



Morse index for the ground state in the energy supercritical Gross–Pitaevskii equation

Dmitry E. Pelinovsky*, Szymon Sobieszek

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada

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Abstract

The ground state of the energy super-critical Gross–Pitaevskii equation with a harmonic potential converges in the energy space to the singular solution in the limit of large amplitudes. The ground state can be represented by a solution curve which has either oscillatory or monotone behavior, depending on the dimension of the system and the power of the focusing nonlinearity. We address here the monotone case for the cubic nonlinearity in the spatial dimensions $d \geq 13$. By using the shooting method for the radial Schrödinger operators, we prove that the Morse index of the ground state is finite and is independent of the (large) amplitude. We also prove that it is equal to the Morse index of the limiting singular solution, which can be computed from numerical approximations. The numerical results suggest that the Morse index of the ground state is equal to one for every $d \geq 13$.

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1. Introduction

We consider the stationary Gross-Pitaevskii equation with a harmonic potential,

$$(-\Delta + |x|^2)u - |u|^{2p}u = \lambda u, \quad (1.1)$$

where $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, and $u \in \mathbb{R}$. Existence of its ground state (a positive and radially decreasing solution) has been addressed before in the energy subcritical [7,13], critical [20], and supercritical

* Corresponding author.

E-mail addresses: dmpeli@math.mcmaster.ca (D.E. Pelinovsky), sobiesz@math.mcmaster.ca (S. Sobieszek).

[21,22] regimes, where the critical exponent is $p = \frac{2}{d-2}$ if $d \geq 3$. Uniqueness of the ground state was proven in the energy subcritical [9,10] and critical [23] cases. We are concerned here with the energy supercritical case of the focusing Gross–Pitaevskii equation. Scattering in the defocussing version of energy supercritical equations was studied in [15–17].

The stationary Gross–Pitaevskii equation (1.1) is the Euler–Lagrange equation for the action functional $\Lambda_\lambda(u) = E(u) - \lambda M(u)$, where $E(u)$ and $M(u)$ are the energy and mass given by

$$M(u) = \int_{\mathbb{R}^d} |u|^2 dx \tag{1.2}$$

and

$$E(u) = \int_{\mathbb{R}^d} \left(|\nabla u|^2 + |x|^2 |u|^2 - \frac{1}{p+1} |u|^{2p+2} \right) dx. \tag{1.3}$$

The energy and mass are formally the conserved quantities in the evolution of the time-dependent Gross–Pitaevskii equation. They are defined in the energy space $\mathcal{E} \cap L_r^{2p+2}$, where

$$\mathcal{E} := \left\{ u \in L_r^2(\mathbb{R}^+) : u' \in L_r^2(\mathbb{R}^+), \quad ru \in L_r^2(\mathbb{R}^+) \right\} \tag{1.4}$$

and L_r^q denotes the space of radially symmetric $L^q(\mathbb{R}^d)$ functions.

The ground state of the stationary equation (1.1) can be obtained variationally in the energy subcritical and critical cases $p(d-2) \leq 2$, but the variational methods are not applicable in the energy supercritical case $p(d-2) > 2$ if $d \geq 3$. In all cases, the ground state forms a family which appears as the curve on the (λ, b) plane, where $b := u(0) \equiv \|u\|_{L^\infty}$ is referred to as *the amplitude*. The large-amplitude limit is the limit as $b \rightarrow \infty$. In the energy supercritical regime, it was proven in [22] that there exists $\lambda_\infty \in (0, d)$ such that $\lambda \rightarrow \lambda_\infty$ as $b \rightarrow \infty$ along the solution curve. It was discovered in our recent work [1] under some non-degeneracy assumptions that there exists $d_*(p)$ such that the solution curve is oscillatory for $2 + \frac{2}{p} < d < d_*(p)$ and monotone for $d > d_*(p)$.

For the sake of simplicity, we have been working with the cubic nonlinearity, $p = 1$, for which $d_*(p = 1) = 8 + 2\sqrt{6}$. We will continue working with the cubic nonlinearity here. Fig. 1.1 shows the dependence of $\lambda(b)$, which is oscillatory for $5 \leq d \leq 12$ and monotone for $d \geq 13$.

A similar duality between the oscillatory and monotone behaviors was discovered for the classical Liouville–Bratu–Gerfand problem in [12] and explored in [2,3], [4], and [18] for the stationary focusing nonlinear Schrödinger equation in a ball and without a harmonic potential. The similarity is explained by the same linearization of the stationary equation near the origin after the Emden–Fowler transformation [6]. Another example of the oscillatory and monotone behaviors was considered in [5] for the Schrödinger–Newton–Hooke model.

Let us define the ground state of the stationary equation (1.1) in radial variable $r = |x|$ as a solution of the following boundary-value problem:

$$\begin{cases} u''(r) + \frac{d-1}{r} u'(r) - r^2 u(r) + \lambda u(r) + u(r)^3 = 0, & r > 0, \\ u(r) > 0, & u'(r) < 0, \\ \lim_{r \rightarrow 0} u(r) < \infty, & \lim_{r \rightarrow \infty} u(r) = 0. \end{cases} \tag{1.5}$$

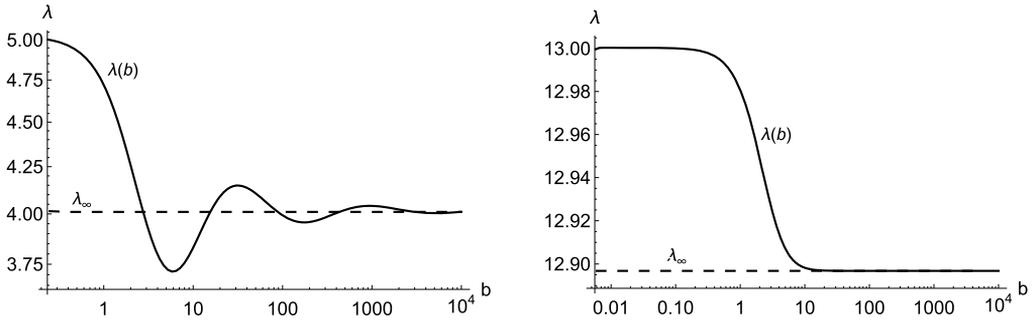


Fig. 1.1. Graph of λ versus b for the ground state of the boundary-value problem (1.5) for $d = 5$ (left) and $d = 13$ (right).

Any solution of the boundary-value problem (1.5) belongs to $\mathcal{E} \cap L^\infty$, since $u \in C^2(0, \infty)$ and $u(r) \rightarrow 0$ decays fast as $r \rightarrow \infty$.

As is well understood since the pioneering work in [12], the ground state of the boundary-value problem (1.5) can be found from the family of solutions to the following initial-value problem:

$$\begin{cases} f_b''(r) + \frac{d-1}{r} f_b'(r) - r^2 f_b(r) + \lambda f_b(r) + f_b(r)^3 = 0, & r > 0, \\ f_b(0) = b, \quad f_b'(0) = 0, \end{cases} \tag{1.6}$$

where $b > 0$ is an arbitrary parameter. By Theorem 1.1 in our previous work [1], for any $b > 0$ and $d \geq 4$, there exists some $\lambda \in (d - 4, d)$ such that the unique solution $f_b \in C^2(0, \infty)$ to the initial-value problem (1.6) is monotonically decaying to zero as $r \rightarrow \infty$, making it a ground state $u \equiv u_b \in \mathcal{E} \cap L^\infty$ of the boundary-value problem (1.5). That value of λ is denoted as $\lambda(b)$. The mapping $b \mapsto \lambda(b)$ defines a solution curve on the (λ, b) plane. Uniqueness of $\lambda(b)$ is an open problem for $d \geq 4$, whereas Fig. 1.1 suggests that $\lambda(b)$ is unique for every $b > 0$.

The purpose of this work is to study Morse index of the ground state u_b denoted by $m(u_b)$. The Morse index $m(u_b)$ is defined as the number of negative eigenvalues of the linearized operator \mathcal{L}_b given by

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d-1}{r} \frac{d}{dr} + r^2 - \lambda(b) - 3u_b^2(r). \tag{1.7}$$

Since \mathcal{E} is the form domain of \mathcal{L}_b , we can write $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$, where \mathcal{E}^* is the dual of \mathcal{E} with respect to the scalar product in L_r^2 .

Assuming C^1 property of u_b in b and differentiating the initial-value problem (1.6) with $\lambda = \lambda(b)$ in b , we can see that $\mathcal{L}_b \partial_b u_b = \lambda'(b) u_b$, where $\partial_b u_b \in \mathcal{E}$. Hence, any value of b for which $\lambda'(b) = 0$ corresponds to zero eigenvalue being in the spectrum of \mathcal{L}_b in L_r^2 . Although the converse is not known, this property implies that the oscillatory case is very different from the monotone case, where the former has infinitely many crossing of zero eigenvalue of \mathcal{L}_b in the parameter continuation in b as $b \rightarrow \infty$ whereas the latter does not have any eigenvalue crossing as $b \rightarrow \infty$, see also Fig. 1.1. This suggests that the Morse index should be well defined in the monotone case, independently of b for large values of b . This is in fact the main result which we formulate as the following theorem.

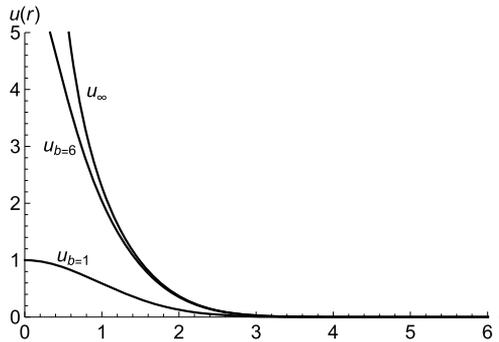


Fig. 1.2. Graph of the ground state u_b for $b = 1$ and $b = 6$ in comparison with the limiting singular solution u_∞ for $d = 13$.

