Averaging of nonlinearity-managed pulses

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We consider the nonlinear Schrödinger equation with the nonlinearity management which describes Bose–Einstein condensates under Feshbach resonance. By using an averaging theory, we derive the Hamiltonian averaged equation and compare it with other averaging methods developed for this problem. The averaged equation is used for analytical approximations of nonlinearity-managed solitons. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1922660]

We have systematically categorized the averaging procedure for the nonlinear Schrödinger (NLS) equation with nonlinearity management. We have derived an averaged equation by using four equivalent methods: (i) near-identity canonical transformations, (ii) asymptotic multi-scale expansion methods based on local transformations, (iii) asymptotic multi-scale expansion methods based on nonlocal transformations, and (iv) direct perturbation series expansions. Stationary solutions of the averaged equation are used to approximate time-dependent solutions of the full NLS equation. Two families of stationary solutions include bright and dark solitons. We show that these solutions exist in an open quadrant of the parameter plane (γ_0,ω), where γ_0 is the averaged nonlinearity coefficient and ω is the frequency of the stationary solutions.

I. INTRODUCTION

In atomic physics, the Feshbach resonance of the scattering length of interatomic interactions is used for control of Bose–Einstein condensates.[1,2] The periodic variation of the scattering length by means of an external magnetic field provides an experimentally realizable protocol for generation of solitary waves in harmonic traps,[3–5] control of periodic waves in optical lattices,[6–9] and persistence of nonlinear structures against collapse phenomena in higher dimensions.[10,11]

We consider the main model for Feshbach resonance given by the NLS equation with the variable nonlinearity coefficient (which we call the nonlinearity management). The averaging theory of nonlinearity-managed pulses can be developed for the NLS equation in the space of any dimension and with an external trapping potential. We simplify details of the paper by considering the model in the space of one dimension and without the external potential:

$$iu_t = -u_{xx} - \gamma_0|u|^2u - \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^2u, \qquad (1.1)$$

where $x\in\mathbb{R}$, $t\in\mathbb{R}_+$, $u\in\mathbb{C}$, γ_0 is parameter, and $\gamma(\tau)$ is a continuous function, such that

$$\gamma(\tau+1)=\gamma(\tau), \quad \int_0^1 \gamma(\tau)d\tau = 0, \quad \max_{0\leq\tau\leq 1}|\gamma(\tau)| = \Gamma. \qquad (1.2)$$

The NLS equation (1.1) has the standard Hamiltonian form with the time-dependent Hamiltonian:

$$iu_t = \frac{\delta H}{\delta \bar{u}},$$

where

$$H[u,\bar{u},\tau] = \int_{\mathbb{R}}\left(|u_x|^2 - \frac{1}{2}\gamma_0|u|^4 - \frac{1}{2\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^4\right)dx. \qquad (1.3)$$

When $\epsilon \ll 1$ and $\Gamma=O(1)$, the nonlinearity management is referred to as *strong*. When $\epsilon\ll 1$ and $\Gamma=O(\epsilon)$, the nonlinearity management is referred to as *weak*.

Two recent publications[4,10] addressed averaging procedures for the NLS equation (1.1) in the asymptotic limit $\epsilon \ll 1$. In Ref 4 [(Eq. (9)], a local (differential) equation was derived in the strong management case by using a nonlocal (integral) transformation. The nonlocal transformation destroys the Hamiltonian form (1.3) of the NLS equation (1.1). In Ref. 10 [Eq.(22)], another local differential equation was derived in the weak management case by using the perturbation series expansions. The local averaged equation has a "quasi"-Hamiltonian form with a nonlinear symplectic structure (see Eq. (23) in Ref. 10). More important, the two averaged equations in Refs. 4 and 10 do not match each other in the limit of common validity.

