SELF-FOCUSING AND TRANSVERSE INSTABILITIES OF SOLITARY WAVES

Yuri S. KIVSHAR[a], Dmitry E. PELINOVSKY[b]

[a] *Optical Sciences Centre, Research School of Physical Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia*
[b] *Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3, Canada*

ELSEVIER

AMSTERDAM – LAUSANNE – NEW YORK – OXFORD – SHANNON – TOKYO

ELSEVIER

Physics Reports 331 (2000) 117–195

PHYSICS REPORTS

www.elsevier.com/locate/physrep

Self-focusing and transverse instabilities of solitary waves

Yuri S. Kivshar[a], Dmitry E. Pelinovsky[b]

[a]Optical Sciences Centre, Research School of Physical Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia

[b]Department of Mathematics, University of Toronto, Toronto, Ont. M5S 3G3, Canada

Received April 1999; editor: A.A. Maradudin

Contents

Abstract

We give an overview of the basic physical concepts and analytical methods for investigating *the symmetry-breaking instabilities* of solitary waves. We discuss self-focusing of spatial optical solitons in diffractive nonlinear media due to either *transverse* (one more unbounded spatial dimension) or *modulational* (induced by temporal wave dispersion) instabilities, in the framework of the cubic nonlinear Schrödinger (NLS) equation and its generalizations. Both *linear* and *nonlinear* regimes of the instability-induced soliton

PII: S0370-1573(99)00106-4

dynamics are analyzed for bright (self-focusing media) and dark (self-defocusing media) solitary waves. For a defocusing Kerr medium, the results of the small-amplitude limit are compared with the theory of the transverse instabilities of the Korteweg–de Vries solitons developed in the framework of the exactly integrable Kadomtsev–Petviashvili equation. We give also a comprehensive summary of different physical problems involving the analysis of the transverse and modulational instabilities of solitary waves including the soliton self-focusing in the discrete NLS equation, the models of parametric wave mixing, the Davey–Stewartson equation, the Zakharov–Kuznetsov and Shrira equations, instabilities of higher-order and ring-like spatially localized modes, the kink stability in the dissipative Cahn–Hilliard equation, etc. Experimental observations of the soliton self-focusing and transverse instabilities for bright and dark solitons in nonlinear optics are briefly summarized as well.

PACS: 42.65. − k

1. Introduction

Wave instabilities are probably the most remarkable physical phenomena that may occur in nonlinear systems (see, e.g., Infeld and Rowlands (1990) and references therein). Modulational instability and breakup of a continuous-wave (c.w.) field of large intensity was first predicted and analyzed in the context of waves in fluids (Benjamin and Feir, 1967; Zakharov, 1968). The similar effect is self-focusing of light in optical media with a nonlinear response (Askar'yan, 1962; Chiao et al., 1964; Talanov, 1964; Kelley, 1965; Ostrovsky, 1966; Bespalov and Talanov, 1966) which is responsible for the appearance of hot spots and associated optical damage in media irradiated by high power laser pulses.

One of the important physical processes associated with the development of modulational instability is the generation of a train of spatially (beams) or temporary (pulses) localized waves (Yuen and Ferguson, 1978), the effect observed experimentally in different physical systems, e.g. in the fluid dynamics (Benjamin and Feir, 1967; Yuen and Lake, 1975; Melville, 1982; Su, 1982), nonlinear beam propagation (Campillo et al., 1973, 1974; Iturbe-Castillo et al., 1995), electrical transmission lines (Mizumura and Noguchi, 1975; Marquié et al., 1994, 1995), optical fibers (Tai et al., 1986), etc.

One of the fundamental models describing the nonlinearity-induced modulational instability is the generalized nonlinear Schrödinger (NLS) equation, written in one spatial dimension as follows:

$$\mathrm{i}\varphi_t + \varphi_{xx} + (r+1)|\varphi|^{2r}\varphi = 0 , \tag{1.1}$$

where r is the power of nonlinearity. In the case $r = 1$, this model describes the propagation of an electric field envelope in an optical waveguide, the famous model known to be integrable by means of the inverse scattering transform (see, e.g., Ablowitz and Segur, 1981, and references therein).

