COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
2016, VOL. 41, NO. 2, 227–255
http://dx.doi.org/10.1080/03605302.2015.1123272

Taylor & Francis
Taylor & Francis Group

L^2 orbital stability of Dirac solitons in the massive Thirring model

Andres Contreras[a], Dmitry E. Pelinovsky[b], and Yusuke Shimabukuro[b]

[a]Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM, United States; [b]Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada

ABSTRACT
We prove L^2 orbital stability of Dirac solitons in the massive Thirring model. Our method uses local well posedness of the massive Thirring model in L^2, conservation of the charge functional, and the auto–Bäcklund transformation. The latter transformation exists because the massive Thirring model is integrable via the inverse scattering transform method.

ARTICLE HISTORY
Received 26 November 2013
Accepted 6 November 2015

KEYWORDS
Auto–Bäcklund transformation; conserved quantities; Dirac solitons; Lax operators; massive Thirring model; orbital stability

MATHEMATICS SUBJECT CLASSIFICATION
35Q41; 35Q51; 35Q75; 37K10; 37K45

1. Introduction

Among one-dimensional nonlinear Dirac equations, the Massive Thirring Model (MTM) is particularly interesting because of its integrability via the inverse scattering transform method [17, 20]. The nonlinear Dirac system arises as a relativistic extension of the nonlinear Schrödinger equation and while they share common features, the Dirac system is more difficult for analytical studies because the classical energy-based methods do not apply to systems with sign-indefinite energy functionals.

We consider the Cauchy problem of the MTM system

$$\begin{cases} i(u_t + u_x) + v + u|v|^2 = 0, \\ i(v_t - v_x) + u + v|u|^2 = 0, \end{cases} \tag{1.1}$$

subject to an initial condition $(u, v)|_{t=0} = (u_0, v_0)$ in $H^s(\mathbb{R})$ for $s \geq 0$. Here the subscripts denote partial derivatives.

The Cauchy problem for the MTM system (1.1) is known to be locally well-posed in $H^s(\mathbb{R})$ for $s > 0$ and globally well-posed for $s > \frac{1}{2}$ [28] (see earlier results in [11]). More pertinent to our study is the global well-posedness in $L^2(\mathbb{R})$ proved in the recent works [6, 14]. The next theorem summarizes the global well-posedness result for the scopes needed in our work.

Theorem 1.1 ([6, 14]). *Let $(u_0, v_0) \in L^2(\mathbb{R})$. There exists a global solution $(u, v) \in C(\mathbb{R}; L^2(\mathbb{R}))$ to the MTM system (1.1) such that the charge is conserved*

$$\|u(\cdot, t)\|_{L^2}^2 + \|v(\cdot, t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 \tag{1.2}$$

CONTACT Dmitry E. Pelinovsky dmpeli@math.mcmaster.ca Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada.

for every $t \in \mathbb{R}$. Moreover, the solution is unique in a certain subspace of $C(\mathbb{R}; L^2(\mathbb{R}))$ and depends continuously on initial data $(u_0, v_0) \in L^2(\mathbb{R})$.

We are interested in orbital stability of Dirac solitons in the MTM system (1.1) given by the explicit expressions

$$\begin{cases} u_\lambda(x,t) = i\delta^{-1}\sin\gamma\ \mathrm{sech}\left[\alpha(x+ct) - i\frac{\gamma}{2}\right]\ e^{-i\beta(t+cx)}, \\ v_\lambda(x,t) = -i\delta\sin\gamma\ \mathrm{sech}\left[\alpha(x+ct) + i\frac{\gamma}{2}\right]\ e^{-i\beta(t+cx)}, \end{cases} \tag{1.3}$$

where λ is an arbitrary complex nonzero parameter that determines $\delta = |\lambda|$, $\gamma = 2\mathrm{Arg}(\lambda)$, as well as

$$c = \frac{\delta^2 - \delta^{-2}}{\delta^2 + \delta^{-2}}, \quad \alpha = \frac{\delta^2 + \delta^{-2}}{2}\sin\gamma, \quad \beta = \frac{\delta^2 + \delta^{-2}}{2}\cos\gamma.$$

Let us now state the main result of our work.

Theorem 1.2. *Let $(u, v) \in C(\mathbb{R}; L^2(\mathbb{R}))$ be a solution of the MTM system (1.1) in Theorem 1.1 and λ_0 be a complex non-zero number. There exists a real positive ϵ_0 such that if the initial value $(u_0, v_0) \in L^2(\mathbb{R})$ satisfies*

$$\epsilon := \|u_0 - u_{\lambda_0}(\cdot, 0)\|_{L^2} + \|v_0 - v_{\lambda_0}(\cdot, 0)\|_{L^2} \le \epsilon_0, \tag{1.4}$$

then for every $t \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}$ such that

$$|\lambda - \lambda_0| \le C\epsilon \tag{1.5}$$

and

$$\inf_{a,\theta\in\mathbb{R}} (\|u(\cdot + a, t) - e^{-i\theta}u_\lambda(\cdot, t)\|_{L^2} + \|v(\cdot + a, t) - e^{-i\theta}v_\lambda(\cdot, t)\|_{L^2}) \le C\epsilon, \tag{1.6}$$

where the positive constant C is independent of ϵ and t.

The proof of Theorem 1.2 relies on the auto-Bäcklund transformation of the MTM system (1.1) and perturbation analysis. Our approach follows the strategy used by Mizumachi and Pelinovsky in [23] for proving L^2-orbital stability of the NLS solitons. Their result was extended by Contreras and Pelinovsky in [9] to multi-solitons of the NLS equation by using a more general dressing transformation. Furthermore, the recent work [10] of Cuccagna and Pelinovsky shows how an asymptotic stability of the NLS solitons can be deduced by combining the auto-Bäcklund transformation and the nonlinear steepest descent method.

Similar ideas have been already applied to other completely integrable systems. We mention for example the work [21] of Merle and Vega where they prove L^2-stability and asymptotic stability of the KdV solitons by using of the Miura transformation that relates the KdV solitons to the kinks of the defocusing modified KdV equation. The work [22] of Mizumachi and Pego makes use of a linearized Bäcklund transformation to establish an asymptotic stability of Toda lattice solitons. Hoffman and Wayne [13] formulated an abstract orbital stability result for soliton solutions of integrable equations that can be achieved via Bäcklund transformations.

In addition to an increasing popularity of the integrability techniques to study nonlinear stability of soliton solutions, we note that such techniques become particularly powerful for the MTM system (1.1). Compared to the NLS equation, proofs of global existence and orbital stability of solitons in the nonlinear Dirac equations (including the MTM system) are complicated by the fact that the quadratic part of the corresponding Hamiltonian is not bounded from neither above nor below. Consequently, there exist two bands of continuous spectrum of the linear Dirac operator for positive and negative energies, which extend to positive and negative infinities. For this reason, proof of orbital stability of Dirac solitons poses a serious difficulty to the application of standard energy arguments. There are, nevertheless, many works that deal with spectral properties of Dirac operators linearized at Dirac solitons [1, 2, 7, 8, 12, 16]. Also, asymptotic stability of small solitary waves in the general nonlinear Dirac equations has been considered [18, 19, 26] (see [3–5] for similar results in the space of three dimensions).

Other than these works, not much is known about the orbital stability of Dirac solitons. The recent work [27] of Pelinovsky and Shimabukuro incorporates the integrability of the MTM system to obtain an additional conserved quantity that can serve as a Lyapunov functional in the proof of H^1-orbital stability of the MTM solitons. The results of [27] are restricted to MTM solitons (1.3) with $\delta = 1$ and γ near $\frac{\pi}{2}$. In the present work, we use the auto-Bäcklund transformation to prove L^2-orbital stability of the MTM solitons (1.3) for every $\delta \in \mathbb{R}^+$ and $\gamma \in (0, \pi)$ by a non-variational method.

Bäcklund transformations are used to generate solutions of a differential system, usually depending on a parameter, from another solution of another differential system. When this transformation relates two different solutions of the same system, it is called the auto-Bäcklund transformation. These transformations, when they exist, can be used to link the orbital stability of a certain class of solutions to that of another class of solutions [13]. In particular, a stable neighborhood of the zero solution can be mapped to a stable neighborhood of one-soliton solution, and vice versa. However, there is no systematic way to find such transformations and, to the best of our knowledge, this is the first appearance of the auto-Bäcklund transformation for the MTM system (1.1) in the literature.

We note in passing that the associated Kaup-Newell spectral problem [17] has been extensively studied and the auto-Bäcklund transformation of other related integrable equations have been reported in the literature. In particular, the work [15] of Imai reports the Darboux transformation of the derivative nonlinear Schrödinger equation and claims that the Darboux transformation of the MTM system (1.1) can be obtained similarly, but no details are given. Furthermore, the Coleman correspondence between the MTM system and the sine-Gordon equation is studied through the inverse scattering transform [17, 24] and this may also yield another derivation of the auto–Bäcklund transformation for the MTM system (1.1) because the auto–Bäcklund transformation of the sine–Gordon equation is well known. For our purposes, we derive the auto–Bäcklund transformation of the MTM system (1.1) by using Ricatti equations and symbolic computations.

The paper is organized as follows. Section 2 introduces the auto-Bäcklund transformation for the MTM system (1.1) and uses it to recover the MTM solitons (1.3) from zero solutions. We also list the Lorentz transformation for the MTM system (1.1) and outline the steps in the proof of Theorem 1.2. Section 3 reports details of the transformation from perturbed one-soliton solutions to small solutions at the initial time $t = 0$. Section 4 describes the transformation from small solutions to perturbed one-soliton solutions for all times $t \in \mathbb{R}$. Section 5 completes the proof of Theorem 1.2.

2. Bäcklund transformation for the MTM system

We begin by introducing the Lax pair and the auto-Bäcklund transformation of the MTM system (1.1) in the laboratory coordinates. Then, we give the Lorentz transformation for the MTM system (1.1) and outline the steps in the proof of Theorem 1.2.

The Lax pair of the MTM system (1.1) is defined in terms of the following two linear operators:

$$L = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \bar{v} \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda}\begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\sigma_3 \tag{2.1}$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \bar{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3. \tag{2.2}$$

The formal compatibility condition $\partial_t\partial_x\vec{\phi} = \partial_x\partial_t\vec{\phi}$ for the system of linear equations

$$\partial_x\vec{\phi} = L\vec{\phi} \quad \text{and} \quad \partial_t\vec{\phi} = A\vec{\phi} \tag{2.3}$$

yields the MTM system (1.1).

Note that the solution (u, v) of the MTM system (1.1) appears as coefficients of differential equations in the linear system (2.3). The auto-Bäcklund transformation relates two solutions of the MTM system (1.1) while preserving the linear system (2.3). Now let us state the auto-Bäcklund transformation.

Proposition 2.1. *Let (u, v) be a C^1 solution of the MTM system (1.1) and $\vec{\phi} = (\phi_1, \phi_2)^t$ be a C^2 nonzero solution of the linear system (2.3) associated with the potential (u, v) and the spectral parameter $\lambda = \delta e^{i\gamma/2}$. Then, the following transformation*

$$\mathbf{u} = -u\frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i\delta^{-1}\sin\gamma\bar{\phi}_1\phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} \tag{2.4}$$

and

$$\mathbf{v} = -v\frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i\delta\sin\gamma\bar{\phi}_1\phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} \tag{2.5}$$

generates a new C^1 solution $(\mathbf{u}, \mathbf{v})$ of the MTM system (1.1). Furthermore, the transformation

$$\psi_1 = \frac{\bar{\phi}_2}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}, \quad \psi_2 = \frac{\bar{\phi}_1}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|} \tag{2.6}$$

yields a new C^2 nonzero solution $\vec{\psi} = (\psi_1, \psi_2)^t$ of the linear system (2.3) associated with the new potential $(\mathbf{u}, \mathbf{v})$ and the same spectral parameter λ.

Proof. Setting $\Gamma = \phi_1/\phi_2$ in the linear system (2.3) with Lax operators (2.1) and (2.2) yields the Riccati equations

$$\begin{cases} \partial_x\Gamma = 2i(\rho_2^2 - \rho_1^2)\Gamma + \frac{i}{2}(|u|^2 - |v|^2)\Gamma + i(\rho_2 v - \rho_1 u)\Gamma^2 - i(\rho_2\bar{v} - \rho_1\bar{u}), \\ \partial_t\Gamma = 2i(\rho_2^2 + \rho_1^2)\Gamma - \frac{i}{2}(|u|^2 + |v|^2)\Gamma + i(\rho_2 v + \rho_1 u)\Gamma^2 - i(\rho_2\bar{v} + \rho_1\bar{u}), \end{cases} \tag{2.7}$$

where $\rho_1 = \frac{1}{2\lambda}$ and $\rho_2 = \frac{\lambda}{2}$. If we choose $\Gamma' := \frac{1}{\Gamma}$, $\mathbf{u} := M(\Gamma;\rho_1)f(\Gamma;u,\rho_1)$, and $\mathbf{v} := M(\Gamma;\rho_2)f(\Gamma;v,\rho_2)$ with

$$M(\Gamma;k) = -\frac{k|\Gamma|^2+\bar{k}}{\bar{k}|\Gamma|^2+k}, \quad f(\Gamma;q,k) = q + \frac{4i\mathrm{Im}(k^2)\overline{\Gamma}}{k|\Gamma|^2+\bar{k}},$$

then the Riccati equations (2.7) remain invariant in variables Γ', $\mathbf{u}$, and $\mathbf{v}$. This invariance has been checked with Wolfram's Mathematica. The transformation formulas above yield representation (2.4) and (2.5). Note that if $\vec{\phi} = \vec{0}$ at one point (x_0,t_0), then $\vec{\phi} = \vec{0}$ for all (x,t). If (u,v) is C^1 in (x,t), $\vec{\phi}$ is C^2 in (x,t), and $\vec{\phi} \neq \vec{0}$, then $(\mathbf{u},\mathbf{v})$ is C^1 for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

The validity of the transformation (2.6) has also been verified with Wolfram's Mathematica. Again, if $\vec{\phi}$ is C^2 in (x,t) and $\vec{\phi} \neq \vec{0}$, then $\vec{\psi}$ is C^2 and $\vec{\psi} \neq \vec{0}$ for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$. □

Let us denote the transformations (2.4)–(2.5) by $\mathcal{B}$, hence

$$\mathcal{B} : (u,v,\vec{\phi},\lambda) \mapsto (\mathbf{u},\mathbf{v}),$$

where $\vec{\phi}$ is a corresponding vector of the linear system (2.3) associated with the potential (u,v) and the spectral parameter λ.

In the simplest example, the MTM soliton (1.3) is recovered by the transformations (2.4) and (2.5) from the zero solution $(u,v) = (0,0)$, that is,

$$\mathcal{B} : (0,0,\vec{\phi},\lambda) \mapsto (u_\lambda, v_\lambda).$$

Indeed, a solution satisfying the linear system (2.3) with $(u,v) = (0,0)$ is given by

$$\begin{cases} \phi_1(x,t) = e^{\frac{i}{4}(\lambda^2-\lambda^{-2})x+\frac{i}{4}(\lambda^2+\lambda^{-2})t}, \\ \phi_2(x,t) = e^{-\frac{i}{4}(\lambda^2-\lambda^{-2})x-\frac{i}{4}(\lambda^2+\lambda^{-2})t}. \end{cases} \tag{2.8}$$

Substituting this expression into (2.4) and (2.5) yields $(\mathbf{u},\mathbf{v}) = (u_\lambda, v_\lambda)$ given by (1.3).

