A spectral transform for the intermediate nonlinear Schrödinger equation

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A new spectral transform system is introduced to solve the initial-value problem for the intermediate nonlinear Schrödinger (INLS) equation describing envelope waves in a deep stratified fluid. The spectral system is a combination of the Zakharov–Shabat linear system and a local Riemann–Hilbert problem in a strip of the complex plane. The inverse scattering transform technique is developed and the Bäcklund–Darboux transformation, soliton solutions and an infinite number of conservation laws are constructed. It is shown that all these properties of the INLS equation are closely related to those of the intermediate long-wave equation. © *1995 American Institute of Physics.*

I. INTRODUCTION

Originally, the inverse scattering transform (IST) technique was elaborated for the solution of several significant nonlinear partial differential equations such as the Korteweg–de Vries (KdV) and the nonlinear Schrödinger (NLS) equations.[1–3] This technique is based on a solution of a linear scattering problem which is expressed by a system of linear partial differential equations and is associated with the nonlinear wave evolution equation by means of compatibility conditions (for a good review, see Ablowitz and Clarkson[4]).

It was discovered a little later that some equations of another type, namely, nonlinear integro-differential equations, are also integrable; the associated linear systems were found and used for the IST scheme in Refs. 5–7. However, in the case of integro-differential equations the linear system defines a differential Riemann–Hilbert problem in the complex plane of the space coordinate x. The simplest example of such an integrable integro-differential equation is the intermediate long-wave (ILW) equation which describes long internal waves in a deep stratified fluid of finite depth.[8] In dimensionless form this equation is

$$u_t+\delta^{-1}u_x+2uu_x+\mathbf{T}(u_{xx})=0, \tag{1.1}$$

where the singular integral operator $\mathbf{T}(u)$ is given by

$$\mathbf{T}(u)=\frac{1}{2\delta}\,\text{p.v.}\int_{-\infty}^{+\infty}\coth\left[\frac{\pi(z-x)}{2\delta}\right]u(z)dz, \tag{1.2}$$

p.v. stands for principal-value integral, and the parameter δ is proportional to the total fluid depth.

The scattering problem for Eq. (1.1) is determined by the following linear system,

$$i\varphi_x^{+}+\left(u-\frac{\mu}{2}\right)\varphi^{+}=\lambda\varphi^{-}, \tag{1.3a}$$

$$i\varphi_t^{\pm}+i(\mu+\delta^{-1})\varphi_x^{\pm}+\varphi_{xx}^{\pm}+[-(\pm i+\mathbf{T})(u_x)+\nu]\varphi^{\pm}=0. \tag{1.3b}$$

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Here the constants μ, λ, and ν are related to a spectral parameter and the functions $\varphi^{\pm}(x)$ are defined by the unique function $\varphi(x)$ according to the following "decomposition formula,"[5]

$$\varphi^{\pm}(x)=\varphi(x\mp i\delta)=\frac{\pm 1-i\mathbf{T}}{2}(\varphi(x)). \tag{1.4}$$

In fact, the functions $\varphi^{\pm}(x)$ represent the boundary values of functions analytic in the horizontal strip of the complex extension of x between $\mathrm{Im}(x)=0$ and $\mathrm{Im}(x)=\pm 2\delta$, and then periodically extended vertically. So, the system (1.3) can be regarded as a differential Riemann–Hilbert problem in a strip of the complex x plane.[5,7] In the limit $\delta\rightarrow 0$ (1.3a) transforms to the second-order Schrödinger equation which is well known as the linear scattering problem for the KdV equation.[1]

Generalizations of the Riemann–Hilbert problem (1.3a) can be achieved in different ways Degasperis *et al.*[9] considered the local matrix 2×2 Riemann–Hilbert problem which transforms in the limit $\delta\rightarrow 0$ to the coupled first-order differential equations referred to as the Zakharov–Shabat spectral problem.[2,3] This spectral problem allowed them to find a new class of integrable integro-differential equations which include the nonlocal analogs of the linear wave equation, the NLS equation and the modified KdV equation. On the other hand, Satsuma *et al.*[10] investigated another kind of nonlocal modified KdV equation and found that its linear associated system consists of two differential first-order Riemann–Hilbert problems like (1.3a).

Another generalization of the problem (1.3a) was proposed by Lebedev and Radul,[11] namely a scalar differential Riemann–Hilbert problem of k order. This approach led to a generalized ILW_k hierarchy of integrable nonlocal equations. Then, Degasperis *et al.*[12,13] elaborated a modification of a dressing method with a noncommutative spectral parameter in order to investigate the ILW_2 and modified ILW_2 hierarchies in detail. It was also found by Lebedev *et al.*[14] that the generalized ILW_k equations are reductions of the KP hierarchy. Recently, Zhang[15] developed this idea and constructed new nonlocal equations which couple the ILW_k equation with a time-dependent Schrödinger equation.

It is important to note that, in spite of this variety of different generalizations of the spectral problem (1.3a), no new physically significant nonlocal integrable equation was obtained which could be regarded as a model for the evolution of nonlinear waves.

However, a new physically significant integro-differential equation has been recently derived for the description of weakly nonlinear wave packets propagating on the interface of a two-layer fluid with a deep lower layer, and a shallow upper layer.[16] This equation can be obtained from (1.1) for $\delta\rightarrow\infty$ by means of the following asymptotic multiscale expansion

$$u=\epsilon^{1/2}(A(\tilde{x},\tilde{t}_1,\tilde{t}_2)\exp[i(x+t)]+\text{c.c.})+O(\epsilon), \tag{1.5}$$

where $\tilde{x}=\epsilon x$, $\tilde{t}_1=\epsilon t$, $\tilde{t}_2=\epsilon^2 t$, $\delta=\tilde{\delta}/\epsilon$, and $\epsilon\ll 1$. In the leading order of the asymptotic expansion, the amplitude A is given by $A=\tilde{A}(\tilde{x}+2\tilde{t}_1,\tilde{t}_2)\exp[i(\tilde{x}/2\tilde{\delta}-\tilde{t}_2/4\tilde{\delta}^2)]$, where $\tilde{A}$ satisfies the equation

$$iA_t=A_{xx}+A(i+\mathbf{T})(|A|^2)_x. \tag{1.6}$$

For convenience, we here and henceforth omit the sign "tilde" for the new variables.

Equation (1.6) plays the same role for the description of these interfacial wave packets as the NLS equation usually does.[17] Moreover, it transforms for $\delta\rightarrow 0$ and $|A|^2\sim O(\delta)$ to the NLS equation in the following form:

$$iA_t=A_{xx}-\delta^{-1}A|A|^2. \tag{1.7}$$

Here we refer to Eq. (1.6) as the intermediate nonlinear Schrödinger (INLS) equation.

In Ref. 16, it was found by bilinear Hirota's method that the INLS equation (1.6) possesses N-soliton solutions with nonvanishing boundary conditions: $|A|\rightarrow\rho$ as $x\rightarrow\infty$, where $\rho\neq 0$ is a real constant. This feature implies that the INLS equation belongs to the class of integrable equations. However, neither a linear associated spectral problem nor other related properties of the INLS equation have yet been found.