Modulational instability can be viewed as the simplest case of the so-called *symmetry-breaking instability*, when a solution of a nonlinear system of a certain dimension (e.g., an uniform c.w. background) is subjected to a broader class of perturbations. Another example is the instability of low-dimensional solitary waves to perturbations involving higher dimensions. This is a typical case of the so-called *transverse instability* of plane solitary waves, first discussed for long-wave solitons of the Korteveg–de Vries (KdV) equation (Kadomtsev and Petviashvili, 1970) and envelope solitons of the NLS equation (Zakharov, 1967; Zakharov and Rubenchik, 1973), and then investigated by different methods for a variety of nonlinear soliton-bearing models (see, e.g., Yajima, 1974; Washimi, 1974; Schmidt, 1975; Spatschek et al., 1975; Katyshev and Makhankov, 1976; Laedke and Spatschek, 1978; Andersen et al., 1979a,b; Ablowitz and Segur, 1979, 1980; Akhmediev et al., 1992; Soto-Crespo et al., 1991, 1992; Kuznetsov and Rasmussen, 1995) with demonstrations in numerical simulations (e.g., Degtyarev et al., 1975; Pereira et al., 1977; Infeld and Rowlands, 1980).

Why this type of soliton instabilities becomes important and deserves a special attention and discussion these days? First of all, the recent progress in developing and employing nonlinear optical materials, led to several important discoveries of self-focusing of light, self-trapped beam propagation, and *spatial optical solitary waves* in different nonlinear materials, including photorefractive (Shih et al., 1995, 1996), quadratic (Torruellas et al., 1995), and saturable non-Kerr (Tikhonenko et al., 1995, 1996b) media. It is expected that the development of novel band-gap materials based on quadratic or cubic nonlinearities will eventually lead to the experimental observation and manipulation of the so-called *light bullets*, self-focused states of light localized in both space and time, the building blocks of the future all-optical photonics devices.

Secondly, different types of symmetry-breaking instabilities have been recently *described theoretically* and *observed experimentally* in nonlinear optics. This includes the observation of a breakup of bright soliton stripes in a bulk photorefractive medium due to the transverse modulational instability and the formation of a sequence of two-dimensional self-trapped beams (Mamaev et al., 1996a,b), the first observation of the generation of pairs of optical vortex solitons due to the transverse instability of dark-soliton stripes (Tikhonenko et al., 1996a; Mamaev et al., 1996c); the theory and the first experimental demonstration of spatial modulational instability in quadratically nonlinear (or $\chi^{(2)}$) optical media (Fuerst et al., 1997a,b; DeRossi et al., 1997a,b); a decay of ring-shape optical beams with nonzero angular momentum into higher-dimensional solitary waves, observed experimentally in a rubidium vapour (a saturable defocusing medium) (Tikhonenko et al., 1995), a quadratic nonlinear crystal (Petrov et al., 1998), and investigated theoretically as a general solitonic effect (Firth and Skryabin, 1997). It is also worth mentioning the pulse compression and all-optical switching in waveguide arrays induced by modulational instability of nonlinear localized modes in discrete systems (e.g., Aceves et al., 1994a,b, 1995). These results, applied to nonlinear systems of a different nature, call for a systematic overview of the basic ideas and analytical methods of the soliton stability theory previously discussed only for particular examples or techniques (Makhankov, 1978; Kuznetsov et al., 1986; Trubnikov and Zhdanov, 1987; Rypdal and Rasmussen, 1989; Infeld and Rowlands, 1990; Kamchatnov, 1997).

The main purpose of this survey is twofold. *First of all*, we give an overview of the basic physical concepts and analytical methods in the theory of the symmetry-breaking instabilities of solitary waves. We analyze in detail linear and nonlinear regimes of self-focusing of planar bright and dark solitons described by *two conventional models*, (i) the elliptic and hyperbolic versions of the cubic NLS equation, and (ii) the Kadomtsev–Petviashvili (KP) equation, the well-known two-dimensional generalization of the KdV equation. *Second*, we discuss all possible scenarios of the instability-induced soliton dynamics and give a summary of the results on the transverse soliton instabilities for different types of physical models and more general types of nonlinearity. In particular, we show that the self-focusing instability leads to either the formation of localized waves stable in higher dimensions with a singular, nonsingular or decaying amplitude evolution, or long-lived periodic oscillations between planar and modulated quasi-plane soliton states. For illustration of the basic asymptotic methods, we compare our results with those for modulational instability of a c.w. background within the generalized NLS equation.