Another important example is a transformation from the MTM solitons (1.3) to the zero solution. We shall only give the explicit expressions of this transformation for the case $|\lambda| = \delta = 1$. By (2.6) and (2.8), we can find the vector $\vec{\psi}$ solving the linear system (2.3) with (u_λ, v_λ) given by (1.3). When $\lambda = e^{i\gamma/2}$, the vector $\vec{\psi}$ is given by

$$\begin{cases} \psi_1(x,t) = e^{\frac{1}{2}x\sin\gamma+\frac{i}{2}t\cos\gamma}\left|\mathrm{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|, \\ \psi_2(x,t) = e^{-\frac{1}{2}x\sin\gamma-\frac{i}{2}t\cos\gamma}\left|\mathrm{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|. \end{cases} \tag{2.9}$$

We note that $\vec{\psi}$ has exponential decay as $|x| \to \infty$ and, therefore, it is an eigenvector of the spectral problem $\partial_x\vec{\psi} = L\vec{\psi}$ for the eigenvalue $\lambda = e^{i\gamma/2}$. Substituting the eigenvector $\vec{\psi}$ into the transformation (2.4) and (2.5), we obtain the zero solution from the MTM soliton, that is,

$$\mathcal{B} : (u_\lambda, v_\lambda, \vec{\psi}, \lambda) \mapsto (0,0).$$

When $|\lambda| = \delta = 1$ for (u_λ, v_λ) given by (1.3), we realize that $c = 0$ and hence the MTM solitons (1.3) are stationary. Travelling MTM solitons with $c \neq 0$ can be recovered from the stationary MTM solitons with $c = 0$ by the Lorentz transformation. Hence, without loss of generality, we can choose $\lambda_0 = e^{i\gamma_0/2}$ for a fixed $\gamma_0 \in (0,\pi)$ in Theorem 1.2. Let us state the Lorentz transformation, which can be verified with the direct substitutions.

Proposition 2.2. *Let (u, v) be a solution of the MTM system (1.1) and let $\vec{\phi}$ be a solution of the linear system (2.3) associated with (u, v) and $\lambda = e^{i\gamma/2}$. Then,*

$$\begin{cases} u'(x,t) := \delta^{-1} u(k_1 x + k_2 t, k_1 t + k_2 x), \\ v'(x,t) := \delta v(k_1 x + k_2 t, k_1 t + k_2 x), \end{cases} \quad k_1 := \frac{\delta^2 + \delta^{-2}}{2}, \quad k_2 := \frac{\delta^2 - \delta^{-2}}{2}, \tag{2.10}$$

is a new solution of the MTM system (1.1), whereas

$$\vec{\phi}'(x,t) := \vec{\phi}(k_1 x + k_2 t, k_1 t + k_2 x), \tag{2.11}$$

is a new solution of the linear system (2.3) associated with (u', v') and $\lambda = \delta e^{i\gamma/2}$.

The stationary MTM solitons at $t = 0$ can be written by using the expressions

$$\begin{cases} u_\gamma(x) = i \sin\gamma \operatorname{sech}\left(x \sin\gamma - i\frac{\gamma}{2}\right), \\ v_\gamma(x) = -i \sin\gamma \operatorname{sech}\left(x \sin\gamma + i\frac{\gamma}{2}\right), \end{cases} \tag{2.12}$$

that depend on the parameter $\gamma \in (0, \pi)$. The time oscillation, gauge translation, and space translation can be included with the help of the transformation

$$\begin{cases} u(x,t) = e^{i\theta - it\cos\gamma} u_\gamma(x + a), \\ v(x,t) = e^{i\theta - it\cos\gamma} v_\gamma(x + a), \end{cases} \tag{2.13}$$

where $\theta, a \in \mathbb{R}$ are two translational parameters of the stationary MTM solitons.

Let us now describe our method for the proof of Theorem 1.2. First we clarify some notations: $(u_{\gamma_0}, v_{\gamma_0})$ denotes one-soliton solution given by (2.12) with a fixed $\gamma_0 \in (0, \pi)$, $\vec{\psi}_{\gamma_0}$ denotes the corresponding eigenvector given by (2.9) for $t = 0$, whereas $L(u, v, \lambda)$ and $A(u, v, \lambda)$ denote the Lax operators L and A that contain (u, v) and a spectral parameter λ.

The main steps for the proof of Theorem 1.2 are the following. First, we fix an initial data $(u_0, v_0) \in H^2(\mathbb{R})$ such that (u_0, v_0) is sufficiently close to $(u_{\gamma_0}, v_{\gamma_0})$ in L^2-norm, according to the bound (1.4).

Step 1: From a perturbed one-soliton solution to a small solution at $t = 0$. In this step, we need to study the vector solution $\vec{\psi}$ of the linear equation

$$\partial_x \vec{\psi} = L(u_0, v_0, \lambda)\vec{\psi} \quad \text{at time } t = 0. \tag{2.14}$$

In addition to proving the existence of an exponentially decaying solution $\vec{\psi}$ of the linear equation (2.14) for an eigenvalue λ, we need to prove that if (u_0, v_0) is close to $(u_{\gamma_0}, v_{\gamma_0})$ in L^2-norm, then $\vec{\psi}$ is close to $\vec{\psi}_{\gamma_0}$ in H^1-norm and λ is close to $e^{i\gamma_0/2}$. Parameter λ in bound (1.5) is now determined by the eigenvalue of the linear equation (2.14).

The earlier example of obtaining the zero solution from the one-soliton solution gives a good insight that the auto-Bäcklund transformation given by Proposition 2.1 produces a function (p_0, q_0) at $t = 0$,

$$\mathcal{B} : (u_0, v_0, \vec{\psi}, \lambda) \mapsto (p_0, q_0), \tag{2.15}$$

such that (p_0, q_0) is small in L^2-norm. Moreover, if $(u_0, v_0) \in H^2(\mathbb{R})$, then $(p_0, q_0) \in H^2(\mathbb{R})$.

Step 2: Time evolution of the transformed solution. By the standard well-posedness theory for Dirac equations [11, 25, 28], there exists a unique global solution $(p, q) \in C(\mathbb{R}; H^2(\mathbb{R}))$ to the MTM system (1.1) such that $(p,q)|_{t=0} = (p_0,q_0)$. Thanks to the L^2-conservation (1.2), the solution $(p(\cdot,t),q(\cdot,t))$ remains small in the L^2-norm for every $t \in \mathbb{R}$.

Step 3: From a small solution to a perturbed one-soliton solution for all times $t \in \mathbb{R}$. In this step, we are interested in the existence problem of the vector function $\vec{\phi}$ that solves the linear system

$$\partial_x \vec{\phi} = L(p,q,\lambda)\vec{\phi}, \quad \partial_t \vec{\phi} = A(p,q,\lambda)\vec{\phi} \tag{2.16}$$

where $(p,q) \in C(\mathbb{R}; H^2(\mathbb{R}))$ is the unique global solution to the MTM system (1.1) starting with the initial data $(p,q)|_{t=0} = (p_0,q_0)$ in $H^2(\mathbb{R})$. Using the vector $\vec{\phi}$ and the auto-Bäcklund transformation given by Proposition 2.1, we obtain a new solution (u,v) to the MTM system (1.1),

$$\mathcal{B} : (p,q,\vec{\phi},\lambda) \mapsto (u,v). \tag{2.17}$$

Moreover, if $(p,q) \in C(\mathbb{R}; H^2(\mathbb{R}))$, then $(u,v) \in C(\mathbb{R}; H^2(\mathbb{R}))$. Some translational parameter a and θ arise as functions of time t in the construction of the most general solution of the linear equation $\partial_x \vec{\phi} = L(p,q,\lambda)\vec{\phi}$ in the system (2.16). Bound (1.6) on the solution (u,v) is found from the analysis of the auto–Bäcklund transformation (2.17).

To summarize, there are three key ingredients in our method: mapping of an L^2-neighborhood of the one-soliton solution to that of the zero solution at $t = 0$, the L^2-conservation of the MTM system, and mapping of an L^2-neighborhood of the zero solution to that of the one-soliton solution for every $t \in \mathbb{R}$. As a result, if the initial data is sufficiently close to the one-soliton solution in L^2 according to the initial bound (1.4), then the solution of the MTM system remains close to the one-soliton solution in L^2 for all times according to the final bound (1.6). A schematic picture is as follows:

$$\begin{array}{ccc} (u_0,v_0) & \cdots\cdots\cdots & (u,v) \\ \Big\downarrow \mathcal{B} & & \mathcal{B} \Big\uparrow \\ (p_0,q_0) & \longrightarrow & (p,q) \end{array}$$

Finally, we can remove the technical assumption that $(u_0,v_0) \in H^2(\mathbb{R})$ by an approximation argument in $L^2(\mathbb{R})$. This is possible because the MTM system (1.1) is globally well-posed in $L^2(\mathbb{R})$ by Theorem 1.1, whereas the bounds (1.5) and (1.6) are found to be uniform for the sequence of approximating solutions of the MTM system (1.1), the initial data of which approximate (u_0,v_0) in $L^2(\mathbb{R})$.

We note that the solution (p,q) to the MTM system (1.1) in a L^2-neighborhood of the zero solution could contain some L^2-small MTM solitons, which are related to the discrete spectrum of the spectral problem (2.14). Sufficient conditions for the absence of the discrete spectrum were derived in [25], and the L^2 smallness of the initial data is not generally sufficient for excluding eigenvalues of the discrete spectrum. If the small solitons occur in the Cauchy problem associated with the MTM system (1.1), asymptotic decay of solutions (u,v) to the MTM solitons given by (1.3) cannot be proved, in other words, (p,q) do not decay to $(0,0)$ in L^∞-norm as $t \to \infty$. Therefore, a more restrictive hypothesis on the initial data is generally needed to establish asymptotic stability of MTM solitons. See [10] for restrictions on initial data of the cubic NLS equation required in the proof of asymptotic stability of NLS solitons.

We also note that modulation equations for parameters a and θ in Theorem 1.2 are not included in our method. This can be viewed as an advantage of the auto-Bäcklund transformation, which does not rely on the global control of the dynamics of a and θ by means of the modulation equations. Values of a and θ are related to arbitrary constants that appear in the construction of $\vec{\phi}$ as a solution of the linear equation $\partial_x\vec{\phi} = L(p, q, \lambda)\vec{\phi}$ in the system (2.16). These values are eliminated in the infimum norm stated in the orbital stability result (1.6) in Theorem 1.2.

3. From a perturbed one-soliton solution to a small solution

Here we use the auto-Bäcklund transformation given by Proposition 2.1 to transform a L^2-neighborhood of the one-soliton solution to that of the zero solution at $t = 0$. Let $(u_0, v_0) \in L^2(\mathbb{R})$ be the initial data of the MTM system (1.1) satisfying bound (1.4) for $\lambda_0 = e^{i\gamma_0/2}$. Let $\vec{\psi}$ be a decaying eigenfunction of the spectral problem

$$\partial_x\vec{\psi} = L(u_0, v_0, \lambda)\vec{\psi}, \tag{3.1}$$

for an eigenvalue λ. First, we show that under the condition (1.4), an eigenvector $\vec{\psi}$ always exists and λ is close to λ_0. Then, we write $\lambda = \delta e^{i\gamma/2}$ and define

$$p_0 := -u_0\frac{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2}{e^{i\gamma/2}|\psi_1|^2 + e^{-i\gamma/2}|\psi_2|^2} + \frac{2i\delta^{-1}\sin\gamma\,\bar{\psi}_1\psi_2}{e^{i\gamma/2}|\psi_1|^2 + e^{-i\gamma/2}|\psi_2|^2} \tag{3.2}$$

and

$$q_0 := -v_0\frac{e^{i\gamma/2}|\psi_1|^2 + e^{-i\gamma/2}|\psi_2|^2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2} - \frac{2i\delta\sin\gamma\,\bar{\psi}_1\psi_2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2}. \tag{3.3}$$

We intend to show that (p_0, q_0) is small in L^2 norm.

When $(u_0, v_0) = (u_{\gamma_0}, v_{\gamma_0})$, the spectral problem (3.1) has exactly one decaying eigenvector $\vec{\psi}$ given by

$$\begin{cases} \psi_1(x) = e^{\frac{1}{2}x\sin\gamma_0}\left|\operatorname{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|, \\ \psi_2(x) = e^{-\frac{1}{2}x\sin\gamma_0}\left|\operatorname{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|, \end{cases} \tag{3.4}$$

which corresponds to the eigenvalue $\lambda = \lambda_0 = e^{i\gamma_0/2}$. The other linearly independent solution $\vec{\xi}$ of the spectral problem (3.1) with $\lambda = \lambda_0$ is given by

$$\begin{cases} \xi_1(x) = e^{\frac{1}{2}x\sin\gamma_0}(e^{-2x\sin\gamma_0} - x\sin(2\gamma_0))\left|\operatorname{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|, \\ \xi_2(x) = -e^{-\frac{1}{2}x\sin\gamma_0}(e^{2x\sin\gamma_0} + 2\cos\gamma_0 + x\sin(2\gamma_0))\left|\operatorname{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|. \end{cases} \tag{3.5}$$

This solution grows exponentially as $|x| \to \infty$. Therefore, $\dim\ker(\partial_x - L(u_{\gamma_0}, v_{\gamma_0}, \lambda_0)) = 1$. For clarity, we denote the decaying eigenvector (3.4) by $\vec{\psi}_{\gamma_0}$.

When (u_0, v_0) is close to $(u_{\gamma_0}, v_{\gamma_0})$ in L^2-norm, we would like to construct a decaying solution $\vec{\psi}$ of the spectral problem (3.1), which is close to the eigenvector $\vec{\psi}_{\gamma_0}$. This is achieved in Lemma 3.1 below. To simplify analysis, we introduce a unitary transformation in the linear equation (3.1),

$$\vec{\psi} = \begin{bmatrix} f & 0 \\ 0 & \bar{f} \end{bmatrix} \vec{\phi}, \tag{3.6}$$

where $f(x) = e^{\frac{i}{4}\int_0^x (|u_0|^2 - |v_0|^2)dx}$ is well defined for any $(u_0, v_0) \in L^2(\mathbb{R})$. Then, the linear equation (3.1) becomes

$$\partial_x \vec{\phi} = M(u_0, v_0, \lambda)\vec{\phi}, \tag{3.7}$$

where

$$M(u_0, v_0, \lambda) := \frac{i}{4}\begin{bmatrix} \lambda^2 - \lambda^{-2} & 2(\bar{u}_0\lambda^{-1} - \bar{v}_0\lambda)\bar{f}^2 \\ 2(u_0\lambda^{-1} - v_0\lambda)f^2 & \lambda^{-2} - \lambda^2 \end{bmatrix}.$$

The following lemma gives the main result of the perturbation theory. Below, $A \lesssim B$ means that there exists a positive ϵ-independent constant C such that $A \leq CB$ for all sufficiently small ϵ.

