Integrable semi-discretization of the massive Thirring system in laboratory coordinates

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Abstract

Several integrable semi-discretizations are known in the literature for the massive Thirring system in characteristic coordinates. We present for the first time an integrable semi-discretization of the massive Thirring system in laboratory coordinates. Our approach relies on the relation between the continuous massive Thirring system and the Ablowitz–Ladik lattice. The Bäcklund transformation for solutions to the Ablowitz–Ladik lattice and the time evolution of the massive Thirring system in laboratory coordinates are combined together in the derivation of the Lax system for the integrable semi-discretization of the massive Thirring system.

1 Introduction

The purpose of this work is to find an integrable semi-discretization of the massive Thirring model (MTM) in laboratory coordinates [29]. In the space of (1+1) dimensions, the MTM can be written as a system of two semi-linear equations for $(u,v) \in \mathbb{C}^2$ in the normalized form:

$$\left\{ \begin{array}{l} i\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) + v = |v|^2 u, \\ i\left(\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x}\right) + u = |u|^2 v. \end{array} \right. \tag{1}$$

The standard Cauchy problem is posed in the time coordinates $t \in \mathbb{R}$ for the initial data (u_0, v_0) extended in the spatial coordinate $x \in \mathbb{R}$. Solutions of the Cauchy problem with $(u_0, v_0) \in L^2(\mathbb{R})$ for the MTM system (1) were studied recently in [7, 8, 13, 14, 15, 28, 36, 37].

Integrability of the MTM follows from the existence of the following Lax operators:

$$L(\lambda; u, v) = \frac{1}{2}(|u|^2 - |v|^2)\sigma_3 + \begin{bmatrix} 0 & \lambda v + \lambda^{-1} u \\ \lambda \bar{v} + \lambda^{-1}\bar{u} & 0 \end{bmatrix} + \frac{1}{2}\left(\lambda^2 - \lambda^{-2}\right)\sigma_3, \tag{2}$$

and

$$A(\lambda; u, v) = -\frac{1}{2}(|u|^2 + |v|^2)\sigma_3 + \begin{bmatrix} 0 & \lambda v - \lambda^{-1} u \\ \lambda \bar{v} - \lambda^{-1}\bar{u} & 0 \end{bmatrix} + \frac{1}{2}\left(\lambda^2 + \lambda^{-2}\right)\sigma_3, \tag{3}$$

where λ is a spectral parameter, $\sigma_3 = \mathrm{diag}(1,-1)$ is the third Pauli spin matrix, and $(\bar{u}, \bar{v})$ stands for the complex conjugate of (u,v). The MTM system (1) is the compatibility condition

$$\frac{\partial^2 \vec{\varphi}}{\partial x \partial t} = \frac{\partial^2 \vec{\varphi}}{\partial t \partial x}, \tag{4}$$

for $\vec{\varphi} \in \mathbb{C}^2$ satisfying the following Lax equations:

$$-2i\frac{\partial \vec{\varphi}}{\partial x} = L(\lambda; u, v)\vec{\varphi} \quad \text{and} \quad -2i\frac{\partial \vec{\varphi}}{\partial t} = A(\lambda; u, v)\vec{\varphi}. \tag{5}$$

Lax operators in the form (2) and (3) were introduced by Mikhailov [21] and then studied in [18, 19]. More recently, the same Lax operators have been used for many purposes, e.g. for the inverse scattering transform [25, 35], for spectral stability of solitary waves [17], and for orbital stability of Dirac solitons [10, 26].

Numerical simulations of nonlinear PDEs rely on spatial semi-discretizations obtained either with finite-difference or spectral methods. Because the energy functional of the MTM system (1) near the zero equilibrium $(u, v) = (0, 0)$ is not bounded, either from above or below, the spatial discretizations of the MTM system (like in many other massive Dirac models) suffer from spurious eigenvalues and other numerical instabilities, see recent studies in [5, 11, 12, 20, 27]. The integrable semi-discretization of the MTM system, if it can be constructed, preserves the integrability scheme and models dynamics of nonlinear waves without such spurious instabilities and other artifacts.

Since the pioneering works of Ablowitz and Ladik on equations related to the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem [1, 2], it is well known that continuous nonlinear evolution equations integrable with the inverse scattering transform can be semi-discretized in spatial coordinates or fully discretized in both spatial and temporal coordinates in such a way as to preserve integrability [16]. Since the literature on this subject is vast, we shall only restrict our attention to the relevant results on the massive Thirring model.