The paper is organized as follows. In Section 2 we introduce our basic models and discuss their physical applications and generalizations, including a brief overview of physical mechanisms for suppressing wave collapse in the NLS model. Then, in Section 3 we discuss the general criteria for the transverse self-focusing of solitary waves in dispersive and diffractive nonlinear media. The asymptotic analysis of this phenomenon is presented in Section 4, where we derive the modulation equations for the parameters of slowly modulated self-focusing solitons. Modifications of the asymptotic approach for small-amplitude (long-scale and short-scale) expansions are discussed in Sections 5 and 6, respectively. Section 7 gives a brief summary of some generalizations of the NLS- and KdV-type models, and it presents also some other types of nonlinear models, including discrete lattices, the Zakharov–Kuznetsov and Shrira equations, the Davey–Stewartson equation, the models of parametric wave mixing in diffractive media with a quadratic optical response, the

Cahn–Hilliard equation, etc. In Section 8 we discuss the experimental observations of self-focusing and soliton instabilities in nonlinear optics, and Section 9 gives a list of some closely related or unsolved problems which might be useful for the future studies.

2. Physical models

Throughout this paper, we distinguish two different phenomena: *self-focusing collapse* of a spatially localized wave and *self-focusing* (*or transverse*) *symmetry-breaking instability* of a planar solitary wave. Collapse (or blowup) occurs when the amplitude of an unstable solitary wave localized in *all* dimensions grows to infinity in a finite time. As a matter of fact, the wave collapse is *a particular scenario* of the instability-induced evolution of a solitary wave under the action of perturbations of the same dimension. *Transverse instability* is an instability of a solitary wave that is localized in one (longitudinal) dimension and nonlocalized but perturbed in other (transverse) dimensions. The latter phenomenon is more generic, it may occur in the systems where the blowup instability is suppressed or it does not happen at all. There are known several different scenarios of the long-term dynamics of the soliton transverse instability. Only in the systems where both blowup and transverse instability coexist, we expect that a perturbation along the soliton front may break a plane soliton into a chain of localized modes, each undergoing a further transformation into a collapsing mode. Here we discuss the basic soliton-bearing models which we consider below for analyzing the soliton transverse instability (Section 2.1), and also some physically important generalizations which are required for the suppression of the blowup instability whenever it may occur (Section 2.2).

2.1. Basic soliton equations

The fundamental model to analyze the soliton transverse self-focusing is the (2 + 1)-dimensional (i.e. two *spatial* and one *temporal* variables) NLS equation which can be written as follows:

$$i\psi_t + \psi_{xx} + \sigma_d \psi_{yy} + 2\sigma_n |\psi|^2 \psi = 0 , \tag{2.1}$$

where $\sigma_n = \pm 1$ defines the type of the cubic nonlinearity, i.e. *focusing* (at $\sigma_n = +1$) or *defocusing* (at $\sigma_n = -1$), and $\sigma_d = \pm 1$ defines the type of the wave dispersion/diffraction.

The most well-known applications of Eq. (2.1) are in nonlinear optics (see, e.g., a number of books, Shen, 1984; Boyd, 1992; Newell and Moloney, 1992; Agrawal, 1995). In particular, in the theory of *spatial optical solitons* (e.g., Boardman and Xie, 1993; Kivshar, 1998a), this model always appears with $\sigma_d = +1$ (diffraction), and it can be derived for the beam propagation in a bulk medium from Maxwell's equations in the so-called paraxial approximation, taking into account two transverse and one longitudinal spatial dimensions. The same equation appears for the case of *temporal modulations* of the (1 + 1)-dimensional solitons in a waveguide with the so-called anomalous dispersion, i.e., $\omega''(k) > 0$, where $\omega = \omega(k)$ is the frequency of a wave which is a function of its wave number. For this latter case, the variable t stands for the propagation coordinate, and the variable y plays a role of the retarded time in the reference frame moving with the group velocity. For $\sigma_d = +1$, the model (2.1) is usually referred to as *the elliptic cubic NLS equation*.

For the so-called normal dispersion, i.e., for $\omega''(k) < 0$, the NLS equation appears with $\sigma_d = -1$, and it is usually called *the hyperbolic cubic NLS equation*. In this latter case, the NLS equation (2.1) is less studied in the literature. However, it also provides a fundamental model for describing the nonlinear dynamics of a broad class of waves, e.g., deep-water gravitational waves (Zakharov, 1968; Martin et al., 1980; Yuen and Lake, 1986), lower-hybrid (Litvak et al., 1979) and cyclotron (Myra and Lin, 1980) waves in magnetized plasmas, etc. In nonlinear optics, Eq. (2.1) with $\sigma_d = -1$ describes the spatio-temporal dynamics of an optical beam in a nonlinear medium with positive Kerr effect which undergoes self-focusing in space and self-modulation in time (see, e.g., Chernev and Petrov, 1992a).