Theorem 1.1. *Under non-degeneracy Assumptions 2.1 and 2.2, for every $d \geq 13$, there exists $b_0 > 0$ such that the Morse index $m(u_b)$ is finite and is independent of b for every $b \in (b_0, \infty)$.*

Remark 1.1. Regarding the Morse index for the ground state in the energy supercritical case, we are only aware of the works [8,14], where the Morse index was estimated in the monotone case for the limiting singular solutions of the Dirichlet problem in a ball. We believe that the conclusion of Theorem 1.1 and the technique behind its proof remain valid for other problems in the monotone case, e.g. for the nonlinear Schrödinger equation in a ball.

Remark 1.2. By the Lyapunov–Schmidt reduction technique (see, e.g., [21]), the solution curve satisfies $\lambda(b) \rightarrow d$ and $u_b \rightarrow 0$ as $b \rightarrow 0$, where the Morse index $m(u_b)$ is equal to one for small $b > 0$. If $m(u_b) = 1$ for $b > b_0$ in Theorem 1.1, then it is quite possible that $m(u_b) = 1$ for every $b \in (0, \infty)$. Since the ground state is energetically stable with respect to the radial perturbation in $\mathcal{E} \cap L_r^4$ if $m(u_b) = 1$ and the mapping of $\lambda \mapsto \|u_b\|_{L_r^2}^2$ is monotonically decreasing (see, e.g., Theorem 4.8 in [19]), it is rather interesting that the transition from the oscillatory case for $5 \leq d \leq 12$ to the monotone case $d \geq 13$ may enforce stability of the ground state.

We shall now explain the strategy to prove Theorem 1.1. We use the limiting singular solution $u_\infty \in \mathcal{E} \cap L_r^4$ which exists for a particular value of $\lambda = \lambda_\infty$ if $d \geq 5$ [22]. It was also established in [22] that $u_b \rightarrow u_\infty$ in \mathcal{E} and $\lambda(b) \rightarrow \lambda_\infty$ as $b \rightarrow \infty$. Uniqueness of λ_∞ is also an open problem, see the discussion in [1].

Fig. 1.2 shows the ground state $u_b(r)$ for two values of b and the limiting singular solution $u_\infty(r)$ for $d = 13$. The discrepancy between the two solutions moves to smaller values of r if the value of b is increased. When $b = 10$, the difference between u_b and u_∞ becomes invisible on the scale used in Fig. 1.2.

Since the limiting singular solution u_∞ satisfies the following divergent behavior

$$u_\infty(r) = \frac{\sqrt{d-3}}{r} \left[1 + \mathcal{O}(r^2) \right], \quad \text{as } r \rightarrow 0, \tag{1.8}$$

it is natural to introduce $F_\infty(r) := ru_\infty(r)$ and define it from the family of solutions to the following initial-value problem

$$\begin{cases} F''(r) + \frac{d-3}{r}F'(r) - \frac{d-3}{r^2}F(r) - r^2F(r) + \lambda F(r) + \frac{1}{r^2}F(r)^3 = 0, & r > 0, \\ F(0) = \sqrt{d-3}, \quad F'(0) = 0. \end{cases} \tag{1.9}$$

By Theorem 1.2 in [1] (based on the earlier work in [22]), for any $d \geq 5$, there exists some $\lambda_\infty \in (d - 4, d)$ such that the unique solution $F_\infty \in C^2(0, \infty)$ to the initial-value problem (1.9) is monotonically decreasing to zero as $r \rightarrow \infty$, making it the limiting singular solution $u_\infty \in \mathcal{E} \cap L^4_r$ by the transformation $u_\infty(r) = r^{-1}F_\infty(r)$.

By Theorem 1.3 in [1] proven under non-degeneracy assumptions, convergence of $\lambda(b) \rightarrow \lambda_\infty$ as $b \rightarrow \infty$ is oscillatory for $5 \leq d \leq 12$ and monotone for $d \geq 13$, see Fig. 1.1. The latter case is the only case we are interested in here.

In order to characterize the Morse index of \mathcal{L}_b , we use the Emden–Fowler transformation [6] for the nonlinear equation in (1.9) and study two families of solutions. One family is obtained from $F_b(r) := ru_b(r)$ and is parametrized by its parameter b from the behavior as $r \rightarrow 0$. The other family is parametrized by another parameter c from the decaying behavior as $r \rightarrow \infty$. The second family is considered in a local neighborhood of the limiting singular solution $F_\infty(r) = ru_\infty(r)$. Both families have C^1 property with respect to their parameters and their derivatives with respect to these parameters are solutions of the homogeneous equation $\mathcal{L}_b v = 0$ after the inverse Emden–Fowler transformation, e.g., $v(r) = r^{-1}\partial_b F_b(r)$. The proof of Theorem 1.1 is achieved from the Sturm’s Oscillation Theorem (see, e.g., Theorem 3.5 in [24]) by showing that the two derivatives have finitely many oscillations and there exists $b_0 > 0$ such that the two derivatives are linearly independent for every $b \in (b_0, \infty)$.

As a by-product of our approach, we establish the equivalence of the Morse indices $m(u_b)$ and $m(u_\infty)$, where $m(u_\infty)$ is defined by the number of negative eigenvalues of the limiting operator $\mathcal{L}_\infty := \lim_{b \rightarrow \infty} \mathcal{L}_b$ computed at the limiting singular solution for $d \geq 5$:

$$\mathcal{L}_\infty := -\frac{d^2}{dr^2} - \frac{d-1}{r} \frac{d}{dr} + r^2 - \lambda_\infty - 3u_\infty^2(r). \tag{1.10}$$

Compared to $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$, where the potential $-3u_b^2(r)$ is bounded from below, the potential $-3u_\infty^2(r)$ is unbounded from below.

The following theorem gives the precise result on the Morse index of the two linear operators.

Theorem 1.2. *Under non-degeneracy Assumptions 2.1 and 2.2, for every $d \geq 13$, there exists $b_0 > 0$ such that $m(u_b) = m(u_\infty)$ for $b \in (b_0, \infty)$.*

Remark 1.3. If the norm convergence of the resolvent for \mathcal{L}_b to the resolvent for \mathcal{L}_∞ can be established as $b \rightarrow \infty$, this would imply the result of Theorem 1.2. We do not study the norm convergence of resolvents here as our methods are based on analysis of differential equations.

Remark 1.4. The result of Theorem 1.2 suggests a simple way to obtain $m(u_b)$ in the monotone case for large b from $m(u_\infty)$, which can be approximated numerically with good accuracy.

Fig. 1.3 shows uniquely normalized solutions $v(r)$ of the homogeneous equations $\mathcal{L}_b v = 0$ with $b = 1$ and $\mathcal{L}_\infty v = 0$ such that $v(r) \rightarrow 0$ as $r \rightarrow \infty$. Both solutions diverge as $r \rightarrow 0$ with different divergence rates. Since there exists only one zero for each solution on $(0, \infty)$, Sturm’s

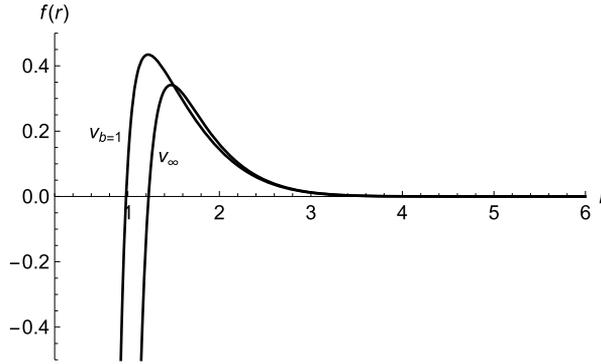


Fig. 1.3. Graph of the uniquely normalized solutions $v(r)$ of $\mathcal{L}_b v = 0$ with $b = 1$ and $\mathcal{L}_\infty v = 0$ satisfying $v(r) \rightarrow 0$ as $r \rightarrow \infty$ for $d = 13$.

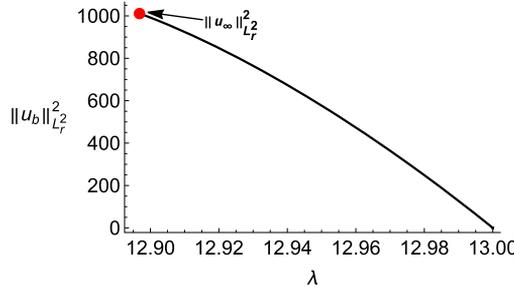


Fig. 1.4. Mass $\|u_b\|_{L_r^2}^2$ of the ground state u_b for $d = 13$ as a function of λ together with the mass $\|u_\infty\|_{L_r^2}^2$ of the limiting singular solution u_∞ .

Oscillation Theorem (Theorem 3.5 in [24]) asserts that $m(u_b) = m(u_\infty) = 1$.