In this paper, we apply the IST technique to solve the initial-value problem for the INLS equation. In addition we find the Bäcklund–Darboux transformation of its solutions, and an infinite set of conservation laws. The starting point of our consideration is the following linear spectral system associated with (1.6),

$$i\varphi_{1x}-\frac{\mu}{2}\varphi_1+A\varphi_2^+=0, \tag{1.8a}$$

$$\varphi_2^+-\lambda\varphi_2^-+A^*\varphi_1=0, \tag{1.8b}$$

$$i\varphi_{1t}+i\mu\varphi_{1x}+\varphi_{1xx}-2iA_x\varphi_2^+ +\nu\varphi_1=0, \tag{1.8c}$$

$$i\varphi_{2t}^{\pm}+i\mu\varphi_{2x}^{\pm}+\varphi_{2xx}^{\pm}+[(\pm i+\mathbf{T})(|A|^2)_x+\nu]\varphi_2^{\pm}=0, \tag{1.8d}$$

where the parameters μ, λ and ν and the functions $\varphi_2^{\pm}(x)=\varphi_2(x\mp i\delta)$ have the same meaning as those for system (1.3a) and (1.3b). However, it is important to point out that there are no analytical requirements here on the function φ_1.

It is easy to check that the commutatibility conditions for the linear system (1.8) give the nonlinear equation (1.6). In Appendix A we show that the spectral problem (1.8) can be derived from (1.3) by means of the same asymptotic multiscale procedure (1.5) as that used for the derivation of the INLS equation from the ILW equation. This is consistent with the hypothesis[18,19] that the asymptotic multiscale expansions of integrable equations allow one to obtain not only a new integrable equation but also a scheme for its integrability.

The system (1.8a) and (1.8b) can be regarded as a combination of the Zakharov–Shabat spectral problem and a local Riemann–Hilbert problem. It represents an unusual, asymmetrical generalization of the differential matrix 2×2 spectral system. Note that a similar type of a linear associated system was independently introduced by Zhang in the recent paper[15] for the coupled ILW_k and Schrödinger equations. We can show that Eqs. (1.8) can be reduced to Zhang's general system [Eqs. (2.8a) and (2.8b) in Ref. 15] for $k=0$, $n=2$.

In the limit $\delta\to 0$, Eqs. (1.8a) and (1.8b) transform to the first-order system

$$i\varphi_{1x}-\frac{\mu}{2}\varphi_1+A\varphi_2=0, \tag{1.9a}$$

$$-i\varphi_{2x}-\frac{\mu}{2}\varphi_2+\frac{A^*}{2\delta}\varphi_1=0, \tag{1.9b}$$

which is the Zakharov–Shabat spectral system for the NLS equation (1.7).[3] For this limiting transition, we suppose that $\lambda=1+\mu\delta+O(\delta^2)$ and neglect in (1.9a) and (1.9b) terms of the order of $O(\delta)$. A similar limiting transition also exists for the time evolution of the linear system (1.8c) and (1.8d). However, throughout this paper we will consider only the x-dependent part of the linear system (1.8a) and (1.8b) because the time evolution of its solutions can then be found by means of a trivial substitution.

Our paper is constructed as follows. In Secs. II–IV we consider the direct and inverse scattering problem for system (1.8a) and (1.8b) for finite δ, and with nonvanishing boundary conditions imposed on $A(x,t)$. Then, in Sec. V we generalize the IST formalism and present the dressing transformation[20] for the INLS equation. A new way of constructing an infinite set of conservation laws is proposed in Sec. VI. The concluding Sec. VII is devoted to discussion.

It should be mentioned that our analysis follows mainly the pioneer work of Refs. 3 and 5. However, in the limit $\delta\rightarrow 0$, our results transform to an alternative version of the IST technique for the NLS equation to that developed by Zakharov and Shabat.[3] This is due to the asymmetry of the linear system (1.8a) and (1.8b) with respect to the functions φ_1, φ_2. On the other hand, the scattering problem for the INLS equation is qualitatively similar to the scattering problem for the ILW equation considered by Kodama *et al.*[5] because of the aforementioned asymptotic correspondence of both equations.

II. THE JOST FUNCTIONS

Here we consider the direct scattering problem for the linear differential-difference system (1.8a) and (1.8b) with finite nonzero δ. We suppose that solutions to the INLS equation tend to constant values at infinity exponentially rapidly: $|A|\rightarrow\rho$ as $|x|\rightarrow\infty$ where ρ is a real positive constant. Of course, a phase shift may appear as $x\rightarrow\pm\infty$, but we will omit this effect here for simplicity.

To consider the direct scattering problem we need to introduce appropriate Jost functions. These behave at infinity like harmonic waves, i.e., $\varphi_j\sim\exp(\pm ikx/2)$, where $j=1,2$ and k is a spectral parameter. For this purpose, it is necessary to relate the constants μ, λ in (1.8a) and (1.8b) to this new parameter k according to the following expressions,

$$\lambda(k)=\frac{k}{k\cosh(k\delta)-\mu\sinh(k\delta)},\tag{2.1a}$$

$$\mu^2(k)+2\rho^2\mu(k)=2\rho^2k\coth(k\delta)+k^2.\tag{2.1b}$$

Note that, in the limit $\delta\rightarrow 0$, $\lambda=1+\mu\delta+O(\delta^2)$, as we supposed in deriving (1.9b). Further, the parameter μ in (1.9a) and (1.9b) is now parametrized as $\mu^2(k)=2\rho^2/\delta+k^2$, where ρ^2 is supposed to have the order of $O(\delta)$. This expression coincides with that for the NLS equation (1.7).[3]

Then, it is convenient to define the modified linear functions

$$W_1(x,k)=\varphi_1(x)\exp\left(\frac{i\mu(k)(x-i\delta)}{2}\right),\quad W_2^{\pm}(x,k)=\varphi_2^{\pm}(x)\exp\left(\frac{i\mu(k)(x\mp i\delta)}{2}\right).\tag{2.2}$$

The system (1.8a) and (1.8b) in these new variables takes the following one-parameter form

$$iW_{1x}+AW_2^+=0,\tag{2.3a}$$

$$W_2^+-\gamma W_2^-+A^*W_1=0,\tag{2.3b}$$

where

$$\gamma(k)=\lambda(k)\exp(\mu(k)\delta)=\gamma(-k).\tag{2.3c}$$

Next, let us introduce the conformal mapping of the complex k plane to the $q_\pm$ plane according to the following formulas

$$q_\pm(k)=\frac{\pm k-\mu(k)}{2},\tag{2.4a}$$

$$q_-(k)=q_+(-k).\tag{2.4b}$$

It is important that parameter $\gamma(k)$ can be expressed as a function only of q_+ or q_-,

$$\gamma(k)=\gamma(q_\pm)=\left(1-\frac{\rho^2}{q_\pm}\right)\exp(-2q_\pm\delta). \tag{2.5}$$

Moreover, the Jost functions also depend only on the spectral parameters q_+ or q_-, instead of k. The Jost functions are defined as solutions to the system (2.3a) and (2.3b) with the following boundary conditions at infinity,

$$\Phi_\pm(x,q_\pm)\to\phi_\pm(x,q_\pm)=\begin{pmatrix}-\rho(q_\pm)^{-1}\\ 1\end{pmatrix}\exp(-iq_\pm x)\quad \text{as } x\to-\infty, \tag{2.6a}$$

$$\Psi_\pm(x,q_\pm)\to\psi_\pm(x,q_\pm)=\begin{pmatrix}-\rho(q_\pm)^{-1}\\ 1\end{pmatrix}\exp(-iq_\pm x)\quad \text{as } x\to+\infty, \tag{2.6b}$$

where

$$\Phi_\pm,\Psi_\pm\equiv\begin{pmatrix}W_1\\ W_2^+\end{pmatrix}.$$

In Appendix B we give a Green's function representation of solutions to (2.3a) and (2.3b), with the boundary conditions (2.6a) and (2.6b). The Green's functions are in fact a matrix 2×2 generalization of the Green's function for the linear problem (1.3a) with the same pole structure in the complex k plane. Hence, we refer to the results of Kodama *et al.*[5] for a proof that the unique Jost functions defined by (2.6a) and (2.6b) do exist for finite δ.