These preliminary publications call for systematic analysis of the averaging procedure for the NLS equation with the nonlinearity management (1.1). The main objective of our paper is to exploit the Hamiltonian form of the NLS equation (1.1) and to derive the following averaged NLS equation:

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$$i w_t = - w_{xx} - \gamma_0 |w|^2 w - \sigma^2((|w|_x^2)^2 + 2|w|^2|w|_{xx}^2)w, \qquad (1.4)$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}_+$, $w \in \mathbb{C}$, the notations $|w|_x^2, |w|_{xx}^2$ stand for $(|w|^2)_x$, $(|w|^2)_{xx}$, and

$$\sigma^2 = \int_0^1 \gamma_{-1}^2(\tau) d\tau,$$

$$\gamma_{-1}(\tau) = \int_0^\tau \gamma(\tau')d\tau' - \int_0^1 \int_0^\tau \gamma(\tau')d\tau' d\tau. \qquad (1.5)$$

The averaged NLS equation (1.4) can be cast to the standard Hamiltonian form with the averaged Hamiltonian:

$$H_0[w,\bar{w}] = \int_{\mathbb{R}} \left(|w_x|^2 - \frac{1}{2}\gamma_0 |w|^4 + \sigma^2 |w|^2 (|w|_x^2)^2 \right) dx. \qquad (1.6)$$

The Hamiltonian averaged NLS equation (1.4) covers both cases of the strong and weak nonlinearity managements, but the parameter σ^2 is small in the latter case. Since the averaged equation (1.4) is different from both previously derived averaged equations in Refs. 4 and 10, it is important to clarify the validity of all three averaging methods for the NLS equation with the nonlinearity management (1.1). This problem is considered in our paper, where we show that the Hamiltonian averaged equation (1.4) is the unique equation in all three averaged methods. The nonlinear bound states in the NLS equation (1.1) are approximated asymptotically by the stationary solutions in the averaged NLS equation (1.4). A short version of this paper is published in Ref. 5.

The paper is structured as follows. Section II is devoted to derivation of the averaged NLS equation (1.4) with the Hamiltonian averaging and the asymptotic multiscale expansion methods. Problems in the other two averaging methods in Refs. 4 and 10 are reviewed and resolved in Sec. III. The local existence of solutions of the averaged equation (1.4) is discussed in Sec. IV. Section V describes stationary solutions of the averaged NLS equation (1.4) and the asymptotic approximations of the bound states of the full NLS equation (1.1).

II. DERIVATION OF THE HAMILTONIAN AVERAGED EQUATION (1.4)

We shall derive the averaged NLS equation (1.4) with two equivalent methods. The first method is based on canonical transformations of the Hamiltonian (1.3). The second method is based on the asymptotic multiscale expansions of the NLS equation (1.1).

A. Hamiltonian formalism

Since the periodic term in the Hamiltonian (1.3) is singular as $\epsilon \to 0$ and $\Gamma = O(1)$, it is natural to remove the periodic term with a canonical transformation. Let us define new dependent variable $v(x,t)$ by

$$u(x,t) = e^{i\gamma_{-1}(\tau)|v|^2(x,t)} v(x,t), \qquad \tau = \frac{t}{\epsilon}, \qquad (2.1)$$

where $\gamma_{-1}(\tau)$ is the mean-zero antiderivative of $\gamma(\tau)$, defined by Eq. (1.5). The canonical transformation from $(u,\bar{u})$ to $(v,\bar{v})$ is given by the generating functional,

$$S[u,v,\tau] = \frac{1}{2\gamma_{-1}(\tau)} \int_{\mathbb{R}} \log^2\left(\frac{v}{u}\right) dx. \qquad (2.2)$$

The symplectic form is preserved by $S[u,v,\tau]$, while the new Hamiltonian is computed as follows:

$$\tilde{H}[v,\bar{v},\tau] = H[u,\bar{u},\tau] + \frac{\partial S}{\partial t}[u,v,\tau]$$
$$= \int_{\mathbb{R}} \left(|v_x + i\gamma_{-1}(\tau)|v|_x^2 v|^2 - \frac{1}{2}\gamma_0 |v|^4 \right) dx. \qquad (2.3)$$

The averaged method for the Hamiltonian $\tilde{H}[v,\bar{v},\tau]$ is based on the sequence of near-identity canonical transformations:[12]

$$F[v,\bar{w},\tau] = \int_{\mathbb{R}} \left(v\bar{w} + \sum_{n=1}^{N+1} \epsilon^n F_n(v,\bar{w},\tau) + O(\epsilon^{N+2}) \right) dx. \qquad (2.4)$$

