_-)\in X_{(r_+,r_-)}$*, which is a natural transformation under the Fourier transform with oscillatory phase* $e^{ix(\omega-\omega^{-1})}$. *This allows us to avoid reproducing the Fourier analysis anew and to apply all the technical results from* [29] *without any changes, as these results generalize the classical results of Deift & Zhou* [11, 37] *obtained for the cubic NLS equation.*

4.4. **Solvability of the Riemann–Hilbert problems.** Let us define the reflection coefficient

$$r(\lambda):=\frac{\beta(\lambda)}{\alpha(\lambda)},\quad \lambda\in\mathbb{R}\cup(i\,\mathbb{R})\backslash\{0\}. \tag{4.20}$$

Recall the relations (3.7), (3.11), (4.2), and (4.7) which yield

$$\lambda^{-1}r(\lambda)=r_+(\omega)=\omega r_-(\omega),\quad \omega\in\mathbb{R}\backslash\{0\}. \tag{4.21}$$

and

$$\lambda r(\lambda)=\widehat{r}_+(z)=z\widehat{r}_-(z),\quad z\in\mathbb{R}\backslash\{0\}. \tag{4.22}$$

Also recall that $z=\lambda^2$ and $\omega=\lambda^{-2}$. By extending the proof of Propositions 2 and 3 in [29], we obtain the following.

Lemma 11. *If* $(r_+,r_-)\in X_{(r_+,r_-)}$*, then*

$$r(\lambda)\in L^{2,1}_\omega(\mathbb{R})\cap L^\infty_\omega(\mathbb{R}),\quad r(\lambda)\in L^{2,1}_z(\mathbb{R})\cap L^\infty_z(\mathbb{R}), \tag{4.23}$$

and

$$\lambda^{-1}r_+(\omega)\in L^\infty_\omega(\mathbb{R}),\quad \lambda\hat{r}_+(z)\in L^\infty_z(\mathbb{R}). \tag{4.24}$$

Proof. Let us prove the embeddings in $L^2_z(\mathbb{R})$ space. The proof of the embeddings in $L^2_\omega(\mathbb{R})$ space is analogous. Relation (4.22) implies $|r(\lambda)|^2=|\hat{r}_+(z)||\hat{r}_-(z)|$ and

$$r(\lambda)=\begin{cases}\lambda^{-1}\hat{r}_+(z), & |z|\ge 1,\\ \lambda\hat{r}_-(z), & |z|\le 1.\end{cases}$$

Since $\hat{r}_+,\hat{r}_-\in L^{2,1}(\mathbb{R})$, Cauchy–Schwarz inequality implies $r(\lambda)\in L^{2,1}_z(\mathbb{R})$. Since $\hat{r}_+\in H^1(\mathbb{R})$ by Remark 7, $r(\lambda)\in L^\infty_z(\mathbb{R}\backslash[-1,1])$. In order to prove that $r(\lambda)\in L^\infty_z([-1,1])$, we will show that

$\lambda \hat{r}_-(z) \in L^\infty_z([-1,1])$. This follows from the representation

$$z\hat{r}_-(z)^2 = \int_0^z \left[\hat{r}_-^2(z) + 2z\hat{r}_-(z)\hat{r}_-'(z)\right] dz$$

and the Cauchy–Schwarz inequality, since $\hat{r}_- \in \dot{H}^{1,1}(\mathbb{R}) \cap L^2(\mathbb{R})$. Similarly, $\lambda \hat{r}_+(z) \in L^\infty(\mathbb{R})$ since $\hat{r}_+ \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$. $\square$

Remark 9. *By using the relations (3.8), we obtain another constraint on $r(\lambda)$:*

$$1 - |r(\lambda)|^2 = \frac{1}{|\alpha(\lambda)|^2} \geq c_0^2 > 0, \quad \lambda \in i\mathbb{R}, \tag{4.25}$$

where $c_0 := \sup_{\lambda \in i\mathbb{R}} |\alpha(\lambda)| < \infty$, which exists thanks to Lemma 6.

