INVERSE SCATTERING FOR THE MASSIVE THIRRING MODEL

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Abstract. We consider the massive Thirring model in the laboratory coordinates and explain how the inverse scattering transform can be developed with the Riemann–Hilbert approach. The key ingredient of our method is to transform the corresponding spectral problem to the equivalent forms: one is suitable for the spectral parameter at the origin and the other one is suitable for the spectral parameter at infinity. Global solutions to the massive Thirring model are recovered from the reconstruction formulae at the origin and at infinity.

1. Introduction

The massive Thirring model (MTM) was derived by Thirring in 1958 [28] in the context of general relativity. It represents a relativistically invariant nonlinear Dirac equation in the space of one dimension. Another relativistically invariant one-dimensional Dirac equation is given by the Gross–Neveu model [8] also known as the massive Soler model [27] when it is written in the space of three dimensions.

It was discovered in 1970s by Kuznetsov and Mikhailov [16], Orfanidis [19], Kaup and Newell [14] that the MTM is integrable with the inverse scattering transform method in the sense that it admits a Lax pair, countably many conserved quantities, the Bäcklund transformation, and other common features of integrable models. If the MTM system is written in the laboratory coordinates,

$$\left\{ \begin{array}{l} i(u_t + u_x) + v + |v|^2 u = 0, \\ i(v_t - v_x) + u + |u|^2 v = 0, \end{array} \right. \tag{1.1}$$

then it appears as the compatibility condition in the Lax representation

$$L_t - A_x + [L, A] = 0, \tag{1.2}$$

where the 2×2-matrices L and A are given by

$$L = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2} \begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{1.3}$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2} \begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda} \begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3. \tag{1.4}$$

Formal inverse scattering results for the linear operators (1.3) and (1.4) can be found in [16]. The first steps towards rigorous developments of the inverse scattering transform for the MTM system (1.1) were made in 1990s by Villarroel [29] and Zhou [33]. In the former work, the treatment of the Riemann–Hilbert problems is sketchy, whereas in the latter work, an abstract method to solve Riemann–Hilbert problems with rational spectral dependence is developed with applications to the sine–Gordon equation in the laboratory coordinates. Although the MTM system (1.1) does not appear in the list of examples in [33], one can show that the abstract method of Zhou is also applicable to the MTM system.

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The present paper relies on recent progress in the inverse scattering transform method for the derivative NLS equation [21, 23]. The key element of our technique is a transformation of the spectral plane λ for the operator L in (1.3) to the spectral plane $z = \lambda^2$ for a different spectral problem. This transformation can be performed uniformly in the λ plane for the Kaup–Newell spectral problem related to the derivative NLS equation [15]. In the contrast, one needs to consider separately the subsets of the λ plane near the origin and near infinity for the operator L in (1.3) due to its rational dependence on λ. Therefore, two Riemann–Hilbert problems are derived for the MTM system (1.1) with the components (u, v): the one near $\lambda = 0$ recovers u and the other one near $\lambda = \infty$ recovers v.

Let $L^{2,m}(\mathbb{R})$ denote the space of square integrable functions with the weight $(1 + x^2)^{m/2}$ and $L^{2,0}(\mathbb{R}) \equiv L^2(\mathbb{R})$. The inverse scattering transform for the linear operators (1.3) and (1.4) can be controlled in the function space $X_{2,1} := H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$, where

$$H^2(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}), \quad \partial_x^2 u \in L^2(\mathbb{R}) \right\} \tag{1.5}$$

and

$$H^{1,1}(\mathbb{R}) = \left\{ u \in L^{2,1}(\mathbb{R}), \quad \partial_x u \in L^{2,1}(\mathbb{R}) \right\}. \tag{1.6}$$

Transformations of the spectral plane employed here allow us to give a sharp requirement on the L^2-based Hilbert spaces, for which the Riemann–Hilbert problem can be solved by using the technique from Deift and Zhou [7, 32]. The reflection coefficients $r_\pm$ which arise in the Riemann–Hilbert problem can be controlled in the function space $X_{1,1} := H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$.

In the application of the inverse scattering transform to the derivative NLS equation, alternative methods were recently developed [12, 17] based on a different (gauge) transformation of the Kaup–Newell spectral problem to the one related to the Gerdjikov–Ivanov equation. Both the potentials and the reflection coefficients were controlled in the same function space $X_{2,2} := H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R})$ [12, 17]. These function spaces are more restrictive compared to the function spaces used in [21, 23].

Unlike the recent literature on the derivative NLS equation, our interest to the inverse scattering for the MTM system (1.1) is not related to the well-posedness problems. Indeed, the local and global existence of solutions to the Cauchy problem for the MTM system (1.1) in the L^2-based Sobolev spaces $H^m(\mathbb{R})$, $m \in \mathbb{N}$ can be proven with the standard contraction and energy methods, see review of literature in [20]. Low regularity solutions in $L^2(\mathbb{R})$ were already obtained for the MTM system by Selberg and Tesfahun [26], Candy [2], Huh [9, 10, 11], and Zhang [30, 31]. The well-posedness results can be formulated as follows.

Theorem 1. *For every $(u_0, v_0) \in H^m(\mathbb{R})$, $m \in \mathbb{N}$, there exists a unique global solution $(u, v) \in C(\mathbb{R}, H^m(\mathbb{R}))$ such that $(u, v)|_{t=0} = (u_0, v_0)$ and the solution (u, v) depends continuously on the initial data (u_0, v_0). Moreover, for every $(u_0, v_0) \in L^2(\mathbb{R})$, there exists a global solution $(u, v) \in C(\mathbb{R}, L^2(\mathbb{R}))$ such that $(u, v)|_{t=0} = (u_0, v_0)$. The solution (u, v) is unique and depends continuously on the initial data (u_0, v_0) in a subspace of $C(\mathbb{R}, L^2(\mathbb{R}))$.*

The inverse scattering transform and the reconstruction formulas for the global solutions (u, v) to the MTM system (1.1) can be used to solve other interesting analytical problems such as long-range scattering to zero [3], orbital and asymptotic stability of the Dirac solitons [5, 22], and an analytical proof of the soliton resolution conjecture. Similar questions have been recently addressed in the context of the cubic NLS equation [4, 6, 24] and the derivative NLS equation [13, 18].

The goal of this paper is to explain how the inverse scattering transform for the linear operators (1.3) and (1.4) can be developed by using the Riemann–Hilbert problem. For simplicity of presentation, we assume that the initial data to the MTM system (1.1) admit no eigenvalues and resonances in the sense of Definition 1 given in Section 3. Note that eigenvalues can be easily adopted by using Bäcklund transformation for the MTM system [5], whereas resonances can be removed by perturbations of initial data [1] (see relevant results in [21]). Applications of the inverse scattering to the long-range scattering problem for small initial data for which the assumption of no eigenvalues or resonances is justified will be developed separately [25]. The following theorem represents informally the main result of this paper.

Theorem 2. *For every $(u_0, v_0) \in X_{2,1}$ admitting no eigenvalues or resonances in the sense of Definition 1, there is a direct scattering transform with the spectral data (r_+, r_-) defined in $X_{1,1}$. The unique solution $(u, v) \in C(\mathbb{R}, X_{2,1})$ to the MTM system (1.1) can be uniquely recovered from the inverse scattering transform for every $t \in \mathbb{R}$.*

The paper is organized as follows. Section 2 describes Jost functions obtained after two transformations of the differential operator L given by (1.3). Section 3 is used to set up scattering coefficients (r_+, r_-) and to introduce the scattering relations between the Jost functions. Section 4 explains how the Riemann–Hilbert problems can be solved and how the potentials (u, v) can be recovered in the time evolution of the MTM system (1.1).

2. Jost functions

The linear operator L in (1.3) can be rewritten in the form:

$$L = Q(\lambda; u, v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3,$$

where

$$Q(\lambda; u, v) = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix}.$$

Here we freeze the time variable t and drop it from the list of arguments. Since $|u(x)| + |v(x)| \to 0$ as $|x| \to \infty$ if $(u, v) \in H^1(\mathbb{R})$, solutions to the spectral problem

$$\psi_x = L\psi \tag{2.1}$$

can be defined by the following asymptotic behavior:

$$\psi_1^{(-)}(x;\lambda) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ix(\lambda^2-\lambda^{-2})/4}, \qquad \psi_2^{(-)}(x;\lambda) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ix(\lambda^2-\lambda^{-2})/4} \qquad \text{as } x \to -\infty$$

and

$$\psi_1^{(+)}(x;\lambda) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ix(\lambda^2-\lambda^{-2})/4}, \qquad \psi_2^{(+)}(x;\lambda) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ix(\lambda^2-\lambda^{-2})/4} \qquad \text{as } x \to +\infty.$$

The *normalized Jost functions*

$$\varphi_\pm(x;\lambda) = \psi_1^{(\pm)}(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}, \qquad \phi_\pm(x;\lambda) = \psi_2^{(\pm)}(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} \tag{2.2}$$

satisfy the constant boundary conditions at infinity:

$$\lim_{x\to\pm\infty} \varphi_\pm(x;\lambda) = e_1 \quad \text{and} \quad \lim_{x\to\pm\infty} \phi_\pm(x;\lambda) = e_2, \tag{2.3}$$

where $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$. The normalized Jost functions are solutions to the following Volterra's integral equations:

$$\varphi_\pm(x;\lambda) = e_1 + \int_{\pm\infty}^x \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} \end{bmatrix} Q(\lambda; u(y), v(y))\, \varphi_\pm(y;\lambda)dy, \tag{2.4a}$$

$$\phi_\pm(x;\lambda) = e_2 + \int_{\pm\infty}^x \begin{bmatrix} e^{\frac{i}{2}(\lambda^2-\lambda^{-2})(x-y)} & 0 \\ 0 & 1 \end{bmatrix} Q(\lambda; u(y), v(y))\, \phi_\pm(y;\lambda)dy. \tag{2.4b}$$

In order to study behaviour of solutions of the Volterra's integral equations (2.4) as $\lambda \to 0$ and $|\lambda| \to \infty$, we transform the spectral problem (2.1) to two equivalent forms. These two transformations, which generalize the exact transformation of the Kaup–Newell spectral problem to the Zakharov–Shabat spectral problem, see [15, 23], form the backbone of the inverse scattering transform for the spectral problem (2.1).