The analyticity properties of the Jost functions in the $q_\pm$ plane can be found from their triangular representation

$$\Phi_\pm(x,q_\pm)=\phi_\pm(x,q_\pm)-\int_{-\infty}^{x}\mathbf{K}_l(x,z)\phi_\pm(z,q_\pm)dz, \tag{2.7a}$$

$$\Psi_\pm(x,q_\pm)=\psi_\pm(x,q_\pm)+\int_{x}^{+\infty}\mathbf{K}_r(x,z)\psi_\pm(z,q_\pm)dz, \tag{2.7b}$$

where $\mathbf{K}_{l/r}$ are matrices with a one-column structure

$$\mathbf{K}_{l/r}=\begin{pmatrix}0 & F_{l/r}(x,z)\\ 0 & H_{l/r}^+(x,z)\end{pmatrix}. \tag{2.8}$$

In what follows we will omit the sign of "left" and "right" functions where it does not lead to ambiguity.

The direct substitution of (2.7a) and (2.7b) into the system (2.3a) and (2.3b) reveals that the functions $F(x,z)$, $H^+(x,z)$ and the associated functions $G(x,z)$, $H^-(x,z)$ satisfy the differential-difference Goursat problem

$$iF_x+A(x)H^+=0, \tag{2.9a}$$

$$-iG_z+\rho H^-=0, \tag{2.9b}$$

$$H^+-H^-+A^*(x)F-\rho G=0, \tag{2.9c}$$

where $H^\pm=H(x\mp i\delta,z\mp i\delta)$. The system (2.9) is determined for "left" functions at $z<x$ under the condition of their vanishing as $z\to-\infty$, and for "right" functions at $z>x$ under their vanishing as $z\to+\infty$. At the characteristic $z=x$, the system (2.9) has the following boundary conditions:

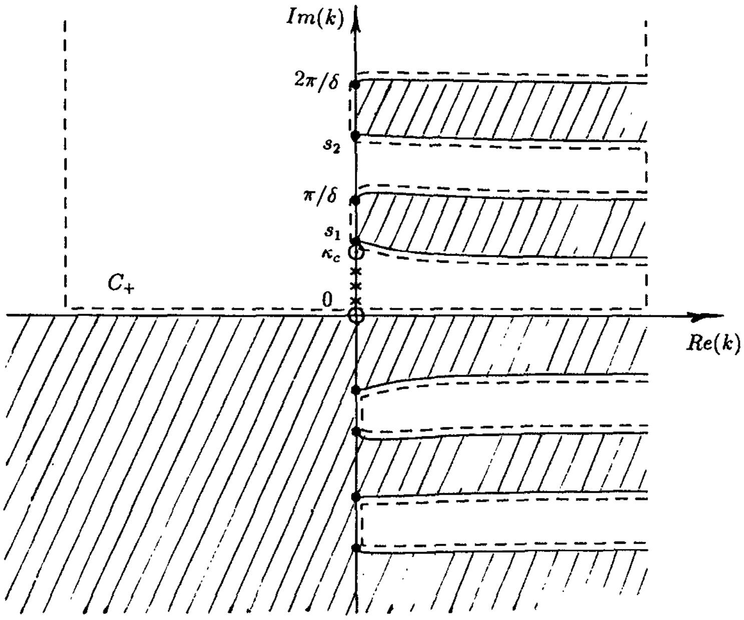

FIG. 1. The complex k plane for the spectral problem (2.3). The bright sheet is characterized by $\mathrm{Im}(q_+)>0$ and the shaded one by $\mathrm{Im}(q_+)<0$. Dashed line depicts contour C_+. Black dots determine the intervals for the branch cuts along the imaginary k axis, while the white dots determine the interval for the discrete spectrum.

$$F(x,x)=i(\rho-A(x)), \tag{2.10a}$$

$$G(x,x)=i(A^*(x)-\rho), \tag{2.10b}$$

$$H^{\pm}(x,x)=\frac{\pm i+\mathbf{T}}{2}(|A|^2-\rho^2)\equiv i[|A|^2-\rho^2]^{\pm}. \tag{2.10c}$$

Because the solutions to (2.9) and (2.10) do not depend on the spectral parameters $q_\pm$, the triangular representation (2.7) implies that the Jost functions $\Phi_\pm$ ($\Psi_\pm$) are analytic functions in the upper (lower) sheet of the complex $q_\pm$ plane. Furthermore, the Jost functions can be expanded in an asymptotic series as $q_\pm\to\infty$ as follows

$$\Phi_\pm=\left[\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{q_\pm}\begin{pmatrix}-A\\ [|A|^2-\rho^2]^+\end{pmatrix}+O\left(\frac{1}{q_\pm^2}\right)\right]\exp(-iq_\pm x),\quad \mathrm{Im}(q_\pm)>0, \tag{2.11a}$$

$$\Psi_\pm=\left[\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{q_\pm}\begin{pmatrix}-A\\ [|A|^2-\rho^2]^+\end{pmatrix}+O\left(\frac{1}{q_\pm^2}\right)\right]\exp(-iq_\pm x),\quad \mathrm{Im}(q_\pm)<0. \tag{2.11b}$$

These expressions give us the boundary conditions for the Jost functions at infinity in the complex $q_\pm$ plane.

Finally, we present in Fig. 1 a complex k plane and its separation into two fundamental sheets of the q_+ plane. [The sheets of the q_- plane can be found by the symmetric transformation (2.4b).] The bright sheet is characterized by $\mathrm{Im}(q_+)>0$ and the shaded one by $\mathrm{Im}(q_+)<0$. Both sheets are connected at real k [negative $\mathrm{Re}(q_+)$] but have branch cuts for some curves in the complex k plane [positive $\mathrm{Re}(q_+)$]. Note that each branch cut of the q_+ plane occurs along the imaginary k axis for the region $s_n \leq \mathrm{Im}(k) \leq \pi n/\delta$, where s_n, $n \geq 1$ are roots of equation $s_n \cot(s_n \delta) = -\rho^2/2$ so that $\pi(n-1)/\delta < s_n < \pi n/\delta$. It is important to mention that a similar spectral k plane exists for the ILW equation.[5] But for the latter equation the branch cuts of the q_+ plane are localized in the imaginary k axis at the points $\mathrm{Im}(k) = \pi n/\delta$.