By the Vakhitov–Kolokolov stability criterion (Theorem 4.8 in [19]), if $m(u_b) = 1$ and the mapping of $\lambda \mapsto \|u_b\|_{L_r^2}^2$ is monotonically decreasing, then the ground state u_b is energetically stable with respect to radial perturbations in $\mathcal{E} \cap L_r^4$. Fig. 1.4 shows the dependence of the mass $M(u_b) = \|u_b\|_{L_r^2}^2$ versus λ for $\lambda = \lambda(b)$. The big dot depicts the finite value of the limiting mass $M(u_\infty) = \|u_\infty\|_{L_r^2}^2$. Since the mapping is monotonically decreasing, the Vakhitov–Kolokolov stability criterion asserts that the ground state u_b is energetically stable for $d = 13$. The energetic stability is equivalent to the orbital stability if the time evolution of the Gross–Pitaevskii equation is locally well-posed in $\mathcal{E} \cap L_r^4$.

The same conclusion holds for other values of d in the monotone case $d \geq 13$. We have also checked other values of b and found no points of bifurcations along the solution family $\lambda(b)$ where \mathcal{L}_b admits zero eigenvalue in L_r^2 . This suggests that the monotone dependence of $\lambda(b)$ with no critical points, where $\lambda'(b)$ vanishes, implies no bifurcation points. This useful property has not been proven in the literature.

The article is organized as follows. Section 2 gives the review of preliminary results. Sections 3 and 4 contain estimates of the derivatives of the two families of solutions described above. The proof of Theorems 1.1 and 1.2 is described in Section 5.

2. Preliminary results

We begin by introducing the Emden-Fowler transformation

$$r = e^t, \quad F(r) = \Psi(t), \quad F'(r) = e^{-t}\Psi'(t), \tag{2.1}$$

which transforms the differential equation

$$F''(r) + \frac{d-3}{r}F'(r) - \frac{d-3}{r^2}F(r) - r^2F(r) + \lambda F(r) + \frac{1}{r^2}F(r)^3 = 0, \quad r > 0 \tag{2.2}$$

to the equivalent form

$$\Psi''(t) + (d-4)\Psi'(t) + (3-d)\Psi(t) + \Psi(t)^3 = -\lambda e^{2t}\Psi(t) + e^{4t}\Psi(t), \quad t \in \mathbb{R}. \tag{2.3}$$

For fixed $d \geq 5$ and $\lambda \in (0, d)$, two one-parameter families of solutions to the second-order differential equation (2.3) have been constructed in [1] according to their asymptotic behaviors as $t \rightarrow -\infty$ and $t \rightarrow +\infty$, respectively.

The first family of solutions to the differential equation (2.3), denoted as $\{\Psi_b\}_{b \in \mathbb{R}}$, corresponds to solutions of the initial-value problem (1.6) after applying the transformation $\Psi_b(t) = e^t f_b(e^t)$. By Lemmas 3.2 and 3.4 in [1], $\Psi_b \in C^2(\mathbb{R})$ satisfies the asymptotic behavior

$$\Psi_b(t) = be^t - (\lambda b + b^3)(2d)^{-1}e^{3t} + \mathcal{O}(e^{5t}), \quad \text{as } t \rightarrow -\infty, \tag{2.4}$$

where the expansion can be differentiated in t . These solutions depend on λ as well, and for $\lambda = \lambda(b)$ and $b > 0$, $\Psi_b(t)$ gives a solution to the boundary-value problem (1.1), after the transformation $u_b(r) = r^{-1}\Psi_b(\log r)$. For other values of λ , $\Psi_b(t)$ generally diverges as $t \rightarrow +\infty$.

The second family of solutions to the differential equation (2.3), denoted as $\{\Psi_c\}_{c \in \mathbb{R}}$, decays to zero as $t \rightarrow +\infty$. By Lemmas 3.3 and 3.4 in [1], $\Psi_c \in C^2(\mathbb{R})$ satisfies the asymptotic behavior

$$\Psi_c(t) \sim ce^{\frac{\lambda-d+2}{2}t} e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \rightarrow +\infty, \tag{2.5}$$

where the sign \sim denotes the asymptotic correspondence which can be differentiated in t . Each $\Psi_c(t)$ generally diverges as $t \rightarrow -\infty$, except when $\lambda = \lambda(b)$ and $c = c(b)$ for some value of $c(b)$ for which it coincides with $\Psi_b(t) = e^t u_b(e^t)$:

$$\lambda = \lambda(b) : \quad \Psi_b(t) = \Psi_{c(b)}(t), \quad \text{for all } t \in \mathbb{R}. \tag{2.6}$$

Each family of solutions is differentiable with respect to parameters λ and either b or c due to smoothness of the differential equation (2.3). Their derivatives decay to zero as $t \rightarrow -\infty$ and $t \rightarrow +\infty$ respectively, but generally diverge at the other infinities.

Let us define linearizations of the second-order equation (2.3) at the two families of solutions:

$$\mathcal{M}_b := \frac{d^2}{dt^2} + (d-4)\frac{d}{dt} + (3-d) + 3\Psi_b^2 + \lambda e^{2t} - e^{4t}, \tag{2.7}$$

$$\mathcal{M}_c := \frac{d^2}{dt^2} + (d-4)\frac{d}{dt} + (3-d) + 3\Psi_c^2 + \lambda e^{2t} - e^{4t}. \tag{2.8}$$

Then, differentiating the second-order equation (2.3) with respect to b and c at fixed λ yields

$$\mathcal{M}_b \partial_b \Psi_b = 0, \quad \mathcal{M}_c \partial_c \Psi_c = 0, \tag{2.9}$$

where $\partial_b \Psi_b(t) \rightarrow 0$ as $t \rightarrow -\infty$ and $\partial_c \Psi_c(t) \rightarrow 0$ as $t \rightarrow +\infty$.

The first family $\{\Psi_b\}_{b \in \mathbb{R}}$ is defined in a neighborhood of a heteroclinic orbit Θ connecting the saddle point $(0, 0)$ and the stable point $(\sqrt{d-3}, 0)$ of the truncated autonomous version of equation (2.3) given by

$$\Theta''(t) + (d - 4)\Theta'(t) + (3 - d)\Theta(t) + \Theta(t)^3 = 0. \tag{2.10}$$

By Lemma 6.1 in [1], there exists a heteroclinic orbit between $(0, 0)$ and $(\sqrt{d-3}, 0)$ which is defined uniquely (module to the translation in t) by the asymptotic behavior

$$\Theta(t) = e^t - (2d)^{-1}e^{3t} + \mathcal{O}(e^{5t}), \quad \text{as } t \rightarrow -\infty. \tag{2.11}$$

The following proposition presents the main result of Lemmas 6.2, 6.5, and 6.8 in [1].

Proposition 2.1. *Fix $d \geq 5$ and $\lambda \in \mathbb{R}$. For every $T > 0$ and $a \in (0, 1)$, there exist (T, a) -independent constants $b_0 > 0$, $C_0 > 0$ and (T, a) -dependent constants $b_{T,a} > 0$, $C_{T,a} > 0$ such that the unique solution Ψ_b to the differential equation (2.3) with the asymptotic behavior (2.4) satisfies for every $b \in (b_0, \infty)$*

$$\sup_{t \in (-\infty, 0]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_0 b^{-2} e^{3t} \tag{2.12}$$

and for every $b \in (b_{T,a}, \infty)$

$$\sup_{t \in [0, T+a \log b]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_{T,a} b^{-2(1-a)}. \tag{2.13}$$

The heteroclinic orbit of the truncated equation (2.10) connects the saddle point $(0, 0)$ associated with the characteristic exponents $\kappa_1 = 1$ and $\kappa_2 = 3 - d$ and the stable point $(\sqrt{d-3}, 0)$ associated with the characteristic exponents κ_+ and κ_- given by

$$\kappa_{\pm} = -\frac{1}{2}(d - 4) \pm \frac{1}{2}\sqrt{d^2 - 16d + 40}. \tag{2.14}$$

For $d \geq 13$, the characteristic exponents are real and satisfy $\kappa_- < \kappa_+ < 0$. We make the following assumption on how the heteroclinic orbit converges to the stable point $(\sqrt{d-3}, 0)$.

Assumption 2.1. Assume that there exists $A_0 \neq 0$ such that

$$\Theta(t) = \sqrt{d-3} + A_0 e^{\kappa_+ t} + \mathcal{O}(e^{\kappa_- t}, e^{2\kappa_+ t}) \quad \text{as } t \rightarrow +\infty. \tag{2.15}$$

Remark 2.1. Assumption 2.1 implies that $\Theta(t)$ converges to $\sqrt{d-3}$ as $t \rightarrow +\infty$ according to the slowest decay rate given by κ_+ . It is not a priori clear why the constant A_0 could not be zero in exceptional cases, for which $\Theta(t)$ converges to $\sqrt{d-3}$ as $t \rightarrow +\infty$ according to the fastest decay rate given by κ_- .

The second family $\{\Psi_c\}_{c \in \mathbb{R}}$ is defined in a neighborhood of the special solution $\Psi_\infty(t) := e^t u_\infty(e^t)$ obtained from the limiting singular solution $u_\infty \in \mathcal{E} \cap L^4_r$. This special solution corresponds to the values of $\lambda = \lambda_\infty$ and $c = c_\infty$ so that

$$\lambda = \lambda_\infty : \quad \Psi_\infty(t) = \Psi_{c_\infty}(t), \quad \text{for all } t \in \mathbb{R}. \tag{2.16}$$

The solution Ψ_∞ satisfies the asymptotic behaviors

$$\Psi_\infty(t) = \sqrt{d-3} \left[1 - \frac{\lambda_\infty}{4d-10} e^{2t} + \mathcal{O}(e^{4t}) \right], \quad \text{as } t \rightarrow -\infty \tag{2.17}$$

and

$$\Psi_\infty(t) \sim c_\infty e^{\frac{\lambda_\infty-d+2}{2}t} e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \rightarrow +\infty. \tag{2.18}$$

The following proposition presents a modification of Lemmas 6.6 and 6.9 in [1]. Since the statement was not proven in [1], we give the precise proof of this result in Appendix A.