2.1. **Transformation of the Jost functions for small λ.** Assume $u \in L^\infty(\mathbb{R})$, $\lambda \neq 0$, and define the transformation matrix by

$$T(u;\lambda) := \begin{pmatrix} 1 & 0 \\ u & \lambda^{-1} \end{pmatrix}. \tag{2.5}$$

Let ψ be a solution of the spectral problem (2.1) and define $\Psi := T\psi$. Straightforward computations show that Ψ satisfies the equivalent linear equation

$$\Psi_x = \mathcal{L}\Psi, \tag{2.6}$$

with new linear operator

$$\mathcal{L} = Q_1(u,v) + \lambda^2 Q_2(u,v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{2.7}$$

where

$$Q_1(u,v) = \begin{pmatrix} -\frac{i}{4}(|u|^2+|v|^2) & \frac{i}{2}\overline{u} \\ u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v & \frac{i}{4}(|u|^2+|v|^2) \end{pmatrix}, \quad Q_2(u,v) = \frac{i}{2}\begin{pmatrix} u\overline{v} & -\overline{v} \\ u+u^2\overline{v} & -u\overline{v} \end{pmatrix}.$$

Let us define $z := \lambda^2$ and introduce the partition $\mathbb{C} = B_0 \cup \mathbb{S}^1 \cup B_\infty$ with

$$B_0 := \{z \in \mathbb{C} : \quad |z| < 1\}, \quad \mathbb{S}^1 := \{z \in \mathbb{C} : \quad |z| = 1\}, \quad B_\infty := \{z \in \mathbb{C} : \quad |z| > 1\}. \tag{2.8}$$

The second term in (2.7) is bounded if $z \in B_0$. The normalized Jost functions associated to the spectral problem (2.6) denoted by $\{m_\pm, n_\pm\}$ can be obtained from the original Jost functions $\{\varphi_\pm, \psi_\pm\}$ by the transformation formulas:

$$m_\pm(x;z) = T(u(x);\lambda)\varphi_\pm(x;\lambda), \qquad n_\pm(x;z) = \lambda\, T(u(x);\lambda)\phi_\pm(x;\lambda), \tag{2.9}$$

subject to the constant boundary conditions at infinity:

$$\lim_{x\to\pm\infty} m_\pm(x;\lambda) = e_1 \quad \text{and} \quad \lim_{x\to\pm\infty} n_\pm(x;\lambda) = e_2. \tag{2.10}$$

The transformed Jost functions are solutions to the following Volterra's integral equations:

$$m_\pm(x;z) = e_1 + \int_{\pm\infty}^x \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(z-z^{-1})(x-y)} \end{bmatrix} \left[Q_1(u(y),v(y)) + zQ_2(u(y),v(y))\right] m_\pm(y;z)dy, \tag{2.11a}$$

$$n_\pm(x;z) = e_2 + \int_{\pm\infty}^x \begin{bmatrix} e^{\frac{i}{2}(z-z^{-1})(x-y)} & 0 \\ 0 & 1 \end{bmatrix} \left[Q_1(u(y),v(y)) + zQ_2(u(y),v(y))\right] n_\pm(y;z)dy. \tag{2.11b}$$

Compared to [23], we have an additional term $\frac{i}{2}z(x-y)$ in the argument of the oscillatory kernel and the additional term $zQ_2(u,v)$ under the integration sign. However, both additional terms are bounded in B_0 where $|z| < 1$. Therefore, the same analysis as in the proof of Lemmas 1 and 2 in [23] yields the following.

Lemma 1. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u_x \in L^1(\mathbb{R})$. For every $z \in (-1,1)$, there exist unique solutions $m_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ and $n_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ satisfying the integral equations (2.11). For every $x \in \mathbb{R}$, $m_\pm(x,\cdot)$ and $n_\mp(x,\cdot)$ are continued analytically in $\mathbb{C}^\pm \cap B_0$. There exist a positive constant C such that*

$$\|m_\pm(\cdot;z)\|_{L^\infty} + \|n_\mp(\cdot;z)\|_{L^\infty} \leq C, \quad z \in \mathbb{C}^\pm \cap B_0. \tag{2.12}$$

Lemma 2. *Under the conditions of Lemma 1, for every $x \in \mathbb{R}$ the normalized Jost functions $m_\pm$ and $n_\pm$ satisfy the following limits as $\mathrm{Im}(z) \to 0$ along a contour in the domains of their analyticity:*

$$\lim_{z\to 0} \frac{m_\pm(x;z)}{m_\pm^\infty(x)} = e_1, \quad \lim_{z\to 0} \frac{n_\pm(x;z)}{n_\pm^\infty(x)} = e_2, \tag{2.13}$$

where

$$m_\pm^\infty(x) = e^{-\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}, \quad n_\pm^\infty(x) = e^{\frac{i}{4}\int_{\pm\infty}^x (|u|^2+|v|^2)dy}.$$

If in addition $u \in C^1(\mathbb{R})$, then

$$\lim_{z\to 0} z^{-1}\left[\frac{m_\pm(x;z)}{m_\pm^\infty(x)} - e_1\right] = \begin{pmatrix} -\int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right] dy \\ 2iu_x + u|v|^2 + v \end{pmatrix}, \tag{2.14a}$$

$$\lim_{z\to 0} z^{-1}\left[\frac{n_\pm(x;z)}{n_\pm^\infty(x)} - e_2\right] = \begin{pmatrix} \overline{u} \\ \int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right] dy \end{pmatrix}, \tag{2.14b}$$

Remark 1. *By Sobolev's embedding of $H^1(\mathbb{R})$ into the space of continuous, bounded, and decaying at infinity functions, if $u \in H^1(\mathbb{R})$, then $u \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u(x) \to 0$ as $|x| \to \infty$. By the embedding of $L^{2,1}(\mathbb{R})$ into $L^1(\mathbb{R})$, if $u \in H^{1,1}(\mathbb{R})$, then $u \in L^1(\mathbb{R})$ and $u_x \in L^1(\mathbb{R})$. Thus, requirements of Lemma 1 are satisfied if $(u,v) \in H^{1,1}(\mathbb{R})$ defined by (1.6). The additional requirement $u \in C^1(\mathbb{R})$ of Lemma 2 is satisfied if $u \in H^2(\mathbb{R})$ defined by (1.5). Hence,*

$$X_{2,1} := H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R}) \tag{2.15}$$

is an optimal L^2-based Sobolev space for direct scattering of the MTM system (1.1).

Remark 2. *Notations $(m_\pm, n_\pm)$ for the Jost functions used here are different from notations $(m_\pm, n_\pm)$ used in* [23]*, where an additional transformation was used to generate $n_\pm$ (denoted by $p_\pm$ in* [23]*). This additional transformation is not necessary for our further work.*

2.2. **Transformation of the Jost functions for large λ.** Assume $v \in L^\infty(\mathbb{R})$ and define the transformation matrix by

$$\widehat{T}(v;\lambda) := \begin{pmatrix} 1 & 0 \\ v & \lambda \end{pmatrix}. \tag{2.16}$$

Let ψ be a solution of the spectral problem (2.1) and define $\widehat{\Psi} := \widehat{T}\psi$. Straightforward computations show that $\widehat{\Psi}$ satisfies the equivalent linear equation

$$\widehat{\Psi}_x = \widehat{\mathcal{L}}\widehat{\Psi}, \tag{2.17}$$

with new linear operator

$$\widehat{\mathcal{L}} = \widehat{Q}_1(u,v) + \frac{1}{\lambda^2}\widehat{Q}_2(u,v) + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{2.18}$$

where

$$\widehat{Q}_1(u,v) = \begin{pmatrix} \frac{i}{4}(|u|^2+|v|^2) & -\frac{i}{2}\overline{v} \\ v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u & -\frac{i}{4}(|u|^2+|v|^2) \end{pmatrix}, \quad \widehat{Q}_2(u,v) = -\frac{i}{2}\begin{pmatrix} \overline{u}v & -\overline{u} \\ v + \overline{u}v^2 & -\overline{u}v \end{pmatrix}.$$

