17 September 1998

PHYSICS LETTERS B

ELSEVIER

Physics Letters B 436 (1998) 117–124

Exact vortex solutions of the complex sine-Gordon theory on the plane

I.V. Barashenkov [1], D.E. Pelinovsky [2, 3]

Department of Mathematics, University of Cape Town, Rondebosch 7701, South Africa

Received 11 May 1998; revised 26 June 1998
Editor: P.V. Landshoff

Abstract

We construct explicit multivortex solutions for the first and second complex sine-Gordon equations. The constructed solutions are expressible in terms of the modified Bessel and rational functions, respectively. The vorticity-raising and lowering Bäcklund transformations are interpreted as the Schlesinger transformations of the fifth Painlevé equation. © 1998 Published by Elsevier Science B.V. All rights reserved.

PACS: 11.27.+d; 03.65.Ge

Motivation. Recently there has been an upsurge of interest in the complex sine-Gordon equation. Originally derived in the reduction of the $O(4)$ nonlinear σ-model [1] and a theory of dual strings interacting through a scalar field [2], this equation reappeared in a number of field-theoretic [3] and fluid dynamical [4] contexts. The equation was shown to be completely integrable [1,5,6], and the multisoliton solutions were constructed in a variety of forms, both over vanishing [7,8] and nonvanishing backgrounds [9,10]. The study of its quantized version started in [7,11,12] and received a new impetus recently [13] when it was realized that the complex sine-Gordon theory may be reformulated in terms of the gauged Wess-Zumino-Witten action and interpreted as an integrably deformed SU(2)/U(1)-coset model [14].

The complex sine-Gordon theory can be conveniently defined by its action functional,

$$E_{\mathrm{SG\text{-}1}} = \int \left[|\nabla\psi|^2 + \left(1 - |\psi|^2\right)^2 \right] \frac{d^2 x}{1 - |\psi|^2}. \tag{1}$$

The subscript 1 serves to distinguish this model from another integrable complexification of the sine-Gordon

[1] Supported by a visiting fellowship from ICTP, Trieste
[2] Now at Department of Mathematics, University of Toronto, 100 St. George street, Toronto, Ontario, Canada M5S 3G3.
[3] Email: dmpeli@math.toronto.edu

PII: S0370-2693(98)00841-7

theory, the so-called complex sine-Gordon-2:

$$E_{\mathrm{SG\text{-}2}} = \int \left[\frac{|\nabla\psi|^2}{1-\frac{1}{2}|\psi|^2} + \tfrac{1}{2}\left(1-|\psi|^2\right)^2 \right] d^2x. \tag{2}$$

The latter system was derived in Ref. [15] as the bosonic limit of a generalized supersymmetric sine-Gordon equation and, independently, in Ref. [16]. Quantum mechanically, the above two complex sine-Gordon models were shown to be the only O(2)-symmetric theories whose S-matrix is factorizable at the tree level [12].

In all previous analyses the complex sine-Gordon equations were considered in the $(1+1)$-dimensional Minkowski space-time. In the present Letter we study these two models in the 2-dimensional Euclidean space. One reason for this is that they define integrable perturbations of Euclidean conformal field theories; more precisely, Eqs. (1), (2) arise as reductions of the $SU(2)_N$ gauged Wess-Zumino-Witten model perturbed by a multiplet of primary fields (by $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively) [14,17]. They are closely related to important two-dimensional lattice systems, viz. Z_N parafermion models perturbed by the first and second thermal operators, respectively [18].

Another motivation for studying solutions of the Euclidean complex sine-Gordon equations comes from a remarkable similarity between Eqs. (1), (2) and several phenomenological lagrangians of condensed matter physics, in particular the Ginsburg-Landau expansion of the free energy in the theory of phase transitions,

$$E_{\mathrm{GL}} = \int \left[|\nabla\psi|^2 + \tfrac{1}{2}\left(1-|\psi|^2\right)^2 \right] d^2x, \tag{3}$$

and the energy of the Heisenberg ferromagnet with easy-plane anisotropy [20]:

$$E_{\mathrm{FM}} = \int \left[(\nabla\alpha)^2 + \sin^2\alpha\,(\nabla\beta)^2 + \cos^2\alpha \right] d^2x. \tag{4}$$