With the rotation of coordinates

$$\tau = \frac{1}{2}(t - x), \quad \xi = -\frac{1}{2}(x + t), \tag{6}$$

the MTM system (1) in laboratory coordinates (x, t) can be written in characteristic coordinates (ξ, τ):

$$\begin{cases} -i\frac{\partial u}{\partial \xi} + v = |v|^2 u, \\ i\frac{\partial v}{\partial \tau} + u = |u|^2 v. \end{cases} \tag{7}$$

The Cauchy problem in the time coordinate τ and the spatial coordinate ξ for the system (7) corresponds to the Goursat problem for the system (1) and vice versa. Therefore, the spatial discretization of the system (7) in the spatial coordinate ξ is not relevant for the Cauchy problem for the system (1). Unfortunately, it is the only spatial discretization available for the MTM up to now, thanks to the relatively simple connection of the Lax operators for the system (7) to the first negative flow of the Kaup–Newell (KN) spectral problem [18].

The first result on the integrable discretizations of the MTM system in characteristic coordinates goes back to the works of Nijhoff *et al.* in [22, 23]. By using the Bäcklund transformation for the continuous equations related to the KN spectral problem [22], integrable semi-discretizations in ξ or full discretizations in ξ and τ were obtained for the MTM system (7) and its equivalent formulations [23]. Since the relevant Bäcklund transformation contains a square root singularity [22], the resulting discretizations inherit a square root singularity [23], which may cause problems because of ambiguity in the choice of square root branches and sign-indefinite expressions under the square root signs. Unlike the continuous system (7), the spatial discretizations constructed in [23] were not written in terms of the cubic nonlinear terms.

In a different direction, Tsuchida in [30] explored a gauge transformation of the KN spectral problem to the AKNS spectral problem and constructed integrable semi-discretizations of nonlinear equations related to the KN spectral problem. The semi-discretization constructed in [30] had cubic nonlinearity but had a limitation of a different kind. The complex conjugate symmetry of the semi-discrete MTM system was related to the lattice shift by half of the lattice spacing, where the variables were not defined.

In the latest work [32], Tsuchida obtained another semi-discretization of the MTM system (7) by generalizing the Ablowitz–Ladik (AL) spectral problem [3] and Bäcklund–Darboux transformations for nonlinear equations related to the AL spectral problem [31, 34, 38]. The new semi-discretization of the MTM system (7) in [32] contains the cubic nonlinearity and the complex conjugate reduction, which resemble those in the continuous system (7).

How to transfer integrable semi-discretizations of the second-order equations in characteristic coordinates (such as the sine–Gordon equation or the MTM system) to the integrable semi-discretizations of these equations in laboratory coordinates has been considered to be an open problem for many years. The main obstacle here is that the rotation of coordinates mixes positive and negative powers of the spectral parameter λ in the Lax operators related to the continuous case, hence the semi-discretization scheme needs to be revised. At the same time, the temporal and spatial coordinates are already different in the semi-discrete case (one is continuous and the other one is discrete) so that the rotation of coordinates produces a complicated difference-differential equation. Since it is the Cauchy problem for the MTM system in laboratory coordinates that is used for most of applications of the MTM system (1), constructing a proper integrable semi-discretization of it becomes relevant and important.

2 Main result

Here we implement the method of Tsuchida from his recent work [32] and obtain the desired integrable semi-discretization of the MTM system in laboratory coordinates. We start with the gauge-modified Lax operators for the MTM system (1) derived by Barashenkov and Getmanov in [4]:

$$\mathcal{L}(\lambda; u, v) = \begin{bmatrix} \lambda^2 - |v|^2 & \lambda v + \lambda^{-1} u \\ \lambda \bar{v} + \lambda^{-1} \bar{u} & \lambda^{-2} - |u|^2 \end{bmatrix}, \tag{8}$$

and

$$\mathcal{A}(\lambda; u, v) = \begin{bmatrix} \lambda^2 - |v|^2 & \lambda v - \lambda^{-1} u \\ \lambda \bar{v} - \lambda^{-1} \bar{u} & -\lambda^{-2} + |u|^2 \end{bmatrix}. \tag{9}$$

The gauge-modified Lax formulation (8)–(9) differs from the classical (zero-trace) Lax formulation (2)–(3) only in the diagonal terms. The MTM system (1) still arises as the compatibility condition (4), where $\vec{\varphi} \in \mathbb{C}^2$ satisfies the Lax equations

$$-2i \frac{\partial \vec{\varphi}}{\partial x} = \mathcal{L}(\lambda; u, v) \vec{\varphi} \quad \text{and} \quad -2i \frac{\partial \vec{\varphi}}{\partial t} = \mathcal{A}(\lambda; u, v) \vec{\varphi}, \tag{10}$$

which are related to the new operators $\mathcal{L}$ and $\mathcal{A}$ in (8)–(9).