We introduce the same variable $z := \lambda^2$ and note that the second term in (2.18) is now bounded for $z \in B_\infty$. The normalized Jost functions associated to the spectral problem (2.6) denoted by $\{\widehat{m}_\pm, \widehat{n}_\pm\}$ can be obtained from the original Jost functions $\{\varphi_\pm, \psi_\pm\}$ by the transformation formulas:

$$\widehat{m}_\pm(x;z) = \widehat{T}(v(x);\lambda)\varphi_\pm(x;\lambda), \qquad \widehat{n}_\pm(x;z) = \lambda^{-1}\widehat{T}(v(x);\lambda)\phi_\pm(x;\lambda), \tag{2.19}$$

subject to the constant boundary conditions at infinity:

$$\lim_{x\to\pm\infty} \widehat{m}_\pm(x;\lambda) = e_1 \quad \text{and} \quad \lim_{x\to\pm\infty} \widehat{n}_\pm(x;\lambda) = e_2. \tag{2.20}$$

The transformed Jost functions are solutions to the following Volterra's integral equations:

$$\widehat{m}_\pm(x;z) = e_1 + \int_{\pm\infty}^x \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i}{2}(z-z^{-1})(x-y)} \end{bmatrix} \left[\widehat{Q}_1(u(y),v(y)) + z^{-1}\widehat{Q}_2(u(y),v(y))\right] \widehat{m}_\pm(y;z)dy, \tag{2.21a}$$

$$\widehat{n}_\pm(x;z) = e_2 + \int_{\pm\infty}^x \begin{bmatrix} e^{\frac{i}{2}(z-z^{-1})(x-y)} & 0 \\ 0 & 1 \end{bmatrix} \left[\widehat{Q}_1(u(y),v(y)) + z^{-1}\widehat{Q}_2(u(y),v(y))\right] \widehat{n}_\pm(y;z)dy. \tag{2.21b}$$

The following two lemmas contain results analogous to Lemmas 1 and 2.

Lemma 3. *Let $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v_x \in L^1(\mathbb{R})$. For every $z \in \mathbb{R}\backslash[-1,1]$, there exist unique solutions $\widehat{m}_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ and $\widehat{n}_\pm(\cdot;z) \in L^\infty(\mathbb{R})$ satisfying the integral equations (2.21). For every $x \in \mathbb{R}$, $\widehat{m}_\pm(x,\cdot)$ and $\widehat{n}_\mp(x,\cdot)$ are continued analytically in $\mathbb{C}^\pm \cap B_\infty$. There exist a positive constant C such that*

$$\|\widehat{m}_\pm(\cdot;z)\|_{L^\infty} + \|\widehat{n}_\mp(\cdot;z)\|_{L^\infty} \le C, \quad z \in \mathbb{C}^\pm \cap B_\infty. \tag{2.22}$$

Lemma 4. *Under the conditions of Lemma 3, for every $x \in \mathbb{R}$ the normalized Jost functions $\widehat{m}_\pm$ and $\widehat{n}_\pm$ satisfy the following limits as $\mathrm{Im}(z) \to \infty$ along a contour in the domains of their analyticity:*

$$\lim_{|z|\to\infty} \frac{\widehat{m}_\pm(x;z)}{\widehat{m}_\pm^\infty(x)} = e_1, \quad \lim_{|z|\to\infty} \frac{\widehat{n}_\pm(x;z)}{\widehat{n}_\pm^\infty(x)} = e_2, \tag{2.23}$$

where

$$\widehat{m}_\pm^\infty(x) = e^{\frac{i}{4}\int_{\pm\infty}^x(|u|^2+|v|^2)dy}, \quad \widehat{n}_\pm^\infty(x) = e^{-\frac{i}{4}\int_{\pm\infty}^x(|u|^2+|v|^2)dy}.$$

If in addition $v \in C^1(\mathbb{R})$, then

$$\lim_{|z|\to\infty} z\left[\frac{\widehat{m}_\pm(x;z)}{\widehat{m}_\pm^\infty(x)} - e_1\right] = \begin{pmatrix} -\int_{\pm\infty}^x \left[\overline{v}(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u) + \frac{i}{2}\overline{u}v\right] dy \\ -2iv_x + |u|^2 v + u \end{pmatrix}, \tag{2.24a}$$

$$\lim_{|z|\to\infty} z\left[\frac{\widehat{n}_\pm(x;z)}{\widehat{n}_\pm^\infty(x)} - e_2\right] = \begin{pmatrix} \overline{v} \\ \int_{\pm\infty}^x \left[\overline{v}(v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u) + \frac{i}{2}\overline{u}v\right] dy \end{pmatrix}. \tag{2.24b}$$

2.3. **Continuation of the transformed Jost functions across $\mathbb{S}^1$.** In Lemmas 1 and 3 we showed the existence of the transformed Jost functions $\{m_\pm(\cdot;z), n_\pm(\cdot;z)\}$ for $z \in B_0$ and $\{\widehat{m}_\pm(\cdot;z), \widehat{n}_\pm(\cdot;z)\}$ for $z \in B_\infty$, respectively, where the partition (2.8) is used. Because both sets of the transformed Jost functions are connected to the set $\{\varphi_\pm, \phi_\pm\}$ of the original Jost functions by the transformation formulas (2.9) and (2.19), respectively, we find the following connection formulas for every $z \in \mathbb{S}^1$:

$$m_\pm(x;z) = \begin{bmatrix} 1 & 0 \\ u(x) - z^{-1}v(x) & z^{-1} \end{bmatrix} \widehat{m}_\pm(x;z), \tag{2.25a}$$

$$n_\pm(x;z) = \begin{bmatrix} z & 0 \\ u(x)z - v(x) & 1 \end{bmatrix} \widehat{n}_\pm(x;z), \tag{2.25b}$$

or in the opposite direction,

$$\widehat{m}_\pm(x;z) = \begin{bmatrix} 1 & 0 \\ v(x) - zu(x) & z \end{bmatrix} m_\pm(x;z), \tag{2.26a}$$

$$\widehat{n}_\pm(x;z) = \begin{bmatrix} z^{-1} & 0 \\ v(x)z^{-1} - u(x) & 1 \end{bmatrix} n_\pm(x;z). \tag{2.26b}$$

By Lemmas 3 and 4, the right-hand sides of (2.25a) and (2.25b) yield analytic continuations of $m_\pm(x;\cdot)$ and $n_\mp(x;\cdot)$ in $\mathbb{C}^\pm \cap B_\infty$ respectively with the following limits as $\mathrm{Im}(z) \to \infty$ along a contour in the domains of their analyticity:

$$\lim_{|z|\to\infty} \frac{m_\pm(x;z)}{\widehat{m}_\pm^\infty(x)} = e_1 + u(x)e_2, \quad \lim_{|z|\to\infty} \frac{n_\pm(x;z)}{\widehat{n}_\pm^\infty(x)} = \bar{v}(x)e_1 + (1+u(x)\bar{v}(x))e_2. \tag{2.27}$$

Analogously, by Lemmas 1 and 2, the right-hand sides of (2.26a) and (2.26b) yield analytic continuations of $\widehat{m}_\pm(x;\cdot)$ and $\widehat{n}_\mp(x;\cdot)$ in $\mathbb{C}^\pm \cap B_0$ respectively with the following limits as $\mathrm{Im}(z) \to 0$ along a contour in the domains of their analyticity:

$$\lim_{z\to 0} \frac{\widehat{m}_\pm(x;z)}{m_\pm^\infty(x)} = e_1 + v(x)e_2, \quad \lim_{z\to 0} \frac{\widehat{n}_\pm(x;z)}{n_\pm^\infty(x)} = \bar{u}(x)e_1 + (1+\bar{u}(x)v(x))e_2. \tag{2.28}$$

By Lemmas 1, 3 and the continuation formulas (2.25), (2.26), we obtain the following result.

Lemma 5. *Let $(u,v) \in H^{1,1}(\mathbb{R})$. For every $x \in \mathbb{R}$ the Jost functions defined by the integral equations (2.11) and (2.21) can be continued such that $m_\pm(x;\cdot)$, $n_\mp(x;\cdot)$, $\widehat{m}_\pm(x;\cdot)$, and $\widehat{n}_\mp(x;\cdot)$ are analytic in $\mathbb{C}^\pm$ and continuous in $\mathbb{C}^\pm \cup \mathbb{R}$ with bounded limits as $z \to 0$ and $|z| \to \infty$ given by (2.13), (2.23), (2.27), (2.28).*

3. Scattering coefficients

In order to define the scattering coefficients between the transformed Jost functions $\{m_\pm, n_\pm\}$ and $\{\widehat{m}_\pm, \widehat{n}_\pm\}$, we go back to the original Jost functions $\{\varphi_\pm, \psi_\pm\}$. For every $\lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}$, we define the standard form of the scattering relation by

$$\varphi_-(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} = \alpha(\lambda)\,\varphi_+(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} + \beta(\lambda)\,\phi_+(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}, \tag{3.1a}$$

$$\phi_-(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4} = \gamma(\lambda)\,\varphi_+(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4} + \delta(\lambda)\,\phi_+(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}. \tag{3.1b}$$

Since the operator L in (1.3) admits the symmetry

$$\overline{\phi_\pm(x;\lambda)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \varphi_\pm\left(x;\overline{\lambda}\right),$$

we obtain

$$\gamma(\lambda) = -\overline{\beta(\overline{\lambda})}, \quad \delta(\lambda) = \overline{\alpha(\overline{\lambda})}, \quad \lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}. \tag{3.2}$$