Lemma 3.1. *For a fixed $\lambda_0 = e^{i\gamma_0/2}$ with $\gamma_0 \in (0, \pi)$, there exists a real positive ϵ such that if*

$$\|u_0 - u_{\gamma_0}\|_{L^2} + \|v_0 - v_{\gamma_0}\|_{L^2} \leq \epsilon, \tag{3.8}$$

then there exists an eigenvector $\vec{\psi} \in H^1(\mathbb{R}; \mathbb{C}^2)$ of the spectral problem (3.1) for an eigenvalue $\lambda \in \mathbb{C}$ such that

$$|\lambda - \lambda_0| + \|\vec{\psi} - \vec{\psi}_{\gamma_0}\|_{H^1} \lesssim \|u_0 - u_{\gamma_0}\|_{L^2} + \|v_0 - v_{\gamma_0}\|_{L^2}. \tag{3.9}$$

Proof. We divide the proof into four steps that accomplish the method of Lyapunov–Schmidt reductions. Step 1 is a set-up for the perturbation theory under the condition (3.8). Step 2 splits the problem into two parts by appropriate projections. In Step 3, we solve the first part of the problem by using the implicit function theorem. In Step 4, we solve the residual equation that determines uniquely $\lambda \in \mathbb{C}$ and $\vec{\psi} \in H^1(\mathbb{R}; \mathbb{C}^2)$ satisfying bound (3.9).

Step 1. Set $u_0 = u_{\gamma_0} + u_s$ and $v_0 = v_{\gamma_0} + v_s$, where $(u_s, v_s) \in L^2(\mathbb{R})$ are remainder terms, which are $\mathcal{O}(\epsilon)$ small in L^2 norm, according to the bound (3.8). We expand $1/\lambda^2$ and $1/\lambda$ around λ_0 in Taylor series. Using the fact $|u_{\gamma_0}| = |v_{\gamma_0}|$, we also expand $u_0 f^2$ and $v_0 f^2$ in Taylor series, e.g.

$$u_0 f^2 = u_0 e^{\frac{i}{2}\int_0^x (|u_0|^2 - |v_0|^2)dx} = (u_{\gamma_0} + u_s)\left(1 + g + \frac{1}{2}g^2 + \mathcal{O}(g^3)\right), \tag{3.10}$$

where

$$g := i\int_0^x \mathrm{Re}\left(u_s \bar{u}_{\gamma_0} - v_s \bar{v}_{\gamma_0}\right) dx + \frac{i}{2}\int_0^x (|u_s|^2 - |v_s|^2)dx.$$

Note that g is well defined for $(u_s, v_s) \in L^2(\mathbb{R})$. From these expansions, the linear equation (3.7) becomes

$$(\partial_x - M_{\gamma_0})\vec{\phi} = \Delta M \vec{\phi}, \tag{3.11}$$

where

$$M_{\gamma_0} = M(u_{\gamma_0}, v_{\gamma_0}, \lambda_0) = \frac{1}{2}\begin{bmatrix} -\sin\gamma_0 & i(e^{-i\gamma_0/2}\bar{u}_{\gamma_0} - e^{i\gamma_0/2}\bar{v}_{\gamma_0}) \\ i(e^{-i\gamma_0/2}u_{\gamma_0} - e^{i\gamma_0/2}v_{\gamma_0}) & \sin\gamma_0 \end{bmatrix}$$

and the perturbation term ΔM applied to any $\vec{\phi} \in H^1(\mathbb{R})$ satisfies the inequality

$$\|\Delta M\vec{\phi}\|_{L^2} \lesssim (|\lambda - \lambda_0| + \|u_s\|_{L^2} + \|v_s\|_{L^2})\|\phi\|_{H^1}, \tag{3.12}$$

thanks to the embedding of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Note that the bound (3.12) cannot be derived in the context of the spectral problem (3.1) without the unitary transformation (3.6), which removes the term $\frac{i}{4}(|u|^2 - |v|^2)\sigma_3$ from the operator L in (2.1). This explains a posteriori why we are using the technical transformation (3.6).

We will later need the explicit computation of the leading order part in the perturbation term ΔM with respect to $(\lambda - \lambda_0)$, that is,

$$\Delta M = \frac{i}{2}(\lambda - \lambda_0)\begin{bmatrix} (\lambda_0 + \lambda_0^{-3}) & -(\bar{u}_{\gamma_0}\lambda_0^{-2} + \bar{v}_{\gamma_0}) \\ -(u_{\gamma_0}\lambda_0^{-2} + v_{\gamma_0}) & -(\lambda_0 + \lambda_0^{-3}) \end{bmatrix} + \mathcal{O}((\lambda - \lambda_0)^2, \|u_s\|_{L^2}, \|v_s\|_{L^2}). \tag{3.13}$$

Step 2. We aim to construct an appropriate projection operator by which we split the linear equation (3.11) into two parts. Recall that $\dim\ker(\partial_x - M_{\gamma_0}) = 1$ and let $\vec{\phi}_{\gamma_0} \in \ker(\partial_x - M_{\gamma_0})$ and $\vec{\eta}_{\gamma_0} \in \ker(\partial_x + M^*_{\gamma_0})$. These null vectors can be obtained explicitly:

$$\vec{\phi}_{\gamma_0}(x) = \begin{bmatrix} e^{\frac{1}{2}x\sin\gamma_0} \\ e^{-\frac{1}{2}x\sin\gamma_0} \end{bmatrix}\left|\operatorname{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|, \quad \vec{\eta}_{\gamma_0}(x) = \begin{bmatrix} e^{-\frac{1}{2}x\sin\gamma_0} \\ -e^{\frac{1}{2}x\sin\gamma_0} \end{bmatrix}\left|\operatorname{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|.$$

We note that $\langle\vec{\eta}_{\gamma_0}, \vec{\phi}_{\gamma_0}\rangle_{L^2} = 0$ but $\langle\sigma_3\vec{\eta}_{\gamma_0}, \vec{\phi}_{\gamma_0}\rangle_{L^2} \neq 0$, where $\sigma_3 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Also note that $\vec{\phi}_{\gamma_0} = \vec{\psi}_{\gamma_0}$ given by (3.4) because $|u_{\gamma_0}| = |v_{\gamma_0}|$.

Let us make the following decomposition:

$$\vec{\phi} = \vec{\phi}_{\gamma_0} + \vec{\phi}_s, \tag{3.14}$$

where $\vec{\phi}_s$ is defined uniquely from the normalization condition $\langle\sigma_3\vec{\eta}_{\gamma_0}, \vec{\phi}\rangle_{L^2} = \langle\sigma_3\vec{\eta}_{\gamma_0}, \vec{\phi}_{\gamma_0}\rangle_{L^2}$, which yields the orthogonality condition $\langle\sigma_3\vec{\eta}_{\gamma_0}, \vec{\phi}_s\rangle_{L^2} = 0$. Then we introduce the projection operator $P_{\gamma_0} : L^2(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2) \cap \operatorname{span}\{\sigma_3\vec{\eta}_{\gamma_0}\}^\perp$ defined by

$$P_{\gamma_0}\vec{\phi} = \vec{\phi} - \frac{\langle\sigma_3\vec{\eta}_{\gamma_0}, \vec{\phi}\rangle_{L^2}}{\langle\sigma_3\vec{\eta}_{\gamma_0}, \vec{\phi}_{\gamma_0}\rangle_{L^2}}\vec{\phi}_{\gamma_0}.$$

Note that $P_{\gamma_0}\vec{\phi}_s = \vec{\phi}_s$ and $P_{\gamma_0}\vec{\phi}_{\gamma_0} = \vec{0}$.

From equations (3.11) and (3.14), we define the operator equation

$$F(\vec{\phi}_s, u_s, v_s, \lambda) := (\partial_x - M_{\gamma_0})\vec{\phi}_s - \Delta M(\vec{\phi}_{\gamma_0} + \vec{\phi}_s) = 0. \tag{3.15}$$

Clearly, since $\dim\ker(\partial_x - M_{\gamma_0}) = 1 \neq 0$, the Fréchet derivative $D_{\vec{\phi}_s}F(0, 0, 0, \lambda_0) = \partial_x - M_{\gamma_0}$ has no bounded inverse. Let $\widehat{P}_{\gamma_0} = \sigma_3 P_{\gamma_0}\sigma_3$ and notice that

$$\widehat{P}_{\gamma_0} : L^2(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2) \cap \operatorname{span}\{\vec{\eta}_{\gamma_0}\}^\perp.$$

We decompose equation (3.15) by the projection $\widehat{P}_{\gamma_0}$ into two equations

$$G(\vec{\phi}_s, u_s, v_s, \lambda) := \widehat{P}_{\gamma_0}F(\vec{\phi}_s, u_s, v_s, \lambda) = 0, \tag{3.16}$$

$$H(\vec{\phi}_s, u_s, v_s, \lambda) := (I - \widehat{P}_{\gamma_0})F(\vec{\phi}_s, u_s, v_s, \lambda) = 0. \tag{3.17}$$

Step 3. First, we note that since $\dim\ker(\partial_x - M_{\gamma_0}) = \dim\ker(\partial_x + M^*_{\gamma_0}) = 1 < \infty$, then $\partial_x - M_{\gamma_0}$ is a Fredholm operator of index zero. Observe that Range(G)

$= L^2(\mathbb{R};\mathbb{C}^2)\cap \text{span}\{\vec{\eta}_{\gamma_0}\}^\perp$, where $\vec{\eta}_{\gamma_0} \in \ker\{\partial_x + M^*_{\gamma_0}\}$. By the Fredholm alternative theorem, $D_{\vec{\phi}_s} G(\vec{0},0,0,\lambda_0) = \widehat{P}_{\gamma_0} D_{\vec{\phi}_s} F(\vec{0},0,0,\lambda_0) = \widehat{P}_{\gamma_0}(\partial_x - M_{\gamma_0})P_{\gamma_0}$ has a bounded inverse operator given by

$$P_{\gamma_0}(\partial_x - M_{\gamma_0})^{-1}\widehat{P}_{\gamma_0} : L^2(\mathbb{R};\mathbb{C}^2)\cap \text{span}\{\vec{\eta}_{\gamma_0}\}^\perp \to H^1(\mathbb{R};\mathbb{C}^2)\cap \text{span}\{\sigma_3\vec{\eta}_{\gamma_0}\}^\perp. \tag{3.18}$$

Next we claim that for some $(u_s, v_s) \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{C}$, there exists a unique $\vec{\phi}_* \in H^1(\mathbb{R};\mathbb{C}^2)$ such that $G(\vec{\phi}_*, u_s, v_s, \lambda) = 0$. This can be done by the implicit function theorem. The function

$$G : H^1(\mathbb{R};\mathbb{C}^2) \times L^2(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C} \to L^2(\mathbb{R};\mathbb{C}^2)\cap \text{span}\{\vec{\eta}_{\gamma_0}\}^\perp$$

is C^1 in u_s, v_s (and their complex conjugates), λ, and $\vec{\phi}_s$. We also find that $G(\vec{0},0,0,\lambda_0) = 0$ and the derivative $D_{\vec{\phi}_s} G(\vec{0},0,0,\lambda_s)$ is invertible with the bounded inverse (3.18). For some $\epsilon, \rho > 0$, let

$$U_\epsilon := \{(u_s, v_s, \lambda) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{C} : \quad \|u_s\|_{L^2} + \|v_s\|_{L^2} + |\lambda - \lambda_0| < \epsilon\}$$

and

$$V_\rho := \{\vec{\phi}_s \in H^1(\mathbb{R};\mathbb{C}^2)\cap \text{span}\{\sigma_3\vec{\eta}_{\gamma_0}\}^\perp : \quad \|\vec{\phi}_s\|_{H^1} \le \rho\}.$$

Then, by the implicit function theorem, for sufficiently small $\epsilon, \rho > 0$, and for each $(u_s, v_s, \lambda) \in U_\epsilon$, there exists a unique $\vec{\phi}_* \in V_\rho$ such that $G(\vec{\phi}_*, u_s, v_s, \lambda) = 0$.

A unique element $\vec{\phi}_*$ depends implicitly on (u_s, v_s, λ), that is, we can write $\vec{\phi}_* := \vec{\phi}_*(u_s, v_s, \lambda)$. From equations (3.15) and (3.16), we have

$$(I - P_{\gamma_0}(\partial_x - M_{\gamma_0})^{-1}\widehat{P}_{\gamma_0}\Delta M)\vec{\phi}_* = P_{\gamma_0}(\partial_x - M_{\gamma_0})^{-1}\widehat{P}_{\gamma_0}\Delta M\vec{\phi}_{\gamma_0} \tag{3.19}$$

and from boundedness of the inverse operator given by (3.18) and inequality (3.12), we obtain

$$\begin{aligned}\|\vec{\phi}_*\|_{H^1} &\lesssim \|P_{\gamma_0}(\partial_x - M_{\gamma_0})^{-1}\widehat{P}_{\gamma_0}\Delta M\vec{\phi}_{\gamma_0}\|_{H^1} \lesssim \|\Delta M\vec{\phi}_{\gamma_0}\|_{L^2} \\ &\lesssim |\lambda - \lambda_0| + \|u_s\|_{L^2} + \|v_s\|_{L^2},\end{aligned} \tag{3.20}$$

if $(u_s, v_s, \lambda) \in U_\epsilon$.

Step 4. Lastly we address the bifurcation equation (3.17) to determine $\lambda \in \mathbb{C}$. From equations (3.15) and (3.17), the bifurcation equation can be written explicitly as

$$I(u_s, v_s, \lambda) := \langle \vec{\eta}_{\gamma_0}, \Delta M(\vec{\phi}_{\gamma_0} + \vec{\phi}_*(u_s, v_s, \lambda))\rangle_{L^2} = 0, \tag{3.21}$$

where $\vec{\phi}_*(u_s, v_s, \lambda)$ is uniquely expressed from (3.19) if $(u_s, v_s, \lambda) \in U_\epsilon$. It follows from (3.12) and (3.20) that $I(0,0,\lambda_0) = 0$.

By using the explicit expression (3.13), we check that $s := \partial_\lambda I(0,0,\lambda_0) \neq 0$, where

$$\begin{aligned}s &= \frac{i}{2}\langle \vec{\eta}_{\gamma_0}, \begin{bmatrix} (\lambda_0 + \lambda_0^{-3}) & -(\bar{u}_{\gamma_0}\lambda_0^{-2} + \bar{v}_{\gamma_0}) \\ -(u_{\gamma_0}\lambda_0^{-2} + v_{\gamma_0}) & -(\lambda_0 + \lambda_0^{-3}) \end{bmatrix} \vec{\phi}_{\gamma_0}\rangle_{L^2} \\ &= ie^{-i\gamma_0/2}\int_{\mathbb{R}} \left(2\cos\gamma_0 \left|\text{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|^2 + \sin^2\gamma_0 \left|\text{sech}\left(x\sin\gamma_0 - i\frac{\gamma_0}{2}\right)\right|^4 \right) dx \\ &= 4ie^{-i\gamma_0/2}\int_{\mathbb{R}} \frac{1 + \cos\gamma_0\cosh(2x\sin\gamma_0)}{(\cosh(2x\sin\gamma_0) + \cos\gamma_0)^2} dx \\ &= \frac{4ie^{-i\gamma_0/2}}{\sin\gamma_0}.\end{aligned}$$

As a result, equation (3.21) can be used to uniquely determine the spectral parameter λ if $(u_s, v_s, \lambda) \in U_\epsilon$. From inequalities (3.12) and (3.20), we obtain that this λ satisfies the bound

$$|\lambda - \lambda_0| \lesssim \|u_s\|_{L^2} + \|v_s\|_{L^2}. \tag{3.22}$$

With inequalities (3.20) and (3.22), the proof of Lemma 3.1 is complete. □

Remark 1. A spectral parameter λ in Lemma 3.1 may not be on the unit circle $|\lambda| = 1$ even if $\lambda_0 = e^{i\gamma_0/2}$ is on the unit circle. In what follows, we develop the theory when λ occurs on the unit circle, hence we write $\lambda = e^{i\gamma/2}$ for some $\gamma \in (0, \pi)$. All results obtained below can be generalized to the case of $|\lambda| \neq 1$ by using the Lorentz transformation in Proposition 2.2.