In the self-focusing medium ($\sigma_n = +1$), the c.w. solution of Eq. (2.1) of the form $\psi(t) = \rho \exp(2i\rho^2 t)$, is *modulationally unstable* to spatial periodic perturbations $\sim e^{i\kappa x}$ with the modulation wave numbers κ selected in a finite band, $0 < \kappa < \kappa_{max}$. As a result of the development of such an instability, we expect the generation of a train of localized waves (beams or pulses), usually called *bright NLS solitons*. An individual planar bright soliton is described by the y-independent localized solution of the NLS equation (2.1), that we write here in a general form,

$$\psi(x,t) = \Phi_b(x - 2vt - s; \omega) e^{i(vx - v^2 t + \omega t + \theta)} , \tag{2.2}$$

where, for the case of the cubic nonlinearity,

$$\Phi_b(x;\omega) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}x) .$$

In Eq. (2.2), the parameters $2v$ and ω stand for the soliton velocity and frequency (or its propagation constant), which represent the translational and oscillatory degrees of freedom of a bright NLS soliton, respectively. The constant parameters s and θ are the soliton initial position and phase, respectively. In many problems of the soliton dynamics, the envelope bright soliton (2.2) is analyzed at rest; its translational degree of freedom is not excited and it can be eliminated by the standard Galilei transformation.

As was first shown by Zakharov and Rubenchik (1973), a plane bright soliton (2.2) is unstable to higher-dimensional perturbations both in the elliptic and hyperbolic cases. This phenomenon is called the soliton *transverse instability*, for the spatial case, or *modulational instability*, for the temporal case.

In *the elliptic case* ($\sigma_d = +1$), the unstable transverse perturbations $\sim e^{ipy}$ have wave numbers within the finite interval $0 < p < p_c$, where $p_c = \sqrt{3\omega}$. It was shown numerically (Degtyarev et al., 1975) that the self-focused beams undergo collapse in a finite time (or at finite propagation distance). To avoid a catastrophic collapse in realistic physical models, an effective nonlinearity saturation should be taking into account, and it leads to a finite wave amplitude (Litvak et al., 1991) (see also Section 2.2 below).

In *the hyperbolic case* ($\sigma_d = -1$), the unstable transverse perturbations also have a finite-interval wave numbers, i.e., $0 < p < p_c$, but the derivation of the cutoff value p_c is a complicated problem which led to contradictory conclusions (Kuznetsov et al., 1986; Rypdal and Rasmussen, 1989). Recently, a simple result was obtained by an asymptotic analysis, $p_c = \sqrt{\omega}$ (Pelinovsky and Sulem, 1999). Numerical simulations (Pereira et al., 1978) revealed a breakup of a planar soliton into localized modes which move apart and spread out due to the action of the wave dispersion. In addition, travelling instabilities of a bright soliton was also discovered numerically for transverse wave numbers $p_{c1} < p < p_{c2}$ (Martin et al., 1980) but this issue remained an open problem until recently (Pelinovsky and Sulem, 1999).

In *a defocusing medium* ($\sigma_n = -1$), the c.w. solution is modulationally stable in the elliptic problem ($\sigma_d = +1$) and localized waves can exist on a nonvanishing background as dips of lower intensity, usually called *dark solitons*,

$$\psi(x,t) = \Phi_d(x - 2vt - s; v)e^{-i(2\rho^2 t - \theta)} , \tag{2.3}$$

where

$$\Phi_d(x;v) = k\tanh(kx) + iv, \quad k = \sqrt{\rho^2 - v^2} .$$

Here the boundary conditions for the function $\psi(x,t)$ are specified by the amplitude ρ of the c.w. background, i.e., $|\psi| \to \rho$ as $x \to \infty$, and the parameter v defines the soliton velocity, $|v| \le \rho$. As was first shown by Kuznetsov and Turitsyn (1988), a dark soliton (2.3) is unstable to transverse perturbations with the wave numbers, $0 < |p| < p_c(v)$, where

$$p_c(v) = [-(\rho^2 + v^2) + 2\sqrt{v^4 - v^2\rho^2 + \rho^4}]^{1/2} . \tag{2.4}$$

Recent numerical (McDonald et al., 1993; Law and Swartzlander, 1993), analytical (Pelinovsky et al., 1995), and experimental (Tikhonenko et al., 1996a) results revealed that the instability leads to the generation of a train of vortex solitons with alternative polarities, as a possible scenario of the self-focusing process in a defocusing medium. In addition, the self-focusing of dark solitons may also display long-lived intermediate oscillations between a quasi-planar soliton and a train of vortex solitons (Pelinovsky et al., 1995).