Proposition 2.2. Fix $d \geq 13$ and $a \in (0, 1)$. There exist constants $b_0 > 0$, $C_0 > 0$, and $\epsilon_0 > 0$, such that for every $\epsilon \in (0, \epsilon_0)$, $b \in (b_0, \infty)$, and $(\lambda, c) \in \mathbb{R}^2$ satisfying

$$|\lambda - \lambda_\infty| + |c - c_\infty| \leq \epsilon b^{\kappa-(1-a)}, \tag{2.19}$$

it is true for every $t \in [(a-1) \log b, 0]$ that

$$|\Psi_c(t) - \Psi_\infty(t)| \leq C_0 \epsilon b^{\kappa-(1-a)} e^{\kappa-t}. \tag{2.20}$$

Remark 2.2. Note that the divergent behavior of $e^{\kappa-t}$ for large negative t in (2.20) is canceled by the decay of $b^{\kappa-(1-a)}$ on any fixed interval $[(a-1) \log b, 0]$. Thus, bound (2.20) implies

$$\sup_{t \in [(a-1) \log b, 0]} |\Psi_c(t) - \Psi_\infty(t)| \leq C_0 \epsilon, \tag{2.21}$$

for every $(\lambda, c) \in \mathbb{R}^2$ satisfying (2.19).

Remark 2.3. Since Ψ_c is smooth in λ and c and has the same decay (2.5) as $t \rightarrow +\infty$ in comparison with (2.18) for Ψ_∞ , it is true for every (λ, c) in a local neighborhood of $(\lambda_\infty, c_\infty)$ that

$$\sup_{t \in [0, \infty)} |\Psi_c(t) - \Psi_\infty(t)| \leq C_0 (|\lambda - \lambda_\infty| + |c - c_\infty|). \tag{2.22}$$

Linearization of the second-order equation (2.3) at Ψ_∞ is given by

$$\mathcal{M}_\infty := \frac{d^2}{dt^2} + (d-4) \frac{d}{dt} + (3-d) + 3\Psi_\infty^2 + \lambda_\infty e^{2t} - e^{4t}. \tag{2.23}$$

Differentiating the second-order equation (2.3) with respect to c at fixed λ and then substituting $c = c_\infty$ and $\lambda = \lambda_\infty$ gives

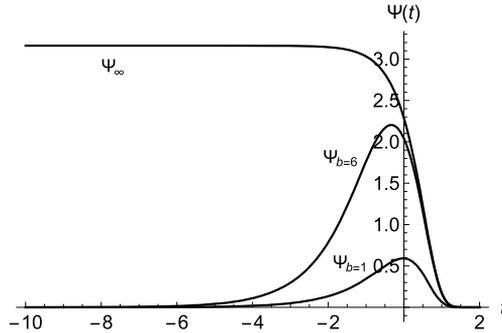


Fig. 2.1. Graph of the solution Ψ_b for $b = 1$ and $b = 6$ in comparison with the solution Ψ_∞ for $d = 13$.

$$\mathcal{M}_\infty \partial_c \Psi_\infty = 0, \tag{2.24}$$

where $\partial_c \Psi_\infty$ is a short notation for $\partial_c \Psi_c|_{(\lambda,c)=(\lambda_\infty,c_\infty)}$. The function $\partial_c \Psi_\infty(t)$ decays fast as $t \rightarrow +\infty$ according to (2.5), but generally diverges as $t \rightarrow -\infty$. Since $\Psi_\infty(t) \rightarrow \sqrt{d-3}$ as $t \rightarrow -\infty$, the divergence of $\partial_c \Psi_\infty(t)$ as $t \rightarrow -\infty$ is defined by the same two characteristic exponents κ_+ and κ_- given by (2.14). We make the following assumption on the divergence of this solution.

Assumption 2.2. Assume that there exists $L_\infty \neq 0$ such that

$$\partial_c \Psi_\infty(t) = L_\infty e^{\kappa_- t} + \mathcal{O}(e^{\kappa_+ t}, e^{(\kappa_- + 2)t}) \quad \text{as } t \rightarrow -\infty. \tag{2.25}$$

Remark 2.4. Assumption 2.2 implies that $\partial_c \Psi_\infty(t)$ diverges as $t \rightarrow -\infty$ with the fastest growth rate given by κ_- . Again, it is not a priori clear why the constant L_∞ could not be zero in exceptional cases, for which $\partial_c \Psi_\infty(t)$ diverges as $t \rightarrow -\infty$ with the slowest growth rate given by κ_+ .

Fig. 2.1 shows $\Psi_b(t)$ for two values of b and $\Psi_\infty(t)$ for $d = 13$. After the inverse Emden-Fowler transformation (2.1) and the transformation $u(r) = r^{-1}F(r)$, these functions correspond to $u_b(r)$ and $u_\infty(r)$ shown in Fig. 1.2.

Fig. 2.2 shows $\partial_c \Psi_{c(b)}(t)$ with $b = 1$ and $\partial_c \Psi_\infty(t)$ for $d = 13$. These functions are solutions of the homogeneous equations $\mathcal{M}_b \partial_c \Psi_{c(b)} = 0$ and $\mathcal{M}_\infty \partial_c \Psi_\infty = 0$. After the transformation $v(r) = r^{-1} \partial_c \Psi_{c(b)}(\log r)$, these functions correspond to solutions of $\mathcal{L}_b v = 0$ and $\mathcal{L}_\infty v = 0$ that decay to zero as $r \rightarrow \infty$ shown in Fig. 1.3. Since $\partial_c \Psi_{c(b)}(t)$ and $\partial_c \Psi_\infty(t)$ have only one zero on \mathbb{R} , the corresponding functions $v(r)$ have only one zero on $(0, \infty)$, so that $m(u_b) = m(u_\infty) = 1$.

3. Derivative of the b -family of solutions

Here we describe the asymptotic behavior of $\partial_b \Psi_b$. The following lemma shows that after translation by $-\log b$, $\partial_b \Psi_b$ converges to $b^{-1}\Theta'$ on the negative t -axis. Moreover, the estimate can be extended from $(-\infty, 0]$ to $[0, T + a \log b]$ for fixed $T \in \mathbb{R}$ and $a \in (0, 1)$ and for sufficiently large b at the expense of slower convergence rate.

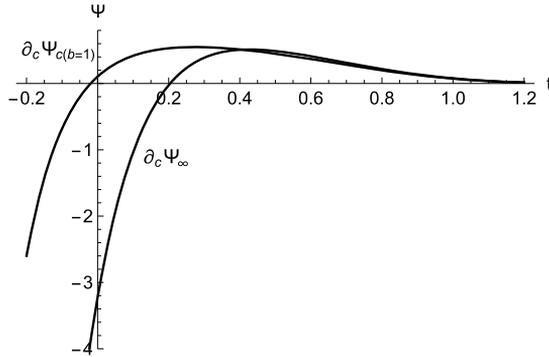


Fig. 2.2. Graph of $\partial_c \Psi_{c(b)}$ with $b = 1$ and $\partial_c \Psi_\infty$ for $d = 13$.

Lemma 3.1. *Let $d \geq 13$ and $\lambda \in \mathbb{R}$. For every $T > 0$ and $a \in (0, 1)$, there exist (T, a) -independent constants $b_0 > 0$, $C_0 > 0$ and (T, a) -dependent constants $b_{T,a} > 0$, $C_{T,a} > 0$ such that $\partial_b \Psi_b$ satisfies for every $b \in (b_0, \infty)$*

$$\sup_{t \in (-\infty, 0]} |\partial_b \Psi_b(t - \log b) - b^{-1} \Theta'(t)| + \sup_{t \in (-\infty, 0]} |\partial_b \Psi'_b(t - \log b) - b^{-1} \Theta''(t)| \leq C_0 b^{-3} \tag{3.1}$$

and for every $b \in (b_{T,a}, \infty)$

$$\begin{aligned} &\sup_{t \in [0, T+a \log b]} |\partial_b \Psi_b(t - \log b) - b^{-1} \Theta'(t)| + \sup_{t \in [0, T+a \log b]} |\partial_b \Psi'_b(t - \log b) - b^{-1} \Theta''(t)| \\ &\leq C_{T,a} b^{-2(1-a)-1}. \end{aligned} \tag{3.2}$$

Proof. We begin by introducing $\gamma(t) := \partial_b \Psi_b(t - \log b) - b^{-1} \Theta'(t)$, where Ψ_b is the unique solution to (2.3) with the asymptotic behavior (2.4) and Θ is the unique solution to (2.10) with the asymptotic behavior (2.11). Since $\partial_b \Psi_b$ satisfies $\mathcal{M}_b \partial_b \Psi_b = 0$ and Θ' satisfies $\mathcal{M}_0 \Theta' = 0$, where \mathcal{M}_b is given by (2.7) and

$$\mathcal{M}_0 := \frac{d^2}{dt^2} + (d - 4) \frac{d}{dt} + (3 - d) + 3\Theta^2, \tag{3.3}$$

the difference term $\gamma(t)$ satisfies the following equation:

$$\mathcal{M}_0 \gamma = f_b(b^{-1} \Theta' + \gamma), \tag{3.4}$$

where

$$f_b(t) := 3(\Theta(t)^2 - \Psi_b(t - \log b)^2) - \lambda b^{-2} e^{2t} + b^{-4} e^{4t}. \tag{3.5}$$

Note that $\mathcal{M}_0 - f_b$ gives \mathcal{M}_b after translation $t \mapsto t + \log b$.