Under Assumption 1 as well as the constraints (4.23) and (4.25), the jump matrices in the Riemann–Hilbert problems 1 and 2 satisfy the same estimates as in Proposition 5 in [29]. Hence these Riemann–Hilbert problems can be solved and estimated with the same technique as in the proofs of Lemmas 7, 8, and 9 in [29]. The following summarizes this result.

Lemma 12. *Under Assumption 1, for every $r(\lambda) \in L^2_\omega(\mathbb{R}) \cap L^\infty_\omega(\mathbb{R})$ satisfying (4.25), there exists a unique solution of the Riemann–Hilbert problem 1 satisfying for every $x \in \mathbb{R}$:*

$$\|M_\pm(x;\omega) - I\|_{L^2_\omega} \leq C\|r(\lambda)\|_{L^2_\omega}, \tag{4.26}$$

where the positive constant C only depends on $\|r(\lambda)\|_{L^\infty_\omega}$. Similarly, under Assumption 1, for every $r(\lambda) \in L^2_z(\mathbb{R}) \cap L^\infty_z(\mathbb{R})$ satisfying (4.25), there exists a unique solution of the Riemann–Hilbert problem 2 satisfying for every $x \in \mathbb{R}$:

$$\|\widehat{M}_\pm(x;z) - I\|_{L^2_z} \leq \widehat{C}\|r(\lambda)\|_{L^2_z} \tag{4.27}$$

where the positive constant $\widehat{C}$ only depends on $\|r(\lambda)\|_{L^\infty_z}$.

The potentials u and v are recovered respectively from M and $\widehat{M}$ by means of the reconstruction formulas (4.5) and (4.10), whereas the derivatives of the potentials u' and v' are recovered from the reconstruction formulas (4.4) and (4.9). At the first order in terms of the scattering coefficient (see, e.g., [3]), we have to analyze the integrals like

$$\lim_{|\omega|\to\infty} \omega[M(x;\omega)]_{12} \sim \frac{i}{2\pi}\int_{\mathbb{R}} \overline{r_-(\omega)} e^{-\frac{i}{2}(\omega-\omega^{-1})x} d\omega \tag{4.28}$$

in the space $H^1_x(\mathbb{R}) \cap L^{2,1}_x(\mathbb{R})$. In order to control the remainder term of the representation (4.28) in $H^1_x(\mathbb{R}) \cap L^{2,1}_x(\mathbb{R})$, we need to generalize Proposition 7 in [29] for the case of the oscillatory factor

$$\Theta(s) = \frac{1}{2}\left(s - \frac{1}{s}\right).$$

The following lemma presents this generalization in the function space

$$X_0 := H^1(\mathbb{R}\backslash[-1,1]) \cap \dot{H}^{1,1}([-1,1]) \cap \dot{L}^{2,-1}([-1,1]).$$

The proof of this lemma is a non-trivial generalization of analysis of the Fourier integrals.

Lemma 13. *There is a positive constant C such that for all $x_0 \in \mathbb{R}_+$ and all $f \in X_0$, we have*

$$\sup_{x\in(x_0,\infty)} \|\langle x\rangle \mathcal{P}^\pm[f(\diamond)e^{\mp ix\Theta(\diamond)}]\|_{L^2(\mathbb{R})} \leq C\|f\|_{X_0} \tag{4.29}$$

where $\langle x\rangle := (1+x^2)^{1/2}$ *and the Cauchy projection operators are explicitly given by*

$$\mathcal{P}^{\pm}[f(\diamond)](z) := \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(s)}{s-(z\pm i\,\varepsilon)}ds, \quad z\in\mathbb{R}.$$

In addition, if $f\in X_0\cap \dot{L}^{2,-1}(\mathbb{R})$, *then*

$$\sup_{x\in\mathbb{R}}\|\mathcal{P}^{\pm}[f(\diamond)e^{\mp ix\Theta(\diamond)}]\|_{L^\infty(\mathbb{R})}\le C\left(\|f\|_{X_0}+\|f\|_{\dot{L}^{2,-1}(\mathbb{R})}\right). \tag{4.30}$$