In Lemma 3.4 below, we will show that a solution $\vec{\phi}$ determined in the proof of Lemma 3.1 can be written explicitly as the perturbed solution around $\vec{\phi}_\gamma$ in suitable function spaces. Then, in Lemma 3.6 below, we will use this representation and the auto-Bäcklund transformation (3.2) and (3.3) to show that (p_0, q_0) is small in L^2 norm.

To develop this analysis, we first prove several technical results. Let $(u, v) = (u_\gamma, v_\gamma)$, $\lambda = e^{i\gamma/2}$ and consider the linear inhomogeneous equation

$$(\partial_x - M_\gamma)\vec{w} = \vec{f}, \tag{3.23}$$

where

$$M_\gamma = \frac{1}{2}\begin{bmatrix} -\sin\gamma & i(e^{-i\gamma/2}\bar{u}_\gamma - e^{i\gamma/2}\bar{v}_\gamma) \\ i(e^{-i\gamma/2}u_\gamma - e^{i\gamma/2}v_\gamma) & \sin\gamma \end{bmatrix}.$$

We introduce Banach spaces $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ such that for $\vec{w} = (w_1, w_2)^t \in X$ and $\vec{f} = (f_1, f_2)^t \in Y$, we have

$$\|\vec{w}\|_X := \|w_1\|_{X_1} + \|w_2\|_{X_2}, \quad \|\vec{f}\|_Y := \|f_1\|_{Y_1} + \|f_2\|_{Y_2},$$

where

$$\begin{aligned}
\|w_1\|_{X_1} := \inf_{w_1 = v_1 + u_1} \Bigg(& \left\| v_1 e^{-\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^\infty_x} \\
& + \left\| u_1 e^{\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^2_x \cap L^\infty_x} \Bigg), \\
\|w_2\|_{X_2} := \inf_{w_2 = v_2 + u_2} \Bigg(& \left\| v_2 e^{\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^\infty_x} \\
& + \left\| u_2 e^{-\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^2_x \cap L^\infty_x} \Bigg)
\end{aligned}$$

and

$$\begin{aligned}
\|f_1\|_{Y_1} := \inf_{f_1 = g_1 + h_1} \Bigg(& \left\| g_1 e^{\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^2_x} \\
& + \left\| h_1 e^{-\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^2_x \cap L^1_x} \Bigg),
\end{aligned}$$

$$\|f_2\|_{Y_2} := \inf_{f_2=g_2+h_2} \left(\left\| g_2 e^{-\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^2_x} \right.$$
$$\left. + \left\| h_2 e^{\frac{x}{2}\sin\gamma} \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right| \right\|_{L^2_x \cap L^1_x} \right).$$

It is obvious that X and Y are continuously embedded into $L^2(\mathbb{R})$. We shall estimate the bound of the operator $P_\gamma(\partial_x - M_\gamma)^{-1}\widehat{P}_\gamma : Y \to X$, where projection operators P_γ and $\widehat{P}_\gamma$ are defined in the proof of Lemma 3.1. First, we will obtain an explicit solution $\vec{w} \in H^1(\mathbb{R};\mathbb{C}^2) \cap \text{span}\{\sigma_3\vec{\eta}_\gamma\}^\perp$ for the linear inhomogeneous equation (3.23) when $\vec{f} \in L^2(\mathbb{R};\mathbb{C}^2) \cap \ker(\partial_x + M_\gamma^*)^\perp$. Then, we will prove that the mapping $Y \ni \vec{f} \mapsto \vec{w} \in X$ is bounded. These goals are achieved in the next two lemmas.

Lemma 3.2. *For any $\vec{f} = (f_1, f_2)^t \in L^2(\mathbb{R};\mathbb{C}^2) \cap \text{span}\{\vec{\eta}_\gamma\}^\perp$, there exists a unique solution $\vec{w} \in H^1(\mathbb{R};\mathbb{C}^2) \cap \text{span}\{\sigma_3\vec{\eta}_\gamma\}^\perp$ of the inhomogeneous equation (3.23) that can be written as*

$$\vec{w}(x) = \frac{1}{4}\vec{\phi}_\gamma(x)\left[k(\vec{f}) + W_-(x) + W_+(x)\right] + \frac{1}{4}\vec{\xi}_\gamma(x)\int_{-\infty}^{x} \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy, \tag{3.24}$$

where

$$W_-(x) := \int_{-\infty}^{x} e^{-\frac{1}{2}y\sin\gamma}(e^{2y\sin\gamma} + 2\cos\gamma + y\sin(2\gamma)) \left|\operatorname{sech}\left(y\sin\gamma - i\frac{\gamma}{2}\right)\right| f_1(y)dy,$$
$$W_+(x) := \int_{x}^{\infty} e^{-\frac{3}{2}y\sin\gamma}(-1 + e^{2y\sin\gamma} y\sin(2\gamma)) \left|\operatorname{sech}\left(y\sin\gamma - i\frac{\gamma}{2}\right)\right| f_2(y)dy,$$

and $k(\vec{f})$ is a continuous linear functional on $L^2(\mathbb{R};\mathbb{C}^2)$.

Proof. Since $\partial_x - M_\gamma : H^1(\mathbb{R};\mathbb{C}^2) \to L^2(\mathbb{R};\mathbb{C}^2)$ is a Fredholm operator of index zero and $\ker(\partial_x + M_\gamma^*) = \text{span}\{\vec{\eta}_\gamma\}$, the inhomogeneous equation (3.23) has a solution in $H^1(\mathbb{R};\mathbb{C}^2)$ if and only if $\vec{f} \in L^2(\mathbb{R};\mathbb{C}^2) \cap \text{span}\{\vec{\eta}_\gamma\}^\perp$. For uniqueness, we add the constraint $\vec{w} \in \text{span}\{\sigma_3\vec{\eta}_\gamma\}^\perp$.

Recall that $U = [\vec{\phi}_\gamma, \vec{\xi}_\gamma]$ is a fundamental matrix of the homogeneous equation $(\partial_x - M_\gamma)U = 0$ and $\vec{\eta}_\gamma$ is a decaying solution of $(\partial_x + M_\gamma^*)\vec{\eta} = \vec{0}$. All functions are known explicitly as

$$\vec{\phi}_\gamma(x) = \begin{bmatrix} e^{\frac{1}{2}x\sin\gamma} \\ e^{-\frac{1}{2}x\sin\gamma} \end{bmatrix} Q(x), \quad \vec{\eta}_\gamma(x) = \begin{bmatrix} e^{-\frac{1}{2}x\sin\gamma} \\ -e^{\frac{1}{2}x\sin\gamma} \end{bmatrix} Q(x),$$

and

$$\vec{\xi}_\gamma(x) = \begin{bmatrix} e^{\frac{1}{2}x\sin\gamma}(e^{-2x\sin\gamma} - x\sin(2\gamma)) \\ -e^{-\frac{1}{2}x\sin\gamma}(e^{2x\sin\gamma} + 2\cos\gamma + x\sin(2\gamma)) \end{bmatrix} Q(x),$$

where

$$Q(x) := \left|\operatorname{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|.$$

From variation of parameters, we have the explicit representation (3.24), where $k(\vec{f})$ is the constant of integration and the other constant is set to zero to ensure that $\vec{w} \in H^1(\mathbb{R};\mathbb{C}^2)$. It remains to prove that every term in the explicit expression (3.24) belongs to $L^2(\mathbb{R};\mathbb{C}^2)$.

Since $|\vec{\phi}_\gamma(x)| \lesssim e^{-\frac{|x|}{2}\sin\gamma}$ and $|Q(x)| \lesssim e^{-|x|\sin\gamma}$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\|W_-\vec{\phi}_\gamma\|_{L^2}\\
&\quad\lesssim \left\| e^{-\frac{1}{2}|x|\sin\gamma} \int_{-\infty}^{x} e^{-\frac{1}{2}y\sin\gamma}(e^{2y\sin\gamma} + 2\cos\gamma + y\sin(2\gamma))Q(y)f_1(y)dy \right\|_{L^2_x}\\
&\quad\lesssim \left\| \int_{-\infty}^{x} e^{\frac{1}{2}(y-x)\sin\gamma}|f_1(y)|dy \right\|_{L^2_x} + \left\| e^{-\frac{1}{2}|x|\sin\gamma} \int_{-\infty}^{x} e^{-\frac{1}{2}|y|\sin\gamma}(2+|y|)|f_1(y)|dy \right\|_{L^2_x}\\
&\quad\lesssim \|\vec{f}\|_{L^2}.
\end{aligned}$$

and

$$\begin{aligned}
\|W_+\vec{\phi}_\gamma\|_{L^2} &\lesssim \left\| e^{-\frac{1}{2}|x|\sin\gamma} \int_{x}^{\infty} e^{-\frac{3}{2}y\sin\gamma}(-1 + e^{2y\sin\gamma} y\sin(2\gamma))Q(y)f_2(y)dy \right\|_{L^2_x}\\
&\lesssim \left\| \int_{x}^{\infty} e^{\frac{1}{2}(x-y)\sin\gamma}|f_2(y)|dy \right\|_{L^2_x} + \left\| e^{-\frac{1}{2}|x|\sin\gamma} \int_{x}^{\infty} e^{-\frac{1}{2}|y|\sin\gamma}|y||f_2(y)|dy \right\|_{L^2_x}\\
&\lesssim \|\vec{f}\|_{L^2},
\end{aligned}$$

where notation $\|f(x)\|_{L^2_x}$ is used in place of $\|f(\cdot)\|_{L^2}$. Since $\vec{f} \in L^2(\mathbb{R};\mathbb{C}^2) \cap \text{span}\{\vec{\eta}_\gamma\}^\perp$, then

$$\int_{x}^{\infty} \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy = -\int_{-\infty}^{x} \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy.$$

Using this equality, we can estimate the last term in the explicit expression (3.24) as follows

$$\begin{aligned}
&\left\| \vec{\xi}_\gamma(x) \int_{-\infty}^{x} \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy \right\|_{L^2_x}\\
&\quad\lesssim \left\| e^{-\frac{1}{2}x\sin\gamma} \int_{-\infty}^{x} \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy \right\|_{L^2_x} + \left\| e^{\frac{1}{2}x\sin\gamma} \int_{x}^{\infty} \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy \right\|_{L^2_x}\\
&\quad\lesssim \left\| \int_{-\infty}^{x} e^{\frac{1}{2}(y-x)\sin\gamma}|\vec{f}(y)|dy \right\|_{L^2_x} + \left\| \int_{x}^{\infty} e^{-\frac{1}{2}(y-x)\sin\gamma}|\vec{f}(y)|dy \right\|_{L^2_x}\\
&\quad\lesssim \|\vec{f}\|_{L^2},
\end{aligned}$$

where $|\vec{f}|$ is the vector norm of the 2-vector $\vec{f}$. Since $\langle\sigma_3\vec{\eta}_\gamma, \vec{\phi}_\gamma\rangle_{L^2} \neq 0$, $k(\vec{f})$ is uniquely determined from the orthogonality condition $\langle\sigma_3\vec{\eta}_\gamma, \vec{w}\rangle_{L^2} = 0$. Since all other terms in (3.24) are in $L^2(\mathbb{R};\mathbb{C}^2)$, $k(\vec{f})$ is bounded for all $\vec{f} \in L^2(\mathbb{R};\mathbb{C}^2)$. Therefore, $k(\vec{f})$ is a continuous linear functional on $L^2(\mathbb{R};\mathbb{C}^2)$. □

Lemma 3.3. *Let $\vec{f} \in Y \cap \text{span}\{\vec{\eta}_\gamma\}^\perp$ and let $\vec{w}$ be a solution of the inhomogeneous equation (3.23) in Lemma 3.2. Then there is a $\vec{f}$-independent constant $C > 0$ such that $\|\vec{w}\|_X \leq C\|\vec{f}\|_Y$.*

Proof. The solution $\vec{w}$ is given by the explicit formula (3.24). We assume now that $\vec{f}$ belongs to the exponentially weighted space Y and prove that $\vec{w}$ belongs to the exponentially weighted space X.

Since $\|a\vec{\phi}_\gamma\|_X \leq 2\|a\|_{L^\infty}$ for any $a \in L^\infty(\mathbb{R})$, $k(\vec{f})$ is a continuous linear functional on $L^2(\mathbb{R};\mathbb{C}^2)$, and Y is embedded into $L^2(\mathbb{R};\mathbb{C}^2)$, we have

$$\|k(\vec{f})\vec{\phi}_\gamma\|_X \lesssim |k(\vec{f})| \lesssim \|\vec{f}\|_{L^2} \lesssim \|\vec{f}\|_Y.$$

The second term in (3.24) is estimated by

$$\begin{aligned}\|W_-\vec{\phi}_\gamma\|_X &\lesssim \left\|\int_{-\infty}^x e^{-\frac{1}{2}y\sin\gamma}(e^{2y\sin\gamma} + 2\cos\gamma + y\sin(2\gamma))Q(y)f_1(y)dy\right\|_{L^\infty_x} \\ &\lesssim \inf_{f_1=g_1+h_1}\left(\left\|g_1 e^{-\frac{1}{2}x\sin\gamma}\left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|\right\|_{L^1_x}\right. \\ &\quad \left. + \left\|h_1 e^{\frac{1}{2}x\sin\gamma}\left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|\right\|_{L^2_x}\right) \\ &\leq \|\vec{f}\|_{Y_1}.\end{aligned}$$

Similarly, the third term in (3.24) is estimated by $\|W_+\vec{\phi}_\gamma\|_X \lesssim \|\vec{f}\|_{Y_2}$. The last term in (3.24) is estimated as follows:

$$\left\|\vec{\xi}_\gamma \int_{-\infty}^x \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy\right\|_X \leq N_1 + N_2 + N_3 + N_4,$$

where

$$\begin{aligned}N_1 &= \left\|e^{-x\sin\gamma}\int_{-\infty}^x \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy\right\|_{L^\infty_x\cap L^2_x}, \\ N_2 &= \left\|x\sin(2\gamma)\int_{-\infty}^x \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy\right\|_{L^\infty_x}, \\ N_3 &= \left\|e^{x\sin\gamma}\int_x^\infty \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy\right\|_{L^\infty_x\cap L^2_x}, \\ N_4 &= \left\|(2\cos\gamma + x\sin(2\gamma))\int_{-\infty}^x \vec{\eta}_\gamma(y)\cdot\vec{f}(y)dy\right\|_{L^\infty_x}.\end{aligned}$$

Since $|\vec{\eta}_\gamma(x)| \lesssim e^{-\frac{|x|}{2}\sin\gamma}$ and $\|e^{\frac{|x|}{2}\sin\gamma}\vec{f}\|_{L^2_x} \lesssim \|\vec{f}\|_Y$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}N_1 &\lesssim \left\|\int_{-\infty}^x e^{-(y-x)\sin\gamma}e^{\frac{1}{2}|y|\sin\gamma}(|f_1| + |f_2|)dy\right\|_{L^\infty_x\cap L^2_x} \\ &\leq \|e^{\frac{1}{2}|x|\sin\gamma}f_1\|_{L^2_x} + \|e^{\frac{1}{2}|x|\sin\gamma}f_2\|_{L^2_x} \\ &\lesssim \|\vec{f}\|_Y.\end{aligned}$$

The other terms N_2, N_3, and N_4 are estimated similarly. Altogether, these estimates justify the bound $\|\vec{w}\|_X \leq C\|\vec{f}\|_Y$ for a $\vec{f}$-independent positive constant C. □

Lemma 3.4. *Under the condition (3.8), assume that $\lambda = e^{i\gamma/2}$ is the eigenvalue of the spectral problem (3.1) for the eigenvector $\vec{\psi} \in H^1(\mathbb{R};\mathbb{C}^2)$ determined in Lemma 3.1. Then,* *the eigenvector can be written in the form (3.6) with*

$$\vec{\phi}(x) = \begin{bmatrix} e^{\frac{1}{2}x\sin\gamma}(1+r_{11}(x)) + e^{-\frac{1}{2}x\sin\gamma} r_{12}(x) \\ e^{\frac{1}{2}x\sin\gamma} r_{21}(x) + e^{-\frac{1}{2}x\sin\gamma}(1+r_{22}(x)) \end{bmatrix} \left| \operatorname{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|, \tag{3.25}$$

where components r_{ij} for $1 \le i,j \le 2$ satisfy the bound

$$\|r_{11}\|_{L^\infty} + \|r_{12}\|_{L^2\cap L^\infty} + \|r_{21}\|_{L^2\cap L^\infty} + \|r_{22}\|_{L^\infty} \lesssim \|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}. \tag{3.26}$$

Proof. Recall the projection operators $P_\gamma : L^2(\mathbb{R};\mathbb{C}^2) \to L^2(\mathbb{R};\mathbb{C}^2) \cap \operatorname{span}\{\sigma_3\vec{\eta}_\gamma\}^\perp$ and $\widehat{P}_\gamma : L^2(\mathbb{R};\mathbb{C}^2) \to L^2(\mathbb{R};\mathbb{C}^2) \cap \operatorname{span}\{\vec{\eta}_\gamma\}^\perp$ introduced in the proof of Lemma 3.1. The existence of the eigenvector $\vec{\phi} \in H^1(\mathbb{R};\mathbb{C}^2)$ of the spectral problem (3.7) for the eigenvalue $\lambda = e^{i\gamma/2}$ has been established in Lemma 3.1. Therefore, we are using operators P_γ and $\widehat{P}_\gamma$ to prove additional properties of the eigenvector $\vec{\phi}$.