III. SCATTERING DATA

Now we are ready to introduce the scattering problem which relates the "left" and "right" Jost functions. The scattering problem for the INLS equation should be considered along the edge of the fundamental sheets where $\mathrm{Im}(q_+)=0$ (Fig. 1). Two different situations appear here because the Jost functions Φ_+, Ψ_+ are defined for different sheets of the complex k plane. As a result, the scattering problem for positive q_+ (complex k) where the sheets have branch cuts is different from that for negative q_+ (real k) where the sheets are connected.

Generally speaking, the differential-difference system (2.3) may admit infinitely many eigenfunctions which correspond to poles of the Green's function [see (B2) of Appendix B and Ref. 5]. Therefore, the two sets of solutions of (2.3) with the boundary conditions (2.6a) and (2.6b) may be independent of each other. However, the Green's function representation for the Jost functions shows that for real k (negative q_+) they are related as follows:

$$\Phi_+(x,k) = a(k)\Psi_+(x,k) + b(k)\Psi_-(x,k), \tag{3.1a}$$

$$\Phi_-(x,k) = \tilde{a}(k)\Psi_-(x,k) + \tilde{b}(k)\Psi_+(x,k), \tag{3.1b}$$

where the coefficients $a(k)$, $b(k)$ can be expressed in terms of the Jost function Φ_+ [(B4) in Appendix B]. The relations (3.1a) and (3.1b) can then be regarded as a scattering problem for the INLS equation, while the sets $a(k)$, $b(k)$ and $\tilde{a}(k)$, $\tilde{b}(k)$ are the scattering data. Using the symmetry condition (2.3c) we obtain the additional relations for the Jost functions with real k,

$$\Phi_-(x,k) = \Phi_+(x,-k), \quad \Psi_-(x,k) = \Psi_+(x,-k). \tag{3.2a}$$

This implies that the sets of scattering data are not independent,

$$\tilde{a}(k) = a(-k), \quad \tilde{b}(k) = b(-k). \tag{3.2b}$$

Therefore, it is sufficient to consider only the scattering problem (3.1) with the scattering data $a(k)$, $b(k)$.

Furthermore, following Kodama *et al.*,[5] we can establish an important relation between the scattering data,

$$|a|^2 + C(k)|b|^2 = 1, \quad C(k) = \frac{1 + 2\delta q_-(1 - q_-/\rho^2)}{1 + 2\delta q_+(1 - q_+/\rho^2)} \exp(2k\delta). \tag{3.3}$$

For the proof of this relation we use the "balance equation,"

$$i(|\Phi_1|^2)_x + \gamma(k)(|\Phi_2|^2(x - i\delta) - |\Phi_2|^2(x + i\delta)) = 0, \tag{3.4}$$

which can be found from (2.3) and the complex conjugate system. Then, integration of (3.4) with the boundary conditions (2.6a), (2.6b), and (3.1a) leads to the formula (3.3). Note that $C(k) \to 1$ as $\delta \to 0$ and Eq. (3.3) transforms to the well-known relation for the Zakharov–Shabat spectral problem.[3]

For positive values of q_+, q_- is complex. Therefore, the Jost function Ψ_- contains an exponentially decaying (growing) term for $\mathrm{Im}(k)>0$ ($\mathrm{Im}(k)<0$) according to (2.4a), (2.6b). However, in this case $b(k)\equiv 0$ and the Jost functions $\Phi_+(x,k)$, $\Psi_+(x,k^*)$ defined on the same sheet in the complex k plane are related by

$$\Phi_+(x,k)=a(k)\Psi_+(x,k^*), \tag{3.5}$$

where

$$a(k)a^*(k^*)=1. \tag{3.6}$$

Inside the fundamental sheet of the k plane we need to know only the analytic properties of the coefficient $a(k)$. It is obvious from (B4a) and (2.6a) that coefficient $a(q_+)$ inherits the properties of the Jost function $\Phi_+(x,q_+)$ and is an analytic function in the upper half-plane of q_+. As $q_+\to\infty$ it has the following asymptotic expansion:

$$a=1-\frac{1}{2i\delta q_+}\int_{-\infty}^{+\infty}(|A|^2-\rho^2)dx+O\left(\frac{1}{q_+^2}\right), \quad \mathrm{Im}(q_\pm)>0. \tag{3.7}$$

Using the technique described by Kodama *et al.*,[5] we can prove that coefficient $a(q_+)$ may have only simple zeros at the points $k=k_n$ such that $\gamma(k_n) = \gamma(k_n^*)$. This equation has the solution $k_n^* = -k_n$, $\mu(k_n^*) = \mu(k_n)$ which implies that k_n has imaginary values $k_n=i\kappa_n$ and $0<\kappa_n<\kappa_c$, where κ_c is a solution of equation $\kappa_c \tan(\kappa_c\delta/2)=\rho^2$. It is important to note that $\kappa_c<s_1$ and, therefore, the discrete spectrum of the linear system (2.3) lies in the complex k plane below the first curve of the branch cut (Fig. 1). We point out also that the range of κ_n is empty for $\rho=0$. It means that there are no soliton solutions to the INLS equation (1.6) for vanishing boundary conditions.

The bound states are defined for the values $k=k_n=i\kappa_n$, for which the Jost functions are localized at $x\to\pm\infty$. In this case, the scattering relation can be rewritten as

$$\Phi_+(x,k_n)=b_n\Psi_-(x,k_n). \tag{3.8}$$

It can be found from the system (2.3)[5] that

$$C_n^{-1}=ib_n^{-1}\left.\frac{da}{dq_+}\right|_{k=k_n}=\int_{-\infty}^{+\infty}|\Psi_{2-}|^2(x,k_n)dx. \tag{3.9}$$

Because C_n is finite for nontrivial functions of the discrete spectrum, the coefficient $a(q_+)$ has only simple zeros at the points $k=k_n$.

Thus, the scattering data $\mathbf{S}$ consist of a continuous spectrum which is determined at the edge of the fundamental sheet of the q_+ plane by coefficients $a(k)$, $b(k)$ and a finite number (N) of simple zeros of the coefficient $a(k)$ at the points $k=k_n=i\kappa_n$. They are given by

$$\mathbf{S}=[a(k),b(k),\{\kappa_n,C_n\}_{n=1}^N]. \tag{3.10}$$

If the Jost functions are found from the solutions of the linear differential-difference system (2.3), then the formulas (B4) and (3.9) allow us to construct the scattering data (3.10) for a given time. As it is well known,[4] the time evolution of the scattering data is trivial. It can be easily found from the time-dependent part of the linear associated system (1.8c) and (1.8d) and is given by the following relations:

$$a(k,t)=a(k,0), \quad b(k,t)=b(k,0)\exp(-ik\mu(k)t), \tag{3.11a}$$

$$\kappa_n(t)=\kappa_n(0), \quad C_n(t)=C_n(0)\exp(\kappa_n\mu(i\kappa_n)t). \tag{3.11b}$$