Furthermore, if $f\in L^{2,1}(\mathbb{R})\cap \dot{L}^{2,-1}(\mathbb{R})$, *then*

$$\sup_{x\in\mathbb{R}}\|\mathcal{P}^{\pm}[(\diamond-\diamond^{-1})f(\diamond)e^{\mp ix\Theta(\diamond)}]\|_{L^2(\mathbb{R})}\le C\left(\|f\|_{L^{2,1}(\mathbb{R})}+\|f\|_{\dot{L}^{2,-1}(\mathbb{R})}\right). \tag{4.31}$$

Proof. Consider the decomposition

$$f(s)e^{\mp ix\Theta(s)} = f(s)e^{\mp ix\Theta(s)}\chi_{\mathbb{R}_-}(s)+f(s)e^{\mp ix\Theta(s)}\chi_{\mathbb{R}_+}(s),$$

where χ_S is a characteristic function on the set $S\subset\mathbb{R}$. Thanks to the linearity of $\mathcal{P}^{\pm}$, it is sufficient to consider separately the functions f that vanish either on $\mathbb{R}_+$ or on $\mathbb{R}_-$. In the following we give an estimate for $\mathcal{P}^{+}[f(\diamond)e^{-ix\Theta(\diamond)}\chi_{\mathbb{R}_+}(\diamond)]$. The other case is handled analogously.

Fix $x>0$ and consider the following decomposition:

$$f(s)e^{-ix\Theta(s)}\chi_{\mathbb{R}_+}(s) = h_I(x,s)+h_{II}(x,s), \tag{4.32}$$

with

$$h_I(x,s) = e^{-ix\Theta(s)}\frac{1}{2\pi}\int_{x/4}^{\infty}e^{ik(s-s^{-1})}\mathfrak{a}[f](k)dk$$

and

$$h_{II}(x,s) = e^{-i\frac{x}{4}(s-s^{-1})}\frac{1}{2\pi}\int_{-\infty}^{x/4}e^{i(k-\frac{x}{4})(s-s^{-1})}\mathfrak{a}[f](k)dk,$$

where

$$\mathfrak{a}[f](k) := \int_0^{\infty}e^{-ik(s-s^{-1})}\frac{1+s^2}{s^2}f(s)ds. \tag{4.33}$$

The following change of coordinates

$$y(s)=s-s^{-1}, \qquad s(y)=\frac{y}{2}+\sqrt{1+\frac{y^2}{4}}, \qquad s'(y)=\frac{1}{2}+\frac{y}{4}\left(\sqrt{1+\frac{y^2}{4}}\right)^{-1}=\frac{s(y)^2}{1+s(y)^2}$$

shows that $\mathfrak{a}[f](k)=\mathfrak{F}[\tilde{f}](k)$, where the function $\tilde{f}$ is given by

$$\tilde{f}(y)=f(s(y)), \quad y\in\mathbb{R}$$

and $\mathfrak{F}$ denotes the Fourier transform

$$\mathfrak{F}[\tilde{f}](k)=\int_{-\infty}^{\infty}e^{-iky}\tilde{f}(y)dy.$$

We obtain

$$\|\tilde{f}\|^2_{L^2(\mathbb{R})}=\int_{\mathbb{R}}|f(s(y))|^2dy=\int_0^{\infty}\frac{1+s^2}{s^2}|f(s)|^2ds\le\|f\|^2_{X_0}$$

and

$$\|\tilde{f}'\|^2_{L^2(\mathbb{R})}=\int_{\mathbb{R}}\left(\frac{s(y)^2}{1+s(y)^2}\right)^2|f'(s(y))|^2dy=\int_0^{\infty}\frac{s^2}{1+s^2}|f'(s)|^2ds\le\|f\|^2_{X_0}.$$

It follows that $\tilde{f} \in H^1(\mathbb{R})$ and thus by Fourier theory $\mathfrak{a}[f](k) \in L^{2,1}_k(\mathbb{R})$. Using the inverse Fourier transform

$$\mathfrak{F}^{-1}[g](y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyk} g(k) dk,$$

we find for $s > 0$:

$$f(s) = \tilde{f}(y(s)) = \mathfrak{F}^{-1}[\mathfrak{a}[f]](y(s)) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik(s-s^{-1})} \mathfrak{a}[f](k) dk. \tag{4.34}$$