Using the projection operator P_γ, we decompose $\vec{\phi} = \vec{\phi}_\gamma + \vec{\phi}_s$ and rewrite the spectral problem (3.7) in the form

$$\left(\partial_x - M_\gamma\right)\vec{\phi}_s = \Delta\widetilde{M}(\vec{\phi}_\gamma + \vec{\phi}_s), \tag{3.27}$$

where $\Delta\widetilde{M}$ is the anti-diagonal matrix that contains the perturbation terms $u_0 - u_\gamma$ and $v_0 - v_\gamma$ only. Because $\vec{\phi}_s \in H^1(\mathbb{R};\mathbb{C}^2)$ exists by Lemma 3.1, we realize that $\Delta\widetilde{M}(\vec{\phi}_\gamma + \vec{\phi}_s) = \widehat{P}\Delta\widetilde{M}(\vec{\phi}_\gamma + \vec{\phi}_s)$, which yields equivalently the constraint

$$\langle \vec{\eta}_\gamma, \Delta\widetilde{M}(\vec{\phi}_\gamma + \vec{\phi}_s)\rangle_{L^2} = 0. \tag{3.28}$$

Therefore, we write the perturbed equation (3.27) in the form

$$\vec{\phi}_s = P_\gamma\left(\partial_x - M_\gamma\right)^{-1}\widehat{P}_\gamma\Delta\widetilde{M}(\vec{\phi}_\gamma + \vec{\phi}_s). \tag{3.29}$$

Note that the operator $\widehat{P}_\gamma$ applies to the sum of the two terms in the right-hand-side of (3.29) thanks to (3.28) and cannot be applied to each term separately.

Since $\Delta\widetilde{M}$ is anti-diagonal, for any $\vec{\zeta} = (\zeta_1,\zeta_2)^t \in X$, we have

$$\|\Delta\widetilde{M}\vec{\zeta}\|_Y = \|(\Delta\widetilde{M})_{1,2}\zeta_2\|_{Y_1} + \|(\Delta\widetilde{M})_{2,1}\zeta_1\|_{Y_2},$$

which is bounded as follows:

$$\|(\Delta\widetilde{M})_{1,2}\zeta_2\|_{Y_1} \lesssim (\|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2})\|\zeta_2\|_{X_2}, \tag{3.30}$$

$$\|(\Delta\widetilde{M})_{2,1}\zeta_1\|_{Y_2} \lesssim (\|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2})\|\zeta_1\|_{X_1}. \tag{3.31}$$

Bound (3.30) follows simply from

$$\begin{aligned}\|(\Delta\widetilde{M})_{1,2}\zeta_2\|_{Y_1} &\le \inf_{\zeta_2=\xi_2+\eta_2}\left(\|(\Delta\widetilde{M})_{1,2}\xi_2 e^{\frac{x}{2}\sin\gamma}R(x)\|_{L^2_x} + \|(\Delta\widetilde{M})_{1,2}\eta_2 e^{-\frac{x}{2}\sin\gamma}R(x)\|_{L^2_x\cap L^1_x}\right)\\ &\lesssim \|(\Delta\widetilde{M})_{1,2}\|_{L^2}\inf_{\zeta_2=\xi_2+\eta_2}\left(\|\xi_2 e^{\frac{x}{2}\sin\gamma}R(x)\|_{L^\infty_x} + \|\eta_2 e^{-\frac{x}{2}\sin\gamma}R(x)\|_{L^\infty_x\cap L^2_x}\right)\\ &= \|(\Delta\widetilde{M})_{1,2}\|_{L^2}\|\zeta_2\|_{X_2},\end{aligned}$$

where $R(x) = \left|\cosh\left(x\sin\gamma - i\frac{\gamma}{2}\right)\right|$. Bound (3.31) is obtained similarly. Because $\vec{\phi}_\gamma \in X$, the bound $\|\vec{w}\|_X \lesssim \|\vec{f}\|_Y$ in Lemma 3.3 and bounds (3.30) and (3.31) imply

$$\begin{aligned}\|P_\gamma\left(\partial_x - M_\gamma\right)^{-1}\widehat{P}_\gamma\Delta\widetilde{M}(\vec{\phi}_\gamma + \vec{\phi}_s)\|_X &\lesssim \|\Delta\widetilde{M}(\vec{\phi}_\gamma + \vec{\phi}_s)\|_Y\\ &\lesssim \left(\|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}\right)(1 + \|\vec{\phi}_s\|_X).\end{aligned}$$

Since $\|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}$ is sufficiently small, the component $\vec{\phi}_s$ in (3.29) satisfies the bound

$$\|\vec{\phi}_s\|_X \lesssim \|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}. \tag{3.32}$$

This completes the proof of the bound (3.26) in the representation (3.25), because the bound (3.32) on $\vec{\phi}_s$ in Banach space X yields the bounds on the components r_{ij} in the corresponding spaces. □

Corollary 3.5. *In addition to the assumptions of Lemma 3.4, assume that $(u_0, v_0) \in H^m(\mathbb{R})$ for an integer $m \geq 0$. Then, r_{ij} for $1 \leq i,j \leq 2$ defined by (3.25) are C^m-functions of x.*

Proof. The statement is proved for $m = 0$ in Lemma 3.4, because r_{ij} are bounded functions according to the bound (3.26) and they are continuous functions since $\vec{\phi} \in H^1(\mathbb{R}; \mathbb{C}^2)$.

For $m = 1$, we differentiate the equation (3.27) with respect to x to get

$$\left(\partial_x - M_\gamma\right) \partial_x \vec{\phi}_s = \vec{r} + R\vec{\phi}_s + \Delta\widetilde{M}\partial_x\vec{\phi}_s, \tag{3.33}$$

where $\vec{r} := \partial_x(\Delta\widetilde{M}\vec{\phi}_\gamma)$ and $R := \partial_x(M_\gamma) + \partial_x(\Delta\widetilde{M})$. Recall that $\vec{\phi}_s \in X$ by Lemma 3.4. If $(u_0, v_0) \in H^1(\mathbb{R})$, then $\vec{r} + R\vec{\phi}_s \in Y$ according to the bounds

$$\|\vec{r}\|_Y \lesssim \|u_0 - u_\gamma\|_{H^1} + \|v_0 - v_\gamma\|_{H^1},$$
$$\|R\vec{\phi}_s\|_Y \lesssim (1 + \|u_0 - u_\gamma\|_{H^1} + \|v_0 - v_\gamma\|_{H^1})\|\vec{\phi}_s\|_X.$$

From bootstrapping of solution of the linear equation (3.29), we have $\partial_x\vec{\phi}_s \in H^1(\mathbb{R})$. Then, since $\vec{r} + R\vec{\phi}_s \in Y$, we have

$$\vec{r} + R\vec{\phi}_s + \Delta\widetilde{M}\partial_x\vec{\phi}_s = \widehat{P}_\gamma(\vec{r} + R\vec{\phi}_s + \Delta\widetilde{M}\partial_x\vec{\phi}_s)$$

Therefore, we can write the derivative equation (3.33) in the form

$$\partial_x\vec{\phi}_s = P_\gamma \left(\partial_x - M_\gamma\right)^{-1} \widehat{P}_\gamma(\vec{r} + R\vec{\phi}_s + \Delta\widetilde{M}\partial_x\vec{\phi}_s). \tag{3.34}$$

Using bounds (3.30) and (3.31) and the smallness of $\|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}$, we obtain

$$\|\partial_x\vec{\phi}_s\|_X \lesssim \|\vec{r} + R\vec{\phi}_s\|_Y < \infty, \tag{3.35}$$

from which it follows that $\partial_x\vec{\phi}_s \in H^1(\mathbb{R}) \cap X$, hence $\partial_x r_{ij} \in C(\mathbb{R})$ for $1 \leq i,j \leq 2$. Note that the bound (3.26) does not hold for $\partial_x r_{ij}$ because $\|u_0 - u_\gamma\|_{H^1} + \|v_0 - v_\gamma\|_{H^1}$ may not be small.

For $m \geq 2$, we differentiate (3.27) m times and obtain the expression

$$\left(\partial_x - M_\gamma\right) \partial_x^m \vec{\phi}_s = \vec{r}_m + \Delta\widetilde{M}\partial_x^m\vec{\phi}_s, \tag{3.36}$$

where $\vec{r}_m := \partial_x^m(\Delta\widetilde{M}\vec{\phi}_\gamma) + [\partial_x^m, M_\gamma + \Delta\widetilde{M}]\vec{\phi}_s$ and we denote $[\partial_x, f]g = \partial_x(fg) - f\partial_x(g)$. We note that the term $[\partial_x^m, M_\gamma + \Delta\widetilde{M}]\vec{\phi}_s$ does not contain the m-th derivative of $\vec{\phi}_s$. By an induction similar to the case $m = 1$, we find that $\vec{r}_m \in Y$ according to the bound

$$\|\vec{r}_m\|_Y \lesssim \|u_0 - u_\gamma\|_{H^m} + \|v_0 - v_\gamma\|_{H^m}.$$

Hence if $(u_0, v_0) \in H^m(\mathbb{R})$, then $\partial_x^m\vec{\phi}_s \in H^1(\mathbb{R}) \cap X$, hence $\partial_x^m r_{ij} \in C(\mathbb{R})$ for $1 \leq i,j \leq 2$. □

Lemma 3.6. *Under the condition (3.8), assume that $\lambda = e^{i\gamma/2}$ is the eigenvalue of the spectral problem (3.1) for the eigenvector $\vec{\psi} \in H^1(\mathbb{R};\mathbb{C}^2)$ determined in Lemma 3.1 and define*

$$p_0 := -u_0 \frac{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2}{e^{i\gamma/2}|\psi_1|^2 + e^{-i\gamma/2}|\psi_2|^2} + \frac{2i\sin\gamma\bar{\psi}_1\psi_2}{e^{i\gamma/2}|\psi_1|^2 + e^{-i\gamma/2}|\psi_2|^2}, \tag{3.37}$$

$$q_0 := -v_0 \frac{e^{i\gamma/2}|\psi_1|^2 + e^{-i\gamma/2}|\psi_2|^2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2} - \frac{2i\sin\gamma\bar{\psi}_1\psi_2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2}. \tag{3.38}$$

Then, $(p_0, q_0) \in L^2(\mathbb{R})$ satisfy the bound

$$\|p_0\|_{L^2} + \|q_0\|_{L^2} \lesssim \|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}. \tag{3.39}$$

If, in addition, $(u_0, v_0) \in H^m(\mathbb{R})$ for an integer $m \geq 1$, then $(p_0, q_0) \in H^m(\mathbb{R})$.

Proof. Let us rewrite equation (3.37) as

$$p_0 S = -u_0 + \frac{2i\sin\gamma\bar{\psi}_1\psi_2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2}, \tag{3.40}$$

where S is a module-one factor given by

$$S := \frac{e^{i\gamma/2}|\psi_1|^2 + e^{-i\gamma/2}|\psi_2|^2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2}.$$

We use the representation (3.6) and (3.25) for the eigenvector $\vec{\psi}$. Substituting $\vec{\psi}$ into the second term of (3.40), we obtain

$$\begin{aligned}\frac{2i\sin\gamma\bar{\psi}_1\psi_2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2} &= \frac{2i\bar{f}^2\sin\gamma\left[1+\epsilon_1+\epsilon_2 e^{x\sin\gamma}+\epsilon_3 e^{-x\sin\gamma}\right]}{e^{x\sin\gamma - i\gamma/2}(1+\epsilon_4) + e^{-x\sin\gamma+i\gamma/2}(1+\epsilon_5)+\epsilon_6} \\ &= i\bar{f}^2\sin\gamma\,\mathrm{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)\left[1+\mathcal{O}(|\epsilon_1|+|\epsilon_4|+|\epsilon_5|+|\epsilon_6|)\right] + \mathcal{O}(|\epsilon_2|+|\epsilon_3|),\end{aligned}$$

where $f(x) = e^{\frac{i}{4}\int_0^x (|u_0|^2 - |v_0|^2)dx}$ and we have defined

$$\begin{aligned}\epsilon_1 &:= \bar{r}_{11} + r_{22} + \bar{r}_{11}r_{22} + \bar{r}_{12}r_{21}, \\ \epsilon_2 &:= r_{21} + \bar{r}_{11}r_{21}, \\ \epsilon_3 &:= \bar{r}_{12} + \bar{r}_{12}r_{22}, \\ \epsilon_4 &:= r_{11} + \bar{r}_{11} + |r_{11}|^2 + e^{i\gamma}|r_{21}|^2, \\ \epsilon_5 &:= r_{22} + \bar{r}_{22} + |r_{22}|^2 + e^{-i\gamma}|r_{12}|^2, \\ \epsilon_6 &:= 2e^{-i\gamma/2}\mathrm{Re}(r_{12} + \bar{r}_{11}r_{12}) + 2e^{i\gamma/2}\mathrm{Re}(r_{21} + r_{21}\bar{r}_{22}).\end{aligned}$$

Bound (3.26) in Lemma 3.4 implies that

$$\|\epsilon_1\|_{L^\infty} + \|\epsilon_2\|_{L^\infty\cap L^2} + \|\epsilon_3\|_{L^\infty\cap L^2} + \|\epsilon_4\|_{L^\infty} + \|\epsilon_5\|_{L^\infty} + \|\epsilon_6\|_{L^\infty\cap L^2} \lesssim \|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}.$$

Since $u_\gamma(x) = i\sin\gamma\,\mathrm{sech}\left(x\sin\gamma - i\frac{\gamma}{2}\right)$ and $|f(x)| = 1$ for all $x \in \mathbb{R}$, we obtain

$$\left\| \frac{2i\sin\gamma\bar{\psi}_1\psi_2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2} - \bar{f}^2 u_\gamma \right\|_{L^2} \lesssim \|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}.$$

Applying the triangle inequality to the representation (3.40), we obtain

$$\|p_0\|_{L^2} = \|p_0 S\|_{L^2} \leq \|u_0 - \bar{f}^2 u_\gamma\|_{L^2} + \left\| \frac{2i \sin\gamma \bar{\psi}_1 \psi_2}{e^{-i\gamma/2}|\psi_1|^2 + e^{i\gamma/2}|\psi_2|^2} - \bar{f}^2 u_\gamma \right\|_{L^2}$$
$$\lesssim \|u_0 - \bar{f}^2 u_\gamma\|_{L^2} + \|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2}.$$

Using the Taylor series expansion (3.10) and the triangle inequality, we obtain

$$\|u_0 - \bar{f}^2 u_\gamma\|_{L^2} \leq \|u_0 - u_\gamma\|_{L^2} + \|u_\gamma\|_{L^2}\|1 - \bar{f}^2\|_{L^\infty}$$
$$\lesssim \|u_0 - u_\gamma\|_{L^2} + \|v_0 - v_\gamma\|_{L^2},$$

which finally yields the bound (3.39) for $\|p_0\|_{L^2}$. The bound (3.39) for $\|q_0\|_{L^2}$ is obtained in exactly the same way.