(Hence we will be using the words ''action'' and ''energy'' interchangeably in what follows.) To see that (1), (2) are relatives of (4), one writes $\psi = \sin\alpha\, e^{i\beta}$ and $\psi = \sqrt{2}\sin(\alpha/2)e^{i\beta}$, transforming Eqs. (1) and (2) into

$$E_{\mathrm{SG\text{-}1}} = \int \left[(\nabla\alpha)^2 + \tan^2\alpha\,(\nabla\beta)^2 + \cos^2\alpha \right] d^2x \tag{5}$$

and

$$E_{\mathrm{SG\text{-}2}} = \frac{1}{2}\int \left[(\nabla\alpha)^2 + 4\tan^2\frac{\alpha}{2}(\nabla\beta)^2 + \cos^2\alpha \right] d^2x, \tag{6}$$

respectively.

The Ginsburg-Landau free energy (3) is minimized by the Gross-Pitaevski vortices originally discovered in the context of superfluidity [19]. These are topological solitons of the form $\psi(x,y) = \Phi_n(r)e^{in\theta}$, where $\Phi_n \to 1$ as $r \to \infty$. Although these important solutions were obtained numerically and in various asymptotic regimes, no analytic expressions for the Gross-Pitaevski vortices are available. Similarly, Eq. (4) is minimized by magnetic vortices [20], and again, these are available only numerically. The aim of this note is to demonstrate that the Euclidean complex sine-Gordon equations also exhibit topological soliton solutions. Unlike the Gross-Pitaevski vortices and unlike their magnetic counterparts, the vortices of Eqs. (1) and (2) can be found exactly, and in a closed analytic form. Consequently, the significance of the complex sine-Gordon equations on the plane stems from the fact that they provide a laboratory for studying analytic properties of vortices and their phenomenology in a wide class of condensed matter models.

We construct these solutions in two different ways: (i) by means of an auto-Bäcklund transformation resulting from the spinor representation of the complex sine-Gordon theory, and (ii) via the Schlesinger transformation of the fifth Painlevé equation,

$$W_{rr} + \frac{1}{r} W_r - \left(\frac{1}{W-1} + \frac{1}{2W} \right) W_r^2 = \frac{(W-1)^2}{r^2} \left(\alpha W + \frac{\beta}{W} \right) + \frac{\gamma}{r} W + \delta W \frac{W+1}{W-1}, \tag{7}$$

which arises in a self-similar reduction of Eqs. (1) and (2).

Vortices via Bäcklund transformation. The complex sine-Gordon-1 equation,

$$\nabla^2 \psi + \frac{(\nabla \psi)^2 \bar{\psi}}{1 - |\psi|^2} + \psi \left(1 - |\psi|^2 \right) = 0, \tag{8}$$

admits an equivalent representation in terms of the Euclidean spinor field, $\Psi = (u,v)^T$ [10]:

$$i \bar{\partial} u + v - |u|^2 v = 0, \tag{9a}$$

$$i \partial v + u - |v|^2 u = 0. \tag{9b}$$

[Here $\partial = \partial / \partial z$, $\bar{\partial} = \partial / \partial \bar{z}$ and $z = (x + iy)/2$.] This is nothing but the Euclidean version of the massive Thirring model; the corresponding action functional has the form

$$\begin{aligned} E_{\text{Th}} &= \int \left[i \Psi^\dagger \gamma_i \partial_i \Psi + \Psi^\dagger \Psi - \tfrac{1}{4} \left(\Psi^\dagger \gamma_i \Psi \right)^2 - 1 + \text{c.c.} \right] d^2 x \\ &= \int \left(i \bar{u}\, \partial v + i \bar{v}\, \bar{\partial} u + |u|^2 + |v|^2 - |uv|^2 - 1 + \text{c.c.} \right) d^2 x. \end{aligned} \tag{10}$$

Since as one can easily check both u and v satisfy Eq. (8), the Thirring model (9) can be regarded as a Bäcklund transformation between two different solutions of Eq. (8). Here we confine ourselves to multivortex solutions of the form $\psi = \Phi_n(r) e^{in\theta}$, where (r, θ) are polar coordinates on the plane and $\Phi_n(r)$ is a real function satisfying

$$\frac{d^2 \Phi_n}{dr^2} + \frac{1}{r} \frac{d\Phi_n}{dr} + \frac{\Phi_n}{1 - \Phi_n^2} \left[\left(\frac{d\Phi_n}{dr} \right)^2 - \frac{n^2}{r^2} \right] + \Phi_n \left(1 - \Phi_n^2 \right) = 0. \tag{11}$$