Since the matrix operator L in (1.3) has zero trace, the Wronskian determinant W of any two solutions to the spectral problem (2.1) for any $\lambda \in \mathbb{C}$ is independent of x. By computing the Wronskian determinants of the solutions $\{\varphi_-, \phi_+\}$ and $\{\varphi_+, \varphi_-\}$ as $x \to +\infty$ and using the scattering relation (3.1) and the asymptotic behavior of the Jost functions $\{\varphi_\pm, \psi_\pm\}$, we obtain

$$\begin{aligned} \alpha(\lambda) &= W\left(\varphi_-(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4}, \phi_+(x;\lambda)e^{-ix(\lambda^2-\lambda^{-2})/4}\right), \\ \beta(\lambda) &= W\left(\varphi_+(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4}, \varphi_-(x;\lambda)e^{ix(\lambda^2-\lambda^{-2})/4}\right). \end{aligned} \tag{3.3}$$

It follows from the asymptotic behavior of $\{\varphi_-, \phi_-\}$ as $x \to -\infty$ that $W(\varphi_-, \phi_-) = 1$. Substituting (3.1) and using the the asymptotic behavior of $\{\varphi_+, \phi_+\}$ as $x \to +\infty$ yield the following constraint on the scattering data:

$$\alpha(\lambda)\delta(\lambda) - \beta(\lambda)\gamma(\lambda) = 1, \quad \lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}. \tag{3.4}$$

In view of the constraints (3.2), the constraint (3.4) can be written as

$$\alpha(\lambda)\overline{\alpha(\overline{\lambda})} + \beta(\lambda)\overline{\beta(\overline{\lambda})} = 1, \quad \lambda \in (\mathbb{R} \cup i\,\mathbb{R}) \setminus \{0\}. \tag{3.5}$$

By using the transformation formulas (2.9) we can rewrite the scattering relation (3.1) in terms of the transformed Jost functions $\{m_\pm, n_\pm\}$. In particular, we apply $T(u;\lambda)$ to (3.1a) and $\lambda T(u;\lambda)$ to (3.1b), so that we obtain for $z \in \mathbb{R}\backslash\{0\}$,

$$m_-(x;z)e^{ix(z-z^{-1})/4} = a(z)m_+(x;z)e^{ix(z-z^{-1})/4} + b_+(z)n_+(x;z)e^{-ix(z-z^{-1})/4}, \tag{3.6a}$$

$$n_-(x;z)e^{-ix(z-z^{-1})/4} = -\overline{b_-(z)}m_+(x;z)e^{ix(z-z^{-1})/4} + \overline{a(z)}n_+(x;z)e^{-ix(z-z^{-1})/4}, \tag{3.6b}$$

where we have recalled $z = \lambda^2$ and defined the scattering coefficients:

$$a(z) := \alpha(\lambda), \quad b_+(z) := \lambda^{-1}\beta(\lambda), \quad b_-(z) := \lambda\beta(\lambda), \quad z \in \mathbb{R}\backslash\{0\}. \tag{3.7}$$

Since $m_\pm(x;z)$ and $n_\pm(x;z)$ depend on $z = \lambda^2$, we deduce that α is even in λ and β is odd in λ for $\lambda \in (\mathbb{R} \cup i\mathbb{R}) \setminus \{0\}$. The latter condition yields $\overline{\lambda}\beta(\overline{\lambda}) = \lambda\beta(\lambda)$, which has been used already in the expression (3.7) for $b_-(z)$. Thanks to the relation (3.5), we have the following constraints

$$\left\{ \begin{array}{ll} |\alpha(\lambda)|^2 + |\beta(\lambda)|^2 = 1, & \lambda \in \mathbb{R}\backslash\{0\}, \\ |\alpha(\lambda)|^2 - |\beta(\lambda)|^2 = 1, & \lambda \in i\mathbb{R}\backslash\{0\}. \end{array} \right. \tag{3.8}$$

Since the matrix operator $\mathcal{L}$ in (2.7) has zero trace, the Wronskian determinant W of any two solutions to the spectral problem (2.6) is also independent of x. As a result, by computing the Wronskian determinant as $x \to +\infty$ and using the asymptotic behavior of the Jost functions $\{m_\pm, n_\pm\}$, we obtain from the scattering relation (3.6) for $z \in \mathbb{R}\backslash\{0\}$:

$$a(z) = W\left(m_-(x;z)e^{ix(z-z^{-1})/4}, n_+(x;z)e^{-ix(z-z^{-1})/4}\right), \tag{3.9a}$$

$$b_+(z) = W\left(m_+(x;z)e^{ix(z-z^{-1})/4}, m_-(x;z)e^{ix(z-z^{-1})/4}\right), \tag{3.9b}$$

$$\overline{b_-(z)} = W\left(n_+(x;z)e^{-ix(z-z^{-1})/4}, n_-(x;z)e^{-ix(z-z^{-1})/4}\right), \tag{3.9c}$$

in accordance with the representation (3.3).

Analogously, by using the transformation formulas (2.19) we can rewrite the scattering relation (3.1) in terms of the transformed Jost functions $\{\widehat{m}_\pm, \widehat{n}_\pm\}$. In particular, we apply $\widehat{T}(u;\lambda)$ to (3.1a) and $\lambda^{-1}\widehat{T}(u;\lambda)$ to (3.1b), so that we obtain for $z \in \mathbb{R}\backslash\{0\}$,

$$\widehat{m}_-(x;z)e^{ix(z-z^{-1})/4} = \widehat{a}(z)\, \widehat{m}_+(x;z)e^{ix(z-z^{-1})/4} + \widehat{b}_+(z)\, \widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4}, \tag{3.10a}$$

$$\widehat{n}_-(x;z)e^{-ix(z-z^{-1})/4} = -\overline{\widehat{b}_-(z)}\, \widehat{m}_+(x;z)e^{ix(z-z^{-1})/4} + \overline{\widehat{a}(z)}\, \widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4}, \tag{3.10b}$$

where we have recalled $z = \lambda^2$ and defined the scattering coefficients

$$\widehat{a}(z) := \alpha(\lambda), \quad \widehat{b}_+(z) := \lambda\beta(\lambda), \quad \widehat{b}_-(z) := \lambda^{-1}\beta(\lambda), \quad z \in \mathbb{R}\backslash\{0\}. \tag{3.11}$$

Since the matrix operator $\widehat{\mathcal{L}}$ in (2.18) has zero trace, we obtain from the scattering relation (3.10) for $z \in \mathbb{R}\backslash\{0\}$:

$$\widehat{a}(z) = W\left(\widehat{m}_-(x;z)e^{ix(z-z^{-1})/4}, \widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4}\right), \tag{3.12a}$$

$$\widehat{b}_+(z) = W\left(\widehat{m}_+(x;z)e^{ix(z-z^{-1})/4}, \widehat{m}_-(x;z)e^{ix(z-z^{-1})/4}\right), \tag{3.12b}$$

$$\overline{\widehat{b}_-(z)} = W\left(\widehat{n}_+(x;z)e^{-ix(z-z^{-1})/4}, \widehat{n}_-(x;z)e^{-ix(z-z^{-1})/4}\right), \tag{3.12c}$$

in accordance with the representation (3.3).

It follows from (3.7) and (3.11) that the two sets of scattering data are actually related by

$$\widehat{a}(z) = a(z), \quad \widehat{b}_+(z) = b_-(z), \quad \widehat{b}_-(z) = b_+(z), \quad z \in \mathbb{R}\backslash\{0\}. \tag{3.13}$$

These relations are in agreement with the continuation formulas (2.25) and (2.26). By using the representations (3.9) and (3.12), as well as Lemma 2, 4, and 5, we obtain the following.

Lemma 6. *Let $(u,v) \in H^{1,1}(\mathbb{R})$. Then, $a = \widehat{a}$ is continued analytically into $\mathbb{C}^-$ with the following limits in $\mathbb{C}^-$:*

$$\lim_{z\to 0} a(z) = e^{-\frac{i}{4}\int_{\mathbb{R}}(|u|^2+|v|^2)dy} =: a_0 \tag{3.14}$$

and

$$\lim_{|z|\to\infty} a(z) = e^{\frac{i}{4}\int_{\mathbb{R}}(|u|^2+|v|^2)dy} =: a_\infty. \tag{3.15}$$

On the other hand, $b_\pm = \widehat{b}_\pm$ are not continued analytically beyond the real line and satisfy the following limits on $\mathbb{R}$:

$$\lim_{z\to 0} b_\pm(z) = \lim_{|z|\to\infty} b_\pm(z) = 0. \tag{3.16}$$

To simplify the inverse scattering transform, we consider the case of no eigenvalues or resonances in the spectral problem (2.1) in the sense of the following definition.