Now if $(u_0, v_0) \in H^m(\mathbb{R})$ for an integer $m \geq 1$, we can differentiate equation (3.40) in x m times and use Corollary 3.5 to conclude that $(p_0, q_0) \in H^m(\mathbb{R})$. □

4. From a small solution to a perturbed one-soliton solution

Here we use the auto-Bäcklund transformation given by Proposition 2.1 to transform a sufficiently smooth solution of the MTM system (1.1) in a L^2-neighborhood of the zero solution to the one in a L^2-neighborhood of the one-soliton solution.

Let $(p_0, q_0) \in H^2(\mathbb{R})$ be the initial data for the MTM system (1.1), which is sufficiently small in L^2 norm. Let $\vec{\phi}$ be a solution of the linear equation

$$\partial_x \vec{\phi} = L(p_0, q_0, \lambda)\vec{\phi} \tag{4.1}$$

with $\lambda = e^{i\gamma/2}$. Two linearly independent solutions of the linear equation (4.1) are constructed in Lemma 4.1 below.

Now, let $(p, q) \in C(\mathbb{R}; H^2(\mathbb{R}))$ be the unique global solution to the MTM system (1.1) such that $(p, q)|_{t=0} = (p_0, q_0)$. This solution exists in $H^2(\mathbb{R})$ by the global well-posedness theory for Dirac equations [11, 25, 28]. The time evolution of the vector function $\vec{\phi}$ in t for every $x \in \mathbb{R}$ is defined by the linear equation

$$\partial_t \vec{\phi} = A(p, q, \lambda)\vec{\phi} \tag{4.2}$$

for the same $\lambda = e^{i\gamma/2}$. Lemma 4.2 characterizes two linearly independent solutions of the linear equation (4.2) for every $t \in \mathbb{R}$.

Lastly, Lemma 4.3 constructs a new solution $(u, v) \in C(\mathbb{R}; H^2(\mathbb{R}))$ to the MTM system (1.1) in a L^2-neighborhood of the one-soliton solution from the auto–Bäcklund transformation involving (p, q) and $\vec{\phi}$ for every $t \in \mathbb{R}$.

Let us introduce the following unitary matrices

$$M_1 = \begin{bmatrix} m_1 & 0 \\ 0 & \bar{m}_1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} \bar{m}_2 & 0 \\ 0 & m_2 \end{bmatrix}, \tag{4.3}$$

where $m_1(x) := e^{\frac{i}{4}\int_{-\infty}^x (|p_0|^2 - |q_0|^2)ds}$ and $m_2(x) := e^{\frac{i}{4}\int_x^{\infty} (|p_0|^2 - |q_0|^2)ds}$. We make substitution

$$\vec{\phi}_1(x) = e^{-\frac{\sin\gamma}{2}x} M_1(x) \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \end{bmatrix} \quad \text{and} \quad \vec{\phi}_2(x) = e^{\frac{\sin\gamma}{2}x} M_2(x) \begin{bmatrix} \chi_1(x) \\ \chi_2(x) \end{bmatrix}, \tag{4.4}$$

into the linear equation (4.1) with $\lambda = e^{i\gamma/2}$ and obtain two boundary value problems:

$$\begin{cases} \varphi_1' = \frac{i}{2}(e^{-i\gamma/2}\bar{p}_0 - e^{i\gamma/2}\bar{q}_0)\bar{m}_1^2\varphi_2, \\ \varphi_2' = \frac{i}{2}(e^{-i\gamma/2}p_0 - e^{i\gamma/2}q_0)m_1^2\varphi_1 + \sin\gamma\varphi_2, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \chi_1' = -\sin\gamma\chi_1 + \frac{i}{2}(e^{-i\gamma/2}\bar{p}_0 - e^{i\gamma/2}\bar{q}_0)m_2^2\chi_2, \\ \chi_2' = \frac{i}{2}(e^{-i\gamma/2}p_0 - e^{i\gamma/2}q_0)\bar{m}_2^2\chi_1, \end{cases} \tag{4.6}$$

subject to the boundary conditions

$$\begin{cases} \lim_{x\to-\infty}\varphi_1(x) = 1, \\ \lim_{x\to\infty} e^{-x\sin\gamma}\varphi_2(x) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \lim_{x\to-\infty} e^{x\sin\gamma}\chi_1(x) = 0, \\ \lim_{x\to\infty}\chi_2(x) = 1. \end{cases} \tag{4.7}$$

The following lemma characterizes solutions of the boundary value problems (4.5), (4.6), and (4.7) if (p_0, q_0) is small in the L^2-norm.

Lemma 4.1. *There exists a real positive δ such that if $\|p_0\|_{L^2} + \|q_0\|_{L^2} \le \delta$, then the boundary value problems (4.5), (4.6), and (4.7) have unique solutions in the class*

$$(\varphi_1, \varphi_2) \in L^\infty(\mathbb{R}) \times (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})), \quad \textit{and} \quad (\chi_1, \chi_2) \in (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})) \times L^\infty(\mathbb{R}),$$

satisfying bounds

$$\|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^2\cap L^\infty} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2} \tag{4.8}$$

and

$$\|\chi_1\|_{L^2\cap L^\infty} + \|\chi_2 - 1\|_{L^\infty} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2}. \tag{4.9}$$

Proof. The boundary value problem (4.5) and (4.7) can be written in the integral form

$$\begin{cases} \varphi_1(x) = T_1(\varphi_1, \varphi_2)(x) := 1 + \frac{i}{2}\int_{-\infty}^{x}\left[e^{-i\gamma/2}\bar{p}_0(y) - e^{i\gamma/2}\bar{q}_0(y)\right]\bar{m}_1^2(y)\varphi_2(y)dy, \\ \varphi_2(x) = T_2(\varphi_1, \varphi_2)(x) := -\frac{i}{2}\int_{x}^{\infty} e^{(x-y)\sin\gamma}\left[e^{-i\gamma/2}p_0(y) - e^{i\gamma/2}q_0(y)\right]m_1^2(y)\varphi_1(y)dy. \end{cases} \tag{4.10}$$

We introduce a Banach space $Z := L^\infty(\mathbb{R}) \times (L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ equipped with the norm

$$\|\vec{u}\|_Z := \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty\cap L^2}$$

and show that $\vec{T} = (T_1, T_2)^t : Z \to Z$ is a contraction mapping. Using the Schwartz inequality, the Younge inequality, and the triangle inequality, we obtain for any $\vec{\varphi}, \widetilde{\vec{\varphi}} \in Z$,

$$\begin{aligned} &\|T_1(\varphi_1, \varphi_2) - T_1(\widetilde{\varphi}_1, \widetilde{\varphi}_2)\|_{L^\infty} \\ &= \sup_{x\in\mathbb{R}}\left|\frac{i}{2}\int_{-\infty}^{x}\left[e^{-i\gamma/2}\bar{p}_0(y) - e^{i\gamma/2}\bar{q}_0(y)\right]\bar{m}_1^2(y)(\varphi_2(y) - \widetilde{\varphi}_2(y))dy\right| \\ &\le \frac{1}{2}\left(\|p_0\|_{L^2} + \|q_0\|_{L^2}\right)\|\varphi_2 - \widetilde{\varphi}_2\|_{L^2} \end{aligned}$$

and

$$\begin{aligned}\|T_2(\varphi_1,\varphi_2) - T_2(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\|_{L^\infty\cap L^2} &\le \frac{1}{2}\|e^{x\sin\gamma}\|_{L^1_x(\mathbb{R}_-)\cap L^2_x(\mathbb{R}_-)}\|e^{-i\gamma/2}p_0 - e^{i\gamma/2}q_0\|_{L^2}\|\varphi_1 - \widetilde{\varphi}_1\|_{L^\infty} \\ &\le \frac{1}{\sin\gamma}\left(\|p_0\|_{L^2} + \|q_0\|_{L^2}\right)\|\varphi_1 - \widetilde{\varphi}_1\|_{L^\infty}.\end{aligned}$$

If $\|p_0\|_{L^2} + \|q_0\|_{L^2} \le \delta$ is sufficiently small such that $\delta < \sin\gamma$ for a fixed $\gamma \in (0,\pi)$, then $\vec{T} = (T_1, T_2)^t$ is a contraction mapping on Z. To prove the inequality (4.8), we have

$$\begin{aligned}\|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^2\cap L^\infty} &\le \|\vec{T}(\varphi_1,\varphi_2) - \vec{T}(0,0)\|_Z \\ &\le \frac{\|p_0\|_{L^2} + \|q_0\|_{L^2}}{\sin\gamma}(1 + \|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^\infty\cap L^2}).\end{aligned}$$

Since $\|p_0\|_{L^2} + \|q_0\|_{L^2} \le \delta < \sin\gamma$, the above estimates yields the inequality (4.8). Repeating similar estimates for the boundary-value problem (4.6) and (4.7), we can prove that $(\chi_1,\chi_2) \in (L^2(\mathbb{R})\cap L^\infty(\mathbb{R}))\times L^\infty(\mathbb{R})$ and the inequality (4.9) holds. □

Let us now define the time evolution of the vector functions $\vec{\phi}_1$ and $\vec{\phi}_2$ in t for every $x\in\mathbb{R}$, according to the linear equation (4.2), where $\lambda = e^{i\gamma/2}$ and $(p,q) \in C(\mathbb{R};H^2(\mathbb{R}))$ is the unique solution of the MTM system (1.1) such that $(p,q)|_{t=0} = (p_0,q_0)$. We also consider the initial data for $\vec{\phi}_1$ and $\vec{\phi}_2$ at $t=0$ given by the two linearly independent solutions (4.4) of the linear equation (4.1). The linear equation (4.2) for $\vec{\phi}_{1,2}$ with $\lambda = e^{i\gamma/2}$ take the form

$$\partial_t\vec{\phi}_{1,2} = \begin{bmatrix} -\frac{i}{4}(|p|^2+|q|^2) + \frac{i}{2}\cos\gamma & -\frac{i}{2}(e^{-i\gamma/2}\bar{p} + e^{i\gamma/2}\bar{q}) \\ -\frac{i}{2}(e^{-i\gamma/2}p + e^{i\gamma/2}q) & \frac{i}{4}(|p|^2+|q|^2) - \frac{i}{2}\cos\gamma \end{bmatrix}\vec{\phi}_{1,2}. \tag{4.11}$$

We set

$$\vec{\phi}_1(x,t) := e^{-\frac{x}{2}\sin\gamma}M_1(x,t)\vec{\varphi}(x,t), \quad \vec{\phi}_2(x,t) := e^{\frac{x}{2}\sin\gamma}M_2(x,t)\vec{\chi}(x,t), \tag{4.12}$$

where $M_1(x,t)$ and $M_2(x,t)$ are given by (4.3) with

$$m_1(x,t) := e^{\frac{i}{4}\int_{-\infty}^x(|p(s,t)|^2-|q(s,t)|^2)ds}, \quad m_2(x,t) := e^{\frac{i}{4}\int_x^\infty(|p(s,t)|^2-|q(s,t)|^2)ds}. \tag{4.13}$$

The following lemma characterizes vector functions $\vec{\varphi}$ and $\vec{\chi}$.

Lemma 4.2. *Let $(p_0,q_0)\in H^2(\mathbb{R})$ and assume that there exists a sufficiently small δ such that $\|p_0\|_{L^2} + \|q_0\|_{L^2} \le \delta$. Let $(p,q)\in C(\mathbb{R};H^2(\mathbb{R}))$ be the unique solution of the MTM system (1.1) such that $(p,q)|_{t=0} = (p_0,q_0)$. Let $\vec{\phi}_1$ and $\vec{\phi}_2$ be solutions of the linear equation (4.11) starting with the initial data given by (4.4). Then, for every $t\in\mathbb{R}$, $\vec{\phi}_1$ and $\vec{\phi}_2$ are given by (4.12), where*

$$(\varphi_1,\varphi_2)(\cdot,t) \in L^\infty(\mathbb{R})\times(L^2(\mathbb{R})\cap L^\infty(\mathbb{R})) \quad \text{and} \quad (\chi_1,\chi_2)(\cdot,t)\in(L^2(\mathbb{R})\cap L^\infty(\mathbb{R}))\times L^\infty(\mathbb{R})$$

satisfy the differential equations

$$\partial_x\vec{\varphi} = \begin{bmatrix} 0 & \frac{i}{2}(e^{-i\gamma/2}\bar{p} - e^{i\gamma/2}\bar{q})\bar{m}_1^2 \\ \frac{i}{2}(e^{-i\gamma/2}p - e^{i\gamma/2}q)m_1^2 & \sin\gamma \end{bmatrix}\vec{\varphi} \tag{4.14}$$

and

$$\partial_x\vec{\chi} = \begin{bmatrix} -\sin\gamma & \frac{i}{2}(e^{-i\gamma/2}\bar{p} - e^{i\gamma/2}\bar{q})m_2^2 \\ \frac{i}{2}(e^{-i\gamma/2}p - e^{i\gamma/2}q)\bar{m}_2^2 & 0 \end{bmatrix}\vec{\chi}, \tag{4.15}$$

subject to the boundary values

$$\begin{cases} \lim_{x\to-\infty} \varphi_1(x,t) = e^{\frac{i}{2}t\cos\gamma}, \\ \lim_{x\to\infty} e^{-x\sin\gamma}\varphi_2(x,t) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \lim_{x\to-\infty} e^{x\sin\gamma}\chi_1(x,t) = 0, \\ \lim_{x\to\infty} \chi_2(x,t) = e^{-\frac{i}{2}t\cos\gamma}. \end{cases} \tag{4.16}$$

Furthermore, for every $t \in \mathbb{R}$, these functions satisfy the bounds

$$\|\varphi_1(\cdot,t) - e^{\frac{i}{2}t\cos\gamma}\|_{L^\infty} + \|\varphi_2(\cdot,t)\|_{L^2\cap L^\infty} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2} \tag{4.17}$$

and

$$\|\chi_1(\cdot,t)\|_{L^2\cap L^\infty} + \|\chi_2(\cdot,t) - e^{-\frac{i}{2}t\cos\gamma}\|_{L^\infty} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2}. \tag{4.18}$$

Proof. By Sobolev embedding of $H^2(\mathbb{R})$ into $C^1(\mathbb{R})$, the x-derivatives of solutions $(p,q) \in C(\mathbb{R}; H^2(\mathbb{R}))$ are continuous and bounded functions of x for every $t \in \mathbb{R}$. Moreover, bootstrapping arguments for the MTM system (1.1) show that the same solution (p,q) exists in $C^1(\mathbb{R}; H^1(\mathbb{R}))$. Therefore, the t-derivatives of solutions (p,q) are also continuous and bounded functions of x for every $t \in \mathbb{R}$. Thus, the technical assumption $(p_0,q_0) \in H^2(\mathbb{R})$ simplifies working with the system of Lax equations (4.1) and (4.2). In particular, we shall prove that $\vec{\varphi}$ satisfies the differential equation (4.14) for every $t \in \mathbb{R}$ if $\vec{\phi}_1$ satisfies the differential equation (4.11) for every $x \in \mathbb{R}$ and the representation (4.12) is used.