Eqs. (9) with u and v of the form $u = -i \Phi_{n-1} e^{i(n-1)\theta}$ and $v = \Phi_n e^{in\theta}$ furnish an equivalent representation for Eq. (11):

$$-\frac{d\Phi_{n-1}}{dr} + \frac{n-1}{r} \Phi_{n-1} = \left(1 - \Phi_{n-1}^2 \right) \Phi_n, \tag{12a}$$

$$\frac{d\Phi_n}{dr} + \frac{n}{r} \Phi_n = \left(1 - \Phi_n^2 \right) \Phi_{n-1}, \tag{12b}$$

where Φ_n and Φ_{n-1} satisfy Eq. (11) with n and $n' = n-1$, respectively. When $n = 1$, Eq. (12a) is solved by $\Phi_0 = 1$ and Eq. (12b) becomes a Riccati equation:

$$\Phi_1' + r^{-1} \Phi_1 = 1 - \Phi_1^2. \tag{13}$$

This equation can be linearized by writing $\Phi_1 = S'/S$, where $S(r)$ satisfies the modified Bessel's equation of zero order: $S'' + S'/r - S = 0$. Selecting $S = I_0(r)$ gives the explicit form of the $n = 1$ vortex solution of the complex sine-Gordon theory:

$$\Phi_1 = \frac{I_1(r)}{I_0(r)}. \tag{14}$$

Here $I_0(r)$ and $I_1 = I_0'(r)$ are the modified Bessel functions of zero and first order, respectively. The vortex is plotted in Fig. 1.

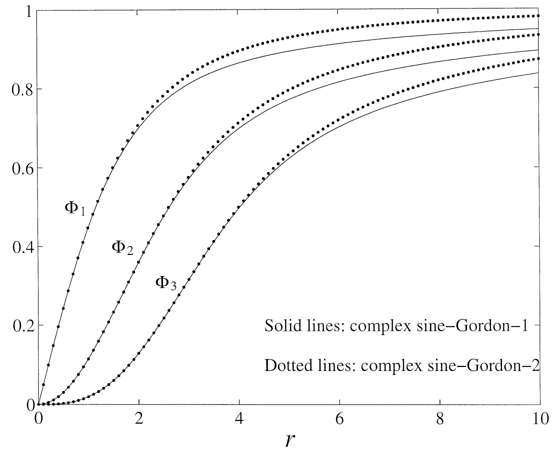

Fig. 1. The vortex solutions with $n=1,2$ and 3.

With the solution Φ_1 at hand, Eqs. (12) yield a recursion relation allowing us to construct solutions with vorticity $n>1$ in a purely algebraic way:

$$\Phi_{n+1}=\frac{-1}{1-\Phi_n^2}\left[\frac{d\Phi_n}{dr}-\frac{n}{r}\Phi_n\right]=\Phi_{n-1}-\frac{2}{1-\Phi_n^2}\frac{d\Phi_n}{dr}, \quad n\geq 1. \tag{15}$$

In particular, the first two higher-order vortices (shown in Fig. 1) are given by

$$\Phi_2=-\frac{I_0I_2-I_1^2}{I_0^2-I_1^2}, \quad \Phi_3=\frac{(I_3-I_1)(I_0^2-I_1^2)+I_1(I_0-I_2)^2}{(I_0-I_2)(I_0I_2-2I_1^2+I_0^2)},$$

where we have eliminated derivatives by means of the well known relation between the modified Bessel functions of different order: $I_{n+1}+I_{n-1}=2I_n'$. The asymptotic behaviour of the vortex with vorticity n is readily found from Eq. (12b):

$$\Phi_n\sim\frac{1}{2^n n!}r^n-\frac{1}{2^{n+2}(n+1)!}r^{n+2}+\mathrm{O}(r^{n+4}) \quad \text{as} \quad r\to 0, \tag{16}$$

$$\Phi_n\sim 1-\frac{n}{2r}-\frac{n^2}{8r^2}+\mathrm{O}(r^{-3}) \quad \text{as} \quad r\to\infty. \tag{17}$$

One consequence of Eq. (17) is that the energy of the vortices diverges [cf. Eq. (19) below], similarly to the energy of the Gross-Pitaevski and easy-plane ferromagnetic vortices [19,20]. (Physically, this fact simply indicates that there is a cut-off radius in the system, for example the radius of the cylindrical superfluid container, or the distance between two adjacent vortex lines.)