Definition 1. *We say that the potential (u,v) admits an eigenvalue at $z_0 \in \mathbb{C}^-$ if $a(z_0) = 0$ and a resonance at $z_0 \in \mathbb{R}$ if $a(z_0) = 0$.*

By taking the limit $x \to +\infty$ in the Volterra's integral equations (2.11a) and (2.21a) for m_- and $\widehat{m}_-$ respectively and comparing it with the scattering relations (3.6a) and (3.10a), we obtain the following equivalent representations for $a = \widehat{a}$:

$$a(z) = 1 - \frac{i}{4}\int_{\mathbb{R}} \left[(|u|^2+|v|^2)m_-^{(1)} - 2\bar{u}m_-^{(2)} - 2z\bar{v}(um_-^{(1)} - m_-^{(2)})\right] dx, \quad z \in B_0 \cap \mathbb{C}^-, \tag{3.17a}$$

$$a(z) = 1 + \frac{i}{4}\int_{\mathbb{R}} \left[(|u|^2+|v|^2)\widehat{m}_-^{(1)} - 2\bar{v}\widehat{m}_-^{(2)} - 2z^{-1}\bar{u}(v\widehat{m}_-^{(1)} - \widehat{m}_-^{(2)})\right] dx, z \in B_\infty \cap \mathbb{C}^-, \tag{3.17b}$$

where the superscripts denote components of the Jost functions. If $(u,v) \in H^{1,1}(\mathbb{R})$ are defined in the ball of radius δ for some $\delta \in (0,1)$, then constants C in (2.12) and (2.22) are independent of δ. Then, it follows from (3.17) that if δ is sufficiently small, then the integrals can be made as small as needed for every $z \in \mathbb{C}^- \cup \mathbb{R}$. This implies the following.

Lemma 7. *Let $(u,v) \in H^{1,1}(\mathbb{R})$ be sufficiently small. Then (u,v) does not admit eigenvalues or resonances in the sense of Definition 1.*

Remark 3. *The result of Lemma 7 was first obtained in Theorem 6.1 in* [20]. *No transformation of the spectral problem (2.1) was employed in* [20]. *Transformations similar to those we are using here were employed later in* [23] *in the context of the derivative NLS equation.*

Remark 4. *The result of Lemma 7 is particularly useful for the study of long-range scattering from small initial data in the forthcoming work* [25]. *Eigenvalues can always be included by using Bäcklund transformation for the MTM system* [5, 21]. *Resonances are structurally unstable and can be removed by perturbations of initial data* [1, 21].

The following equality is deduced from the present formalism and will be useful in the forthcoming work [25].

Lemma 8. *Suppose that $(u,v) \in H^{1,1}(\mathbb{R})$ does not admit eigenvalues or resonances in the sense of Definition 1. Then, the following identity holds:*

$$\int_{\mathbb{R}} (|u(x)|^2 + |v(x)|^2)dx = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\log(|a(s)|^2)}{s} ds. \tag{3.18}$$

Proof. It follows from Lemma 6 that the scalar function A defined by

$$A(z) := \begin{cases} a_\infty^{-1} \left(\overline{a(\overline{z})}\right)^{-1}, & z \in \mathbb{C}^+, \\ a_\infty^{-1} a(z), & z \in \mathbb{C}^-, \end{cases}$$

satisfies the following scalar Riemann–Hilbert problem:

(1) A is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ with continuous boundary values

$$A_\pm(z) = \lim_{\varepsilon \downarrow 0} A(z \pm i\varepsilon), \quad z \in \mathbb{R}.$$

(2) $A \to 1$ as $|z| \to \infty$.
(3) The boundary values $A_\pm$ on $\mathbb{R}$ satisfy the jump relation

$$A_+(z) = A_-(z)|a(z)|^{-2}, \quad z \in \mathbb{R}\,.$$

By the same arguments as in the proof of Lemma 5.6 in [17], the exact solution of this Riemann-Hilbert problem is given by

$$A(z) = \exp\left(-\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log(|a(s)|^2)}{s-z} ds\right). \tag{3.19}$$

Taking the limit $z \to 0$ from $z \in \mathbb{C}^-$ and using the limit (3.14) yield

$$\exp\left(-\frac{i}{2}\int_{\mathbb{R}} (|u(x)|^2 + |v(x)|^2) dx\right) = \exp\left(-\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log(|a(s)|^2)}{s} ds\right). \tag{3.20}$$

If $(u,v) = (0,0)$, then $\beta(\lambda) = 0$. Relations (3.8) imply $|a(z)|^2 = |\alpha(\lambda)|^2 = 1$. Hence, inverting the exponential functions in (3.20) yields the relation (3.18). □

4. Riemann–Hilbert problems

We will derive two Riemann–Hilbert problems. The first problem is formulated for the transformed Jost functions $\{m_\pm, n_\pm\}$, whereas the second problem is formulated for the transformed Jost functions $\{\widehat{m}_\pm, \widehat{n}_\pm\}$. Thanks to the asymptotic representations (2.14) and (2.24), the first problem is useful for reconstruction of the component u as $z \to 0$, whereas the second problem is useful for reconstruction of the component v as $|z| \to \infty$, both components satisfy the MTM system (1.1). This pioneering idea has first appeared on a formal level in [29]. The following assumption is used to simplify solutions to the Riemann–Hilbert problems.

Assumption 1. *Assume that the scattering coefficient a admits no zeros in $\mathbb{C}^- \cup \mathbb{R}$. Moreover, there is $A > 0$ such that $|a(z)| \geq A$ for every $z \in \mathbb{R}$.*

Assumption 1 corresponds to the initial data $(u_0, v_0) \in H^{1,1}(\mathbb{R})$ which admit no eigenvalues or resonances in the sense of Definition 1.

4.1. Riemann-Hilbert problem for the potential u.

The asymptotic limit (2.27) presents a challenge to use $\{m_\pm, n_\pm\}$ for reconstruction of (u,v) as $|z| \to \infty$. On the other hand, the reconstruction formula for (u,v) in terms of $\{m_\pm, n_\pm\}$ is available from the asymptotic limit (2.14) as $z \to 0$. In order to avoid this complication, we use the inversion transformation $\omega = 1/z$, which maps 0 to ∞ and vice versa. The analyticity regions swap under the inversion transformation so that $\{m_-, n_+\}$ become analytic in $\mathbb{C}^+$ for ω and $\{m_+, n_-\}$ become analytic in $\mathbb{C}^-$ for ω.

Let us define matrices $P_\pm(x;\omega) \in \mathbb{C}^{2\times 2}$ for every $x \in \mathbb{R}$ and $\omega \in \mathbb{R}$ by

$$P_+(x;\omega) := \left[\frac{m_-(x;\omega^{-1})}{a(\omega^{-1})}, n_+(x;\omega^{-1})\right], \quad P_-(x;\omega) := \left[m_+(x;\omega^{-1}), \frac{n_-(x;\omega^{-1})}{\overline{a(\omega^{-1})}}\right], \tag{4.1}$$

and two reflection coefficients

$$r_\pm(\omega) = \frac{b_\pm(\omega^{-1})}{a(\omega^{-1})}, \quad \omega \in \mathbb{R}, \tag{4.2}$$

The scattering relation (3.6) can be rewritten as the following jump condition for the Riemann–Hilbert problem:

$$P_+(x;\omega) = P_-(x;\omega)\begin{bmatrix} 1 + r_+(\omega)\overline{r_-(\omega)} & \overline{r_-(\omega)}e^{-\frac{i}{2}(\omega-\omega^{-1})x} \\ r_+(\omega)e^{\frac{i}{2}(\omega-\omega^{-1})x} & 1 \end{bmatrix}$$

If the scattering coefficient a satisfies Assumption 1, then $P_\pm(x;\cdot)$ for every $x \in \mathbb{R}$ are continued analytically in $\mathbb{C}^\pm$ by Lemma 5. We denote these continuations by the same letters. Asymptotic limits (2.13) and (3.14) yield the following behavior of $P_\pm(x;\omega)$ for large $|\omega|$ in the domains of their analyticity:

$$P_\pm(x;\omega) \to \begin{bmatrix} m_+^\infty(x) & 0 \\ 0 & n_+^\infty(x) \end{bmatrix} =: P^\infty(x) \quad \text{as } |\omega| \to \infty.$$

Since we prefer to work with x-independent boundary conditions, we normalize the boundary conditions by defining

$$M_\pm(x;\omega) := [P^\infty(x)]^{-1}\, P_\pm(x;\omega), \quad \omega \in \mathbb{C}^\pm. \tag{4.3}$$

The following Riemann-Hilbert problem is formulated for the function $M(x;\cdot)$.

Riemann-Hilbert problem 1. *For each $x \in \mathbb{R}$, find a 2×2-matrix valued function $M(x;\cdot)$ such that*

(1) *$M(x;\cdot)$ is piecewise analytic in $\mathbb{C}\setminus\mathbb{R}$ with continuous boundary values*
$$M_\pm(x;z) = \lim_{\varepsilon\downarrow 0} M(x;z \pm i\varepsilon), \quad z \in \mathbb{R}.$$
(2) *$M(x;\omega) \to I$ as $|\omega| \to \infty$.*
(3) *The boundary values $M_\pm(x;\cdot)$ on $\mathbb{R}$ satisfy the jump relation*
$$M_+(x;\omega) - M_-(x;\omega) = M_-(x;\omega)R(x;\omega), \quad \omega \in \mathbb{R},$$
where
$$R(x;\omega) := \begin{bmatrix} r_+(\omega)\overline{r_-(\omega)} & \overline{r_-(\omega)}e^{-\frac{i}{2}(\omega-\omega^{-1})x} \\ r_+(\omega)e^{\frac{i}{2}(\omega-\omega^{-1})x} & 0 \end{bmatrix}.$$

It follows from the asymptotic limits (2.14) and the normalization (4.3) that the components (u, v) of the MTM system (1.1) can be recovered from the solution of the Riemann–Hilbert problem 1 by using the following reconstruction formulas:

$$\left[2iu'(x) + u(x)|v(x)|^2 + v(x)\right] e^{\frac{i}{2}\int_x^{+\infty} |u|^2+|v|^2 dy} = \lim_{|\omega|\to\infty} \omega[M(x;\omega)]_{21} \tag{4.4}$$

and

$$\overline{u}(x)e^{-\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy} = \lim_{|\omega|\to\infty} \omega[M(x;\omega)]_{12}, \tag{4.5}$$

where the subscript denotes the element of the 2×2 matrix M.