Although the transverse instability of dark solitons has been considered also for the hyperbolic NLS equation (Rypdal and Rasmussen, 1989), we notice that $\sigma_d = -1$ the c.w. background is modulationally unstable for both signs of σ_n. This implies that the wave background is likely to be destroyed by growing perturbations and experimental observations of dark solitons and their self-focusing dynamics becomes overshadowed by the background instability. Therefore, we omit the case of dark solitons in the hyperbolic NLS equation.

In the small-amplitude asymptotic limit, the transverse self-focusing of a plane dark soliton is described by a rather universal model known as the KP equation (Kadomtsev and Petviashvili, 1970). The analytical approximation resulting in the KP equation corresponds to small-amplitude long-wave modulations of the c.w. background within the asymptotic expansion:

$$\psi = [\rho - \tfrac{1}{2}\varepsilon^2 u(X + 2\rho T, Y, \tau) + O(\varepsilon^4)]\exp\{-2i\rho^2 t + i\varepsilon R(X + 2\rho T, Y, \tau)\} , \tag{2.5}$$

where $X = \varepsilon x$, $T = \varepsilon t$, $Y = \varepsilon^2 y$, $\tau = \varepsilon^3 t$, and $\varepsilon \ll 1$. As follows from Eq. (2.1) for $\sigma_n = -1$ and $\sigma_d = +1$, the function $u = R_X$ satisfies the KP equation with the positive dispersion,

$$(4u_t + 12uu_x + u_{xxx})_x = 4u_{yy} , \tag{2.6}$$

where we have used the conventional notations (x, y, t) for the stretched variables (X, Y, τ) and also put $\rho = 1$. A plane *KdV soliton* is described by a steady-state solution of Eq. (2.6):

$$u = U(x - vt - s; v) , \tag{2.7}$$

where v is the soliton velocity, s is the soliton initial coordinate, and the soliton shape is described by a sech function,

$$U(x;v) = v\,\mathrm{sech}^2(\sqrt{v}x) .$$

The KP equation (2.6) has been studied in many papers devoted to the soliton self-focusing, because it commonly appears in different physical applications, and it is integrable by means of the inverse scattering transform (see, e.g., Ablowitz and Segur, 1981). In particular, we would like to mention a number of analytical (Murakami and Tajiri, 1992; Pelinovsky and Stepanyants, 1993) and numerical (Infeld et al., 1994, 1995) results which showed that, under a periodic transverse perturbation, a plane KdV soliton transforms into a chain of two-dimensional KP solitons.

The NLS equation (2.1) and the KP equation (2.6) are two basic models we have selected in this paper to demonstrate different asymptotic approaches in the theory of the symmetry-breaking soliton instabilities. These models are generic, and many of the techniques we discuss here can be further employed for the analysis of more complicated nonlinear systems.

2.2. *Suppression of wave collapse: physical mechanisms*

Physical models discussed above are drastically simplified. Therefore, they do not take into account all possible processes which may occur in realistic physical systems. This is a general feature of any theoretical model, that holds as a good approximation provided it does not display some exotic properties or singular dynamics. However, there exist several physical models where smooth and localized initial conditions develop singularities in finite time (or at finite distances). For example, the power-law KdV equation $u_t + u^p u_x + u_{xxx} = 0$ describes the development of a singularity from localized initial conditions for $p > 4$; the NLS equation $\mathrm{i}\psi_t + \nabla_D^2\psi + |\psi|^{2r}\psi = 0$ has collapsing solutions for $Dr \geq 2$, where D stands for the space dimension. There exist some other examples, including the modified KP equation, the Boussinesq equation, the model for dispersionless three-wave interaction, the continuum limit of the Toda lattice with a transverse degree of freedom, the subcritical Ginzburg–Landau equation, etc. (see, e.g., Zakharov, 1991; Turitsyn, 1993b; Bergé, 1998a; Sulem and Sulem, 1999).