The generating functional defines $\bar{v}$ and w by variational derivatives in v and $\bar{w}$, while correction terms $F_n(v,\bar{w},\tau)$ are periodic, mean-zero functions of τ. The new averaged Hamiltonian takes the form:

$$H_{\text{new}}[w,\bar{w},\tau] = \tilde{H}[v,\bar{v},\tau] + i\frac{\partial F}{\partial t}[v,\bar{w},\tau]$$
$$= H_N[w,\bar{w}] + O(\epsilon^{N+1}), \qquad (2.5)$$

where $H_N[w,\bar{w}]$ is the Nth order averaged Hamiltonian. Since $\gamma_{-1}(\tau)$ has zero mean, the leading-order averaged Hamiltonian is found immediately in the form (1.6), where

$$H_0[w,\bar{w}] = \int_0^1 \tilde{H}[w,\bar{w},\tau] d\tau. \qquad (2.6)$$

The first-order correction term $F_1(w,\bar{w},\tau)$ is then found explicitly as

$$F_1(w,\bar{w},\tau) = (\gamma_{-1})_{-1}(\tau) \int_{\mathbb{R}} |w|_x^2 (\bar{w}w_x - \bar{w}_x w) dx$$
$$+ i(\gamma_{-1}^2(\tau) - \sigma^2)_{-1} \int_{\mathbb{R}} |w|^2 (|w|_x^2)^2 dx, \qquad (2.7)$$

where the notation $a_{-1}(\tau)$ stands for the mean-zero antiderivative of the mean-zero periodic function $a(\tau)$. The sequence of near-identity canonical transformations can be continued to any order of $O(\epsilon^{N+1})$, $N \geqslant 0$ in a formal algorithmic procedure.[12] Similar Hamiltonian averaging method is applied to the harmonically driven pendulum (the Kapitza pendulum) and its infinite-dimensional counterpart (the sine-Gordon equation)[13] (see also the review in Ref. 14).

B. Asymptotic multiscale expansion method

Using the transformation (2.1), we replace the NLS equation (1.1) with the equivalent equation:

$$iv_t - \gamma_{-1}|v|_t^2 v = -v_{xx} - \gamma_0|v|^2 v - i\gamma_{-1}(2|v|_x^2 v_x + |v|_{xx}^2 v) + \gamma_{-1}^2(|v|_x^2)^2 v. \quad (2.8)$$

It follows from Eq. (2.8) that

$$|v|_t^2 = i(\bar{v}v_x - \bar{v}_x v)_x - 2\gamma_{-1}(|v|^2|v|_x^2)_x. \quad (2.9)$$

As a result, the transformed equation (2.8) is written in the standard NLS form,

$$iv_t = -v_{xx} - \gamma_0|v|^2 v - 2i\gamma_{-1}(v^2\bar{v}_{xx} + 2|v_x|^2 v + v_x^2\bar{v}) - \gamma_{-1}^2((|v|_x^2)^2 + 2|v|_{xx}^2|v|^2)v. \quad (2.10)$$

The NLS equation (2.10) is related to the Hamiltonian (2.3) by the standard Hamiltonian form. We look for the solution $v(x,t)$ with the asymptotic multiscale expansion series:

$$v(x,t) = w(x,T) + \sum_{n=1}^{N+1} \epsilon^n v_n(x,\tau,T) + O(\epsilon^{N+2}), \quad (2.11)$$

where τ is fast time and T is the vector of slow time scales, needed for continuation of the multiscale series (2.11):

$$\tau = \frac{t}{\epsilon}, \quad T = (t, \epsilon t, \epsilon^2 t, \ldots). \quad (2.12)$$

The leading-order equation for $w(x,t)$ and $v_1(x,\tau,t)$ is derived from Eq. (2.10) by truncating the order of $O(\epsilon^2)$ in Eq. (2.11). The averaged NLS equation (1.4) removes secularly growing terms from $v_1(x,\tau,t)$, which can then be found explicitly as

$$v_1 = -2(\gamma_{-1})_{-1}(w^2\bar{w}_{xx} + 2|w_x|^2 w + w_x^2\bar{w}) - i(\gamma_{-1}^2 - \sigma^2)_{-1}((|w|_x^2)^2 + 2|w|_{xx}^2|w|^2)w, \quad (2.13)$$

such that v_1 is related to the negative variational derivative of $F_1(w,\bar{w},\tau)$ in $\bar{w}$. The formal asymptotic multiscale expansion series (2.11) can be continued to any power order of ϵ^{N+1}, $N \geqslant 0$, in full correspondence with the near-identity canonical transformations (2.4) and (2.5).