Remark 5. *The gauge factors in (4.4)–(4.5) appear because of the normalization (4.3) and the asymptotic limits (2.14). A different approach was utilized in* [12, 17] *to avoid these gauge factors. The inverse scattering transform was developed to a different spectral problem, which was obtained from the original spectral problem (2.1) after a gauge transformation.*

4.2. **Riemann-Hilbert problem for the potential** v**.** Let us define matrices $\widehat{P}_\pm(x;z) \in \mathbb{C}^{2\times 2}$ for every $x \in \mathbb{R}$ and $z \in \mathbb{R}$ by

$$\widehat{P}_+(x;z) := \left[\widehat{m}_+(x;z), \frac{\widehat{n}_-(x;z)}{\overline{\widehat{a}(z)}}\right], \quad \widehat{P}_-(x;z) := \left[\frac{\widehat{m}_-(x;z)}{\widehat{a}(z)}, \widehat{n}_+(x;z)\right], \tag{4.6}$$

and two reflection coefficients by

$$\widehat{r}_\pm(z) = \frac{\widehat{b}_\pm(z)}{\widehat{a}(z)} = \frac{b_\mp(z)}{a(z)}, \quad z \in \mathbb{R}, \tag{4.7}$$

where the relations (3.13) have been used. The scattering relation (3.10) can be rewritten as the following jump condition for the Riemann–Hilbert problem:

$$\widehat{P}_+(x;z) = \widehat{P}_-(x;z) \left[\begin{array}{cc} 1 & -\overline{\widehat{r}_-(z)} e^{\frac{i}{2}(z-z^{-1})x} \\ -\widehat{r}_+(z) e^{-\frac{i}{2}(z-z^{-1})x} & 1 + \widehat{r}_+(z)\overline{\widehat{r}_-(z)} \end{array}\right]$$

If the scattering coefficient a satisfies Assumption 1, then $\widehat{P}_\pm(x;\cdot)$ for every $x \in \mathbb{R}$ are continued analytically in $\mathbb{C}^\pm$ by Lemma 5. We denote these continuations by the same letters. Asymptotic limits (2.23) and (3.15) yield the following behavior of $\widehat{P}(x;z)$ for large $|z|$ in the domains of their analyticity:

$$\widehat{P}_\pm(x;z) \to \left[\begin{array}{cc} \widehat{m}_+^\infty(x) & 0 \\ 0 & \widehat{n}_+^\infty(x) \end{array}\right] =: \widehat{P}^\infty(x), \quad \text{as } |z| \to \infty.$$

In order to normalize the boundary conditions, we define

$$\widehat{M}_\pm(x;z) := \left[\widehat{P}^\infty(x)\right]^{-1} \widehat{P}_\pm(x;z), \quad z \in \mathbb{C}^\pm. \tag{4.8}$$

The following Riemann-Hilbert problem is formulated for the function $\widehat{M}(x;\cdot)$.

Riemann-Hilbert problem 2. *For each $x \in \mathbb{R}$, find a 2×2-matrix valued function $\widehat{M}(x;\cdot)$ such that*

(1) *$\widehat{M}(x;\cdot)$ is piecewise analytic in $\mathbb{C}\setminus\mathbb{R}$ with continuous boundary values*

$$\widehat{M}_\pm(x;z) = \lim_{\varepsilon\downarrow 0} \widehat{M}(x;z \pm i\varepsilon), \quad z \in \mathbb{R}.$$

(2) *$\widehat{M}(x;z) \to I$ as $|z| \to \infty$.*

(3) *The boundary values $\widehat{M}_\pm(x;\cdot)$ on $\mathbb{R}$ satisfy the jump relation*

$$\widehat{M}_+(x;z) - \widehat{M}_-(x;z) = \widehat{M}_-(x;z)\widehat{R}(x;z),$$

where

$$\widehat{R}(x;z) := \left[\begin{array}{cc} 0 & -\overline{\widehat{r}_-(z)} e^{\frac{i}{2}(z-z^{-1})x} \\ -\widehat{r}_+(z) e^{-\frac{i}{2}(z-z^{-1})x} & \widehat{r}_+(z)\overline{\widehat{r}_-(z)} \end{array}\right].$$

It follows from the asymptotic limit (2.24) and the normalization (4.8) that the components (u,v) of the MTM system (1.1) can be recovered from the solution of the Riemann–Hilbert problem 2 by using the following reconstruction formulas:

$$\left[-2iv'(x)+|u(x)|^2v(x)+u(x)\right]e^{-\frac{i}{2}\int_x^{+\infty}|u|^2+|v|^2dy}=\lim_{|z|\to\infty}z\left[\widehat{M}(x;z)\right]_{21} \tag{4.9}$$

and

$$\overline{v}(x)e^{\frac{i}{2}\int_x^{+\infty}(|u|^2+|v|^2)dy}=\lim_{|z|\to\infty}z\left[\widehat{M}(x;z)\right]_{12}, \tag{4.10}$$

where the subscript denotes the element of the 2×2 matrix M.

Remark 6. *It follows from the limit (3.16) that $R(x;0)=\widehat{R}(x;0)=0$ implying $M_+(x;0)=M_-(x;0)$ and $\widehat{M}_+(x;0)=\widehat{M}_-(x;0)$. Note that due to differences between (2.13) and (2.23) with (2.27) and (2.28), it follows that $M_\pm(x;0)\neq I$ and $\widehat{M}_\pm(x;0)\neq I$.*

4.3. **Estimates on the reflection coefficients.** In order to be able to solve the Riemann–Hilbert problems 1 and 2, we need to derive estimates on the reflection coefficients $r_\pm$ and $\widehat{r}_\pm$ defined by (4.2) and (4.7). We start with the Jost functions. In order to exclude ambiguity in notations, we write $m_\pm(x;z)\in H^1_z(\mathbb{R})$ for the same purpose as $m_\pm(x;\cdot)\in H^1(\mathbb{R})$.

Thanks to the Fourier theory reviewed in Proposition 1 in [23], the Volterra's integral equations (2.11) and (2.21) with the oscillation factors $e^{\frac{i}{2}(z-z^{-1})x}$ are estimated in the limits $z\to 0$ and $|z|\to\infty$ similarly to what was done in the proof of Lemma 3 in [23]. As a result, we obtain the following.

Lemma 9. *Let $(u,v)\in H^{1,1}(\mathbb{R})$. Then for every $x\in\mathbb{R}^\pm$, we have*

$$m_\pm(x;z)-m_\pm^\infty(x)e_1\in H^1_z(-1,1),\quad n_\pm(x;z)-n_\pm^\infty(x)e_2\in H^1_z(-1,1) \tag{4.11}$$

and

$$\widehat{m}_\pm(x;z)-\widehat{m}_\pm^\infty(x)e_1\in H^1_z(\mathbb{R}\backslash[-1,1]),\quad \widehat{n}_\pm(x;z)-\widehat{n}_\pm^\infty(x)e_2\in H^1_z(\mathbb{R}\backslash[-1,1]). \tag{4.12}$$

If in addition $(u,v)\in H^2(\mathbb{R})$, then

$$z^{-1}\left[\frac{m_\pm(x;z)}{m_\pm^\infty(x)}-e_1\right]-\begin{pmatrix}-\int_{\pm\infty}^x\left[\overline{u}(u_x-\frac{i}{2}u|v|^2-\frac{i}{2}v)-\frac{i}{2}u\overline{v}\right]dy\\ 2iu_x+u|v|^2+v\end{pmatrix}\in H^1_z(-1,1), \tag{4.13a}$$

$$z^{-1}\left[\frac{n_\pm(x;z)}{n_\pm^\infty(x)}-e_2\right]-\begin{pmatrix}\overline{u}\\ \int_{\pm\infty}^x\left[\overline{u}(u_x-\frac{i}{2}u|v|^2-\frac{i}{2}v)-\frac{i}{2}u\overline{v}\right]dy\end{pmatrix}\in H^1_z(-1,1) \tag{4.13b}$$

and

$$z\left[\frac{\widehat{m}_\pm(x;z)}{\widehat{m}_\pm^\infty(x)}-e_1\right]-\begin{pmatrix}-\int_{\pm\infty}^x\left[\overline{v}(v_x+\frac{i}{2}|u|^2v+\frac{i}{2}u)+\frac{i}{2}\overline{u}v\right]dy\\ -2iv_x+|u|^2v+u\end{pmatrix}\in H^1_z(\mathbb{R}\backslash[-1,1]), \tag{4.14a}$$

$$z\left[\frac{\widehat{n}_\pm(x;z)}{\widehat{n}_\pm^\infty(x)}-e_2\right]-\begin{pmatrix}\overline{v}\\ \int_{\pm\infty}^x\left[\overline{v}(v_x+\frac{i}{2}|u|^2v+\frac{i}{2}u)+\frac{i}{2}\overline{u}v\right]dy\end{pmatrix}\in H^1_z(\mathbb{R}\backslash[-1,1]). \tag{4.14b}$$

Estimates (4.11) and (4.13) can be extended by means of the transformation $z=\omega^{-1}$ since the inversion does not change the oscillation factors $e^{\frac{i}{2}(z-z^{-1})x}$ in the integral equations (2.11) except of the sign flip. As a result, these estimates can be rewritten in variable ω, which is useful for the Riemann–Hilbert problem 1. These equivalent estimates are given by the following.