In the framework of the elliptic NLS equation, a spatially localized wave can develop a singularity in a finite time provided its total power exceeds a certain critical (threshold) value (e.g., Rypdal and Rasmussen, 1989). However, from the physical point of view, it is clear that such a catastrophic self-focusing wave collapse cannot proceed indefinitely. Moreover, the critical collapse that corresponds to the marginal condition $Dr = 2$, should be sensitive to small structural perturbations of the NLS equation. There are known many different physical mechanisms that can suppress or even completely eliminate collapse. Therefore, the models possessing the collapse dynamics are valid for describing only the initial and intermediate regimes of the beam/pulse self-focusing, and they should be modified for a later stage of the instability-induced dynamics. Below, we give a summary of some physical mechanisms which appear mostly in applications to nonlinear optics problems (see also Goldman, 1984; Bergé, 1998a). It is worth mentioning that the modulation theory developed by Fibich and Papanicolaou (1998, 1999) may allow to analyze the effects of a rather general class of perturbations on critical self-focusing.

2.2.1. Nonlinearity saturation

One of the well-known ways of stabilizing the blowup instability of optical beams above the self-focusing power threshold comes from the fact that in practice the refractive index of an optical medium saturates at high powers of optical beams. Several different models of the saturable nonlinearity were used to demonstrate the existence of stable self-trapped beams in two and three dimensions, including *exponential saturation* (e.g., Wilcox and Wilcox, 1975; Kaw et al., 1975; Vidal and Johnston, 1996), *two-level type model* (e.g., Marburger and Dawes, 1968; Gustafson et al., 1968), *cubic–quintic nolinearity* with a defocusing contribution of the next-order quintic nonlinear response (e.g., Zakharov et al., 1971; Wright et al., 1995; Josserand and Rica, 1997), the so-called *threshold nonlinearity* (e.g., Snyder et al., 1991), *logarithmic nonlinearity* (Bialynicki-Birula and Mycielski, 1979; Snyder and Mitchell, 1997a). A similar mechanism is produced by the so-called cavitation effect due to repulsion of the ions from the region of the energy concentration in an ultra-intense laser beam (Komashko et al., 1995).

The physical mechanism behind the stabilizing action of the nonlinearity saturation is very simple: When a beam increases its amplitude the effective action of nonlinearity decreases therefore preventing the further self-focusing and collapse. In most of the cases, this stabilizing mechanism is associated with the existence of *stable multi-dimensional solitary waves*.

2.2.2. Nonparaxial and vector focusing

In applications to nonlinear optics, the NLS equation appears as a result of several approximations. The most important one is the so-called *paraxial approximation* which allows to derive the NLS equation for the beam dynamics from the scalar wave (or Helmholtz) equation. Presenting the field in the form $E(\boldsymbol{r}_\perp, z) = \psi(\boldsymbol{r}_\perp, z)\,e^{ikz}$, this approximation means that the resulting scalar equation for ψ in an isotropic medium,

$$i\psi_z + \nabla^2\psi + |\psi|^2\psi + \varepsilon\psi_{zz} = 0\ , \tag{2.8}$$

can be considered in neglection of the last term with the small parameter ε which is defined by a ratio of the initial beam radius to the diffraction length. Eq. (2.8) is ill-posed as a Cauchy problem and, therefore, its direct numerical integration can meet some problems (see, e.g., Sheppard and Haelterman, 1998).

Beginning with Feit and Fleck (1988), it was argued that *no singularity forms* if beam nonparaxiality is included. This result was further supported by asymptotic analysis of Fibich (1996) who demonstrated for the equation above that the critical self-focusing is indeed arrested for any small ε (see also Fibich and Papanicolaou, 1998, 1999).

More accurate models of nonparaxial optical self-focusing should include *vectorial effects* of the coupling between TE and TM components and backscattering, both the effects lead to additional power losses. Vectorial coupling becomes significant when a beam focuses to a peak which has the width comparable with the wavelength. As a result, the weak-guidance approximation, based on the assumption that the term $\nabla(\nabla \boldsymbol{E})$ can be dropped from the Maxwell wave equations, becomes invalid. Vectorial effects (see, e.g., Milsted and Contrell, 1996, and references therein) support the existence of stable self-trapped beams with the coupled transverse and longitudinal fields of the same order (e.g., Eleonskii et al., 1973), and, therefore, the effective coupling of TE and TM components can also arrest collapse (Chi and Guo, 1995).