Proof of the bound (3.1). Two linearly independent solutions of $\mathcal{M}_0 \gamma = 0$ are given by $\Theta'(t)$ and another function $\Xi(t)$, which can be found from the Wronskian relation

$$W(\Theta', \Xi)(t) = \Theta'(t)\Xi'(t) - \Theta''(t)\Xi(t) = W_0e^{(4-d)t}, \tag{3.6}$$

for some constant $W_0 \neq 0$. We take $W_0 = 1$ in order to normalize $\Xi(t)$ uniquely. Since $\Theta'(t) \rightarrow 0$ as $t \rightarrow -\infty$ according to the asymptotic expansion

$$\Theta'(t) = e^t + \mathcal{O}(e^{3t}), \quad \text{as } t \rightarrow -\infty, \tag{3.7}$$

we have $\Xi(t) \rightarrow \infty$ as $t \rightarrow -\infty$ according to the asymptotic expansion

$$\Xi(t) = (2 - d)^{-1}e^{(3-d)t} + \mathcal{O}(e^{(5-d)t}), \quad \text{as } t \rightarrow -\infty. \tag{3.8}$$

In order to estimate the supremum-norm of $\gamma(t)$ for $t \in (-\infty, 0]$, we first rewrite the differential equation (3.4) as an integral equation

$$\gamma(t) = \int_{-\infty}^t e^{(d-4)t'}[\Theta'(t')\Xi(t) - \Theta'(t)\Xi(t')]f_b(t')[b^{-1}\Theta'(t') + \gamma(t')]dt', \tag{3.9}$$

where the free solution $c_1\Theta'(t) + c_2\Xi(t)$ is set to zero from the requirement that $\gamma(t) = \mathcal{O}(e^{3t})$ as $t \rightarrow -\infty$. The integral kernel in (3.9) becomes bounded if we introduce the transformation $\tilde{\gamma}(t) = e^{-t}\gamma(t)$. The integral equation corresponding to $\tilde{\gamma}$ is

$$\tilde{\gamma}(t) = \int_{-\infty}^t K_1(t, t')f_b(t')[b^{-1}e^{-t'}\Theta'(t') + \tilde{\gamma}(t')]dt', \tag{3.10}$$

where

$$K_1(t, t') := [e^{-t'}\Theta'(t')][e^{(d-3)t}\Xi(t)]e^{(d-2)(t'-t)} - [e^{-t}\Theta'(t)][e^{(d-3)t'}\Xi(t')]. \tag{3.11}$$

Since $\Theta'(t) = \mathcal{O}(e^t)$ and $\Xi(t) = \mathcal{O}(e^{(3-d)t})$ as $t \rightarrow -\infty$, the integral kernel $K_1(t, t')$ is a bounded function for all $-\infty < t' \leq t \leq 0$.

By using bound (2.12) of Proposition 2.1, we obtain from (3.5) that

$$|f_b(t)| \leq C_0b^{-2}e^{2t}, \quad t \in (-\infty, 0], \tag{3.12}$$

where the constant C_0 is independent of b for sufficiently large b and may change from one line to another line. Boundedness of K_1 in the integral equation (3.10) for all $-\infty < t' \leq t \leq 0$ and the estimate (3.12) allow us to estimate the supremum norm of $\tilde{\gamma}(t)$ on $(-\infty, 0]$ as follows

$$\|\tilde{\gamma}\|_\infty \leq C_0b^{-2}(b^{-1}\|e^{-t}\Theta'\|_\infty + \|\tilde{\gamma}\|_\infty). \tag{3.13}$$

Due to smallness of b^{-2} and boundness of b -independent $\|e^{-t}\Theta'\|_\infty$, this estimate implies that

$$\sup_{t \in (-\infty, 0]} |\tilde{\gamma}(t)| \leq C_0b^{-3}. \tag{3.14}$$

Since $|\gamma(t)| \leq |\tilde{\gamma}(t)|$ for all $t \in (-\infty, 0]$, we obtain the first part of bound (3.1). Since $\tilde{\gamma} \in C^1(-\infty, 0)$, we obtain the second part of bound (3.1) by differentiating equation (3.10) and using (3.14).

Proof of the bound (3.2). In order to estimate $|\gamma(t)|$ for sufficiently large positive t , we need to define solutions to $\mathcal{M}_0\gamma = 0$ from their behavior as $t \rightarrow +\infty$. Since $(\sqrt{d-3}, 0)$ is a stable node of the nonlinear equation (2.10), we can pick two linearly independent solutions $\gamma_1(t)$ and $\gamma_2(t)$ from their decaying behavior

$$\gamma_1(t) = \mathcal{O}(e^{\kappa_- t}), \quad \gamma_2(t) = \mathcal{O}(e^{\kappa_+ t}) \quad \text{as } t \rightarrow +\infty, \tag{3.15}$$

where $\kappa_- < \kappa_+ < 0$ are given by (2.14). The Liouville’s formula yields the Wronskian relation

$$W(\gamma_1, \gamma_2)(t) = \gamma_1(t)\gamma_2'(t) - \gamma_1'(t)\gamma_2(t) = W_0 e^{(4-d)t}, \tag{3.16}$$

for some constant $W_0 \neq 0$, and by normalizing $\gamma_1(t)$ and $\gamma_2(t)$ we can assume that $W_0 = 1$. In order to derive supremum-norm estimates for $\gamma(t)$, we once again rewrite differential equation (3.4) as an integral equation

$$\begin{aligned} \gamma(t) = & \gamma(0)[\gamma_1(t)\gamma_2'(0) - \gamma_1'(0)\gamma_2(t)] + \gamma'(0)[\gamma_1(0)\gamma_2(t) - \gamma_1(t)\gamma_2(0)] \\ & + \int_0^t e^{(d-4)t'} [\gamma_1(t')\gamma_2'(t) - \gamma_1'(t)\gamma_2(t')] f_b(t') [b^{-1}\Theta'(t') + \gamma(t')] dt', \end{aligned} \tag{3.17}$$

this time for $t \in [0, T + a \log b]$. From bound (3.1) we obtain existence of a constant $C_0 > 0$ and $b_0 > 0$, such that

$$|\gamma(0)| + |\gamma'(0)| \leq C_0 b^{-3}, \quad \text{for all } b \geq b_0. \tag{3.18}$$

Due to the decay of $\gamma_1(t)$ and $\gamma_2(t)$ as $t \rightarrow +\infty$, the kernel of the integral equation (3.17) behaves like $e^{\kappa_+(t-t')}$ and $e^{\kappa_-(t-t')}$, and is thus bounded as $t \rightarrow +\infty$ since $t' \leq t$. By using bound (2.13) of Proposition 2.1, we obtain from (3.5) that

$$\sup_{t \in [0, T+a \log b]} |f_b(t)| \leq C_{T,a} b^{-2(1-a)}, \tag{3.19}$$

where the b -independent constant $C_{T,a}$ may change from one line to another line. Using estimates (3.15), (3.18), and (3.19), we obtain from the integral equation (3.17) the following bound on the supremum-norm of $\gamma(t)$ on $[0, T + a \log b]$:

$$\|\gamma\|_\infty \leq C_{T,a} \left(b^{-3} + b^{-2(1-a)-1} \|\Theta'\|_{L^1} + (T + a \log b) b^{-2(1-a)} \|\gamma\|_\infty \right). \tag{3.20}$$

Due to smallness of $(T + a \log b) b^{-2(1-a)}$ and boundedness of b -independent $\|\Theta'\|_{L^1}$, this estimate implies that

$$\sup_{t \in [0, T+a \log b]} |\gamma(t)| \leq C_{T,a} b^{-2(1-a)-1}. \tag{3.21}$$

By differentiating equation (3.17) and using (3.21), we obtain a similar bound on $\gamma'(t)$, which together with (3.21) gives us bound (3.2). \square

Bound (3.2) of Lemma 3.1 and the expansion (2.15) imply the following important representation of $\partial_b \Psi_b(t)$ at $t = T + (a - 1) \log b$.

Corollary 3.1. *Under Assumption 2.1, there exist some constant $a_0 \in (0, 0.5)$ such that for every $a \in (0, a_0)$ and $T > 0$, there exist some (T, a) -dependent constants $b_{T,a} > 0$ and $C_{T,a} > 0$ such that for every $b \in (b_{T,a}, \infty)$ we have*

$$|\partial_b \Psi_b(T + (a - 1) \log b) - A_0 \kappa_+ b^{a\kappa_+ - 1} e^{\kappa_+ T}| \leq C_{T,a} \max \left\{ b^{-2(1-a)-1}, b^{a\kappa_- - 1}, b^{2a\kappa_+ - 1} \right\}. \tag{3.22}$$

Proof. Bound (3.22) follows from the expansion (2.15) at large positive $t = T + a \log b$, the bound (3.2) at $t = T + (a - 1) \log b$, and the triangle inequality if

$$b^{a\kappa_+ - 1} \gg b^{-2(1-a)-1}.$$

This constraint is satisfied if $a \in (0, a_0)$, where

$$a_0 := \frac{2}{2 + |\kappa_+|} = \frac{4}{d - \sqrt{d^2 - 16d + 40}} = \frac{d + \sqrt{d^2 - 16d + 40}}{2(2d - 5)}. \tag{3.23}$$

Note that $a_0 \in (0, 0.5)$ for every $d \geq 13$. \square

Remark 3.1. Lemma 3.1 suggests that both bounds (2.12) and (2.13) of Proposition 2.1 can be differentiated in b after translation: $t \mapsto t + \log b$. This C^1 property would also follow from applications of Banach fixed-point theorem in [1] due to contraction of integral operators and smoothness of the vector fields. Since the C^1 property was not written explicitly in [1], we provided the precise proof of Lemma 3.1.