Lemma 10. *Let $(u,v)\in H^{1,1}(\mathbb{R})$. Then for every $x\in\mathbb{R}^\pm$, we have*

$$m_\pm(x;\omega^{-1})-m_\pm^\infty(x)e_1\in H^1_\omega(\mathbb{R}\backslash[-1,1]),\quad n_\pm(x;\omega^{-1})-n_\pm^\infty(x)e_2\in H^1_\omega(\mathbb{R}\backslash[-1,1]). \tag{4.15}$$

If in addition $(u,v) \in H^2(\mathbb{R})$, *then*

$$\omega\left[\frac{m_\pm(x;\omega^{-1})}{m_\mp^\infty(x)} - e_1\right] - \left(\begin{array}{c} -\int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right] dy \\ 2iu_x + u|v|^2 + v \end{array}\right) \in H^1_\omega(\mathbb{R}\setminus[-1,1]), \tag{4.16a}$$

$$\omega\left[\frac{n_\pm(x;\omega^{-1})}{n_\mp^\infty(x)} - e_2\right] - \left(\begin{array}{c} \overline{u} \\ \int_{\pm\infty}^x \left[\overline{u}(u_x - \frac{i}{2}u|v|^2 - \frac{i}{2}v) - \frac{i}{2}u\overline{v}\right] dy \end{array}\right) \in H^1_\omega(\mathbb{R}\setminus[-1,1]). \tag{4.16b}$$

The estimates of Lemmas 9 and 10 on the Jost functions are still insufficient to yield estimates on the reflection coefficients $r_\pm$ and $\hat{r}_\pm$ given by (4.2) and (4.7) on the entire line $\mathbb{R}$. However, the set $[-1,1]$ is compact and the Jost functions are connected by the analytic continuation (2.25) and (2.26). Formulas (2.25a) and (2.26a) present no singularities on $[-1,1]$ for $\omega = z^{-1}$ and z respectively and the z-independent limits $m_\mp^\infty(x)e_1$ and $\widehat{m}_\mp^\infty(x)e_1$ present no divergence on the compact interval $[-1,1]$. As a result, the estimates (4.12) and (4.15) on $\widehat{m}_\pm$ and $m_\pm$ respectively are extended in the following.

Lemma 11. *Let* $(u,v) \in H^{1,1}(\mathbb{R})$. *Then for every* $x \in \mathbb{R}^\pm$, *we have*

$$m_\pm(x;\omega^{-1}) - m_\mp^\infty(x)e_1 \in H^1_\omega(\mathbb{R}), \quad \widehat{m}_\pm(x;z) - \widehat{m}_\mp^\infty(x)e_1 \in H^1_z(\mathbb{R}). \tag{4.17}$$

On the other hand, the continuation formula (2.25b) presents a singularity as $\omega = z^{-1} \to 0$ as is seen from this explicit representation

$$n_\pm(x;\omega^{-1}) = \left[\begin{array}{c} 1 \\ u(x) \end{array}\right]\omega^{-1}\widehat{n}_\pm^{(1)}(x;\omega^{-1}) + \left[\begin{array}{cc} 0 & 0 \\ -v(x) & 1 \end{array}\right]\widehat{n}_\pm(x;\omega^{-1}). \tag{4.18}$$

Similarly, the continuation formula (2.26b) presents a singularity as $z \to 0$ as is seen from this explicit representation

$$\widehat{n}_\pm(x;z) = \left[\begin{array}{c} 1 \\ v(x) \end{array}\right] z^{-1}n_\pm^{(1)}(x;z) + \left[\begin{array}{cc} 0 & 0 \\ -u(x) & 1 \end{array}\right] n_\pm(x;z). \tag{4.19}$$

However, $z^{-1}n_\pm^{(1)}(x;z) \in H^1_z(-1,1)$ thanks to the estimates (4.13b). Similarly, $\omega^{-1}\widehat{n}_\pm^{(1)}(x;\omega^{-1}) \in H^1_\omega(-1,1)$. The estimates (4.12) and (4.15) on $\widehat{n}_\pm$ and $n_\pm$ respectively are extended thanks to compactness of $[-1,1]$ in the following.

Lemma 12. *Let* $(u,v) \in H^{1,1}(\mathbb{R})$. *Then for every* $x \in \mathbb{R}^\pm$, *we have*

$$n_\pm(x;\omega^{-1}) - n_\mp^\infty(x)e_1 \in H^1_\omega(\mathbb{R}), \quad \widehat{n}_\pm(x;z) - \widehat{n}_\mp^\infty(x)e_1 \in H^1_z(\mathbb{R}). \tag{4.20}$$

Remark 7. *If in addition* $(u,v) \in H^2(\mathbb{R})$, *then estimates (4.14) and (4.16) can be extended to the full line* $\mathbb{R}$ *in* z *and* ω *respectively thanks again to compactness of* $[-1,1]$ *so that* $|z| \leq 1$ *and* $|\omega| \leq 1$ *in the left-hand sides of (4.14) and (4.16) respectively.*

We are now ready to transfer the estimates to the scattering coefficients a, $b_\pm$, and to the reflection coefficients $r_\pm$ and $\widehat{r}_\pm$. The following result follows from Lemmas 11 and 12 by using the same analysis as in the proof of Lemma 4 in [23].

Lemma 13. *Let* $(u,v) \in H^{1,1}(\mathbb{R})$. *Then,*

$$a(\omega^{-1}) - a_0, \;\; b_+(\omega^{-1}), \;\; b_-(\omega^{-1}) \in H^1_\omega(\mathbb{R}), \tag{4.21}$$

and

$$a(z) - a_\infty, \;\; b_+(z), \;\; b_-(z) \in H^1_z(\mathbb{R}). \tag{4.22}$$

If in addition $(u,v) \in H^2(\mathbb{R})$, *then*

$$b_+(\omega^{-1}), \;\; b_-(\omega^{-1}) \in L^{2,1}_\omega(\mathbb{R}), \tag{4.23}$$

and

$$b_+(z),\ \ b_-(z) \in L^{2,1}_z(\mathbb{R}). \tag{4.24}$$

Remark 8. *Under Assumption 1, if $(u,v) \in X_{2,1}$, where $X_{2,1}$ is given by (2.15), then $(r_+, r_-) \in X_{1,1}$ and $(\hat{r}_+, \hat{r}_-) \in X_{1,1}$, where*

$$X_{1,1} := H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}). \tag{4.25}$$

The space $X_{1,1}$ is an optimal L^2-based Sobolev space for inverse scattering of the MTM system (1.1).

4.4. **Solvability of the Riemann–Hilbert problems.** In order to solve the Riemann–Hilbert problems 1 and 2, we shall define the reflection coefficient

$$r(\lambda) := \frac{\beta(\lambda)}{\alpha(\lambda)}, \quad \lambda \in \mathbb{R} \cup (i\,\mathbb{R})\backslash\{0\}. \tag{4.26}$$

It follows from the relations (3.7), (3.11), (4.2), and (4.7) that

$$\lambda^{-1} r(\lambda) = r_+(\omega) = \omega r_-(\omega), \quad \omega \in \mathbb{R}\backslash\{0\}. \tag{4.27}$$

and

$$\lambda r(\lambda) = \widehat{r}_+(z) = z\widehat{r}_-(z), \quad z \in \mathbb{R}\backslash\{0\}. \tag{4.28}$$

By the same proof as in Proposition 2 in [23], we have

$$\text{if} \quad r_\pm(\omega) \in H^1_\omega(\mathbb{R}) \cap L^{2,1}_\omega(\mathbb{R}), \quad \text{then} \quad r(\lambda) \in L^{2,1}_\omega(\mathbb{R}) \cap L^\infty_\omega(\mathbb{R}) \tag{4.29}$$

and

$$\text{if} \quad \hat{r}_\pm(z) \in H^1_z(\mathbb{R}) \cap L^{2,1}_z(\mathbb{R}), \quad \text{then} \quad r(\lambda) \in L^{2,1}_z(\mathbb{R}) \cap L^\infty_z(\mathbb{R}). \tag{4.30}$$

Similarly, by the same proof as in Proposition 3 in [23], we also have

$$\lambda^{-1} r_+(\omega) \in L^\infty_\omega(\mathbb{R}) \quad \text{and} \quad \lambda \hat{r}_+(z) \in L^\infty_z(\mathbb{R}). \tag{4.31}$$

In addition, by using the relations (3.8), we obtain the following constraint:

$$1 - |r(\lambda)|^2 = \frac{1}{|\alpha(\lambda)|^2} \geq c_0^2 > 0, \quad \lambda \in i\mathbb{R}, \tag{4.32}$$

where $c_0 := \sup_{\lambda \in i\,\mathbb{R}} |\alpha(\lambda)| < \infty$, which exists thanks to Lemma 6.