By Lemma 4.1, $\vec{\varphi}$ is a bounded function of x for $t = 0$ and by bootstrapping arguments, $\vec{\varphi} \in C(\mathbb{R})$ for $t = 0$. We now claim that the differential equation (4.11) preserves this property for every $t \in \mathbb{R}$. From the differential equation (4.11) and the representation (4.12), we obtain

$$\begin{aligned} \partial_t(|\varphi_1|^2 + |\varphi_2|^2) &= \sin\left(\frac{\gamma}{2}\right)\left[(\bar{q} - \bar{p})\bar{m}_1^2\bar{\varphi}_1\varphi_2 + (q - p)m_1^2\varphi_1\bar{\varphi}_2\right] \\ &\le (|p| + |q|)(|\varphi_1|^2 + |\varphi_2|^2). \end{aligned}$$

By Gronwall's inequality, for any $T > 0$, we obtain

$$|\varphi_1(x,t)|^2 + |\varphi_2(x,t)|^2 \le e^{\alpha_T T}(|\varphi_1(x,0)|^2 + |\varphi_2(x,0)|^2) \quad x \in \mathbb{R}, \quad t \in [-T,T], \tag{4.19}$$

where

$$\alpha_T := \sup_{t\in[-T,T]} \sup_{x\in\mathbb{R}} \left(|p(x,t)| + |q(x,t)|\right).$$

Since the exponential factor remains bounded for any finite time $T > 0$, then it follows that $\vec{\varphi}(\cdot,t) \in L^\infty(\mathbb{R})$ for every $t \in \mathbb{R}$. Bootstrapping then yields $\vec{\varphi}(\cdot,t) \in C(\mathbb{R})$ for every $t \in \mathbb{R}$.

Since coefficients of the linear system (4.11) are continuous functions of (x,t), we have $\partial_t\vec{\varphi}(\cdot,t) \in C(\mathbb{R})$ for every $t \in \mathbb{R}$. Now, if (p,q) are C^1 functions of x and t, then a similar method shows that $\partial_x\vec{\varphi}, \partial_t\partial_x\vec{\varphi}, \partial_t^2\vec{\varphi} \in C(\mathbb{R})$ for every $t \in \mathbb{R}$.

We shall now establish the validity of the differential equation (4.14). For $\vec{\phi}_1$ in (4.12), we write this equation in the abstract form $\partial_x\vec{\phi}_1 = L\vec{\phi}_1$. We also write the differential equation (4.11) for $\vec{\phi}_1$ in the abstract form $\partial_t\vec{\phi}_1 = A\vec{\phi}_1$. To establish (4.14) for every $t \in \mathbb{R}$, we construct the residual function $\vec{F} := \partial_x\vec{\phi}_1 - L\vec{\phi}_1$. This function is zero for every $x \in \mathbb{R}$ and $t = 0$. We shall prove that $\vec{F}$ is zero for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

The compatibility condition $\partial_x A - \partial_t L + [A, L] = 0$ is satisfied for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$, if (p,q) is a C^1 solution of the MTM system (1.1). After differentiating $\vec{F}$ with respect to t, we obtain

$$\begin{aligned}\partial_t \vec{F} &= \partial_t \partial_x \vec{\phi}_1 - (\partial_t L)\vec{\phi}_1 - L\partial_t \vec{\phi}_1 \\ &= \partial_x (A\vec{\phi}_1) - (\partial_t L)\vec{\phi}_1 - LA\vec{\phi}_1 \\ &= (\partial_x A - \partial_t L + [A, L])\vec{\phi}_1 + A\vec{F} \\ &= A\vec{F}.\end{aligned}$$

Let $\vec{F} = (F_1, F_2)^t$. From the linear evolution $\partial_t \vec{F} = AF$, we again obtain

$$\begin{aligned}\partial_t (|F_1|^2 + |F_2|^2) &= \sin\left(\frac{\gamma}{2}\right)\left[(\bar{q} - \bar{p})\bar{F}_1 F_2 + (q - p)F_1 \bar{F}_2\right] \\ &\leq (|p| + |q|)(|F_1|^2 + |F_2|^2),\end{aligned}$$

which yields with Gronwall's inequality for any $T > 0$

$$|F_1(x,t)|^2 + |F_2(x,t)|^2 \leq e^{\alpha_T T}(|F_1(x,0)|^2 + |F_2(x,0)|^2), \quad x \in \mathbb{R}, \quad t \in [-T, T],$$

with the same definition of α_T. Since $\vec{F}(x,0) = \vec{0}$, then the above inequality yields $\vec{F}(x,t) = \vec{0}$ for every $x \in \mathbb{R}$ and $t \in [-T, T]$. Hence, $\vec{\varphi}$ satisfies the differential equation (4.14).

We have shown that $\vec{\varphi}(\cdot, t) \in L^\infty(\mathbb{R})$ for every $t \in \mathbb{R}$. We now show that $\varphi_2(\cdot, t) \in L^2(\mathbb{R})$ for every $t \in \mathbb{R}$. It follows from the differential equation (4.11) and the representation (4.12) that

$$\begin{aligned}\partial_t (|\varphi_2|^2) &\leq (|p| + |q|)|\bar{\varphi}_1 \varphi_2| \\ &\lesssim |\varphi_2|^2 + (|p|^2 + |q|^2)|\varphi_1|^2.\end{aligned}$$

Using Gronwall's inequality and the previous bound (4.19), we have for any $T > 0$

$$\begin{aligned}|\varphi_2(x,t)|^2 &\leq e^T \left[|\varphi_2(x,0)|^2 + \int_{-T}^{T} (|p(x,s)|^2 + |q(x,s)|^2)|\varphi_1(x,s)|^2 ds\right] \\ &\leq e^T |\varphi_2(x,0)|^2 + e^{(1+\alpha_T)T} \int_{-T}^{T} (|p(x,s)|^2 + |q(x,s)|^2)\left(|\varphi_1(x,0)|^2 + |\varphi_2(x,0)|^2\right) ds,\end{aligned}$$

where $x \in \mathbb{R}$ and $t \in [-T, T]$. Therefore, we have

$$\begin{aligned}\|\varphi_2(\cdot,t)\|_{L^2}^2 &\leq e^T \|\varphi_2(\cdot,0)\|_{L^2}^2 \\ &\quad + e^{(1+\alpha_T)T}\left(\|\varphi_1(\cdot,0)\|_{L^\infty}^2 + \|\varphi_2(\cdot,0)\|_{L^\infty}^2\right)\int_{-T}^{T} (\|p(\cdot,s)\|_{L^2}^2 + \|q(\cdot,s)\|_{L^2}^2) ds.\end{aligned}$$

Since the right-hand side of this inequality remains bounded for any finite time $T > 0$, then it follows that $\varphi_2(\cdot, t) \in L^2(\mathbb{R})$ for every $t \in \mathbb{R}$.

It remains to prove the boundary values for $\vec{\varphi}_1(x,t)$ as $x \to \pm\infty$ in (4.16). The second boundary condition

$$\lim_{x \to \infty} e^{-x \sin\gamma} \varphi_2(x,t) = 0$$

follows from the fact that $\varphi_2(\cdot, t) \in L^\infty(\mathbb{R})$ for every $t \in \mathbb{R}$. To prove the first boundary condition, we use Duhamel's formula to write the differential equation (4.11) in the integral form:

$$\vec{\phi}_1(x,t) = e^{\frac{i}{2} t \sigma_3 \cos\gamma} \vec{\phi}_1(x,0) + \int_0^t e^{\frac{i}{2}(t-s)\sigma_3 \cos\gamma} A_1(x,s)\vec{\phi}_1(x,s) ds,$$

where

$$A_1(x,t) := \begin{bmatrix} -\frac{i}{4}(|p|^2+|q|^2) & -\frac{i}{2}(e^{-i\gamma/2}\bar{p}+e^{i\gamma/2}\bar{q}) \\ -\frac{i}{2}(e^{-i\gamma/2}p+e^{i\gamma/2}q) & \frac{i}{4}(|p|^2+|q|^2) \end{bmatrix}.$$

Using the representation (4.12), we have for $t \in \mathbb{R}$

$$|M_1\vec{\varphi}(x,t) - e^{\frac{i}{2}t\sigma_3\cos\gamma}M_1\vec{\varphi}(x,0)| \le \int_0^{|t|} |A_1(x,s)M_1\vec{\varphi}(x,s)|ds,$$

where $|\vec{f}|$ denotes the vector norm of the 2-vector $\vec{f}$. Since $\vec{\varphi}(\cdot,t) \in L^\infty(\mathbb{R}) \times (L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}))$ for every $t \in \mathbb{R}$ and $p(\cdot,t), q(\cdot,t) \in H^2(\mathbb{R})$, we claim that

- $|A_1(x,s)M_1\vec{\varphi}(x,s)|$ is bounded by some s-independent constant for every $x \in \mathbb{R}$ and $|s| \le |t|$
- $\lim_{|x|\to\infty} A_1(x,s)M_1\vec{\varphi}(x,s) = \vec{0}$ pointwise for every $|s| \le |t|$.

Then, the dominated convergence theorem gives

$$\lim_{x\to-\infty} |M_1(x,t)\vec{\varphi}(x,t) - e^{\frac{i}{2}t\sigma_3\cos\gamma}M_1(x,0)\vec{\varphi}(x,0)| = 0, \quad t \in \mathbb{R}.$$

Since $\vec{\varphi}(x,0) \to (1,0)^t$ as $x \to -\infty$ and $M_1(x,t) \to I$ as $x \to -\infty$ for every $t \in \mathbb{R}$, the above limit recovers the first boundary condition

$$\lim_{x\to-\infty} \varphi_1(x,t) = e^{\frac{i}{2}t\cos\gamma}.$$

The proof of the differential equation (4.15) and the boundary condition for $\vec{\chi}$ in (4.16) is analogous. Finally, since the L^2 norm of solutions of the MTM system (1.1) is constant in time t, according to (1.2), the proof of bounds (4.17) and (4.18) is analogous to the proof in Lemma 4.1. □

Lemma 4.3. *Let $(p_0,q_0) \in H^2(\mathbb{R})$ and assume that there exists a sufficiently small δ such that $\|p_0\|_{L^2} + \|q_0\|_{L^2} \le \delta$. Let $(p,q) \in C(\mathbb{R}; H^2(\mathbb{R}))$ be the unique solution to the MTM system (1.1) such that $(p,q)|_{t=0} = (p_0,q_0)$. Using solutions $\vec{\varphi}$ and $\vec{\chi}$ in Lemma 4.2, let us define*

$$\begin{bmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{bmatrix} := c_1(t)e^{-\frac{x}{2}\sin\gamma}M_1(x,t)\vec{\varphi}(x,t) + c_2(t)e^{\frac{x}{2}\sin\gamma}M_2(x,t)\vec{\chi}(x,t), \tag{4.20}$$

where $c_1(t) := e^{(a+i\theta)/2}$, $c_2(t) := e^{-(a+i\theta)/2}$ are given in terms of the real coefficients a, θ, which may depend on t. Then, the auto–Bäcklund transformation

$$u := -p\frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i\sin\gamma\bar{\phi}_1\phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} \tag{4.21}$$

and

$$v := -q\frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i\sin\gamma\bar{\phi}_1\phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} \tag{4.22}$$

generates a new solution $(u,v) \in C(\mathbb{R}; H^2(\mathbb{R}))$ to the MTM system (1.1) satisfying the bound

$$\left\| u(x,t) - ie^{-i\theta - it\cos\gamma}\sin\gamma\,\mathrm{sech}\left(x\sin\gamma - i\frac{\gamma}{2} - a\right)\right\|_{L^2_x} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2} \tag{4.23}$$

and

$$\left\| v(x,t) + ie^{-i\theta - it\cos\gamma} \sin\gamma \operatorname{sech}\left(x\sin\gamma + i\frac{\gamma}{2} - a\right)\right\|_{L^2_x} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2} \tag{4.24}$$

for every $t \in \mathbb{R}$.

Proof. Let us introduce $\vec{\psi} = (\psi_1, \psi_2)^t$ by

$$\psi_1 := \frac{\bar{\phi}_2}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}, \quad \psi_2 := \frac{\bar{\phi}_1}{|e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2|}. \tag{4.25}$$

The inequalities (4.17) and (4.18) imply that (u, v) and $\vec{\psi}$ are bounded for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$. If (p, q) are C^1 functions of (x, t) and $\vec{\phi}$ is a C^2 function of (x, t), then (u, v) are C^1 functions of (x, t) and $\vec{\psi}$ is a C^2 function of (x, t). Proposition 2.1 states that $\vec{\psi}$ given by (4.25) satisfies the evolution equations

$$\partial_x \vec{\psi} = L(u, v, \lambda)\vec{\psi}, \quad \partial_t \vec{\psi} = A(u, v, \lambda)\vec{\psi},$$

for $\lambda = e^{i\gamma/2}$. As a result, the compatibility condition $\partial_x\partial_t\vec{\psi} = \partial_t\partial_x\vec{\psi}$ for every $x \in \mathbb{R}$ and $t \in \mathbb{R}$ yields the MTM system (1.1) for the functions (u, v).