4. Derivative of the c -family of solutions

Here we describe the asymptotic behavior of $\partial_c \Psi_c$. Since Ψ_c is smooth in λ and c and has the same decay as $t \rightarrow +\infty$ as the limiting solution Ψ_∞ according to (2.5) and (2.18), $\partial_c \Psi_c$ converges to $\partial_c \Psi_\infty$ on $[0, \infty)$. To be precise, there exists a constant $C_0 > 0$ such that for every (λ, c) in a local neighborhood of $(\lambda_\infty, c_\infty)$, we have

$$\sup_{t \in [0, \infty)} |\partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)| + \sup_{t \in [0, \infty)} |\partial_c \Psi'_c(t) - \partial_c \Psi'_\infty(t)| \leq C_0 (|\lambda - \lambda_\infty| + |c - c_\infty|). \tag{4.1}$$

The following lemma extends the estimate on the difference $|\partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)|$ from $[0, \infty)$ to $[(a - 1) \log b, 0]$ for fixed $a \in (0, 1)$ and for sufficiently large b provided that (λ, c) are sufficiently close to $(\lambda_\infty, c_\infty)$.

Lemma 4.1. Fix $d \geq 13$. For fixed $a \in (0, 1)$, there exist $b_0 > 0$, $C_0 > 0$, and $\epsilon_0 > 0$, such that for every $\epsilon \in (0, \epsilon_0)$ and for every $(\lambda, c) \in \mathbb{R}^2$ satisfying

$$|\lambda - \lambda_\infty| + |c - c_\infty| \leq \epsilon b^{\kappa_-(1-a)}, \tag{4.2}$$

it is true for every $b \geq b_0$ and every $t \in [(a - 1) \log b, 0]$ that

$$|\partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)| + |\partial_c \Psi'_c(t) - \partial_c \Psi'_\infty(t)| \leq C_0 \epsilon e^{\kappa_- t}. \tag{4.3}$$

Proof. Let $r(t) := \partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)$. Since $\partial_c \Psi_c$ satisfies $\mathcal{M}_c \partial_c \Psi_c = 0$ and $\partial_c \Psi_\infty$ satisfies $\mathcal{M}_\infty \partial_c \Psi_\infty = 0$, the difference term r satisfies the following equation:

$$\mathcal{M}_\infty r = f_c + g_c r, \tag{4.4}$$

where

$$f_c(t) := 3(\Psi_\infty(t)^2 - \Psi_c(t)^2) \partial_c \Psi_\infty(t) - (\lambda - \lambda_\infty) e^{2t} \partial_c \Psi_\infty(t), \tag{4.5}$$

$$g_c(t) := 3(\Psi_\infty(t)^2 - \Psi_c(t)^2) + (\lambda_\infty - \lambda) e^{2t}. \tag{4.6}$$

Note that $\mathcal{M}_c = \mathcal{M}_\infty - g_c$.

As in Appendix A, we pick two linearly independent solutions r_1, r_2 to $\mathcal{M}_\infty r = 0$ such that

$$r_1(t) = \mathcal{O}(e^{\kappa_- t}), \quad r_2(t) = \mathcal{O}(e^{\kappa_+ t}), \quad \text{as } t \rightarrow -\infty, \tag{4.7}$$

where $\kappa_- < \kappa_+ < 0$ are given by (2.14). Using the method of variation of parameters, we rewrite the differential equation (4.4) as an integral equation for every $t \in [(a - 1) \log b, 0]$:

$$\begin{aligned} r(t) = & r(0)[r_1(t)r'_2(0) - r'_1(0)r_2(t)] + r'(0)[r_1(0)r_2(t) - r_1(t)r_2(0)] \\ & + \int_t^0 e^{(d-4)t'} [r_1(t)r_2(t') - r_1(t')r_2(t)] [f_c(t') + g_c(t')r(t')] dt', \end{aligned} \tag{4.8}$$

where we have used the normalization of the Wronskian $W(r_1, r_2)(t) = e^{-(d-4)t}$ between the two solutions r_1 and r_2 as in (A.3).

In order to eliminate the divergent behavior of the kernel in (4.8) as $t \rightarrow -\infty$, we introduce the transformation $\tilde{r}(t) = e^{-\kappa_- t} r(t)$, which results in the following integral equation for \tilde{r} :

$$\begin{aligned} \tilde{r}(t) = & r(0)e^{-\kappa_- t} [r_1(t)r'_2(0) - r'_1(0)r_2(t)] + r'(0)e^{-\kappa_- t} [r_1(0)r_2(t) - r_1(t)r_2(0)] \\ & + \int_t^0 K_2(t, t') [e^{-\kappa_- t'} f_c(t') + g_c(t')\tilde{r}(t')] dt', \end{aligned} \tag{4.9}$$

where the kernel $K_2(t, t')$ is the same as in (A.6):

$$K_2(t, t') := e^{\kappa_-(t'-t) - (\kappa_+ + \kappa_-)t'} [r_1(t)r_2(t') - r_1(t')r_2(t)]. \tag{4.10}$$

The kernel is bounded for every $-\infty < t \leq t' \leq 0$ as in (A.7). It follows from (4.1) that

$$|r(0)| + |r'(0)| \leq C_0(|\lambda - \lambda_\infty| + |c - c_\infty|) \leq C_0 \epsilon b^{\kappa_-(1-a)}, \tag{4.11}$$

where the (ϵ, b) -independent constant C_0 can change from one line to another line. It follows from the expansion (2.25) in Assumption 2.2 that $\partial_c \Psi_\infty(t) = \mathcal{O}(e^{\kappa_-t})$ as $t \rightarrow -\infty$. Therefore, we get by using bounds (2.20) and (2.21):

$$\int_t^0 e^{-\kappa_-t'} |f_c(t')| dt' \leq C_0 \left(\epsilon b^{\kappa_-(1-a)} \int_{(a-1)\log b}^0 e^{\kappa_-t'} dt' + |\lambda - \lambda_\infty| \right) \leq C_0 \epsilon. \tag{4.12}$$

On the other hand, for every $\tilde{r} \in L^\infty((a-1)\log b, 0)$, we get by using bounds (2.20) and (2.21):

$$\int_{(a-1)\log b}^0 |g_c(t') \tilde{r}(t')| dt' \leq C_0 \|\tilde{r}\|_\infty. \tag{4.13}$$

Putting estimates (4.7), (4.11), (4.12), and (4.13) together in the integral equation (4.9) yields

$$\sup_{t \in [(a-1)\log b, 0]} |\tilde{r}(t)| \leq C_0 \epsilon, \tag{4.14}$$

which is the first part of bound (4.3) after going back to the original variable $r(t)$. The second part of bound (4.3) is obtained by differentiating (4.9) in t and using bound (4.14). \square

Bound (4.3) of Lemma 4.1 and the expansion (2.25) imply the following important representation of $\partial_c \Psi_c(t)$ at $t = T + (a-1)\log b$.

Corollary 4.1. *Under Assumption 2.2, for every $a \in (0, 1)$ and $T > 0$, there exist some (T, a) -dependent constants $b_{T,a} > 0$ and $C_{T,a} > 0$ such that for every $b \in (b_{T,a}, \infty)$ we have*

$$\begin{aligned} & |\partial_c \Psi_c(T + (a-1)\log b) - L_\infty e^{\kappa_-T} b^{-\kappa_-(1-a)}| \\ & \leq C_{T,a} \max\{\epsilon b^{-\kappa_-(1-a)}, b^{-\kappa_+(1-a)}, b^{-(2+\kappa_-)(1-a)}\}. \end{aligned} \tag{4.15}$$

Proof. Bound (4.15) follows from the bound (4.3) at $t = T + (a-1)\log b$ for fixed $T > 0$, $a \in (0, 1)$, and sufficiently large $b > 0$ after $\partial_c \Psi_\infty(t)$ for large negative t is expressed from the expansion (2.25). \square

Remark 4.1. Lemma 4.1 can be obtained from the C^1 property of Ψ_c in (λ, c) after some transformations. It follows from the proof of Proposition 2.2 in Appendix A that

$$|\Psi_c(t) - \Psi_\infty(t)| = \mathcal{O}(\epsilon) \quad t \in [(a-1)\log b, 0],$$

and the asymptotic expansion can be differentiated in ϵ . Parameter ϵ determines the size of the distance $|\lambda - \lambda_\infty|$ and $|c - c_\infty|$ so that we can write $c - c_\infty = \mathcal{O}(\epsilon b^{-\kappa_-(1-a)})$ and differentiate it in ϵ . By taking derivative in c and using the chain rule, this yields

$$|\partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)| = \mathcal{O}(\epsilon b^{-\kappa_-(1-a)}) \quad t \in [(a - 1) \log b, 0],$$

which is equivalent to the bound (4.3). Since taking derivatives in c and using the chain rule are not obvious from the proof of Proposition 2.2, we provided the precise proof of Lemma 4.1.

5. Proofs of the main results

We recall that $\mathcal{M}_b = \mathcal{M}_{c(b)}$ for $\lambda = \lambda(b)$ since $\Psi_b(t) = \Psi_{c(b)}(t)$ for every $t \in \mathbb{R}$. Hence, both $\partial_b \Psi$ and $\partial_c \Psi_{c(b)}$ are solutions of the same homogeneous equation $\mathcal{M}_b \gamma = 0$ for $\lambda = \lambda(b)$. The following lemma shows that these two solutions are linearly independent for sufficiently large values of b .