Bogomol'nyi bound. An important question is whether the vortex renders the action a minimum. Let $n=1$ and rewrite Eq. (1) as

$$E_{\text{SG-1}}=\int\left|\partial\psi+|\psi|^2-1\right|^2\frac{d^2x}{1-|\psi|^2}+\int\nabla\cdot\boldsymbol{A}d^2x, \tag{18}$$

where $\boldsymbol{A}$ is a real vector field with components

$$A_i=\ln(1-|\psi|^2)\,\epsilon_{ij}\partial_j\mathrm{Arg}\,\psi+2\psi_i; \quad i=1,2,$$

and $\psi = \psi_1 + i\psi_2$. Assume our fields are such that $|\psi|^2 < 1$; then the first term in (18) attains its minimum at solutions to the ''Bogomol'nyi equation'' $\partial\psi = 1 - |\psi|^2$. This is exactly our Eq. (9b) with $v = \psi$ and $u = -i$; its vortex solution is given by Eq. (14). The second integral in (18) represents the divergent part of the action; it can be written as a flux through a circle of the radius $R \to \infty$. Perturbing the vortex $\psi = \Phi_1(r)e^{i\theta}$ by a function $\delta\psi$ decaying faster than $1/r$ at infinity will not affect this part; the flux is uniquely determined by the vortex asymptotes:

$$\oint_{C_R} \boldsymbol{A} \cdot \boldsymbol{n}\, dl = 2\pi(2R - \ln R - 1) + \mathscr{O}(R^{-1}). \tag{19}$$

Consequently, the $n=1$ vortex saturates the minimum of the action in the class of functions with $|\psi|^2 < 1$.

The importance of the last inequality should be specially emphasized. Without the condition $|\psi|^2 < 1$ being imposed, one could construct a perturbation $\tilde{\psi}(x,y)$ of the vortex satisfying $|\tilde{\psi}| = 1$, $\nabla\tilde{\psi} = 0$ on some closed curve on the (x,y)-plane which does not enclose the origin. Taking then $|\tilde{\psi}| \gg 1$ in the interior of this contour, the action (1) could be made arbitrarily negative.

It is interesting to note that the first-order Eqs. (9) with generic u and v can also be interpreted as the Bogomol'nyi limit for some more general system with twice as many degrees of freedom. The corresponding action functional is

$$E[u,v] = \int \left(\frac{|\nabla u|^2}{1-|u|^2} + 1 - |u|^2 \right) d^2x + \int \left(\frac{|\nabla v|^2}{1-|v|^2} + 1 - |v|^2 \right) d^2x + E_{\mathrm{Th}}, \tag{20}$$

where E_{Th} is the Thirring action (10). Clearly, any solution to (9) is automatically a solution to the second-order system (20). The action (20) can be written as

$$E[u,v] = \int \frac{|i\,\bar{\partial}u + v(1-|u|^2)|^2}{1-|u|^2} d^2x + \int \frac{|i\,\partial v + u(1-|v|^2)|^2}{1-|v|^2} d^2x + \int \nabla \cdot \boldsymbol{A} d^2x, \tag{21}$$

where $A_i = \ln(1-|v|^2)\epsilon_{ij}\partial_j \mathrm{Arg}\, v - \ln(1-|u|^2)\epsilon_{ij}\partial_j \mathrm{Arg}\, u$. Assuming, again, that $|u|^2, |v|^2 < 1$, the lower bound of the action (20), (21) is saturated by solutions to Eqs. (9).