IV. INVERSE SCATTERING FORMALISM

The final step of the IST scheme is to reconstruct a solution to the INLS equation from the scattering data (3.10) which have now been obtained for any time. This can be done using an appropriate integration in the complex q_+ plane of the scattering problem (3.1a), which is convenient to rewrite in the following form:

$$\frac{\Phi_+(x,k)}{a(k)}=\Psi_+(x,k)+\frac{b(k)}{a(k)}\Psi_-(x,k). \tag{4.1}$$

Let us consider the contour $C_+(C_-)$ going along the edge of the fundamental sheet inside of the upper (lower) half-plane of q_+ (contour C_+ is shown in Fig. 1). Because the fundamental sheet has branch cuts, we need to close the contours appropriately. Taking into account the boundary conditions of the Jost functions Φ_+ and Ψ_+ (2.11) and of the coefficient $a(k)$ (3.7) in the complex q_+ plane as well as the fact that coefficient $b(k)$ vanishes at the edge of the sheet for complex values of k, we can close the contour C_+ (C_-) at infinity in the region where $\mathrm{Im}(q_+)>0$ $(\mathrm{Im}(q_+)<0)$. So, we have the integration formula

$$\frac{1}{2\pi}\int_{C_+}dq_+\frac{\Phi_+(x,q_+)}{a(q_+)}\exp(iq_+z)=\frac{1}{2\pi}\int_{C_-}dq_+\Psi_+(x,q_+)\exp(iq_+z)+\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\frac{dq_+}{dk}\frac{b(k)}{a(k)}\Psi_-(x,k)\exp(iq_+z), \quad z>x. \tag{4.2}$$

The left-hand side of (4.2) is easily to calculate from the residue theorem because the functions $\Phi_+(x,q_+)$ and $a(q_+)$ are analytic inside of contour C_+. On the other hand, for the calculation of right-hand side of (4.2) we have to use the triangular representation (2.7) of the Jost functions $\Psi_\pm$. As a result of this integration, we get a system of the Gelfand–Levitan–Marchenko (GLM) integral equations for the functions $F(x,z)$ and $H^+(x,z)$. Then, using (2.9b) and (1.4) we can obtain the GLM integral equations for the associated functions $G(x,z)$ and $H^-(x,z)$. Thus the following GLM equations form a complete system:

$$F(x,z)+f(x,z)+\int_x^{+\infty}F(x,s)h^+(s,z)ds=0, \tag{4.3a}$$

$$G(x,z)+g(x,z)+\int_x^{+\infty}H^-(x,s)g(s,z)ds=0, \tag{4.3b}$$

$$H^\pm(x,z)+h^\pm(x,z)+\int_x^{+\infty}H^\pm(x,s)h^\pm(s,z)ds=0, \tag{4.3c}$$

where $z>x$ [i.e., we deal with the "right" triangular representation (2.7b)]. The functions $f(x,z)$, $g(x,z)$ and $h^\pm(x,z)$ are expressed in terms of the scattering data (3.10). We give the expressions only for functions f, h^+ because the other functions can be easily reconstructed,

$$f(x,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\hat{f}(k)\exp(-iq_-x+iq_+z)-\rho\sum_{n=1}^{N}(q_{-n})^{-1}C_n\exp(-iq_{-n}x+iq_{+n}z), \tag{4.4a}$$

$$h^{+}(x,z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\hat{h}(k)\exp(-iq_{-}x+iq_{+}z)+\sum_{n=1}^{N}C_{n}\exp(-iq_{-n}x+iq_{+n}z), \quad (4.4b)$$

with the Fourier coefficients

$$\hat{f}(k)=-\frac{dq_{+}}{dk}\frac{b(k)}{a(k)}\frac{\rho}{q_{-}(k)}, \quad \hat{h}(k)=\frac{dq_{+}}{dk}\frac{b(k)}{a(k)}, \quad (4.5)$$

and $q_{\pm n}=q_{\pm}(k=i\kappa_n)$. For given $f(x,z)$, $g(x,z)$ and $h^{\pm}(x,z)$, the solutions of the GLM integral equations (4.3a)–(4.3c) at $z>x$ generate solutions to the INLS equation from the relations (2.10) on the characteristic $z=x$.

Thus Secs. II–IV give a closed scheme for the solution of the initial-value problem of the INLS equation (1.6), which is a standard IST scheme for integrable one-dimensional evolution equations. It consists of the solution of the linear associated problem (2.3) [or (2.9)] for the given initial value $A(x,0)$, the calculation of the scattering data by formulas (B4) and (3.9), of their time evolution by (3.11), and the final reconstruction of $A(x,t)$ for any time by a solution of the integral equations (4.3). Nevertheless, we would like to point out that even in the limit $\delta\to 0$, the triangular representation (2.9) and the GLM equations (4.3) are different from those considered by Zakharov and Shabat[3] for the NLS equation. This new IST scheme for the NLS equation seems simpler than the former version because the triangular matrix has a remarkable one-column form (2.8). We employ this scheme in the next section to obtain a Bäcklund–Darboux transformation for the INLS equation.

V. THE DRESSING TRANSFORMATION FOR THE INLS EQUATION

In Sec. II–IV we presented the IST scheme for the Jost functions $\Phi_{\pm}$, $\Psi_{\pm}$ which are defined as solutions of the linear spectral problem (2.3) with the boundary conditions (2.6a) and (2.6b). In this definition the functions $\phi_{\pm}$, $\psi_{\pm}$ can be regarded as Jost functions for the system (2.3) with A replaced by the constant coefficient ρ. Now we reformulate this scheme to construct a general transformation of solutions both for the INLS equation, and for the linear associated problem for an arbitrary initial solution $A=A_0(x)$.

We start from the integral GLM equations (4.3) which relate two sets of functions $f(x,z)$, $g(x,z)$, $h^{\pm}(x,z)$ and $F(x,z)$, $G(x,z)$, $H^{\pm}(x,z)$. Let one set of functions satisfy the following linear system for a given solution to the INLS equation $A=A_0(x)$,

$$if_x+A_0(x)h^{+}=0, \quad (5.1a)$$

$$-ig_z+A_0^{*}(z)h^{-}=0, \quad (5.1b)$$

$$h^{+}-h^{-}+A_0^{*}(x)f-A_0(z)g=0. \quad (5.1c)$$

Then, we can readily show that the other set of functions being a solution of the GLM equations (4.3) generates a new solution to the INLS equation according to the relations

$$A=A_0+iF(x,x), \quad (5.2a)$$

$$A^{*}=A_0^{*}-iG(x,x), \quad (5.2b)$$

$$|A|^2=|A_0|^2-i(H^{+}(x,x)-H^{-}(x,x)). \quad (5.2c)$$

Note that for a trivial solution $A_0=\rho$ the relations (5.2) reduce to (2.10) and the Fourier representation of functions $f(x,z)$, $g(x,z)$, and $h^{\pm}(x,z)$ (4.4) gives a general solution for the linear equations (5.1).