Lemma 5.1. Fix $d \geq 13$. Under Assumptions 2.1 and 2.2, for every $a \in (0, a_0)$ with a_0 given by (3.23), there exists $b_0 > 0$ such that for every $b \in (b_0, \infty)$, there exists no $C \in \mathbb{R}$ such that

$$C \partial_c \Psi_{c(b)}(t) = \partial_b \Psi_b(t), \quad \text{for all } t \in \mathbb{R}. \tag{5.1}$$

Proof. In order to get a contradiction, suppose that relation (5.1) holds for some constant $C \in \mathbb{R}$. The results of Corollaries 3.1 and 4.1 apply for $t = T + (a - 1) \log b$ for fixed $a \in (0, a_0)$, $T > 0$, and sufficiently large b . Substituting bounds (3.22) and (4.15) into (5.1) yields

$$\begin{aligned} C L_\infty e^{\kappa_- T} b^{-\kappa_-(1-a)} \left[1 + \mathcal{O}(\epsilon, b^{-(\kappa_+ - \kappa_-)(1-a)}, b^{-2(1-a)}) \right] \\ = A_0 \kappa_+ b^{a\kappa_+ - 1} e^{\kappa_+ T} \left[1 + \mathcal{O}(b^{-2(1-a) - a\kappa_+}, b^{-a(\kappa_+ - \kappa_-)}, b^{a\kappa_+}) \right], \end{aligned} \tag{5.2}$$

where we recall that $2(1 - a) + a\kappa_+ > 0$ if $a \in (0, a_0)$, where a_0 is given by (3.23) in Corollary 3.1. Since $A_0 \neq 0$ and $L_\infty \neq 0$ by Assumptions 2.1 and 2.2, we obtain from (5.2) that

$$\begin{aligned} C \left[1 + \mathcal{O}(\epsilon, b^{-(\kappa_+ - \kappa_-)(1-a)}, b^{-2(1-a)}) \right] \\ = L_\infty^{-1} A_0 \kappa_+ b^{a(\kappa_+ - \kappa_-) + \kappa_- - 1} e^{(\kappa_+ - \kappa_-)T} \left[1 + \mathcal{O}(b^{-2(1-a) - a\kappa_+}, b^{-a(\kappa_+ - \kappa_-)}, b^{a\kappa_+}) \right]. \end{aligned} \tag{5.3}$$

Since the remainder terms on both sides of (5.3) are smaller than the leading-order terms and $\kappa_+ \neq \kappa_-$, this gives a T -dependent coefficient C , which is a contradiction with the relation (5.1) for all $t \in \mathbb{R}$ and hence for all $T > 0$. \square

From Lemma 5.1, we can now prove Theorem 1.1 which states that the Morse index $m(u_b)$ is finite and is independent of b for every $b \in (b_0, \infty)$.

Proof of Theorem 1.1. For every $b \in (0, \infty)$, the potential $-3u_b^2(r)$ in \mathcal{L}_b is bounded from below on $[0, \infty)$. The Schrödinger operator $-\Delta_r + r^2 : \mathcal{E} \mapsto \mathcal{E}^*$ is strictly positive with a purely discrete spectrum. Since $\mathcal{L}_b = -\Delta_r + r^2 - \lambda(b) - 3u_b^2(r)$ is bounded from below, the number of

negative eigenvalues (the Morse index) of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ is finite by Theorem 10.7 in [11]. Thus, $m(u_b) < \infty$. It remains to show that $m(u_b)$ is independent of b for every $b \in (b_0, \infty)$.

Let us recall the Emden–Fowler transformation (2.1), which relates solutions of $\mathcal{L}_b v = 0$ with solutions of $\mathcal{M}_b \gamma = 0$ by $v(r) = r^{-1} \gamma(\log r)$. The spectrum of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ includes the zero eigenvalue if and only if there exists $v \in \mathcal{E}$ satisfying $\mathcal{L}_b v = 0$. This is impossible due to Lemma 5.1 according to the following arguments.

As $t \rightarrow -\infty$, there exist two linearly independent solutions to $\mathcal{M}_b \gamma = 0$ and the decaying solution is

$$\partial_b \Psi_b(t) = e^t + \mathcal{O}(e^{3t}), \quad \text{as } t \rightarrow -\infty.$$

The other solution is growing as $e^{(3-d)t}$ which corresponds to $v(r) \sim r^{2-d}$ so that

$$r^{d-1} |v(r)|^2 \sim r^{3-d}$$

is not integrable near $r = 0$ for $d \geq 4$. Hence, the corresponding $v(r)$ is not in L^2_r and if there exists nonzero $v \in \mathcal{E}$ satisfying $\mathcal{L}_b v = 0$, then there exists a constant $C_- \neq 0$ such that

$$v(r) = C_- r^{-1} \partial_b \Psi_b(\log r).$$

As $t \rightarrow +\infty$, there exist two linearly independent solutions to $\mathcal{M}_b \gamma = 0$ and the decaying solution is

$$\partial_c \Psi_{c(b)}(t) \sim e^{\frac{\lambda(b)-d+2}{2}t} e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \rightarrow \infty.$$

The other solution is growing as $e^{\frac{1}{2}e^{2t}}$, which corresponds to $v(r) \sim e^{\frac{1}{2}r^2}$, clearly not in L^2_r . If there exists nonzero $v \in \mathcal{E}$ satisfying $\mathcal{L}_b v = 0$, then there exists a constant $C_+ \neq 0$ such that

$$v(r) = C_+ r^{-1} \partial_c \Psi_{c(b)}(\log r).$$

Since $C_-, C_+ \neq 0$, if there exists nonzero $v \in \mathcal{E}$, then $\partial_b \Psi_b$ and $\partial_c \Psi_{c(b)}$ are linearly dependent, which results in a contradiction with Lemma 5.1 for every $b \in (b_0, \infty)$. Hence $0 \notin \sigma(\mathcal{L}_b)$ for every $b \in (b_0, \infty)$. Since eigenvalues of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ are continuous in b , this implies that $m(u_b)$ is independent of b for every $b \in (b_0, \infty)$. ■

By Lemma 4.1, $\partial_c \Psi_c(t)$ converges to $\partial_c \Psi_\infty$ on $[(a - 1) \log b, \infty)$ as $(\lambda, c) \rightarrow (\lambda_\infty, c_\infty)$. Each zero of either $\partial_c \Psi_c$ or $\partial_c \Psi_\infty$ is simple since they are solutions of the second-order linear homogeneous equations $\mathcal{M}_c \partial_c \Psi_c = 0$ and $\mathcal{M}_\infty \partial_c \Psi_\infty = 0$. Consequently, the number of nodal domains of $\partial_c \Psi_c$ in $[(a - 1) \log b, \infty)$ coincides with that of $\partial_c \Psi_\infty$ in $[(a - 1) \log b, \infty)$.

The following lemma shows that $\partial_c \Psi_{c(b)}$ does not have additional nodal domains in the interval $(-\infty, (a - 1) \log b)$ for sufficiently large b .

Lemma 5.2. Fix $d \geq 13$. Under Assumption 2.2, for every $a \in (0, 1)$, there exists $b_0 > 0$ such that for every $b \in (b_0, \infty)$, there exists $C_0 > 0$ such that

$$e^{(d-3)t} |\partial_c \Psi_{c(b)}(t)| \geq C_0, \quad t \in (-\infty, (a - 1) \log b). \tag{5.4}$$

Proof. Recall that $\mathcal{M}_b = \mathcal{M}_{c(b)}$ for $\lambda = \lambda(b)$ and $\mathcal{M}_b \partial_b \Psi_b = 0$ with

$$\partial_b \Psi_b(t) = e^t + \mathcal{O}(e^{3t}), \quad \text{as } t \rightarrow -\infty. \tag{5.5}$$

Similar to the proof of Lemma 3.1, we denote the second linearly independent solution of $\mathcal{M}_b \gamma = 0$ by Φ and normalize it such that

$$\Phi(t) = (2 - d)^{-1} e^{(3-d)t} + \mathcal{O}(e^{(5-d)t}), \quad \text{as } t \rightarrow -\infty. \tag{5.6}$$

We are interested in the behavior of $\partial_c \Psi_{c(b)}$ for $t \in (-\infty, t_0)$, where $t_0 := (a - 1) \log b$. It follows from (2.25) and (4.3) that for sufficiently large b , we have

$$\partial_c \Psi_{c(b)}(t_0) = L_\infty e^{\kappa_- t_0} \left[1 + \mathcal{O}\left(\epsilon, e^{(\kappa_+ - \kappa_-)t_0}, e^{2t_0}\right) \right], \tag{5.7}$$

$$\partial_c \Psi'_{c(b)}(t_0) = L_\infty \kappa_- e^{\kappa_- t_0} \left[1 + \mathcal{O}\left(\epsilon, e^{(\kappa_+ - \kappa_-)t_0}, e^{2t_0}\right) \right]. \tag{5.8}$$