2.2.3. Dissipation and diffusion

Dissipation of the beam energy during its self-focusing is one of the important physical mechanisms that should prevent its collapse. The effect of a linear damping $\sim -i\nu\psi$ in the NLS equation has been analyzed numerically (Goldman et al., 1980), by means of the collective coordinates (Rasmussen et al., 1994), and by the moment method (Pérez-García et al., 1995). In general, the linear damping increases the length of self-focusing collapse but *it does not remove it completely*. For a given input power P, there exists a critical damping $\nu_c(P)$ such that for $\nu < \nu_c(P)$ the beam collapses completely even in the presence of the energy dissipation.

Another important mechanism of the energy absorption is the thermal self-action of light due to *heat diffusion* which can be described by the coupling between the NLS equation and the heat diffusion equation (e.g., Litvak et al., 1975; Bertolotti et al., 1997, and references therein). The effect of the heat diffusion is more dramatic, and the beam amplitude remains finite even for the powers larger then the critical power of self-focusing. Similar effects are produced by a nonlocal response of the medium (see, e.g., Suter and Blasberg, 1993) and by a dissipative interaction of the focused beam with stimulated Brillouin scattering which results in the self-focusing suppression and modification of the beam structure (Rubenchik et al., 1995).

2.2.4. Nonlocal interaction

The effective saturation effect in the soliton self-focusing is also produced by nonlocality of the nonlinear interaction, which can be described by the nonlocal NLS equation,

$$i\psi_t + \nabla_D^2\psi + \psi\int V(|\boldsymbol{r}-\boldsymbol{r}'|)|\psi(\boldsymbol{r}')|^2\,d\boldsymbol{r}' = 0\ , \tag{2.9}$$

where $V(|x|)$ is the potential of the interaction between the particles. The standard NLS model follows from Eq. (2.9) under the assumption of the delta-like point interaction. As was demonstrated by Turitsyn (1985) for a few important physical types of the potential $V(|x|)$, Eq. (2.9) does not possess collapsing solutions, so that collapse can be indeed arrested by higher-order derivative terms produced by the nonlocal interaction. The physical mechanism of the stabilizing action of nonlocality is even more clear in the critical limit of highly nonlocal media when Eq. (2.9) becomes a linear equation for an effective quantum oscillator (Snyder and Mitchell, 1997b).

2.2.5. Higher-order dispersion and discreteness

The similar stabilizing action of nonlocal nonlinearity is produced solely by a nonlocal linear dispersion (Gaididei et al., 1996). In the simplest case, the problem can be reduced, by keeping only the lowest-order term, to the NLS equation with the fourth-order dispersion,

$$i\psi_t + \nabla_D^2\psi + |\psi|^{2r}\psi + \frac{\gamma}{2}\nabla^4\psi = 0\ , \tag{2.10}$$

with small $\gamma \neq 0$. The effect produced by this higher-order dispersion depends on the sign of γ, so that for $\gamma > 0$ it leads to resonant radiation of linear waves and defocusing of a localized wave. For

$\gamma < 0$, Eq. (2.10) can possess *stable localized solutions* of the form $\psi(\boldsymbol{r}, t) = \phi(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}$, provided

$$(Dr - 2)P < 2r\omega\left(\frac{\mathrm{d}P}{\mathrm{d}\omega}\right),$$

where $P = \int |\psi|^2\,\mathrm{d}^D r$ is the total beam power (Karpman, 1996), the result not yet verified numerically.

Now, it becomes clear that the discreteness of physical models may also lead to the arrest of collapse and/or *stable localized states* (e.g., Bang et al., 1994; Aceves et al., 1995; Christiansen et al., 1996a,b) because it generates a higher-order dispersion term of a proper sign. Indeed, if we use the perturbation theory to approximate the lattice discreteness by a continuum model taking some higher-order terms in the corresponding Taylor expansion (a small parameter is a ratio of the lattice spacing to the characteristic length of the wave), the resulting forth-order dispersion produces a stabilizing effect on the localized waves. In the opposite case of a strong discreteness, the size of the self-focused beam is limited by the lattice spacing, so that no singularity can develop in principle (see, e.g., Laedke et al., 1994).