We shall now prove inequality (4.23). The proof of inequality (4.24) is analogous. First, we write (4.21) in the form of

$$R := \frac{2i\sin\gamma\,\bar{\phi}_1\phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} = u + p\frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}. \tag{4.26}$$

Explicit substitutions of (4.20) into (4.26) yield

$$R := \frac{2i\sin\gamma\left(\bar{m}_1 m_2 e^{-i\theta}\bar{\varphi}_1\chi_2 + R_1\right)}{e^{i\gamma/2 + a - x\sin\gamma}|\varphi_1|^2 + e^{-i\gamma/2 - a + x\sin\gamma}|\chi_2|^2 + R_2},$$

where

$$R_1 := \bar{m}_1^2 e^{a - x\sin\gamma}\bar{\varphi}_1\varphi_2 + \bar{m}_1 m_2 e^{i\theta}\varphi_2\bar{\chi}_1 + m_2^2 e^{-a + x\sin\gamma}\bar{\chi}_1\chi_2$$

and

$$\begin{aligned} R_2 := {} & e^{i\gamma/2 - a + x\sin\gamma}|\chi_1|^2 + e^{-i\gamma/2 + a - x\sin\gamma}|\varphi_2|^2 + 2e^{i\gamma/2}\mathrm{Re}[m_1 m_2 e^{i\theta}\varphi_1\bar{\chi}_1] \\ & + 2e^{-i\gamma/2}\mathrm{Re}[\bar{m}_1\bar{m}_2 e^{i\theta}\varphi_2\bar{\chi}_2]. \end{aligned}$$

By bounds (4.17) and (4.18) in Lemma 4.2, we have $|\varphi_1|, |\chi_2| \sim 1$ and $|\varphi_2|, |\chi_1| \sim 0$, so that for $a - x\sin\gamma \le 0$,

$$R = \frac{2i\sin\gamma\,\bar{m}_1 m_2 e^{-i\theta + a - x\sin\gamma}\bar{\varphi}_1\chi_2}{e^{i\gamma/2 + 2(a - x\sin\gamma)}|\varphi_1|^2 + e^{-i\gamma/2}|\chi_2|^2} + \mathcal{O}(|\varphi_2| + |\chi_1|) \tag{4.27}$$

and for $a - x\sin\gamma \ge 0$,

$$R = \frac{2i\sin\gamma\,\bar{m}_1 m_2 e^{-i\theta - a + x\sin\gamma}\bar{\varphi}_1\chi_2}{e^{i\gamma/2}|\varphi_1|^2 + e^{-i\gamma/2 - 2(a - x\sin\gamma)}|\chi_2|^2} + \mathcal{O}(|\varphi_2| + |\chi_1|). \tag{4.28}$$

Combining (4.27) and (4.28), we get

$$\left|R - \frac{2i\sin\gamma e^{-i\theta - it\cos\gamma}}{e^{i\gamma/2+a-x\sin\gamma} + e^{-i\gamma/2} - a + x\sin\gamma}\right|$$
$$\lesssim e^{-|a-x\sin\gamma|}(|\varphi_1 - e^{it\frac{\cos\gamma}{2}}| + |\chi_2 - e^{-it\frac{\cos\gamma}{2}}| + |m_1 - 1| + |m_2 - 1|) + |\varphi_2| + |\chi_1|$$

Since $m_1 = e^{\frac{i}{4}\int_{-\infty}^{x}(|p|^2-|q|^2)ds}$ and $m_2 = e^{\frac{i}{4}\int_{x}^{\infty}(|p|^2-|q|^2)ds}$, we obtain the bounds

$$\|m_{1,2}(\cdot,t) - 1\|_{L^\infty} \lesssim \|p\|_{L^2}^2 + \|q\|_{L^2}^2,$$

provided that $\|p\|_{L^2}$ and $\|q\|_{L^2}$ are sufficiently small. Then, by Lemma 4.2 and the L^2 conservation law (1.2), the previous estimate yields

$$\left\|R(x,t) - ie^{-i\theta-it\cos\gamma}\sin\gamma\,\mathrm{sech}\left(x\sin\gamma - i\frac{\gamma}{2} - a\right)\right\|_{L_x^2} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2}. \tag{4.29}$$

Using the definition (4.26), the bound (4.29), and the triangle inequality, we obtain inequality (4.23).

Lastly, if $(p,q) \in C(\mathbb{R}; H^2(\mathbb{R}))$, we can differentiate equations (4.27) and (4.28) in x twice to show from (4.21) and (4.22) that $(u,v) \in C(\mathbb{R}; H^2(\mathbb{R}))$. □

5. Proof of theorem 1.2

Thanks to the Lorentz transformation given by Proposition 2.2, we may choose $\lambda_0 = e^{i\gamma_0/2}$, $\gamma_0 \in (0,\pi)$ in Theorem 1.2. For a given initial data (u_0, v_0) satisfying the inequality (1.4) for sufficiently small ϵ, we map a L^2-neighborhood of one-soliton solution to that of the zero solution. To do so, we use Lemma 3.1 and obtain an eigenvector $\vec{\psi}$ of the spectral problem (3.1) for an eigenvalue $\lambda \in \mathbb{C}$ satisfying

$$|\lambda - e^{i\gamma_0/2}| \lesssim \|u_0 - u_{\gamma_0}\|_{L^2} + \|v_0 - v_{\gamma_0}\|_{L^2} =: \epsilon. \tag{5.1}$$

We should note that the same Lorentz transformation cannot be used twice to consider the cases of $\lambda_0 = e^{i\gamma_0/2}$ and $\lambda = e^{i\gamma/2}$ simultaneously; the assumption $\lambda_0 = e^{i\gamma_0/2}$ implies that λ is not generally on the unit circle, and vice versa. Hence, if $\lambda_0 = e^{i\gamma_0/2}$ is set, all formulas in Section 3 below Remark 1 must in fact be generalized for a general λ. However, this generalization is straightforward thanks again to the existence of the Lorentz transformation given by Proposition 2.2. In what follows, we then use the general MTM solitons (u_λ, v_λ) given by (1.3).

By Lemma 3.6, the auto–Bäcklund transformation (3.2) and (3.3) with $\vec{\psi}$ in Lemma 3.1 yields an initial data $(p_0, q_0) \in L^2(\mathbb{R})$ of the MTM system (1.1) satisfying the estimate

$$\begin{aligned}\|p_0\|_{L^2} + \|q_0\|_{L^2} &\lesssim \|u_0 - u_\lambda(\cdot,0)\|_{L^2} + \|v_0 - v_\lambda(\cdot,0)\|_{L^2}\\ &\lesssim \|u_0 - u_{\gamma_0}\|_{L^2} + \|v_0 - v_{\gamma_0}\|_{L^2} + \|u_\lambda(\cdot,0) - u_{\gamma_0}\|_{L^2} + \|v_\lambda(\cdot,0) - v_{\gamma_0}\|_{L^2}\\ &\lesssim \|u_0 - u_{\gamma_0}\|_{L^2} + \|v_0 - v_{\gamma_0}\|_{L^2} =: \epsilon,\end{aligned} \tag{5.2}$$

where we have used the triangle inequality and the bound (5.1).

Since the time evolution in Section 4 is well-defined if $(p_0, q_0) \in H^2(\mathbb{R})$, let us first assume that the initial data $(u_0, v_0) \in L^2(\mathbb{R})$ satisfying the inequality (1.4) also satisfies $(u_0, v_0) \in H^2(\mathbb{R})$. Then, $(p_0, q_0) \in H^2(\mathbb{R})$ by Lemma 3.6. Let $(p, q) \in C(\mathbb{R}; H^2(\mathbb{R}))$ be the unique solution of the MTM system (1.1) such that $(p, q)|_{t=0} = (p_0, q_0)$. Next we will map a L^2-neighborhood of the zero solution to that of one-soliton solution for all $t \in \mathbb{R}$.

By Lemma 4.2, we construct a solution of the Lax equations

$$\partial_x \vec{\phi} = L(p, q, \lambda)\vec{\phi} \quad \text{and} \quad \partial_t \vec{\phi} = A(p, q, \lambda)\vec{\phi} \tag{5.3}$$

for the same eigenvalue λ as in (5.1). Let

$$k_1(\lambda) := \frac{i}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right), \quad k_2(\lambda) := \frac{1}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right).$$

The solution of the Lax system (5.3) is constructed in the form

$$\vec{\phi}(x, t) = c_1(t) M_1(x, t) e^{x k_1(\lambda)} \vec{\varphi}(x, t) + c_2(t) M_2(x, t) e^{-x k_1(\lambda)} \vec{\chi}(x, t), \tag{5.4}$$

where unitary matrices M_1 and M_2 are given in (4.3) with m_1 and m_2 given by (4.13), whereas the vectors $\vec{\varphi}$ and $\vec{\chi}$ satisfy the estimates

$$\|\varphi_1(\cdot, t) - e^{itk_2(\lambda)}\|_{L^\infty} + \|\varphi_2(\cdot, t)\|_{L^2 \cap L^\infty} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2} \tag{5.5}$$

and

$$\|\chi_1(\cdot, t)\|_{L^2 \cap L^\infty} + \|\chi_2(\cdot, t) - e^{-itk_2(\lambda)}\|_{L^\infty} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2}. \tag{5.6}$$

The coefficients c_1 and c_2 of the linear superposition (5.4) can be parameterized by parameters a and θ as follows:

$$c_1 = e^{(a+i\theta)/2}, \quad c_2 = e^{-(a+i\theta)/2},$$

where parameters a and θ may depend on the time variable t but not on the space variable x. These parameters determine the spatial and gauge translations of the MTM solitons according to the transformation (2.13).

By Lemma 4.3, the auto–Bäcklund transformation generates a new solution (u, v) of the MTM system (1.1) satisfying the bound for every $t \in \mathbb{R}$,

$$\begin{aligned} &\inf_{a,\theta \in \mathbb{R}} \left(\|u(\cdot + a, t) - e^{-i\theta} u_\lambda(\cdot, t)\|_{L^2} + \|v(\cdot + a, t) - e^{-i\theta} v_\lambda(\cdot, t)\|_{L^2}\right) \\ &\quad \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2}. \end{aligned} \tag{5.7}$$

Theorem 1.2 is proved if $(u_0, v_0) \in H^2(\mathbb{R})$. To obtain the same result for $(u_0, v_0) \in L^2(\mathbb{R})$ but $(u_0, v_0) \notin H^2(\mathbb{R})$, we construct an approximating sequence $(u_{0,n}, v_{0,n}) \in H^2(\mathbb{R})$ $(n \in \mathbb{N})$ that converges as $n \to \infty$ to $(u_0, v_0) \in L^2(\mathbb{R})$ in the L^2-norm. For a sufficiently small $\epsilon > 0$, we let

$$\|u_{0,n} - u_{\gamma_0}\|_{L^2} + \|v_{0,n} - v_{\gamma_0}\|_{L^2} \le \epsilon, \quad \text{for every } n \in \mathbb{N}.$$

Under this condition, for each $(u_{0,n}, v_{0,n}) \in H^2(\mathbb{R})$, we obtain inequalities (5.1), (5.2), and (5.7) independently of n. Therefore, there is a subsequence of solutions $(u_n, v_n) \in$

$C(\mathbb{R}; H^2(\mathbb{R}))$ ($n \in \mathbb{N}$) of the MTM system (1.1) such that it converges as $n \to \infty$ to a solution $(u, v) \in C(\mathbb{R}; L^2(\mathbb{R}))$ of the MTM system (1.1) satisfying inequalities (1.5) and (1.6). The proof of Theorem 1.2 is now complete.

References

[1] Berkolaiko, G., Comech, A. (2012). On spectral stability of solitary waves of nonlinear Dirac equations in 1D. *Math. Model. Nat. Phenom.* 7:13–31.

[2] Berkolaiko, G., Comech, A., Sukhtayev, A. (2015). Vakhitov–Kolokolov and energy vanishing conditions for linear instability of solitary waves in models of classical self-interacting spinor fields. *Nonlinearity* 28:577–592.

[3] Boussaid, N. (2006). Stable directions for small nonlinear Dirac standing waves. *Comm. Math. Phys.* 268:757–817.

[4] Boussaid, N. (2008). On the asymptotic stability of small nonlinear Dirac standing waves in a resonant case. *SIAM J. Math. Anal.* 40:1621–1670.

[5] Boussaid, N., Cuccagna, S. (2012). On stabibity of standing waves of nonlinear Dirac equations. *Comm. Part. Diff. Eqs.* 37:1001–1056.

[6] Candy, T. (2011). Global existence for an L^2-critical nonlinear Dirac equation in one dimension. *Adv. Diff. Eqs.* 16:643–666.

[7] Chugunova, M., Pelinovsky, D. (2006). Block-diagonalization of the symmetric first-order coupled-mode system. *SIAM J. Appl. Dyn. Syst.* 5:66–83.

[8] Comech, A., Guan, M., Gustafson, S. (2014). On linear instability of solitary waves for the nonlinear Dirac equation. *Annales de lÍnstitute Henri Poincaré* 31:639–654.

[9] Contreras, A., Pelinovsky, D.E. (2014). Stability of multi-solitons in the cubic NLS equation. *J. Hyper. Diff. Eqs.* 11:329–353.

[10] Cuccagna, S., Pelinovsky, D.E. (2014). The asymptotic stability of solitons in the cubic NLS equation on the line. *Appl. Anal.* 93:791–822.

[11] Delgado, V. (1978). Global solutions of the Cauchy problem for the (classical) coupled Maxwell–Dirac and other nonlinear Dirac equations in one space dimension. *Proc. AMS* 69:289–296.

[12] Derks, G., Gottwald, G.A. (2005). A robust numerical method to study oscillatory instability of gap solitary waves. *SIAM J. Appl. Dyn. Syst.* 4:140–158.

[13] Hoffman, A., Wayne, C.E. (2013). Orbital stability of localized structures via Backlund transformations. *Diff. Int. Eqs.* 26:303–320.

[14] Huh, H., Moon, B. (2015). Low regularity well-posedness for Gross–Neveu equations. *Comm. Pure Appl. Anal.* 14:1903–1913.

[15] Imai, K. (1999). Generalization of Kaup-Newell inverse scattering formulation and Darboux transformation. *J. Phys. Soc. Japan.* 68:355–359.

[16] Kaup, D.J., Lakoba, T.I. (1996). The squared eigenfunctions of the massive Thirring model in laboratory coordinates. *J. Math. Phys.* 37:308–323.

[17] Kaup, D.J., Newell, A.C. (1977). On the Coleman correspondence and the solution of the Massive Thirring model. *Lett. Nuovo Cimento* 20:325–331.

[18] Komech, A., Komech, A. (2010). Global attraction to solitary waves for a nonlinear Dirac equation with mean field interaction. *SIAM J. Math. Anal.* 42:2944–2964.

[19] Kopylova, E.A. (2011). Weighted energy decay for 1D Dirac equation. *Dynamics PDE* 8:113–125.

[20] Kuznetzov, E.A., Mikhailov, A.V. (1977). On the complete integrability of the two-dimensional classical Thirring model. *Theor. Math. Phys.* 30:193–200.

[21] Merle, F., Vega, L. (2003). L^2 stability of solitons for KdV equation. *Int. Math. Res. Not.* 13:735–753.

[22] Mizumachi, T., Pego, R.L. (2008). Asymptotic stability of Toda lattice solitons. *Nonlinearity* 21:2099–2111.

[23] Mizumachi, T., Pelinovsky, D.E. (2012). Backlund transformation and L^2-stability of NLS solitons. *Int. Math. Res. Not.* 9:2034–2067.

[24] Orfanidis, S.J. (1976). Soliton solutions of the massive Thirring model and the inverse scattering transform. *Phys. Rev. D* 14:472–478.

[25] Pelinovsky, D.E. (2011). Survey on global existence in the nonlinear Dirac equations in one dimension. *RIMS Kokyuroku Bessatsu* B26:37–50.
[26] Pelinovsky, D.E., Stefanov, A. (2012). Asymptotic stability of small gap solitons in nonlinear Dirac equations. *J. Math. Phys.* 53:073705.
[27] Pelinovsky, D.E., Shimabukuro, Y. (2014). Orbital stability of Dirac solitons. *Lett. Math. Phys.* 104:21–41.
[28] Selberg, S., Tesfahun, A. (2010). Low regularity well-posedness for some nonlinear Dirac equations in one space dimension. *Diff. Integral Eqs.* 23:265–278.