Addressing the decomposition (4.32), we obtain for the functions h_I thanks to $s'(y) < 1$:

$$\|h_I(x,\cdot)\|^2_{L^2(\mathbb{R}_+)} \leq \left\| \frac{1}{2\pi} \int_{x/4}^{\infty} e^{iky} \mathfrak{a}[f](k) dk \right\|_{L^2_y(\mathbb{R})} = \int_{x/4}^{\infty} |\mathfrak{a}[f](k)|^2 dk \leq \frac{C}{1+x^2} \|\mathfrak{a}[f]\|^2_{L^{2,1}(\mathbb{R})}. \tag{4.35}$$

On the other hand, the function $h_{II}(x,\cdot)$ is analytic in the domain $\{\mathrm{Im}(s) < 0\}$ and additionally for $s = -i\xi$ with $\xi \in \mathbb{R}_+$ we have

$$|h_{II}(x,s)| \leq C \|\mathfrak{a}[f]\|_{L^{2,1}(\mathbb{R})} e^{-\frac{x}{4}(\xi+\xi^{-1})}.$$

Therefore, $\|h_{II}(x,\cdot)\|_{L^2(i\,\mathbb{R}_-)}$ is decaying exponentially as $x \to \infty$. Now we have

$$\|\mathcal{P}^+[f(\diamond)e^{-ix\Theta(\diamond)}\chi_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})} \leq \|\mathcal{P}^+[h_I(x,\diamond)\chi_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})} + \|\mathcal{P}^+[h_{II}(x,\diamond)\chi_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})}$$

Since $\mathcal{P}^+$ is a bounded operator $L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ it follows by (4.35) that

$$\|\mathcal{P}^+[h_I(x,\diamond)\chi_{\mathbb{R}_+}(\diamond)]\|_{L^2(\mathbb{R})} \leq \|h_I(x,\cdot)\|^2_{L^2(\mathbb{R}_+)} \leq C\langle x\rangle^{-1} \|f\|^2_{X_0}.$$

Using a suitable path of integration and the analyticity of h_{II} we find that

$$\mathcal{P}^+[h_{II}(x,\diamond)](z) = -\mathcal{P}_{i\,\mathbb{R}_-}[h_{II}(x,\diamond)](z),$$

where

$$\mathcal{P}_{i\,\mathbb{R}_-}[h](z) := \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{h(is)}{is - z} ds, \quad z \in \mathbb{R},$$

for a function $h : i\,\mathbb{R}_- \to \mathbb{C}$. Since $\mathcal{P}_{i\,\mathbb{R}_-}$ is a bounded operator $L^2(i\,\mathbb{R}_-) \to L^2(\mathbb{R})$ (see, e.g., estimate (23.11) in [4]) and because $\|h_{II}(x,\cdot)\|_{L^2(i\,\mathbb{R}_-)}$ is decaying exponentially as $x \to \infty$, the proof of the estimate (4.29) is complete.

In order to prove the estimate (4.30), we first note that for $z \leq 0$

$$\begin{aligned} |\mathcal{P}^+[e^{-ix\Theta(\diamond)} f(\diamond)\chi_{\mathbb{R}_+}(\diamond)](z)| &\leq \int_0^{\infty} \frac{|f(s)|}{s} ds \\ &\leq \left(\int_0^1 \frac{|f(s)|^2}{s^2} ds\right)^{1/2} + \left(\int_1^{\infty} \frac{1}{s^2} ds\right)^{1/2} \left(\int_1^{\infty} |f(s)|^2 ds\right)^{1/2} \\ &\leq C\left(\|f\|_{X_0} + \|f\|_{\dot{L}^{2,-1}(\mathbb{R})}\right). \end{aligned} \tag{4.36}$$

Thus it remains to estimate $|\mathcal{P}^+[e^{-ix\Theta(\diamond)} f(\diamond)\chi_{\mathbb{R}_+}(\diamond)](z)|$ for $z > 0$. First, we will derive a bound for the special case $x = 0$ and by (4.39) below we will see that the same bound holds for any $x \in \mathbb{R}$. Therfore, using (4.34) we decompose

$$f(s) = h_+(s) + h_-(s), \qquad h_\pm(s) := \pm\frac{1}{2\pi} \int_0^{\pm\infty} e^{ik(s-s^{-1})} \mathfrak{a}[f](k) dk,$$

where $h_\pm$ has an analytic extension within the domain $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0, \pm\mathrm{Im}(s) > 0\}$ and for $\xi > 0$ we have

$$|h_\pm(\pm i\xi)| \le C\|e^{-k(\xi+\xi^{-1})}\|_{L^2(\mathbb{R}_+)}\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)} = \frac{C}{\sqrt{2}}\sqrt{\frac{\xi}{1+\xi^2}}\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)}. \tag{4.37}$$