Next, we suppose that the linear problem (2.3) for $A=A_0(x)$ has no discrete spectrum. Our aim is to construct a transformation which would give birth to N bound states of the linear system for a new solution $A=A(x)$. We look for solutions to (5.1) and (4.3) by separating variables x and z,

$$f(x,z)=\sum_{n=1}^{N} f_n(x)h_n^{*+}(z), \quad g(x,z)=\sum_{n=1}^{N} h_n^{-}(x)f_n^{*}(z), \quad h^{\pm}(x,z)=\sum_{n=1}^{N} h_n^{\pm}(x)h_n^{*\pm}(z). \tag{5.3}$$

The substitution of (5.3) into (5.1) reveals that the functions f_n, $h_n^{\pm}$ satisfy the linear associated problem (2.3) for $A=A_0(x)$ and represent its partial solutions for some values of k, for instance, for $k=k_n=i\kappa_n$, $0<\kappa_n<\kappa_c$, $n=1,\ldots,N$. It is necessary to note that the GLM system (4.3) is written for the "right" Jost functions with the boundary conditions (2.6b). Therefore, we impose the following boundary conditions for each set of the solutions to (2.3a) and (2.3b),

$$\psi_n=\begin{pmatrix} f_n \\ h_n^{+} \end{pmatrix} \to \begin{pmatrix} -\rho(q_{-n})^{-1} \\ 1 \end{pmatrix} \exp(-iq_{-n}x) \quad \text{as } x\to+\infty. \tag{5.4}$$

Under these conditions, the functions ψ_n are localized as $x\to+\infty$ because $\mathrm{Im}(q_{-n})=-\kappa_n<0$. Of course, they diverge as $x\to-\infty$ since the system (2.3) at $A=A_0(x)$ has no bound states. Further, solving the integral equations (4.3) we get a new solution to the INLS equation in the form,

$$A=A_0-i\sum_{n,k=1}^{N} \frac{\Delta_{nk}^{+}}{\Delta_N^{+}} f_n(x)h_k^{*+}(x), \quad |A|^2=|A_0|^2-i\frac{\partial}{\partial x}\ln\left(\frac{\Delta_N^{+}}{\Delta_N^{-}}\right), \tag{5.5}$$

where

$$\Delta_N^{\pm}=\det\left(\delta_{nk}+\int_x^{+\infty} h_n^{\pm}(s)h_k^{*\pm}(s)ds\right)_{1\le n,k\le N},$$

Δ_{nk}^{+} is a cofactor of an element (n,k) of the determinant Δ_N^{+}, and δ_{nk} is the Kronecker symbol. Thus, solutions to the linear associated problem (2.3) generate a transformation of solutions to the INLS equation $A_0\to A$ according to (5.5). Moreover, we can also find a transformation of the Jost functions $\psi_{\pm}\to\Psi_{\pm}$ which are defined by solutions to the system (2.3a) and (2.3b) for $A_0(x)$ and $A(x)$, respectively, with the boundary conditions (2.6b). This transformation follows immediately from the triangular representation (2.7b)

$$\Psi_{\pm}=\psi_{\pm}-\sum_{n,k=1}^{N} \frac{\Delta_{nk}^{+}}{\Delta_N^{+}} \psi_n \int_x^{+\infty} \psi_{2\pm}^{+}(s)h_k^{*+}(s)ds. \tag{5.6}$$

Furthermore, calculating the asymptotics of the formula (5.6) as $x\to-\infty$ and using the relations $a^{-1}=\lim_{x\to-\infty}[\Psi_{2+}^{+}\exp(iq_{+}x)]$, $a_0^{-1}=\lim_{x\to-\infty}[\psi_{2+}^{+}\exp(iq_{+}x)]$ we can also find a transformation of the scattering coefficient $a_0\to a$,

$$a(q_{+})=a_0(q_{+})\prod_{n=1}^{N} \frac{q_{+}-q_{+n}}{q_{+}-q_{-n}}. \tag{5.7}$$

Here we have supposed that the coefficient a_0 has no zeros for $\mathrm{Im}(q_+)>0$. However, the formula (5.7) shows that the transformation (5.5), (5.6) gives birth to N zeros of the coefficient a for the linear problem (2.3) with $A=A(x)$. These N zeros correspond to N bound states of the linear problem and the expressions (5.6) at $\psi_-=\psi_n$, $n=1,2,\ldots,N$ enable us to find the Jost functions Ψ_- corresponding to the bound states (3.8) in an explicit form

$$\Psi_n=-\sum_{k=1}^{N}\frac{\Delta_{kn}^+}{\Delta_N^+}\psi_k. \tag{5.8}$$

Thus a set of solutions ψ_n to the linear problem (2.3) without solitons generates solutions both to the INLS equation and to the linear associated system with N solitons. The transformation constructed above is referred to as the Bäcklund–Darboux transformation.[21] Note that a pure N-soliton solution appears for $A_0=\rho$, with the solutions ψ_n then specified in the exponential form (5.4). For this case the soliton solutions, the Jost functions, the scattering coefficients and the bound states can be found in an explicit form from the formulas (5.5)–(5.8). For instance, a single-soliton solution is given by

$$A=\rho\,\frac{1+[(v-i\kappa)/(v+i\kappa)]\exp[-\kappa(\xi-i\delta)]}{1+\exp[-\kappa(\xi-i\delta)]},\quad |A|^2=\rho^2-\frac{\kappa\sin(\kappa\delta)}{\cosh(\kappa\xi)+\cos(\kappa\delta)}, \tag{5.9}$$

where $\xi=x-vt-x_0$, x_0, κ are arbitrary parameters, and the soliton speed v is determined by a quadratic equation

$$v^2+2\rho^2v+\kappa^2=2\rho^2\kappa\cot(\kappa\delta).$$

VI. CONSERVATION LAWS

The characteristic feature of all integrable equations is the existence of an infinite number of conservation laws. Here we show that the INLS equation also possesses this feature and we present a new way of constructing a complete set of conservation laws by means of solving a linear recursion relation.

First, the time evolution of the scattering coefficients (3.11) implies that the coefficient $a(q_+)$ is a constant in time. Furthermore, the asymptotic expansion of $a(q_+)$ for $q_+\to\infty$ (3.7) would give an explicit form of the conserved quantities if it is possible to calculate the higher-order terms of the asymptotic series.

Next, we note that parameter ρ determines only a normalization of conserved quantities with respect to a homogeneous (vacuum) state $A=\rho$. Therefore, for simplification of all intermediate formulas we put $\rho\equiv0$. However, we shall include this parameter in the final expressions for the conserved quantities [formulas (6.8)]. For the case $\rho=0$, the local Riemann–Hilbert problem (2.3b) has a simple solution

$$M_2^+(x,q_+)=1-\frac{1-i\mathbf{T}}{2}(A^*M_1)-\frac{i}{4\delta}\int_{-\infty}^{+\infty}A^*M_1\,dx, \tag{6.1}$$

where $M_1=\Phi_{1+}\exp(iq_+x)$, $M_2^+=\Phi_{2+}^+\exp(iq_+x)$. The last constant in (6.1) is introduced in order to satisfy the boundary condition $\lim_{x\to-\infty}M_2^+(x,q_+)=1$ because the operator $\mathbf{T}$ has the following asymptotic property[11]

$$\lim_{x\to\pm\infty}\mathbf{T}(u)=\mp\frac{1}{2\delta}\int_{-\infty}^{+\infty}u\,dx.$$

The other limiting transition for the function M_2^+ determines the coefficient $a(q_+)$,

$$a(q_+) = \lim_{x\to+\infty} M_2^+(x,q_+) = 1 - \frac{i}{2\delta}\int_{-\infty}^{+\infty} A^* M_1\, dx. \tag{6.2}$$

Substitution of (6.1) into (2.3a) gives an equation for M_1,

$$M_1 = \frac{1}{q_+}\left[-A - iM_{1x} + A\,\frac{1-i\mathbf{T}}{2}\,(A^*M_1) + \frac{i}{4\delta} A \int_{-\infty}^{+\infty} A^* M_1\, dx \right]. \tag{6.3}$$