III. ALTERNATIVE AVERAGING METHODS FOR THE NLS EQUATION (1.1)

We shall review two alternative averaging methods for the NLS equation (1.1), which were used recently in Refs. 4 and 10. In both cases, we shall outline the problems in the computations of the formal asymptotic multiscale expansion series. Resolving these problems, we will show that the same Hamiltonian averaged NLS equation (1.4) gives the leading-order averaged equation in both the methods.

A. Averaging method based on the nonlocal transformation

A local transformation (2.1) is used in Sec. II to remove the large periodic term of the NLS equation (1.1). There exists an equivalent nonlocal transformation, which serves the same purpose:

$$u(x,t) = e^{i\phi(x,t)}v(x,t), \quad \phi(x,t) = \frac{1}{\epsilon}\int_0^t \gamma\left(\frac{t'}{\epsilon}\right)|v|^2(x,t')dt'. \quad (3.1)$$

By using the nonlocal transformation (3.1), one can reduce the NLS equation (1.1) to the system:

$$iv_t = -v_{xx} - 2i\phi_x v_x - i\phi_{xx}v + (\phi_x)^2 v, \quad (3.2)$$

$$\phi_t = \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|v|^2, \quad \phi(x,0) = 0. \quad (3.3)$$

In Ref. 4, the variable $\phi(x,t)$ was eliminated from the system (3.2) and (3.3) and the solution $v(x,t)$ of a nonlocal (integral) scalar equation was sought to be the asymptotic multiscale expansion series (2.11). The leading-order equation for $w(x,t)$ and $v_1(x,\tau,t)$ can be derived from Eq. (3.2) by truncating the order of $O(\epsilon^2)$. The secularly growing terms for $v_1(x,\tau,t)$ are removed if $w(x,t)$ satisfies the leading-order averaged equation:

$$iw_t = -w_{xx} - \gamma_0|w|^2 w - i\nu_1(2|w|_x^2 w_x + |w|_{xx}^2 w) + (\nu_1^2 + \sigma^2)(|w|_x^2)^2 w, \quad (3.4)$$

where σ^2 is introduced in Eq. (1.5) and

$$\nu_1 = \int_0^1 \int_0^\tau \gamma(\tau')d\tau' d\tau.$$

It can be checked that the averaged equation (3.4) has no Hamiltonian form (1.3) and cannot be reduced to the Hamiltonian averaged NLS equation (1.4). We will show that, although the averaged equation (3.4) gives the necessary and sufficient condition for existence of the bounded function $v_1(x,\tau,t)$, a continuation of the asymptotic multiscale expansion series (2.11) to the next order $O(\epsilon^2)$ is not yet possible.

The correction term $v_1(x,\tau,t)$ can be found in the explicit form:

$$v_1(x,\tau,t) = -(\gamma_{-1})_{-1}(2|w|_x^2 w_x + |w|_{xx}^2 w) - i(\gamma_{-1}^2 - \sigma^2)_{-1}(|w|_x^2)^2 w, \quad (3.5)$$

such that

$$\int_0^1 \gamma(\tau)(\bar{w}(x,t)v_1(x,\tau,t) + w(x,t)\bar{v}_1(x,\tau,t))d\tau = 2\sigma^2(|w|^2|w|_x^2)_x \neq 0. \quad (3.6)$$

As a result, the right-hand side of the inhomogeneous equation for $v_2(x,\tau,T)$ contains terms with the linear growth in τ, which cannot be removed by constraints on $w(x,T)$. We conclude that *the averaging procedure with the asymptotic multiscale expansion series fail for differential equations with nonlocal (integral) terms.*

In order to resolve this obstacle, we shall work with the system of differential equations, avoiding any nonlocal (integral) terms and extending artificially the second-order time-evolution problem (1.1) to the third-order time-evolution system (3.2) and (3.3). We look for the asymptotic multiscale expansion series for the functions $v(x,t)$ and $\phi(x,t)$:

$$v(x,t) = w(x,T) + \sum_{n=1}^{N+1} \epsilon^n v_n(x,\tau,T) + O(\epsilon^{N+2}), \tag{3.7}$$

$$\phi(x,t) = \gamma_{-1}(\tau)|w|^2 + \varphi(x,T) + \sum_{n=1}^{N+1} \epsilon^n \phi_n(x,\tau,T) + O(\epsilon^{N+2}), \tag{3.8}$$