Since $\partial_c \Psi_{c(b)}$ is a linear combination of $\partial_b \Psi_b$ and Φ by the linear superposition principle, we can express $\partial_c \Psi_{c(b)}$ as

$$\begin{aligned} \partial_c \Psi_{c(b)}(t) = e^{(d-4)t_0} & \left[(\partial_c \Psi_{c(b)}(t_0) \Phi'(t_0) - \partial_c \Psi'_{c(b)}(t_0) \Phi(t_0)) \partial_b \Psi_b(t) \right. \\ & \left. + \left(\partial_c \Psi'_{c(b)}(t_0) \partial_b \Psi_b(t_0) - \partial_c \Psi_{c(b)}(t_0) \partial_b \Psi'_b(t_0) \right) \Phi(t) \right], \end{aligned} \tag{5.9}$$

where we have used the normalization $W(\partial_b \Psi_b, \Phi) = e^{(4-d)t}$ of the Wronskian between the two solutions $\partial_b \Psi_b$ and Φ . Since $t_0 \rightarrow -\infty$ as $b \rightarrow \infty$, we can use asymptotics (5.5) and (5.6) as well as the boundary conditions (5.7) and (5.8) to obtain for every $t \in (-\infty, t_0)$:

$$\begin{aligned} \partial_c \Psi_{c(b)}(t) = (d - 2)^{-1} L_\infty e^{\kappa_- t} & \left[(\kappa_- + d - 3) e^{t-t_0} + (1 - \kappa_-) e^{(3-d)(t-t_0)} \right] \\ & \times \left[1 + \mathcal{O}\left(\epsilon, e^{(\kappa_+ - \kappa_-)t_0}, e^{2t_0}\right) \right] \left[1 + \mathcal{O}(e^{2t}) \right], \end{aligned} \tag{5.10}$$

where $1 - \kappa_- > 0$ and

$$\kappa_- + d - 3 = \frac{1}{2} \left(d - 2 - \sqrt{d^2 - 16d + 40} \right) > 0$$

for every $d \geq 13$. Thus, the sign of $\partial_c \Psi_{c(b)}(t)$ for every $t \in (-\infty, t_0)$ coincides with the sign of L_∞ . Multiplying (5.10) by $e^{(d-3)t}$ yields the bound (5.4). \square

Remark 5.1. It is interesting to know that $\partial_c \Psi_{c(b)}$ and $\partial_c \Psi_\infty$ diverge as $t \rightarrow -\infty$ with different growth rates: $d - 3$ for the former and $|\kappa_-|$ for the latter, as is seen from (2.25) and (5.10). This difference is explained by the different behavior of the t -dependent coefficients of $\mathcal{M}_{c(b)}$ and \mathcal{M}_∞ as $t \rightarrow -\infty$ in (2.8) and (2.23).

From Lemmas 4.1 and 5.2, we can now prove Theorem 1.2 which states that $m(u_b) = m(u_\infty)$ for every $b \in (b_0, \infty)$.

Proof of Theorem 1.2. By Sturm’s Oscillation Theorem (see, e.g., Theorem 3.5 in [24]), the Morse index $m(u_b)$ coincides with the number of zeros of the function $v(r)$ on $(0, \infty)$ satisfying $\mathcal{L}_b v = 0$ and $v(r) \rightarrow 0$ as $r \rightarrow \infty$. Due to the Emden–Fowler transformation (2.1), the number of zeros of $v(r)$ on $(0, \infty)$ coincides with the number of zeros of $\partial_c \Psi_{c(b)}(t)$ on \mathbb{R} since $\mathcal{M}_b \partial_c \Psi_{c(b)} = 0$ and $\partial_c \Psi_{c(b)}(t) \rightarrow 0$ as $t \rightarrow \infty$.

By Lemma 5.2, all zeros of $\partial_c \Psi_{c(b)}$ are located in the interval $[(a - 1) \log b, \infty)$ for fixed $a \in (0, 1)$ and sufficiently large $b > 0$. By Lemma 4.1 and simplicity of the zeros of $\partial_c \Psi_{c(b)}$ and $\partial_c \Psi_\infty$, the number of zeros of $\partial_c \Psi_{c(b)}$ and $\partial_c \Psi_\infty$ in $[(a - 1) \log b, \infty)$ coincides since $\lambda(b) \rightarrow \lambda_\infty$ and $c(b) \rightarrow c_\infty$ as $b \rightarrow \infty$. All zeros of $\partial_c \Psi_\infty$ are located in the interval $[(a - 1) \log b, \infty)$ by Assumption 2.2 with the expansion (2.25) and give $m(u_\infty) = m(u_b)$ for every $b \in (b_0, \infty)$. ■

Appendix A. Proof of Proposition 2.2

Let $\Sigma(t) := \Psi_c(t) - \Psi_\infty(t)$. It follows from (2.3) that Σ satisfies the following equation:

$$\mathcal{M}_\infty \Sigma = \mathcal{F}(\Sigma)(t), \tag{A.1}$$

where \mathcal{M}_∞ is defined by (2.23) and

$$\mathcal{F}(\Sigma)(t) := -(\lambda - \lambda_\infty)e^{2t}(\Psi_\infty(t) + \Sigma(t)) - 3\Psi_\infty(t)\Sigma(t)^2 - \Sigma(t)^3.$$

Since $\Psi_\infty(t) \rightarrow \sqrt{d-3}$ as $t \rightarrow -\infty$, as it follows from (2.17), we can pick two linearly independent solutions r_1, r_2 to $\mathcal{M}_\infty r = 0$ such that

$$r_1(t) = \mathcal{O}(e^{\kappa_- t}), \quad r_2(t) = \mathcal{O}(e^{\kappa_+ t}), \quad \text{as } t \rightarrow -\infty, \tag{A.2}$$

where $\kappa_- < \kappa_+ < 0$ are given by (2.14). Using the Liouville’s formula, we normalize the Wronskian according to the relation:

$$W(r_1, r_2)(t) = r_1(t)r_2'(t) - r_1'(t)r_2(t) = e^{-(d-4)t}. \tag{A.3}$$

By the variation of parameters method, we rewrite the differential equation (A.1) as an integral equation for every $t \in [(a - 1) \log b, 0]$:

$$\begin{aligned} \Sigma(t) = & \Sigma(0) [r_1(t)r_2'(0) - r_1'(0)r_2(t)] + \Sigma'(0) [r_1(0)r_2(t) - r_1(t)r_2(0)] \\ & + \int_t^0 e^{(d-4)t'} [r_1(t)r_2'(t') - r_1'(t')r_2(t)] \mathcal{F}(\Sigma)(t') dt'. \end{aligned} \tag{A.4}$$

In order to use Banach fixed-point iterations, we introduce $\tilde{\Sigma}(t) := e^{-\kappa_- t} \Sigma(t)$, which satisfies $\tilde{\Sigma}(t) = \mathcal{A}(\tilde{\Sigma})(t)$, where

$$\begin{aligned}
 \mathcal{A}(\tilde{\Sigma})(t) &= \Sigma(0)e^{-\kappa-t} [r_1(t)r_2'(0) - r_1'(0)r_2(t)] + \Sigma'(0)e^{-\kappa-t} [r_1(0)r_2(t) - r_1(t)r_2(0)] \\
 &\quad - (\lambda - \lambda_\infty) \int_t^0 K_2(t, t')e^{2t'} [e^{-\kappa-t'}\Psi_\infty(t') + \tilde{\Sigma}(t')] dt' \\
 &\quad - \int_t^0 K_2(t, t') [3e^{\kappa-t'}\Psi_\infty(t')\tilde{\Sigma}^2(t') + e^{2\kappa-t'}\tilde{\Sigma}^3(t')] dt',
 \end{aligned} \tag{A.5}$$

where the kernel $K_2(t, t')$ is defined as

$$K_2(t, t') := e^{\kappa-(t'-t)-(κ_++κ_-)t'} [r_1(t)r_2(t') - r_1(t')r_2(t)]. \tag{A.6}$$

It follows from (A.2) that $|K_2(t, t')| \leq C (1 + e^{(\kappa_+ - \kappa_-)(t-t')})$ for $-\infty < t \leq t' \leq 0$, which means that there exists some constant $K_0 > 0$, such that

$$\sup_{-\infty < t \leq t' \leq 0} |K_2(t, t')| \leq K_0. \tag{A.7}$$

It follows from (2.22) that there exists some constant $C_0 > 0$ such that

$$|\Sigma(0)| + |\Sigma'(0)| \leq C_0 (|\lambda - \lambda_\infty| + |c - c_\infty|).$$

The integral operator $\mathcal{A}(\tilde{\Sigma})$ in (A.5) is estimated for every $\tilde{\Sigma} \in L^\infty((a - 1) \log b, 0)$ as

$$\begin{aligned}
 \|\mathcal{A}(\tilde{\Sigma})\|_\infty &\leq C_0 [|\lambda - \lambda_\infty| + |c - c_\infty| + |\lambda - \lambda_\infty| \|\tilde{\Sigma}\|_\infty \\
 &\quad + b^{-\kappa-(1-a)} \|\tilde{\Sigma}\|_\infty^2 + b^{-2\kappa-(1-a)} \|\tilde{\Sigma}\|_\infty^3].
 \end{aligned} \tag{A.8}$$

Similar estimate applies to $\|\mathcal{A}(\tilde{\Sigma}_1) - \mathcal{A}(\tilde{\Sigma}_2)\|_\infty$. The estimates show that the integral operator $\mathcal{A}(\tilde{\Sigma})$ is closed and is a contraction in the ball $B_\delta \subset L^\infty((a - 1) \log b, 0)$ of the small radius $\delta := 2C_0\epsilon b^{\kappa-(1-a)}$, provided that (λ, c) satisfy the bound (2.19) and $\epsilon > 0$ is sufficiently small. By the Banach fixed-point theorem, there exists a unique fixed point of $\mathcal{A}(\tilde{\Sigma})$ satisfying

$$\|\tilde{\Sigma}\|_\infty \leq 2C_0\epsilon b^{\kappa-(1-a)}, \tag{A.9}$$

which proves the bound (2.20) after going back to the original variable Σ and redefining C_0 .

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