Using a residue calculation we obtain for $z > 0$

$$\begin{aligned} \mathcal{P}^+[f(\diamond)\chi_{\mathbb{R}_+}(\diamond)](z) &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_0^\infty \frac{h_+(s)+h_-(s)}{s-(z\pm i\,\varepsilon)}ds \\ &= \mathcal{P}_{i\,\mathbb{R}_+}[h_+](z) - \mathcal{P}_{i\,\mathbb{R}_-}[h_-](z) + h_+(z). \end{aligned}$$

Thanks to the bound (4.37), the summands $\mathcal{P}_{i\,\mathbb{R}_+}[h_+](z)$ and $\mathcal{P}_{i\,\mathbb{R}_-}[h_-](z)$ are estimated in the following way,

$$\begin{aligned} \sup_{z\in\mathbb{R}_+}|\mathcal{P}_{i\,\mathbb{R}_\pm}[h_\pm](z)| &\le \int_0^\infty \frac{|h_\pm(\pm i\xi)|}{\xi}d\xi \\ &\le C\int_0^\infty \frac{1}{\sqrt{\xi}\sqrt{1+\xi^2}}d\xi\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)} \\ &\le C\|\mathfrak{a}[f]\|_{L^2_k(\mathbb{R}_\pm)}. \end{aligned}$$

In addition, for $z > 0$ we have $|h_+(z)| \le \|\mathfrak{a}[f]\|_{L^1_k(\mathbb{R}_+)}$ so that the triangle inequality implies:

$$\sup_{z\in\mathbb{R}_+}|\mathcal{P}^+[f(\diamond)\chi_{\mathbb{R}_+}(\diamond)](z)| \le C\left(\|\mathfrak{a}[f]\|_{L^1(\mathbb{R})} + \|\mathfrak{a}[f]\|_{L^2(\mathbb{R})}\right). \tag{4.38}$$

Now, let us reinsert the factor $e^{-ix\Theta(s)}$. From the definition of $\mathfrak{a}$ it follows that multiplication by $e^{-ix\Theta(s)}$ is equivalent of a shift of $\mathfrak{a}[f](k)$ in the k-variable,

$$\mathfrak{a}[e^{-ix\Theta(\diamond)}f(\diamond)](k) = \mathfrak{a}[f(\diamond)]\left(k+\frac{x}{2}\right). \tag{4.39}$$

Thus, the $L^1(\mathbb{R})\cap L^2(\mathbb{R})$-norm with respect to k of $\mathfrak{a}[e^{-ix\Theta(\diamond)}f(\diamond)](k)$ does not depend on x. Therefore, (4.38) yields

$$\begin{aligned} \sup_{z\in\mathbb{R}_+}|\mathcal{P}^+[e^{-ix\Theta(\diamond)}f(\diamond)\chi_{\mathbb{R}_+}(\diamond)](z)| &\le C\|\mathfrak{a}[e^{-ix\Theta(\diamond)}f(\diamond)]\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} \\ &= C\|\mathfrak{a}[f]\|_{L^1(\mathbb{R})\cap L^2(\mathbb{R})} \le C\|f\|_{X_0}, \end{aligned} \tag{4.40}$$

which, together with (4.36), completes the proof of (4.30).