where τ and T are the same as in (2.12). The leading-order functions $w(x,T)$ and $\varphi(x,T)$ are yet to be defined from continuations of the asymptotic series (3.7) and (3.8). By truncating the order of $O(\epsilon^2)$, we have the system of equations:

$$\begin{aligned} iw_t + iv_{1\tau} = &- w_{xx} - 2i\varphi_x w_x - i\varphi_{xx} w + (\varphi_x)^2 w \\ &- i\gamma_{-1}(|w|_x^2 w_x + |w|_{xx}^2 w) + 2\gamma_{-1}\varphi_x |w|_x^2 w \\ &+ \gamma_{-1}^2 (|w|_x^2)^2 w \end{aligned} \tag{3.9}$$

and

$$\varphi_t + \phi_{1\tau} = - \gamma_{-1}|w|_\tau^2 + \gamma(\bar{w}v_1 + w\bar{v}_1). \tag{3.10}$$

Terms with the nonzero mean lead to secular growth of solutions $v_1(x,\tau,t)$ and $\phi_1(x,\tau,t)$ in τ. The leading-order functions $w(x,t)$ and $\varphi(x,t)$ are defined by removing secular terms from the system (3.9) and (3.10). Removing secular terms in the first equation (3.9), we derive an averaged equation for $w(x,t)$:

$$iw_t = - w_{xx} - 2i\varphi_x w_x - i\varphi_{xx} w + (\varphi_x)^2 w + \sigma^2(|w|_x^2)^2 w. \tag{3.11}$$

This equation recovers the previous result (3.4) for $\varphi(x,t) = 0$ and $\nu_1 = 0$. However, it is clear that the constraint $\varphi(x,t) = 0$ cannot be set arbitrarily, since the second equation (3.10) also has secular terms. In order to identify these terms, we find explicitly

$$\begin{aligned} v_1(x,\tau,t) = &- (\gamma_{-1})_{-1}(2|w|_x^2 w_x + |w|_{xx}^2 w + 2i\varphi_x |w|_x^2 w) \\ &- i(\gamma_{-1}^2 - \sigma^2)_{-1}(|w|_x^2)^2 w. \end{aligned} \tag{3.12}$$

Removing the secular terms in the second equation (3.10), we derive an averaged equation for $\varphi(x,t)$:

$$\varphi_t = 2\sigma^2(|w|^2|w|_x^2)_x. \tag{3.13}$$

Under the condition (3.13), a bounded solution can be found for $\phi_1(x,\tau,t)$ from Eq. (3.10). Thus, an obstacle (3.6) in the nonlocal averaging procedure is removed in the local averaging procedure by using the renormalization of $\phi(x,t)$ in the series (3.8). Similarly, the asymptotic multiscale expansion series (3.7) and (3.8) can be continued algorithmically to any power order of ϵ^{N+1}, $N \geqslant 0$.

The leading-order averaging theory for the NLS equation (1.1) with the nonlocal transformation (3.1) is given by the third-order time-evolution system (3.11) and (3.13). We will show that this system is equivalent to the Hamiltonian averaged NLS equation (1.4) by using the gauge transformation:

$$\tilde{w}(x,t) = w(x,t)e^{i\varphi(x,t)}. \tag{3.14}$$

The transformation (3.14) establishes a complete equivalence between the nonlocal solution (3.1), (3.7), and (3.8) at the leading order and the local solution (2.1) and (2.11) at the leading order. In order to prove the gauge transformation (3.14), we use the polar form $w = \rho e^{i\theta}$, where $\rho(x,t)$ and $\theta(x,t)$ are real functions, and we rewrite the system (3.11) and (3.13) as follows:

$$\rho_t = - 2\rho_x(\theta + \varphi)_x - \rho(\theta + \varphi)_{xx}, \tag{3.15}$$

$$- \rho\theta_t = - \rho_{xx} + \rho(\theta_x + \varphi_x)^2 - \gamma_0\rho^3 + 4\sigma^2\rho^3\rho_x^2, \tag{3.16}$$

$$\varphi_t = 4\sigma^2(\rho^3\rho_x)_x. \tag{3.17}$$

The third-order system (3.15)–(3.17) reduces in the variables ρ and $\psi = \theta + \varphi$ to the second-order system:

$$\rho_t = - 2\rho_x\psi_x - \rho\psi_{xx}, \tag{3.18}$$

$$- \rho\psi_t = - \rho_{xx} + \rho\psi_x^2 - \gamma_0\rho^3 - 8\sigma^2\rho^3\rho_x^2 - 4\sigma^2\rho^4\rho_{xx}. \tag{3.19}$$