The main result of this work is a derivation of the following spatial discretization of the MTM system (1) suitable for the Cauchy problem in the laboratory coordinates:

$$\left\{ \begin{array}{l} 4i \frac{dU_n}{dt} + Q_{n+1} + Q_n + \frac{2i}{h}(R_{n+1} - R_n) + U_n^2(\bar{R}_n + \bar{R}_{n+1}) \\ \qquad -U_n(|Q_{n+1}|^2 + |Q_n|^2 + |R_{n+1}|^2 + |R_n|^2) - \frac{ih}{2} U_n^2 (\bar{Q}_{n+1} - \bar{Q}_n) = 0, \\ -\frac{2i}{h}(Q_{n+1} - Q_n) + 2U_n - |U_n|^2 (Q_{n+1} + Q_n) = 0, \\ R_{n+1} + R_n - 2U_n + \frac{ih}{2} |U_n|^2 (R_{n+1} - R_n) = 0, \end{array} \right. \tag{11}$$

where h is the lattice spacing parameter and n is the discrete lattice variable. In the limit $h \to 0$ the slowly varying solutions between the lattice nodes can be represented by

$$U_n(t) = U(x = hn, t), \quad R_n(t) = R(x = hn, t), \quad Q_n(t) = Q(x = nh, t),$$

where the field variables satisfy the continuum limit of the system (11) given by

$$\left\{ \begin{array}{l} 2i \frac{\partial U}{\partial t} + i \frac{\partial R}{\partial x} + Q + U^2 \bar{R} - U(|Q|^2 + |R|^2) = 0, \\ -i \frac{\partial Q}{\partial x} + U - |U|^2 Q = 0, \\ R - U = 0. \end{array} \right. \tag{12}$$

The system (12) in variables $U(x,t) = u(x, t-x)$ and $Q(x,t) = v(x, t-x)$ yields the MTM system (1). Therefore, we can claim that the system (11) is a proper integrable spatial semi-discretization of the continuous MTM system in laboratory coordinates.

Note that the last two difference equations of the system (11) are uncoupled between the components $\{R_n\}_{n\in\mathbb{Z}}$ and $\{Q_n\}_{n\in\mathbb{Z}}$ and, moreover, they are linear with respect to $\{R_n\}_{n\in\mathbb{Z}}$ and $\{Q_n\}_{n\in\mathbb{Z}}$. If the sequence $\{U_n\}_{n\in\mathbb{Z}}$ decays to zero as $|n|\to\infty$ sufficiently fast, then one can express R_n and Q_n in an explicit form by

$$R_n = 2\sum_{k=-\infty}^{n-1}(-1)^{n-k-1}U_k\frac{\prod_{m=k+1}^{n-1}(1-ih|U_m|^2/2)}{\prod_{m=k}^{n-1}(1+ih|U_m|^2/2)}, \tag{13}$$

and

$$Q_n = -ih\sum_{k=-\infty}^{n-1}U_k\frac{\prod_{m=k+1}^{n-1}(1+ih|U_m|^2/2)}{\prod_{m=k}^{n-1}(1-ih|U_m|^2/2)}. \tag{14}$$

The time evolution of U_n is obtained by solving the first differential equation of the system (11). The representation (13) and (14) simplifies the numerical construction of the solutions of the semi-discrete system (11).

As follows from our work, the semi-discretization of the MTM system (11) is integrable in the sense of the following Lax equations:

$$\vec{\varphi}_{n+1} = \left[\begin{array}{cc} \lambda + 2ih^{-1}\lambda^{-1}\frac{1+ih|U_n|^2/2}{1-ih|U_n|^2/2} & \frac{2U_n}{1-ih|U_n|^2/2} \\ \frac{2\bar{U}_n}{1-ih|U_n|^2/2} & -\lambda\frac{1+ih|U_n|^2/2}{1-ih|U_n|^2/2} + 2ih^{-1}\lambda^{-1} \end{array}\right]\vec{\varphi}_n, \tag{15}$$

and

$$-2i\left(\frac{d\vec{\varphi}_n}{dt}\right)_n = \begin{bmatrix} \lambda^2 - |R_n|^2 & \lambda R_n - \lambda^{-1}Q_n \\ \lambda\bar{R}_n - \lambda^{-1}\bar{Q}_n & -\lambda^{-2} + |Q_n|^2 \end{bmatrix}\vec{\varphi}_n. \tag{16}$$