Finally, the bound (4.31) follows from $\|\mathcal{P}^\pm\|_{L^2\to L^2} = 1$ and the fact that $(s-s^{-1})f(s) \in L^2_s(\mathbb{R})$ if $f \in L^{2,1}(\mathbb{R}) \cap \dot{L}^{2,-1}(\mathbb{R})$. □

The first term in (4.28) is estimated with a similar change of coordinates $y := \omega - \omega^{-1}$ and further analysis in the proof of Lemma 13. However, it is controlled $H^1_x(\mathbb{R}) \cap L^{2,1}_x(\mathbb{R})$ if the scattering coefficient r_- is defined in $X_{(r_+,r_-)}$ according to the bound

$$\left|\int_{\mathbb{R}}\overline{r_-(\omega)}e^{-\frac{i}{2}(\omega-\omega^{-1})x}d\omega\right|_{H^1_x(\mathbb{R})\cap L^{2,1}_x(\mathbb{R})} \le C\|r_-\|_{X_{(r_+,r_-)}}. \tag{4.41}$$

By using the estimate (4.41) and the estimates of Lemma 13, we can proceed similarly to Lemmas 10, 11, and 12 in [29]. The following lemma summarize the estimates on the potential (u, v) obtained from the reconstruction formulas (4.4)–(4.5) and (4.9)–(4.10).

Lemma 14. *Under Assumption 1, for every $(r_+, r_-) \in X_{(r_+,r_-)}$ and $(\widehat{r}_+, \widehat{r}_-) \in X_{(r_+,r_-)}$, the components $(u, v) \in X_{(u,v)}$ satisfy the bound*

$$\|u\|_{H^2\cap H^{1,1}} + \|v\|_{H^2\cap H^{1,1}} \leq C \left(\|r_+\|_{X_{(r_+,r_-)}} + \|r_-\|_{X_{(r_+,r_-)}} + \|\hat{r}_+\|_{X_{(r_+,r_-)}} + \|\hat{r}_-\|_{X_{(r_+,r_-)}} \right), \tag{4.42}$$

where the positive constant C depends on $\|r_\pm\|_{X_{(r_+,r_-)}}$ and $\|\hat{r}_\pm\|_{X_{(r_+,r_-)}}$.

Lemma 9 proves the first assertion of Theorem 2. Lemma 14 proves the second assertion of Theorem 2 at $t = 0$. It remains to prove the second assertion of Theorem 2 for every $t \in \mathbb{R}$.

4.5. **Time evolution of the spectral data.** Thanks to the well-posedness result of Theorem 1 and standard estimates in weighted L^2-based Sobolev spaces, there exists a global solution $(u, v) \in C(\mathbb{R}, X_{(u,v)})$ to the MTM system (1.1) for any initial data $(u, v)|_{t=0} = (u_0, v_0) \in X_{(u,v)}$. For this global solution, the normalized Jost functions (2.2) can be extended for every $t \in \mathbb{R}$:

$$\left\{ \begin{array}{l} \varphi_\pm(t, x; \lambda) = \psi_1^{(\pm)}(t, x; \lambda) e^{-ix(\lambda^2-\lambda^{-2})/4 - it(\lambda^2+\lambda^{-2})/4}, \\ \phi_\pm(t, x; \lambda) = \psi_2^{(\pm)}(t, x; \lambda) e^{ix(\lambda^2-\lambda^{-2})/4 + it(\lambda^2+\lambda^{-2})/4}. \end{array} \right. \tag{4.43}$$

where $(\varphi_\pm, \phi_\pm)$ still satisfy the same boundary conditions (2.3). Introducing the scattering coefficients in the same way as in Section 3, we obtain the time evolution of the scattering coefficients:

$$\alpha(t, \lambda) = \alpha(0, \lambda), \quad \beta(t, \lambda) = \beta(0, \lambda) e^{-it(\lambda^2+\lambda^{-2})/2}, \quad \lambda \in \mathbb{R} \cup (i\,\mathbb{R}) \backslash \{0\}. \tag{4.44}$$

Transferring the scattering coefficients to the reflection coefficients with the help of (3.7), (3.11), (4.2), and (4.7) yields the time evolution of the reflection coefficients:

$$r_\pm(t, \omega) = r_\pm(0, \omega) e^{-it(\omega+\omega^{-1})/2}, \quad \omega \in \mathbb{R} \backslash \{0\} \tag{4.45}$$

and

$$\hat{r}_\pm(t, z) = \hat{r}_\pm(0, z) e^{-it(z+z^{-1})/2}, \quad z \in \mathbb{R} \backslash \{0\}. \tag{4.46}$$

It is now clear that if $r_\pm$ and $\hat{r}_\pm$ are in $X_{(r_+,r_-)}$ at the initial time $t = 0$, then they remain in $X_{(r_+,r_-)}$ for every $t \in \mathbb{R}$. Thus, the recovery formulas of Lemma 14 for the global solution $(u, v) \in C(\mathbb{R}, X_{(u,v)})$ to the MTM system (1.1) hold for every $t \in \mathbb{R}$. This proves the second assertion of Theorem 2 for every $t \in \mathbb{R}$. Hence Theorem 2 is proven.