It is easy to show that if $\rho(x,t)$ and $\psi(x,t)$ solve the system (3.18) and (3.19), then the complex-valued function $\tilde{w} = \rho e^{i\psi}$ solves the Hamiltonian averaged NLS equation (1.4), such that the gauge transformation (3.14) is proved. The gauge transformation reduces the system of averaged equations (3.11) and (3.13) to the Hamiltonian averaged equation (1.4). Thus, the averaged method based on the nonlocal transformation represents the same result as the Hamiltonian averaged method developed in Sec. II. Another averaging method for nonlocal Maxwell equations based on the local differential system of equations was developed recently in Ref. 15 for justification of the NLS approximation for optical pulses.

B. Averaging method based on the perturbation series

When the nonlinearity management is weak, such that Γ is order of ϵ in Eq. (1.2), the asymptotic multiscale expansion method can be applied directly to the NLS equation (1.1) without removing the periodic term by either local or nonlocal transformations (2.1) and (3.1). This method was pioneered in Ref. 16 and was applied in Ref. 10 to the NLS equation with the nonlinearity management (1.1). We will show that an accurate application of this averaging method results in the same Hamiltonian averaged equation (1.4).

We rescale $\gamma(\tau) \mapsto \epsilon\gamma(\tau)$ in the case of the weak nonlinearity management and rewrite the NLS equation (1.1) in the form:

$$iu_t = - u_{xx} - \gamma_0|u|^2 u - \gamma(\tau)|u|^2 u, \quad \tau = \frac{t}{\epsilon}. \tag{3.20}$$

The solution is sought to be the asymptotic multiscale expansion series:

$$u(x,t) = w(x,T) + \sum_{n=1}^{N+1} \epsilon^n u_n(x,\tau,T) + O(\epsilon^{N+2}). \quad (3.21)$$

By truncating the order of $O(\epsilon^2)$, the leading-order averaged equation is

$$iw_t = - w_{xx} - \gamma_0 |w|^2 w \quad (3.22)$$

and the first-order correction term is

$$u_1(x,\tau,t) = i\gamma_{-1}(\tau)|w|^2 w. \quad (3.23)$$

By truncating the order of $O(\epsilon^3)$, we have the inhomogeneous equation:

$$iw_{T_1} + iu_{1t} + iu_{2\tau} = - u_{1xx} - (\gamma_0 + \gamma(\tau))(2|w|^2 u_1 + w^2 \bar{u}_1), \quad (3.24)$$

where $T_1 = \epsilon t$. Since the right-hand side of Eq. (3.24) has zero mean, we set $w_{T_1} = 0$ and find the second-order correction term:

$$u_2(x,\tau,t) = -(\gamma_{-1})_{-1}(\tau)((|w|^2 w)_{xx} + |w|^4 w + i(|w|^2 w)_t) - \tfrac{1}{2}\gamma_{-1}^2(\tau)|w|^4 w. \quad (3.25)$$

By truncating the order of $O(\epsilon^4)$, we have the following inhomogeneous equation:

$$iw_{T_2} + iu_{2t} + iu_{3\tau} = - u_{2xx} - (\gamma_0 + \gamma(\tau))(2|w|^2 u_2 + w^2 \bar{u}_2 + 2|u_1|^2 w + u_1^2 \bar{w}), \quad (3.26)$$

where $T_2 = \epsilon^2 t$. Removing the terms with the nonzero mean and using the explicit forms (3.23) and (3.25), and the leading-order averaged equation (3.22), we simplify the second-order averaged equation to the form:

$$iw_{T_2} = - \sigma^2 ((|w|_x^2)^2 + 2|w|^2 |w|_{xx}^2) w. \quad (3.27)$$

Combined together w_t and $\epsilon^2 w_{T_2}$, the averaged equations (3.22) and (3.27) recover the same Hamiltonian averaged NLS equation (1.4) with the scaling transformation: $\sigma^2 \mapsto \epsilon^2 \sigma^2$.