The semi-discrete MTM system (11) is the compatibility condition

$$\frac{d}{dt}\vec{\varphi}_{n+1} = \left(\frac{d\vec{\varphi}}{dt}\right)_{n+1}, \tag{17}$$

for $\vec{\varphi}\in\mathbb{C}^2$ on $n\in\mathbb{Z}$ and $t\in\mathbb{R}$. In particular, the trivial zero solution satisfies the MTM system (11) and reduces (15) and (16) to uncoupled equations for components of $\vec{\varphi}$ which are readily solvable. Non-trivial solutions of the semi-discrete MTM system (11) will be constructed in future work.

3 Proof of the main result

Here we derive the system (11) as an integrable semi-discretization of the MTM system (1). This follows the work [32] with suitable modifications.

The MTM system in characteristic coordinates (7) can be written as the compatibility condition of the following Lax equations

$$i\frac{\partial\vec{\varphi}}{\partial\xi} = \left[\begin{array}{cc} \lambda^2 - |v|^2 & \lambda v \\ \lambda\bar{v} & 0 \end{array}\right]\vec{\varphi} \quad \text{and} \quad i\frac{\partial\vec{\varphi}}{\partial\tau} = \left[\begin{array}{cc} 0 & \lambda^{-1}u \\ \lambda^{-1}\bar{u} & \lambda^{-2} - |u|^2 \end{array}\right]\vec{\varphi}. \tag{18}$$

This can be checked by direct differentiation from the compatibility condition

$$\frac{\partial^2\vec{\varphi}}{\partial\xi\partial\tau} = \frac{\partial^2\vec{\varphi}}{\partial\tau\partial\xi}.$$

Consequently, by using the transformation (6) in the Lax equations (18), we obtain the Lax equations (2)–(3) for the MTM in laboratory coordinates (1).

The Lax equations (18) with triangular matrices are different from the classical Lax equations with zero-trace matrices [21, 19]:

$$i\frac{\partial\vec{\varphi}}{\partial\xi} = \left[\begin{array}{cc} \frac{1}{2}(\lambda^2 - |v|^2) & \lambda v \\ \lambda\bar{v} & -\frac{1}{2}(\lambda^2 - |v|^2) \end{array}\right]\vec{\varphi}, \quad i\frac{\partial\vec{\varphi}}{\partial\tau} = \left[\begin{array}{cc} -\frac{1}{2}(\lambda^{-2} - |u|^2) & \lambda^{-1}u \\ \lambda^{-1}\bar{u} & \frac{1}{2}(\lambda^{-2} - |u|^2) \end{array}\right]\vec{\varphi}. \tag{19}$$

where the ξ-dependent problem is referred to as Kaup–Newell spectral problem [18] and the τ-dependent problem is the first negative flow of the Kaup–Newell hierarchy. As shown in [4], the two formulations are gauge-equivalent by using the conservation law

$$\frac{\partial|v|^2}{\partial\tau} = \frac{\partial|u|^2}{\partial\xi}, \tag{20}$$

which holds for the MTM system in characteristic coordinates (7). Adding $\frac{1}{2}(\lambda^2 - |v|^2)I_{2\times 2}$ to the ξ-dependent problem and $\frac{1}{2}(\lambda^{-2} - |u|^2)I_{2\times 2}$ to the τ-dependent problem in the classical Lax equations (19) with zero-trace matrices, where $I_{2\times 2}$ is the 2×2 identity matrix, yields the Lax equations (18) with triangular matrices. The MTM system (7) is invariant under the gauge transformation of the Lax operators [4].

The gauge-modified Lax formulation (18) of the MTM system (7) is related to the following Ablowitz–Ladik (AL) lattice:

$$\left\{ \begin{array}{l} \frac{dQ_m}{dt} = a(1-Q_mR_m)(Q_{m+1}-Q_{m-1}) + ib(1-Q_mR_m)(Q_{m+1}+Q_{m-1}), \\ \frac{dR_m}{dt} = a(1-Q_mR_m)(R_{m+1}-R_{m-1}) - ib(1-Q_mR_m)(R_{m+1}+R_{m-1}), \end{array} \right. \quad m\in\mathbb{Z}, \tag{21}$$

where $a, b \in \mathbb{R}$ are parameters of the model. In order to show how the AL lattice (21) is related to the MTM system (7), we write the Lax equations for the AL lattice:

$$\vec{\varphi}_{m+1} = \left[\begin{array}{cc} \lambda & Q_m \\ R_m & \lambda^{-1} \end{array} \right] \vec{\varphi}_m, \tag{22}$$

and

$$\left(\frac{d\vec{\varphi}}{\partial t}\right)_m = (a+ib)\left[\begin{array}{cc} \lambda^2 - Q_mR_{m-1} & \lambda Q_m \\ \lambda R_{m-1} & 0 \end{array} \right]\vec{\varphi}_m + (a-ib)\left[\begin{array}{cc} 0 & \lambda^{-1}Q_{m-1} \\ \lambda^{-1}R_m & \lambda^{-2} - Q_{m-1}R_m \end{array} \right]\vec{\varphi}_m. \tag{23}$$

The compatibility condition

$$\frac{d}{dt}\vec{\varphi}_{m+1} = \left(\frac{d\vec{\varphi}}{dt}\right)_{m+1}, \tag{24}$$

for the system (22)–(23) yields the AL lattice (21). With the correspondence

$$Q_{m-1} = u, \quad Q_m = v, \quad R_{m-1} = \bar{v}, \quad R_m = \bar{u},$$

the two matrix operators in the linear combination of the time evolution equation (23) can be used in the Lax equations (18). In this context, the index m can be dropped and the variables (u, v) satisfy the MTM system (7) from commutativity of the Lax equations (18).

It is known [34, 38] that a new solution $\{\tilde{Q}_m, \tilde{R}_m\}_{m\in\mathbb{Z}}$ of the AL lattice (21) can be obtained from another solution $\{Q_m, R_m\}_{m\in\mathbb{Z}}$ of the same lattice by the Bäcklund–Darboux transformation. The Bäcklund–Darboux transformation also relates the eigenvectors $\vec{\varphi}_m$ and $\tilde{\vec{\varphi}}_m$ satisfying the Lax equations (22) and (23) for the same spectral parameter λ. The relation between $\vec{\varphi}_m$ and $\tilde{\vec{\varphi}}_m$ can be written in the form used in [32]:

$$\tilde{\vec{\varphi}}_m = \left[\begin{array}{cc} \alpha\lambda + \delta\lambda^{-1} & 0 \\ 0 & \gamma\lambda + \beta\lambda^{-1} \end{array} \right]\vec{\varphi}_m + (\alpha\beta - \gamma\delta)\left[\begin{array}{cc} \gamma\lambda & U_m \\ V_m & \delta\lambda^{-1} \end{array} \right]^{-1}\vec{\varphi}_m, \tag{25}$$

where $(\alpha, \beta, \gamma, \delta)$ are arbitrary parameters such that $\alpha\beta - \gamma\delta \neq 0$ and $\{U_m, V_m\}_{m\in\mathbb{Z}}$ are some potentials. For the purpose of the Bäcklund–Darboux transformation for the AL lattice, the potentials $\{U_m, V_m\}_{m\in\mathbb{Z}}$ are expressed in terms of eigenfunctions satisfying the spectral problem (22) at a fixed value of the spectral parameter λ. However, for the purpose of the integrable semi-discretization of the MTM system, we specify constraints on $\{U_m, V_m\}_{m\in\mathbb{Z}}$ from the commutativity condition below.

Let us drop the index m and forget about the AL lattice (21) and the spectral problem (22). If the Bäcklund–Darboux transformation (25) is iterated in new index n, we can introduce the new spectral problem

$$\vec{\varphi}_{n+1} = \left[\begin{array}{cc} \alpha\lambda + \delta\lambda^{-1} & 0 \\ 0 & \gamma\lambda + \beta\lambda^{-1} \end{array} \right] \vec{\varphi}_n + (\alpha\beta - \gamma\delta) \left[\begin{array}{cc} \gamma\lambda & U_n \\ V_n & \delta\lambda^{-1} \end{array} \right]^{-1} \vec{\varphi}_n. \tag{26}$$

This spectral problem is coupled with the time evolution problem

$$-2i \left(\frac{d\vec{\varphi}}{\partial t} \right)_n = \begin{bmatrix} \lambda^2 - |R_n|^2 & \lambda R_n - \lambda^{-1} Q_n \\ \lambda \bar{R}_n - \lambda^{-1} \bar{Q}_n & -\lambda^{-2} + |Q_n|^2 \end{bmatrix} \vec{\varphi}_n, \tag{27}$$

which is obtained from the time evolution problem for the MTM system in characteristic coordinates, see equations (9) and (10). In the Lax equations (26) and (27), we have unknown potentials $\{U_n, V_n, R_n, Q_n\}_{n\in\mathbb{Z}}$ and arbitrary parameters $(\alpha, \beta, \gamma, \delta)$ such that $\alpha\beta - \gamma\delta \neq 0$.