Expanding the solutions to this equation in an asymptotic series as $q_+ \to \infty$, we get

$$M_1(x,q_+) = \sum_{n=1}^{\infty} \frac{m_n(x)}{q_+^n}, \tag{6.4}$$

where $m_n(x)$ satisfy the following recursion relation,

$$m_{n+1} = -im_{nx} + A\,\frac{1-i\mathbf{T}}{2}\,(A^*m_n) + \frac{i}{4\delta} A\int_{-\infty}^{+\infty} A^* m_n\, dx, \quad m_1 = -A. \tag{6.5}$$

The functions $m_n(x)$ generate the higher-order terms of asymptotic series for $a(q_+)$ according to the relations,

$$a(q_+) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{q_+^n}, \quad a_n = \frac{1}{2i\delta}\int_{-\infty}^{+\infty} (A^* m_n)\,dx. \tag{6.6}$$

All the quantities a_n are conservation laws for the INLS equation. The first three quantities are given by the following explicit expressions:

$$a_1 = \frac{i}{2\delta} I_1, \quad a_2 = \frac{i}{2\delta} I_2 - \frac{1}{8\delta^2} I_1^2, \quad a_3 = \frac{i}{2\delta} I_3 - \frac{1}{4\delta^2} I_1 I_2 - \frac{i}{48\delta^3} I_1^3, \tag{6.7}$$

where

$$I_1 = \int_{-\infty}^{+\infty} (|A|^2 - \rho^2)\,dx, \tag{6.8a}$$

$$I_2 = \int_{-\infty}^{+\infty} \left(-iA_x A^* + \frac{1}{2}(|A|^4 - \rho^4) \right) dx, \tag{6.8b}$$

$$I_3 = \int_{-\infty}^{+\infty} \left(-A_{xx}A^* - iA_x A^* |A|^2 + \frac{1}{3}(|A|^6 - \rho^6) - \frac{1}{2}|A|^2 \mathbf{T}(|A|^2)_x \right) dx. \tag{6.8c}$$

For the simplification of these expressions we have used the following properties of the operator $\mathbf{T}$:[11]

$$\int_{-\infty}^{+\infty} v\mathbf{T}(u)\,dx = -\int_{-\infty}^{+\infty} u\mathbf{T}(v)\,dx,$$

$$\int_{-\infty}^{+\infty} u\mathbf{T}[u\mathbf{T}(u)]\,dx = -\frac{1}{3}\int_{-\infty}^{+\infty} u^3\, dx - \frac{1}{12\delta^2}\left(\int_{-\infty}^{+\infty} u\, dx \right)^3.$$

These first conservation laws I_1, I_2, I_3 correspond to the conservation of the number of particles, the momentum and the energy, respectively. For $\delta \to 0$ and $|A|^2 \sim O(\delta)$ they transform to the corresponding quantities for the NLS equation.[3] However, we would like to point out that the recursion relation (6.5) is different from that for the NLS equation. In the latter case, the recurrence relation is quadratic, while in the former case it is linear. However, here the conserved quantities a_n are a superposition of the homogeneous quantities I_n. On the other hand, (6.7) suggests that there exists the following remarkable expansion in terms of the homogeneous quantities I_n,

$$\ln(a(q_+)) = \frac{i}{2\delta} \sum_{n=1}^{\infty} \frac{I_n}{q_+^n}. \tag{6.9}$$

A proof of the validity of this expansion in the general case, and a construction of the recursion relation for I_n needs further investigation.

Finally, we present a Hamiltonian form of the INLS equation generated from I_3,

$$\begin{pmatrix} A \\ A^* \end{pmatrix}_t = \mathbf{J}_3 \begin{pmatrix} \delta_A I_3 \\ \delta_{A^*} I_3 \end{pmatrix}, \quad \mathbf{J}_3 = \begin{pmatrix} -A \int A\, dx & i + A \int A^*\, dx \\ -i + A^* \int A\, dx & -A^* \int A^*\, dx \end{pmatrix}, \tag{6.10}$$

where $\delta_A I_3$ denotes a functional derivative of I_3 on A and the matrix operator $\mathbf{J}_3$ satisfies the condition $\mathbf{J}_3 = -\mathbf{J}_3^t$. Note that the operator $\mathbf{J}_3$ is nonlocal, and in this respect (6.10) differs from the Hamiltonian form of the NLS equation.[4] Moreover, the conserved quantities I_n starting with $n=2$ cannot be rewritten in the form of local conservation laws.

VII. DISCUSSION

In this paper we have considered the IST scheme for the INLS equation with finite δ, where the linear spectral problem (1.8) is a new differential-difference generalization of the Zakharov–Shabat spectral problem. In this case the scattering data consist of a standard set with a continuous spectrum and a finite number of bound states. It is known for the ILW equation[6,7] that the IST technique becomes more interesting as the parameter δ tends to infinity. In this case, a new type of bound states appears and a new method for a nonlocal Riemann–Hilbert problem *in a spectral space* should be applied.

The other interesting problem which is beyond the scope of this paper is the IST scheme for vanishing boundary conditions (when $\rho=0$). In this case, the linear problem (2.3a) and (2.3b) can be rewritten as a nonlocal Riemann–Hilbert problem *in a coordinate space*. This type of a spectral problem has not yet appeared in IST theory. Note, however, that the wave dynamics with vanishing boundary conditions is rather trivial because the bound states and solitons are absent for $\rho=0$.

Finally, we would like to emphasize that the IST scheme for the INLS equation represents a new generalization of that for the ILW equation for the matrix 2×2 case. The consideration of this new spectral problem might be interesting for a construction of a hierarchy of nonlocal equations related to this spectral problem and investigation of their Hamiltonian structure. The recent paper[15] reveals that work in this direction is currently under way.

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APPENDIX A: DERIVATION OF LINEAR SYSTEM (1.8)

Here we show how to derive the linear spectral problem (1.8) associated with the INLS equation (1.6) from linear system (1.3) by means of a regular asymptotic expansion. Let us start from the differential-difference equation (1.3a) which contains the potential $u(x,t)$ and substitute the asymptotic expansion

$$u=\sum_{n=1}^{\infty} \epsilon^{n/2} u_n(x,t;\tilde{x},\tilde{t}_1,\tilde{t}_2,\ldots). \tag{A1}$$

The first few terms were found in Ref. 16, and they are given by

$$u_1=A \exp[i(x+t)]+A^* \exp[-i(x+t)], \tag{A2a}$$

$$u_2=-|A|^2+A^2 \exp[2i(x+t)]+A^{*2} \exp[-2i(x+t)], \tag{A2b}$$

$$u_3=A^3 \exp[3i(x+t)]+A^{*3} \exp[-3i(x+t)]. \tag{A2c}$$

Here $A=A(\tilde{x},\tilde{t}_1,\tilde{t}_2,\ldots)$, $\tilde{x}=\epsilon x$, $\tilde{t}_n=\epsilon^n t$, $n=1,2,\ldots$, and $\delta=\tilde{\delta}/\epsilon$ as in (1.5) and in the following discussion.