A different averaged equation was derived with the same method in Ref. 10 [see their Eq. (22)]. The different averaged equation has a nonlinear symplectic structure with a different averaged Hamiltonian [see Eqs. (23)-(24) in Refs. 10]. However, there exists a near-identity transformation,

$$w = A + \tfrac{1}{2}\epsilon^2 \sigma^2 |A|^4 A + O(\epsilon^4), \quad (3.28)$$

that transforms Eq. (22) for A in Ref. 10 to our main equations (3.22) and (3.27) for w, when the terms of the order of $O(\epsilon^4)$ are truncated. The asymptotic transformation (3.28) was used in Ref. 8 in a similar content of weak nonlinearity management in discrete NLS lattices. The main averaged equation was transformed there to the Lagrangian form [see Eqs. (21) and (22) in Ref. 8]. If the Legendre transformation is employed, the Lagrangian form in Ref. 8 produces Hamiltonian of the discrete averaged NLS equation, which is a spatial discretization of the Hamiltonian (1.6) of the continuous averaged NLS equation. We note that, in contrast to the asymptotic transformation (3.28) truncated at the order of $O(\epsilon^4)$, the averaged Hamiltonian (1.6) is valid uniformly both for weak and strong nonlinearity managements.

The explicit solutions (3.23) and (3.25) can be simplified with the use of the local transformation (2.1), which is valid for any magnitude of Γ. When Γ is of the order of ϵ, the complex exponential factor in the transformation (2.1) is expanded into the Taylor series, which increases the number of correction terms in the asymptotic solution $u(x,t)$, compared to the asymptotic solution $v(x,t)$. Besides this complication, the asymptotic multiscale expansion method here repeats the same algorithm as the one used in Sec. II.

IV. LOCAL EXISTENCE OF SOLUTIONS

We shall analyze local existence of solutions of the averaged equation (1.4), which is of quasilinear type. The local existence theory for this class of equations has been developed by a number of authors (see Ref. 17 for an extensive list of references). We use the result by Poppenberg,[18] which states the conditions under which the PDE of the class

$$iu_t = - u_{xx} + V(x)u + G(u,u_x,u_{xx})u \quad (4.1)$$

possesses a unique solution $u(x,t) \in C^1([0,T],H^\infty)$ starting with $u(x,0) = \phi \in H^\infty$ for some $T > 0$, where $H^\infty = \cap_{n \geq 0} H^n$. The averaged equation (1.4) satisfies the conditions in Ref. 18 and therefore we obtain a local well-posedness result.

Proposition: *Let $w(x,0) = \phi \in H^\infty$. There exists $T > 0$, such that the averaged equation (1.4) possesses a unique solution $w(x,t) \in C^1([0,T],H^\infty)$.*

The local existence holds in the space H^∞, rather than in the energy space H^1. If a local existence of solutions in H^1 can be established, it would imply immediately the global existence of solutions in the same space. Indeed, using Sobolev inequality for $w \in H^1$ with $\|w\|_{L^2} = \text{constant}$, we have

$$\int_{\mathbb{R}} |w|^4 dx \leq C \left(\int_{\mathbb{R}} |w_x|^2 dx \right)^{1/2}. \quad (4.2)$$

Since the Hamiltonian is constant during the evolution, we have for $\gamma_0 > 0$:

$$H = \int_{\mathbb{R}} \left(|w_x|^2 - \frac{1}{2}\gamma_0 |w|^4 + \sigma^2 |w|^2 (|w|_x^2)^2 \right) dx \geq \int_{\mathbb{R}} |w_x|^2 dx - C\gamma_0 \left(\int_{\mathbb{R}} |w_x|^2 dx \right)^{1/2},$$

which implies that $\|w\|_{H^1}$ is uniformly bounded as long as the solution $w(x,t)$ exists. Together with local existence this estimate would imply global existence in H^1. Analysis of local well-posedness of solutions of the averaged equation (1.4) in H^1 is beyond the scopes of this paper.