In what follows, we consider the compatibility condition (17) and obtain the constraints on these unknown potentials. The compatibility condition (17) can be written as the following Lax equation

$$-2i \frac{d}{dt} N_n = P_{n+1} N_n - N_n P_n, \tag{28}$$

where we have introduced

$$N_n := \left[\begin{array}{cc} \alpha\lambda + \delta\lambda^{-1} & 0 \\ 0 & \gamma\lambda + \beta\lambda^{-1} \end{array} \right] + (\alpha\beta - \gamma\delta) \left[\begin{array}{cc} \gamma\lambda & U_n \\ V_n & \delta\lambda^{-1} \end{array} \right]^{-1},$$

and

$$P_n := \begin{bmatrix} \lambda^2 - |R_n|^2 & \lambda R_n - \lambda^{-1} Q_n \\ \lambda \bar{R}_n - \lambda^{-1} \bar{Q}_n & -\lambda^{-2} + |Q_n|^2 \end{bmatrix}.$$

Since the spectral parameter λ is independent of t, we obtain

$$\frac{d}{dt} N_n = -(\alpha\beta - \gamma\delta) \left[\begin{array}{cc} \gamma\lambda & U_n \\ V_n & \delta\lambda^{-1} \end{array} \right]^{-1} \left[\begin{array}{cc} 0 & \frac{dU_n}{dt} \\ \frac{dV_n}{dt} & 0 \end{array} \right] \left[\begin{array}{cc} \gamma\lambda & U_n \\ V_n & \delta\lambda^{-1} \end{array} \right]^{-1}.$$

For N_n in the first term of the right-hand side of the Lax equation (28), we can use the equivalent representation

$$N_n = \left[\begin{array}{cc} \alpha\lambda(\gamma\lambda + \beta\lambda^{-1}) & U_n(\alpha\lambda + \delta\lambda^{-1}) \\ V_n(\gamma\lambda + \beta\lambda^{-1}) & \beta\lambda^{-1}(\alpha\lambda + \delta\lambda^{-1}) \end{array} \right] \left[\begin{array}{cc} \gamma\lambda & U_n \\ V_n & \delta\lambda^{-1} \end{array} \right]^{-1}.$$

For N_n in the second term of the right-hand side of the Lax equation (28), we can use the equivalent representation

$$N_n = \left[\begin{array}{cc} \gamma\lambda & U_n \\ V_n & \delta\lambda^{-1} \end{array} \right]^{-1} \left[\begin{array}{cc} \alpha\lambda(\gamma\lambda + \beta\lambda^{-1}) & U_n(\gamma\lambda + \beta\lambda^{-1}) \\ V_n(\alpha\lambda + \delta\lambda^{-1}) & \beta\lambda^{-1}(\alpha\lambda + \delta\lambda^{-1}) \end{array} \right].$$

Substituting these expressions to the Lax equation (28) yields the following two constraints arising at different powers of λ in the $(1,1)$ entries:

$$\alpha\gamma(|R_{n+1}|^2 - |R_n|^2) + \gamma(U_n \bar{R}_n - V_n R_{n+1}) + \alpha(V_n R_n - U_n \bar{R}_{n+1}) = 0, \tag{29}$$

$$U_n V_n(|Q_n|^2 - |Q_{n+1}|^2) + \alpha(U_n \bar{Q}_{n+1} - V_n Q_n) + \gamma(V_n Q_{n+1} - U_n \bar{Q}_n) = 0 \tag{30}$$

and the following two constraints arising at different powers of λ in the $(2,2)$ entries:

$$U_n V_n(|R_{n+1}|^2 - |R_n|^2) + \beta(U_n \bar{R}_n - V_n R_{n+1}) + \delta(V_n R_n - U_n \bar{R}_{n+1}) = 0, \tag{31}$$

$$\beta\delta(|Q_n|^2 - |Q_{n+1}|^2) + \delta(U_n \bar{Q}_{n+1} - V_n Q_n) + \beta(V_n Q_{n+1} - U_n \bar{Q}_n) = 0. \tag{32}$$