Remark 10. *Adding the time dependence to the Riemann-Hilbert problem 1 we find the time-dependent jump relation $M_+(x, t; \omega) - M_-(x, t; \omega) = M_-(x, t; \omega) R(x, t; \omega)$, where*

$$R(x, t; \omega) := \left[\begin{array}{cc} r_+(\omega)\overline{r_-(\omega)} & \overline{r_-(\omega)} e^{-\frac{i}{2}(\omega-\omega^{-1})x + \frac{i}{2}(\omega+\omega^{-1})t} \\ r_+(\omega) e^{\frac{i}{2}(\omega-\omega^{-1})x - \frac{i}{2}(\omega+\omega^{-1})t} & 0 \end{array} \right].$$

The same phase function as in $R(x, t; \omega)$ appears in the inverse scattering theory for the sine-Gordon equation. A Riemann-Hilbert problem with such a phase function was studied in [7]*, where the long-time behavior of the sine-Gordon equation was analyzed.*

Remark 11. *In the context of the MTM system (1.1), it is more natural to address global solutions in weighted H^1 space such as $H^{1,1}(\mathbb{R})$ and drop the requirement $(u, v) \in H^2(\mathbb{R})$. The scattering coefficients $r_\pm$ and $\hat{r}_\pm$ are then defined in the space X_0. However, there are two obstacles to develop the inverse scattering transform for such a larger class of initial data. First, the asymptotic limits (2.14a) and (2.23a) are not justified, therefore, the recovery formulas (4.4) and (4.9) cannot be utilized. Second, without requirement $r_\pm, \hat{r}_\pm \in L^{2,1}(\mathbb{R})$, the time evolution (4.45)–(4.46) is not closed in X_0 since $r_-, \hat{r}_- \in L^{2,-2}(\mathbb{R})$ cannot be verified. In this sense, the space $X_{(u,v)}$ for (u, v) and $X_{(r_+,r_-)}$ for (r_+, r_-) and $(\widehat{r}_+, \widehat{r}_-)$ are optimal for the inverse scattering transform of the MTM system (1.1).*

5. Conclusion

We gave functional-analytical details on how the direct and inverse scattering transforms can be applied to solve the initial-value problem for the MTM system in laboratory coordinates. We showed that initial data $(u_0, v_0) \in X_{(u,v)}$ admitting no eigenvalues or resonances defines uniquely the spectral data (r_+, r_-) in $X_{(r_+,r_-)}$. With the time evolution added, the spectral data (r_+, r_-) remain in the space $X_{(r_+,r_-)}$ and determine uniquely the solution (u, v) to the MTM system (1.1) in the space $X_{(u,v)}$.

We conclude the paper with a list of open questions.

The long-range scattering of solutions to the MTM system (1.1) for small initial data for which the assumption of no eigenvalues or resonances is justified can be considered based on the inverse scattering transform presented here. This will be the subject of the forthcoming work, where the long-range scattering results in [6] obtained by regular functional-analytical methods are to be improved.

The generalization of the inverse scattering transform in the case of eigenvalues is easy and can be performed similarly to what was done for the derivative NLS equation in [27]. However, it is not so easy to include resonances and other spectral singularities in the inverse scattering transform. In particular, the case of algebraic solitons [20] corresponds to the spectral singularities of the scattering coefficients due to slow decay of (u, v) and analysis of this singular case is an open question.

Finally, another interesting question is to consider the inverse scattering transform for the initial data decaying to constant (nonzero) boundary conditions. The MTM system (1.1) admits solitary waves over the nonzero background [2] and analysis of spectral and orbital stability of such solitary waves is at the infancy stage.

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