V. ASYMPTOTIC APPROXIMATIONS FOR NONLINEAR BOUND STATES

We approximate nonlinear bound states of the NLS equation (1.1) with the stationary solutions of the Hamiltonian averaged NLS equation (1.4). Using the standard ansatz for stationary solutions $w(x,t) = \Phi(x) e^{i\omega t}$, we find the ODE problem for $\Phi(x)$:

$$-\Phi'' + \omega\Phi - \gamma_0\Phi^3 - 4\sigma^2(2\Phi^3(\Phi')^2 + \Phi^4\Phi'') = 0, \quad (5.1)$$

where $x \in \mathbb{R}$ and $\Phi \in \mathbb{R}$. Due to the Hamiltonian form of the averaged NLS equation (1.4), the first integral of the ODE (5.1) exists in the form:

$$E = -(\Phi')^2 + \omega\Phi^2 - \tfrac{1}{2}\gamma_0\Phi^4 - 4\sigma^2\Phi^4(\Phi')^2. \quad (5.2)$$

Stationary solutions of the ODE (5.1) approximate the time-dependent solutions of the NLS equation (1.1). The leading-order approximation follows from Eqs. (2.1) and (2.11):

$$u(x,t) = \Phi(x)e^{i\gamma_{-1}(\tau)\Phi^2(x)+i\omega t} + O(\epsilon). \quad (5.3)$$

Two types of the stationary solutions include bright and dark solitons. In the presence of external potentials (e.g., harmonic magnetic traps or periodic optical lattices), these solutions correspond to matter-wave solitary waves trapped in the Bose–Einstein condensates.[6,7] Stability of stationary solutions of the averaged equation (1.4) is an open problem for further studies. Numerical computations[4] indicate dynamical stability of time-dependent solutions (5.3) in the original NLS equation with the nonlinearity management (1.1).

A. Bright solitons

Bright solitons are given by stationary solutions $\Phi(x)$ that decay to zero as $|x| \to \infty$. It follows from the energy conservation (5.2) that $E=0$ for bright solitons, such that the function $\Phi(x)$ is found from the Gaussian quadrature:

$$(\Phi')^2 = \frac{(2\omega - \gamma_0\Phi^2)}{2(1+4\sigma^2\Phi^4)}\Phi^2. \quad (5.4)$$

The zero solution $\Phi=0$ is a saddle point for $\omega > 0$, while the turning point exists for $\gamma_0 > 0$, such that the solution $\Phi(x)$ is a homoclinic orbit with the properties:

$$\Phi(x_0 - x) = \Phi(x - x_0), \quad \Phi(x_0) = \max_{x\in\mathbb{R}}\Phi(x) = \left(\frac{2\omega}{\gamma_0}\right)^{1/2}. \quad (5.5)$$

Bright solitons $\Phi(x)$ exist in the open quadrant $\omega > 0$ and $\gamma_0 > 0$.

B. Dark solitons

Dark solitons are given by stationary solutions $\Phi(x)$ that approach to nonzero boundary conditions as $|x| \to \infty$:

$$\lim_{|z|\to\infty} |\Phi(z)| = \Phi_\infty. \quad (5.6)$$

It follows from the energy conservation (5.2) for dark solitons that

$$E = \omega\Phi_\infty^2 - \tfrac{1}{2}\gamma_0\Phi_\infty^4,$$

such that the function $\Phi(x)$ is found from the Gaussian quadrature:

$$(\Phi')^2 = \frac{(\gamma_0\Phi_\infty^2 - 2\omega + \gamma_0\Phi^2)}{2(1+4\sigma^2\Phi^4)}(\Phi_\infty^2 - \Phi^2). \quad (5.7)$$

The constant solutions $\Phi = \pm\Phi_\infty$ are saddle points if $\omega = \gamma_0\Phi_\infty^2$ and $\gamma_0 < 0$. In this case, the solution $\Phi(x)$ satisfies the ODE:

$$(\Phi')^2 = \frac{|\gamma_0|}{2(1+4\sigma^2\Phi^4)}(\Phi_\infty^2 - \Phi^2)^2, \quad (5.8)$$

such that the solution $\Phi(x)$ is a heteroclinic orbit with the properties:

$$\Phi(x_0 - x) = -\Phi(x - x_0), \quad \Phi(x_0) = \min_{x\in\mathbb{R}}|\Phi(x)| = 0,$$

$$\lim_{x\to\pm\infty}\Phi(x) = \pm\Phi_\infty. \quad (5.9)$$

Dark solitons $\Phi(x)$ exist in the open quadrant $\omega < 0$ and $\gamma_0 < 0$.

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