We will show that the constraints (29)–(32) are equivalent to the following constraints:

$$\begin{aligned} \gamma(\alpha\beta - U_n V_n) R_{n+1} - \alpha(\delta\gamma - U_n V_n) R_n &= (\alpha\beta - \delta\gamma) U_n, && (33) \\ \alpha(\gamma\delta - U_n V_n) \bar{R}_{n+1} - \gamma(\alpha\beta - U_n V_n) \bar{R}_n &= -(\alpha\beta - \delta\gamma) V_n, && (34) \end{aligned}$$

and

$$\begin{aligned} \beta(\gamma\delta - U_n V_n) Q_{n+1} - \delta(\alpha\beta - U_n V_n) Q_n &= -(\alpha\beta - \delta\gamma) U_n, && (35) \\ \delta(\alpha\beta - U_n V_n) \bar{Q}_{n+1} - \beta(\gamma\delta - U_n V_n) \bar{Q}_n &= (\alpha\beta - \delta\gamma) V_n. && (36) \end{aligned}$$

To show the equivalence, we use linear combinations of (29) and (31) and obtain

$$\begin{aligned} \alpha(\gamma\delta - U_n V_n)(|R_{n+1}|^2 - |R_n|^2) - (\alpha\beta - \gamma\delta)(U_n \bar{R}_n - V_n R_{n+1}) &= 0, \\ \gamma(\alpha\beta - U_n V_n)(|R_{n+1}|^2 - |R_n|^2) + (\alpha\beta - \gamma\delta)(V_n R_n - U_n \bar{R}_{n+1}) &= 0. \end{aligned}$$

These relations are regrouped as follows:

$$\begin{aligned} R_{n+1}\left[\alpha(\gamma\delta - U_n V_n)\bar{R}_{n+1} + (\alpha\beta - \gamma\delta)V_n\right] - \bar{R}_n\left[\alpha(\gamma\delta - U_n V_n)R_n + (\alpha\beta - \gamma\delta)U_n\right] &= 0, \\ \bar{R}_{n+1}\left[\gamma(\alpha\beta - U_n V_n)R_{n+1} - (\alpha\beta - \gamma\delta)U_n\right] - R_n\left[\gamma(\alpha\beta - U_n V_n)\bar{R}_n - (\alpha\beta - \gamma\delta)V_n\right] &= 0, \end{aligned}$$

where each constraint is satisfied if and only if constraints (33) and (34) hold. The equivalence of constraints (30) and (32) to constraints (35) and (36) is established by similar operations.

At different powers of λ in the $(1,2)$ entries of the Lax equation (28), we obtain two constraints

$$\begin{aligned} \alpha\gamma U_n(|R_{n+1}|^2 - |R_n|^2) + (\alpha\beta - \gamma\delta)U_n + \alpha\gamma(\delta R_n - \beta R_{n+1}) + U_n^2(\gamma\bar{R}_n - \alpha\bar{R}_{n+1}) &= 0, && (37) \\ \beta\delta U_n(|Q_n|^2 - |Q_{n+1}|^2) + (\alpha\beta - \gamma\delta)U_n + \beta\delta(\gamma Q_{n+1} - \alpha Q_n) + U_n^2(\delta\bar{Q}_{n+1} - \beta\bar{Q}_n) &= 0, && (38) \end{aligned}$$

and the evolution equation

$$\begin{aligned} & 2i(\alpha\beta - \gamma\delta)\frac{dU_n}{dt} + \alpha\gamma(\beta Q_{n+1} - \delta Q_n) + \beta\delta(\alpha R_n - \gamma R_{n+1}) + U_n^2(\alpha\bar{Q}_{n+1} - \gamma\bar{Q}_n) \\ & + U_n^2(\beta\bar{R}_n - \delta\bar{R}_{n+1}) + U_n(\gamma\delta|Q_n|^2 - \alpha\beta|Q_{n+1}|^2) + U_n(\gamma\delta|R_{n+1}|^2 - \alpha\beta|R_n|^2) = 0. && (39) \end{aligned}$$

We show that the two constraints (37) and (38) are redundant in view of the constraints (33)–(36). Indeed, substituting

$$\alpha\gamma(\beta R_{n+1} - \delta R_n) - (\alpha\beta - \delta\gamma)U_n = U_n V_n(\gamma R_{n+1} - \alpha R_n)$$

from (33) into (37) and dividing the result by U_n, we obtain

$$\alpha\gamma(|R_{n+1}|^2 - |R_n|^2) - V_n(\gamma R_{n+1} - \alpha R_n) + U_n(\gamma\bar{R}_n - \alpha\bar{R}_{n+1}) = 0,$$

which is nothing but (29). Similar transformations apply to (38) with (35) to end up at (32). Hence the constraints (37) and (38) are satisfied if the constraints (33)–(36) hold.