Some properties of the complex sine-Gordon vortices receive a natural interpretation when the equation is reformulated as a σ-model on a two-dimensional surface Σ embedded in a three-dimensional space (n_1, n_2, n_3). The metric on Σ is $ds^2 = d\alpha^2 + \tan^2\alpha\, d\beta^2$ [see Eq. (5)]. In order for Σ to be smooth, the space (n_1, n_2, n_3) has to be pseudoeuclidean and the surface noncompact; in fact it looks like an asymptotically conical infinite bowl:

$$n_1 + in_2 = \tan\alpha\, e^{i\beta}; \quad n_3 = \frac{1}{q} - \tanh^{-1} q, \quad q = \frac{\cos\alpha}{(1+\cos^2\alpha)^{1/2}}.$$

Here $0 \le \alpha < \pi/2$, $0 \le \beta < 2\pi$. In terms of n_i, the lagrangian (1) reads

$$E_{\mathrm{SG\text{-}1}} = \int \left[(\nabla n_1)^2 + (\nabla n_2)^2 - (\nabla n_3)^2 + \left(1 + n_1^2 + n_2^2\right)^{-1} \right] d^2x.$$

As $r \to \infty$, all three components of the vortex field, n_1, n_2 and n_3, tend to infinity. Consequently, the vortices map a noncompactified (x,y)-plane onto a noncompact surface – this accounts for their infinite energy. We also acknowledge the role of the condition $|\psi|^2 < 1$, which characterizes solutions admitting the σ-model interpretation.

Reductions to the Painlevé-V. The transformation

$$\Phi_n = \frac{1+W}{1-W}$$

reduces Eq. (11) to the fifth Painlevé Eq. (7) with coefficients

$$\alpha = n^2/8, \quad \beta = -n^2/8, \quad \gamma = 0, \quad \delta = -2. \tag{22}$$

For $\gamma = c(1-a-b)$, where $a^2 = 2\alpha$, $b^2 = -2\beta$, and $c^2 = -2\delta$, Eq. (7) admits a reduction [21] to a Riccati equation

$$W_r = r^{-1}(W-1)(aW+b) + cW. \tag{23}$$

The above relation between the coefficients is in place for $n=1$; in terms of the vortex modulus Φ_1, Eq. (23) turns out to be nothing but our Eq. (13). Next, the Schlesinger transformations of the Painlevé-V to itself [22,23] have the form

$$W_r = r^{-1}(W-1)(aW+b) + cW\frac{1+\hat{W}}{1-\hat{W}}, \tag{24a}$$

$$-\hat{W}_r = r^{-1}(\hat{W}-1)(\hat{a}\hat{W}+\hat{b}) + c\hat{W}\frac{1+W}{1-W}. \tag{24b}$$

Here W and $\hat{W}$ satisfy Eq. (7) with the coefficients $(\alpha,\beta,\gamma,\delta)$ and $(\hat{\alpha},\hat{\beta},\hat{\gamma},\delta)$, respectively, where $\hat{a}^2 = 2\hat{\alpha}$, $\hat{b}^2 = -2\hat{\beta}$, $\hat{\gamma} = c(b-a)$, and $2\hat{a} = a+b-1-\gamma/c$, $2\hat{b} = a+b-1+\gamma/c$. With α, β and γ as in Eq. (22), Eqs. (24) amount to the vorticity-raising transformations (12).

We conclude the discussion of the complex sine-Gordon-1 equation by mentioning that it would be natural to expect its vortex solutions (confined to a finite region on the plane) to arise as degenerate cases of its N-soliton solutions [10] (which have the form of N intersecting infinite folds). This kind of correspondence between two-dimensionally localized ‘‘lumps’’ and one-dimensional multisolitons exists, for example, in the Kadomtsev-Petviashvili equation [24]. Surprisingly, the only two-dimensionally localized bounded solution resulting from the ‘‘degeneration’’ of the generic two-soliton solution of Eq. (8) is discontinuous at the origin: $\psi = (X^2 - \sinh^2 Y)(X^2 + \sinh^2 Y)^{-1}$. Here $X + iY = e^{i\alpha}(x+iy)$, and α is an arbitrary constant angle.