First, we note that integral operator $\mathbf{T}$ (1.2) acts on x-dependent and x-independent terms in a different manner. For the former terms, this operator becomes the Hilbert operator

$$\mathbf{H}(u)=\frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{+\infty} \frac{u(z)}{z-x}\, dz,$$

which does not act to leading order on the slow variable $\tilde{x}$. However, for the latter terms the operator $\mathbf{T}$ keeps the form (1.2) but now rewritten for the new variable $\tilde{x}$ and new parameter $\tilde{\delta}$.

Keeping this property into mind, we obtain a solution to (1.3a) in the form of another asymptotic series

$$\varphi=\varphi_1+\sum_{n=1}^{\infty} \epsilon^{n/2} \varphi_{n+1}(x,t;\tilde{x},\tilde{t}_1,\tilde{t}_2,\ldots), \tag{A3a}$$

$$\mu=-2+\epsilon\tilde{\mu}. \tag{A3b}$$

The first few terms of the expansion (A3a) are given by

$$\varphi_1=\tilde{\varphi}_1 \exp[i(x+t)], \tag{A4a}$$

$$\varphi_2=\tilde{\varphi}_2+A\tilde{\varphi}_1 \exp[2i(x+t)], \tag{A4b}$$

$$\varphi_3=A^2\tilde{\varphi}_1 \exp[3i(x+t)]+A^*\tilde{\varphi}_1 \exp[-i(x+t)]. \tag{A4c}$$

Then, on removing secular terms [i.e., those growing like x, $x \exp(ix)$] we obtain (1.8a) and (1.8b) for the coefficients $\tilde{\varphi}_{1,2}$ of the asymptotic series which are supposed to be functions of the slow variables. It is important to note that the expression (1.4) determines now the function φ^+ (φ^-) as an analytic function in the upper (lower) x-plane. It implies that the function φ^+ (φ^-) contains only positive (negative) powers of $\exp(ix)$ with coefficients having arbitrary analytical properties with respect to a new variable $\tilde{x}$. However, the first term of (A4b) does not depend on x. Therefore, the operator (1.4) in the new variable $\tilde{x}$ decomposes $\tilde{\varphi}_2$ into the functions $\tilde{\varphi}_2^{\pm}$. This accounts for the asymmetry of the linear system (1.8a) and (1.8b) with respect to $\tilde{\varphi}_1$, $\tilde{\varphi}_2$.

Finally, substitution of the series (A3) into (1.3b) gives us (1.8c) and (1.8d) for the functions $\tilde{\varphi}_1(\tilde{x}+2\tilde{t}_1,\tilde{t}_2)$ and $\tilde{\varphi}_2^{\pm}(\tilde{x}+2\tilde{t}_1,\tilde{t}_2)$.

APPENDIX B: GREEN'S FUNCTIONS REPRESENTATION

Here we discuss the Green's function representation of the Jost functions and scattering coefficients. The solution to (2.3a) and (2.3b) with the boundary conditions (2.6a) and (2.6b) can be rewritten as solutions of the integral equations,

$$\Phi_{\pm}(x,k)=\phi_{\pm}(x,k)+\int_{-\infty}^{+\infty}\mathbf{G}_{+}(x-z,k)\mathbf{Q}(z)\Phi_{\pm}(z,k)dz, \tag{B1a}$$

$$\Psi_{\pm}(x,k)=\psi_{\pm}(x,k)+\int_{-\infty}^{+\infty}\mathbf{G}_{-}(x-z,k)\mathbf{Q}(z)\Psi_{\pm}(z,k)dz, \tag{B1b}$$

where

$$\mathbf{Q}(x)=\begin{pmatrix} 0 & A(x)-\rho \\ A^*(x)-\rho & 0 \end{pmatrix}$$

and the matrix Green's functions $\mathbf{G}_{\pm}$ can be found from the Fourier transforms,

$$\mathbf{G}_{\pm}(x,k)=\frac{1}{2\pi}\int_{C_{\pm}}dp\ \exp(-ipx-2p\delta)\begin{pmatrix} \gamma(k)\exp(2p\delta)-1 & \rho \\ \rho & -p \end{pmatrix}\frac{1}{p(\gamma(p)-\gamma(k))}. \tag{B2}$$

Here the contour C_+ (C_-) goes along the edge of the fundamental sheet in the upper (lower) half-plane of q_+. The denominator of the Green's function has an infinite number of zeros because the function $\gamma(p)$ is multivalued. Nevertheless, only two zeros are important: $p=q_+$ and $p=q_-$ because they correspond to the chosen eigenfunctions (2.6a) and (2.6b). Other zeros lie on the branch cuts of the fundamental sheet (see Ref. 5). For the integration of (B2) along contours $C_{\pm}$ they give an identical contribution to the Green's function $\mathbf{G}_{\pm}$. Therefore, we can rewrite (B2a) and (B2b) in a composite Volterra–Fredholm form,

$$\Phi_{\pm}(x,k)=\phi_{\pm}(x,k)-\int_{-\infty}^{x}\mathbf{g}(x-z,k)\mathbf{Q}(z)\Phi_{\pm}(z,k)dz+\int_{-\infty}^{+\infty}\mathbf{g}_0(x-z,k)\mathbf{Q}(z)\Phi_{\pm}(z,k)dz, \tag{B3a}$$

$$\Psi_{\pm}(x,k)=\psi_{\pm}(x,k)+\int_{x}^{+\infty}\mathbf{g}(x-z,k)\mathbf{Q}(z)\Psi_{\pm}(z,k)dz+\int_{-\infty}^{+\infty}\mathbf{g}_0(x-z,k)\mathbf{Q}(z)\Psi_{\pm}(z,k)dz, \tag{B3b}$$

where $\mathbf{g}_0(x,k)=\mathbf{G}_{\pm}(x,k)\mp\mathbf{g}(x,k)\Theta(\pm x)$, Θ is the unit Heaviside function, $\mathbf{g}(x,k)=\mathbf{g}'(x,q_+)+\mathbf{g}'(x,q_-)$, and

$$\mathbf{g}'(x,q_{\pm})=\frac{iq_{\pm}\exp(-iq_{\pm}x)}{\rho^2[1+2\delta q_{\pm}(1-q_{\pm}/\rho^2)]}\begin{pmatrix} -\rho^2/q_{\pm} & \rho \\ \rho & -q_{\pm} \end{pmatrix}.$$

The representation (B3a) and (B3b) implies that the scattering relations (3.1) exist for real k. Using the standard technique,[4] we can express the scattering data $a(k),b(k)$ by the Jost function Φ_+

$$a(k)=1+\frac{iq_+}{\rho^2[1+2\delta q_+(1-q_+/\rho^2)]}\int_{-\infty}^{+\infty}dx\ \exp(iq_+x)\Upsilon(x,k,q_+), \tag{B4a}$$

$$b(k)=\frac{iq_-}{\rho^2[1+2\delta q_-(1-q_-/\rho^2)]}\int_{-\infty}^{+\infty}dx\ \exp(iq_-x-k\delta)\Upsilon(x,k,q_-), \tag{B4b}$$

where

$$\Upsilon(x,k,q_\pm)=q_\pm(A^*(x)-\rho)\Phi_{1+}(x,k)-\rho(A(x)-\rho)\Phi_{2+}^+(x,k).$$

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