Under Assumption 1 as well as the constraints (4.29), (4.30), and (4.32), the jump matrices in the Riemann–Hilbert problems 1 and 2 satisfy the same estimates as in Proposition 5 in [23]. Hence these Riemann–Hilbert problems can be solved and estimated with the same technique as in the proofs of Lemmas 7, 8, and 9 in [23]. The following summarizes this result.

Lemma 14. *Under Assumption 1, for every $r_\pm \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ there exists a unique solution of the Riemann–Hilbert problem 1 satisfying for every $x \in \mathbb{R}$:*

$$\|M_\pm(x;\cdot) - I\|_{L^2} \leq C(\|r_+\|_{L^2} + \|r_-\|_{L^2}), \tag{4.33}$$

where the positive constant C only depends on $\|r_\pm\|_{L^\infty}$. Similarly, under Assumption 1, for every $\widehat{r}_\pm \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ there exists a unique solution of the Riemann–Hilbert problem 2 satisfying for every $x \in \mathbb{R}$:

$$\|\widehat{M}_\pm(x;\cdot) - I\|_{L^2} \leq \widehat{C}(\|\widehat{r}_+\|_{L^2} + \|\widehat{r}_-\|_{L^2}), \tag{4.34}$$

where the positive constant $\widehat{C}$ only depends on $\|\widehat{r}_\pm\|_{L^\infty}$.

Along the same lines, under Assumption 1 as well as the constraints (4.29), (4.30), (4.31) and (4.32), Lemmas 10, 11, and 12 in [23] can be proven for the estimates of the potential (u,v) in the reconstruction formulas (4.4)–(4.5) and (4.9)–(4.10). The following summarizes this result.

Lemma 15. *Under Assumption 1, for every $r_\pm \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ and every $\widehat{r}_\pm \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, the components $(u,v) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$ satisfy the bound*

$$\|u\|_{H^2\cap L^{2,1}} + \|v\|_{H^2\cap L^{2,1}} \leq C(\|r_+\|_{H^1\cap L^{2,1}} + \|r_-\|_{H^1\cap L^{2,1}} + \|\hat{r}_+\|_{H^1\cap L^{2,1}} + \|\hat{r}_-\|_{H^1\cap L^{2,1}}), \tag{4.35}$$

where the positive constant C depends on $\|r_\pm\|_{H^1\cap L^{2,1}}$ and $\|\hat{r}_\pm\|_{H^1\cap L^{2,1}}$.

Lemma 13 proves the first assertion of Theorem 2. Lemma 15 proves the second assertion of Theorem 2 at $t=0$. It remains to prove the second assertion of Theorem 2 for every $t \in \mathbb{R}$.

4.5. **Time evolution of the spectral data.** Thanks to the well-posedness result of Theorem 1 and the weighted estimates in L^2-based Sobolev spaces, there exists a global solution $(u,v) \in C(\mathbb{R}, X_{2,1})$ to the MTM system (1.1) for any initial data $(u,v)|_{t=0} = (u_0,v_0) \in X_{2,1}$. For this global solution, the normalized Jost functions (2.2) can be extended for every $t \in \mathbb{R}$:

$$\left\{ \begin{array}{l} \varphi_\pm(t,x;\lambda) = \psi_1^{(\pm)}(t,x;\lambda) e^{-ix(\lambda^2-\lambda^{-2})/4 - it(\lambda^2+\lambda^{-2})/4}, \\ \phi_\pm(t,x;\lambda) = \psi_2^{(\pm)}(t,x;\lambda) e^{ix(\lambda^2-\lambda^{-2})/4 + it(\lambda^2+\lambda^{-2})/4}. \end{array} \right. \tag{4.36}$$

where $(\varphi_\pm, \phi_\pm)$ still satisfy the same boundary conditions (2.3). Introducing the scattering coefficients in the same way as in Section 3, we obtain the time evolution of the scattering coefficients:

$$\alpha(t,\lambda) = \alpha(0,\lambda), \quad \beta(t,\lambda) = \beta(0,\lambda) e^{-it(\lambda^2+\lambda^{-2})/2}, \quad \lambda \in \mathbb{R} \cup (i\,\mathbb{R}) \backslash \{0\}. \tag{4.37}$$

Transferring the scattering coefficients to the reflection coefficients with the help of (3.7), (3.11), (4.2), and (4.7) yields the time evolution of the reflection coefficients:

$$r_\pm(t,\omega) = r_\pm(0,\omega) e^{-it(\omega+\omega^{-1})/2}, \quad \omega \in \mathbb{R} \backslash \{0\} \tag{4.38}$$

and

$$\hat{r}_\pm(t,z) = \hat{r}_\pm(0,z) e^{-it(z+z^{-1})/2}, \quad z \in \mathbb{R} \backslash \{0\}. \tag{4.39}$$

The following shows that if $r_\pm$ and $\hat{r}_\pm$ are in $X_{1,1}$ at the initial time $t=0$, then they remain in $X_{1,1}$ for every $t \in \mathbb{R}$.

Lemma 16. *Assume that $r_\pm(0,\cdot) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ and $\hat{r}_\pm(0,\cdot) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$. Then, $r_\pm(t,\cdot) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ and $r_\pm(t,\cdot) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$ for every $t \in \mathbb{R}$.*

Proof. Since the time evolution of the reflection coefficients in (4.38) and (4.39) is given by the oscillatory factor of modulus one, we have the following identities for $\hat{r}_\pm$ (expressions for $r_\pm$ are similar):

$$\left\{ \begin{array}{l} |\hat{r}_\pm(t,z)| = |\hat{r}_\pm(0,z)|, \\ |\partial_z \hat{r}_\pm(t,z)| = \left|\partial_z \hat{r}_\pm(0,z) - \frac{i}{2} t(1-z^{-2}) \hat{r}_\pm(0,z)\right|, \end{array} \right. \quad z \in \mathbb{R}.$$

Since $\hat{r}_\pm(0,\cdot) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, the assertion of the lemma is proven if we can prove that $\hat{r}_\pm(0,\cdot) \in L^{2,-2}(\mathbb{R})$. To prove this, let us use the relation

$$\hat{r}_\pm(0,z) = r_\mp(0,z^{-1}), \quad z \in \mathbb{R}$$

and the chain rule $\omega = z^{-1}$ in the following integration:

$$\int_{\mathbb{R}} z^{-4} |\hat{r}_\pm(0,z)|^2 dz = \int_{\mathbb{R}} \omega^2 |r_\mp(0,\omega)|^2 d\omega.$$

Hence, if $r_\mp(0,\cdot) \in L^{2,1}(\mathbb{R})$, then $\hat{r}_\pm(0,\cdot) \in L^{2,-2}(\mathbb{R})$ and vice versa. The assertion of the lemma is proven. $\square$

Remark 9. *The chain rule $\omega = z^{-1}$ can also be used in the following integration:*

$$\int_{\mathbb{R}} |\partial_z \hat{r}_\pm(0,z)|^2 dz = \int_{\mathbb{R}} \omega^2 |\partial_\omega r_\mp(0,\omega)|^2 d\omega.$$

Therefore, if $r_\mp(0,\cdot) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$, then $\hat{r}_\pm(0,\cdot) \in H^{1,1}(\mathbb{R}) \cap L^{2,-2}(\mathbb{R})$ and vice versa. Moreover, thanks to the relations (4.27) and (4.28), we also have $r_-(0,\cdot), \hat{r}_-(0,\cdot) \in L^{2,2}(\mathbb{R})$ and $r_+(0,\cdot), \hat{r}_+(0,\cdot) \in L^{2,-3}(\mathbb{R})$.

The recovery formulas of Lemma 15 for the global solution $(u,v) \in C(\mathbb{R}, X_{2,1})$ to the MTM system (1.1) hold for every $t \in \mathbb{R}$ thanks to the result of Lemma 16. This proves the second assertion of Theorem 2 for every $t \in \mathbb{R}$. Hence Theorem 2 is proven.

Remark 10. *In the context of the MTM system (1.1), it is more natural to address global solutions in weighted H^1 space such as $H^{1,1}(\mathbb{R})$ given by (1.6) and drop the requirement $(u,v) \in H^2(\mathbb{R})$. The scattering coefficients $r_\pm$ and $\hat{r}_\pm$ are then defined in the space $H^1(\mathbb{R})$. However, there are two obstacles to complete inverse scattering for such a larger class of initial data. First, the asymptotic limits (2.14a) and (2.24a) are not justified, therefore, the recovery formulas (4.4) and (4.9) cannot be utilized. Second, without requirement $r_\pm, \hat{r}_\pm \in L^{2,1}(\mathbb{R})$, the time evolution (4.38)–(4.39) is not closed in $H^1(\mathbb{R})$ since $r_-, \hat{r}_- \in L^{2,-2}(\mathbb{R})$ cannot be verified. In this sense, the space $X_{2,1}$ for (u,v) and $X_{1,1}$ for $r_\pm$ and $\hat{r}_\pm$ are optimal for the inverse scattering transform of the MTM system (1.1).*

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