Vortices of the complex sine-Gordon-2. The complex sine-Gordon-2 results from the variation of Eq. (2):

$$\nabla^2\psi + \frac{(\nabla\psi)^2\,\overline{\psi}}{2-|\psi|^2} + \tfrac{1}{2}\psi\left(1-|\psi|^2\right)\left(2-|\psi|^2\right) = 0. \tag{25}$$

The multivortex Ansatz $\psi = \Phi_n(r)e^{in\theta} = Q_n^{1/2}(r)e^{in\theta}$ takes it to

$$\frac{d^2Q_n}{dr^2} + \frac{1}{r}\frac{dQ_n}{dr} + \frac{1-Q_n}{Q_n(Q_n-2)}\left(\frac{dQ_n}{dr}\right)^2 + Q_n(1-Q_n)(2-Q_n) + \frac{\left(a_n^2-b_n^2\right)Q_n}{r^2(2-Q_n)} + \frac{4a_n^2(1-Q_n)}{r^2Q_n(2-Q_n)} = 0, \tag{26}$$

where $a_n = 0$ and $b_n = -2n$. Next, the substitution

$$Q_n = 2(1-W)^{-1}$$

transforms Eq. (26) to the Painlevé Eq. (7) with coefficients

$$\alpha = 0, \quad \beta = -2n^2, \quad \gamma = 0, \quad \delta = 2.$$

This time, in order to construct the multivortex solutions we apply the Schlesinger transformation (24) twice. This leads to a recurrent relation

$$Q^{(k-1)} = \left(2 - Q^{(k)}\right)\left\{1 - \frac{2(a_k + b_k - 1)Q^{(k)}\left[rQ_r^{(k)} - 2a_k + (a_k + b_k)Q^{(k)}\right]}{\left[rQ_r^{(k)} - 2a_k + (a_k + b_k)Q^{(k)}\right]^2 + r^2 Q^{(k)2}\left(2 - Q^{(k)}\right)^2}\right\}, \tag{27}$$

where $Q^{(k)}$ and $Q^{(k-1)}$ satisfy Eq. (26) with the parameters (a_k, b_k) and (a_{k-1}, b_{k-1}), respectively. Here $a_{k-1} = a_k - 1$ and $b_{k-1} = b_k - 1$. Starting with a trivial solution $Q^{(0)} = 1$ arising for $a_0 = -b_0 = n$, and using Eq. (27) n times, we end up with a solution $Q_n = Q^{(-n)}$ which satisfies Eq. (26) with $a_n = 0$ and $b_n = -2n$ and the boundary condition $Q_n \to 1$ as $r \to \infty$. These solutions are given by rational functions; in particular, the first three multivortices (see Fig. 1) read

$$Q_1 = \frac{r^2}{r^2 + 4}; \quad Q_2 = \frac{r^4(r^2 + 24)^2}{r^8 + 64r^6 + 1152r^4 + 9216r^2 + 36864};$$

$$Q_3 = r^6(r^6 + 144r^4 + 5760r^2 + 92160)^2 D_3^{-1},$$

$$D_3 = r^{18} + 324r^{16} + 41472r^{14} + 2820096r^{12} + 114130944r^{10} + 2919628800r^8$$
$$+ 50960793600r^6 + 611529523200r^4 + 4892236185600r^2 + 19568944742400.$$

The energy of the complex sine-Gordon-2 vortices is logarithmically divergent.

Concluding remarks. The Ginsburg-Landau expansion (3) is regarded as a central postulate in the phenomenological theory of phase transitions; however, for some systems Eqs. (1), (2) may happen to provide a more adequate description. In fact, the difference is not as big as one might think. Assuming, for instance, $|\psi|^2 \le 1$, Eq. (2) can be rewritten as

$$E_{\text{SG-2}} \approx \int \left[|\nabla\psi|^2 + \tfrac{1}{2}\left(1 - |\psi|^2\right)^2 + \frac{|\nabla\psi|^2 |\psi|^2}{2} + \dots \right] d^2x; \tag{28}$$

this is different from (3) only in the third term which is small both when $\psi \sim 0$ and when $|\psi| \sim 1, \nabla\psi \sim 0$. More importantly, the complex sine-Gordon models provide a unique opportunity for studying a number of analytic properties which are common to a wide class of vortex-bearing systems. These include the correct Ansatz for two spatially separated vortices, the vortex-phonon scattering matrix and so on; our present construction of coaxial multivortices is hopefully but a first step in this direction. Finally, one may see the complex sine-Gordon vortices as a starting point in the *perturbative* construction of the corresponding solutions of the Ginsburg-Landau and ferromagnet models.