Similarly, at different powers of λ in the $(2,1)$ entries of the Lax equation (28), we obtain two constraints

$$\begin{aligned} \alpha\gamma V_n(|R_{n+1}|^2 - |R_n|^2) - (\alpha\beta - \gamma\delta)V_n + \alpha\gamma(\beta\bar{R}_n - \delta\bar{R}_{n+1}) + V_n^2(\alpha R_n - \gamma R_{n+1}) &= 0, && (40) \\ \beta\delta V_n(|Q_n|^2 - |Q_{n+1}|^2) - (\alpha\beta - \gamma\delta)V_n + \beta\delta(\alpha\bar{Q}_{n+1} - \gamma\bar{Q}_n) + V_n^2(\beta Q_{n+1} - \delta Q_n) &= 0, && (41) \end{aligned}$$

and the evolution equation

$$\begin{aligned} & 2i(\alpha\beta - \gamma\delta)\frac{dV_n}{dt} + \alpha\gamma(\delta\bar{Q}_{n+1} - \beta\bar{Q}_n) + \beta\delta(\gamma\bar{R}_n - \alpha\bar{R}_{n+1}) + V_n^2(\gamma Q_{n+1} - \alpha Q_n) \\ & + V_n^2(\delta R_n - \beta R_{n+1}) + V_n(\alpha\beta|Q_n|^2 - \gamma\delta|Q_{n+1}|^2) + V_n(\alpha\beta|R_{n+1}|^2 - \gamma\delta|R_n|^2) = 0. && (42) \end{aligned}$$

The two constraints (40) and (41) are once again redundant again in view of the constraints (33)–(36). The proof of this is achieved by transformations similar to the previous computations. Thus, we have shown that the Lax equation (28) is satisfied under the four constraints (33)–(36) and the two evolution equations (39) and (42).

The constraints (35)–(36) and the time evolution equations (39) and (42) with $R_n = 0$ are obtained in [32], where the Bäcklund–Darboux transformation (26) is coupled with the time flow given by the negative powers of λ. Similarly, the constraints (33)–(34) and the time evolution equations (39) and (42) with $Q_n = 0$ are obtained when the Bäcklund–Darboux transformation (26) is coupled with the time flow given by the positive powers of λ. All four constraints(33)–(36) and the full system of time evolution equations (39) and (42) arise in the full time flow (27), which is relevant for the MTM system in laboratory coordinates.

The complex-conjugate symmetry $V_n = \bar{U}_n$ between the constraints (33) and (35) on one side and the constraints (34) and (36) on the other side as well as between the evolution equations (39) and (42) is preserved if $\alpha = \bar{\gamma}$ and $\beta = \bar{\delta}$. Without loss of generality, we normalize $\alpha = \gamma = 1$ and introduce the parameter $h \in \mathbb{R}$ by $\beta = -\delta = 2i/h$. Then, equations (33), (35), and (39) divided by $2i/h$ give respectively the third, second, and first equations of the system (11). Thus, integrability of the system (11) is verified from the Lax equation (28) with the Lax operators written explicitly in the system (15) and (16).

4 Conclusion

We have derived an integrable semi-discretization of the MTM system in laboratory coordinates (1). Integrability of the semi-discrete MTM system (11) with the Lax operators (15) and (16) is a starting point for derivation of conserved quantities, multi-soliton solutions, and other useful facts of the semi-discrete MTM in laboratory coordinates. It is also a starting point for numerical simulations of the integrable semi-discretizations of the continuous MTM system (1).

We conclude by mentioning some relevant works on the semi-discretizations of other integrable nonlinear evolution equations.

By the semi-discretization of the Hirota bilinear method, one can obtain an integrable semi-discretization of many continuous nonlinear equations as done in [9] for the coupled Yajima–Oikawa system. One can verify [24] that this technique applied to the Chen–Lee–Liu system yields the same semi-discretization as the one obtained by Tsuchida [32] from coupling the Bäcklund transformation for the AL lattice and the time evolution of the Chen–Lee–Liu system.

From a different point of view, Lax operators for the AL-type lattices with quadratic and higher-order polynomial dependence were considered by Vakhnenko (see the recent review in [33] and earlier references therein). This approach brings many interesting semi-discretizations of coupled nonlinear Schrödinger equations, but these equations are not related to the semi-discretizations of the massive Thirring model [6].

We conclude that the present contribution contains the only integrable semi-discretization of the MTM system in the laboratory coordinates available in the present time.

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