Acknowledgements

We are grateful to M. Bogdan, B. Dubrovin, B. Ivanov, A. Kapaev, P. Winternitz and the referee for useful remarks, and to N. Alexeeva and A. Harin for their numerical assistance at various stages of this work. One of the authors (I.B.) thanks S. Randjbar-Daemi for the hospitality at ICTP. This research was supported by the FRD of South Africa and URC of UCT.

References

[1] K. Pohlmeyer, Commun. Math. Phys. 46 (1976) 207. For the generalization of these results see J. Tafel, J. Math. Phys. 34 (1993) 1892, where the complex sine-Gordon equation is derived as a symmetry reduction of the self-dual Yang-Mills equations on a self-dual manifold.

[2] F. Lund, T. Regge, Phys. Rev. D14 (1976) 1524.

[3] A. Neveu, N. Papanicolaou, Commun. Math. Phys. 58 (1978) 31.

[4] Y. Fukumoto, M. Miyajima, J. Phys. A: Math. Gen. 29 (1996) 8025.
[5] F. Lund, Phys. Rev. Lett. 38 (1977) 1175; Ann. Phys. 115 (1978) 251.
[6] B.S. Getmanov, Pis'ma Zh. Eksp. Teor. Fiz. 25 (1977) 132 [JETP Lett. 25 (1977) 119]
[7] H.J. de Vega, J.M. Maillet, Phys. Rev. D28 (1983) 1441.
[8] I.V. Barashenkov, B.S. Getmanov, Commun. Math. Phys. 112 (1987) 423.
[9] B.S. Getmanov, Teor. Mat. Fiz. 38 (1979) 186 [Theor. Math. Phys. 38 (1979) 124]
[10] I.V. Barashenkov, B.S. Getmanov, V.E. Kovtun, J. Math. Phys. 34 (1993) 3039; I.V. Barashenkov, B.S. Getmanov, ibid. 3054.
[11] H.J. de Vega, J.M. Maillet, Phys. Lett. B101 (1981) 302; J.M. Maillet, Phys. Rev. D26 (1982) 2755.
[12] G. Bonneau, Phys. Lett. B133 (1983) 341; G. Bonneau, F. Delduc, Nucl. Phys. B250 (1985) 561.
[13] N. Dorey, T.J. Hollowood, Nucl. Phys. B440 (1995) 215; Q-H. Park, H.J. Shin, Phys. Lett. B 359 (1995) 125; T.J. Hollowood, J.L. Miramontes and Q-H. Park, Nucl. Phys. B445 (1995) 451.
[14] I. Bakas, Int. J. Mod. Phys. A9 (1994) 3443; Q-H. Park, Phys. Lett. B328 (1994) 329.
[15] S. Sciuto, Phys. Lett. B 90 (1980) 75.
[16] B.S. Getmanov, Teor. Mat. Fiz. 48 (1981) 13.
[17] V.A. Brazhnikov, Nucl. Phys. B 501 [FS] (1997) 685.
[18] V.A. Fateev, Int. J. Mod. Phys. A 6 (1991) 2109.
[19] V.L. Ginsburg, L.P. Pitaevskii, Sov. Phys. JETP 34 (1958) 858; E.M. Lifshitz, L.P. Pitaevskii, Statistical Physics, Part 2 (Pergamon Press, Oxford, 1980)
[20] A.M. Kosevich, V.P. Voronov, I.V. Manzhos, Sov. Phys. JETP 57 (1983) 86; V.G. Bar'yakhtar, B.A. Ivanov, Sov. Sci. Rev. A 16 (1992) 1; B.A. Ivanov, A.K. Kolezhuk, Low. Temp. Phys. 22 (1995) 275.
[21] N.A. Lukashevich, Diff. Eqns. 4 (1968) 732.
[22] B.I. Gromak, Diff. Eqns. 12 (1976) 519.
[23] A.S. Fokas, U. Mugan, M.J. Ablowitz, Physica D 30 (1988) 247.
[24] L.A. Bordag, A.R. Its, V.B. Matveev, S.V. Manakov, V.E. Zakharov, Phys. Lett. A63 (1979) 205.