2.2.6. Normal dispersion

As was suggested by Strickland and Corkum (1994), the normal group-velocity dispersion increases the peak power of two-dimensional self-focusing of short light pulses, therefore it should also provide a limiting (saturating) mechanism of the wave collapse. Numerical simulations (e.g., Chernev and Petrov, 1992a,b; Rothenberg, 1992a,b; Luther et al., 1994a,b,c; Ryan and Agrawal, 1995) demonstrated that the normal dispersion tends to ease self-focusing by spreading the pulse along the propagation direction, and therefore increases the threshold for the two-dimensional collapse through a temporal pulse-splitting process (see discussions and references in Bergé and Rasmussen, 1996a,b; Bergé, 1998a). Normal dispersion removes the critical self-similarity of the wave self-focusing as is seen through the analysis of modulation equations (Luther et al., 1994c; Fibich et al., 1995) leading to a non-self-similar one peak splitting and, as is observed in numerical simulations, a fragmentation process first defined as *fractal collapse* by Zharova et al. (1986). In general, the virial-type arguments and self-similar analysis (Bergé et al., 1996, 1998) suggest that a pulse does not develop a spatial singularity being splitted into small-size beams ('cells') which are sequentially formed and finally spread out with a power lower than the critical power of self-focusing. The similar behaviour is basically preserved for a general power-law nonlinearity $\sim |\psi|^{2r}\psi$, so that it can be shown (Bergé et al., 1996c) that in the critical case $D = 2/r$ no collapse can occur and the longitudinal extension of the solution never reaches zero for any D.

Recently, an extensive numerical simulations of the different regimes of the pulse propagation and self-focusing in Argon have been carried out by Mlejnek et al. (1998) on the basis of the model of the hyperbolic NLS equation coupled to a rate equation describing plasma generation. By varying the pressure p as a control parameter, they observed numerically a number of different scenarios, from a low pressure regime ($p < 1$ atm) to full blowup arrested by normal dispersion at high pressures ($p > 100$ atm).

In all the examples discussed above, we have seen that even small additional terms in the NLS equation may have a large effect on the global dynamics of self-focusing. However, in many numerical studies it was shown that there is very little difference between self-focusing described by the cubic NLS equation and the extended models *during the first focusing cycle until the collapse is arrested*. For that reason, the cubic NLS equation still serves as the fundamental model equation for self-focusing in the prefocal region even though it neglects higher-order effects.

3. Criteria for soliton self-focusing

Fundamental physical mechanisms leading to the transverse self-focusing of solitary waves in different types of dispersive and diffractive nonlinear systems are similar to the mechanisms governing self-focusing and modulational instabilities of small-amplitude quasi-harmonic wave packets (see, e.g., Kadomtsev, 1976; Rabinovich and Trubetskov, 1984). The instability occurs when transverse modulations along the front of a planar solitary wave decrease a local value of the soliton energy. There exist two basic analytical methods to study the soliton self-focusing, the *geometric optics approach* and *linear stability analysis*. Here we briefly overview both these approaches (Sections 3.1 and 3.2) and show a link between the methods for the analysis of transverse instabilities of planar solitons and modulational instability of c.w. modes in dispersive nonlinear systems (Section 3.3).

3.1. Geometric optics approach

The geometric optics approach in the theory of linear and nonlinear waves is based on the assumption that a transversely modulated plane wave remains locally close to its steady-state profile, so that each individual segment of the wave evolves along an individual ray [see, e.g., the book by Anile et al. (1993) for many different applications of this method]. In the dynamics of quasi-plane solitons, this assumption is valid provided the soliton remains *stable against longitudinal perturbations preserving its symmetry*.

The geometric optics method for the soliton transverse self-focusing was first developed by Ostrovsky and Shrira (1976) who considered one-parametric solitons propagating in an isotropic medium (see also Shrira, 1980). To present the basic results of this method, we introduce an orthogonal system of coordinates (α, β) and consider the evolution of the local soliton velocity v, the width of a way tube Δ, and the phase angle θ that the ray makes with the x-axis, see Fig. 1(a). Geometrically, these three parameters are connected through the following kinematic relations,

$$\frac{\partial s}{\partial \alpha} + \frac{1}{\Delta}\frac{\partial v}{\partial \beta} = 0, \quad \frac{\partial s}{\partial \beta} - \frac{1}{v}\frac{\partial \Delta}{\partial \alpha} = 0 \ . \tag{3.1}$$

This system is closed with the help of the energy conservation law for a ray tube,

$$\Delta W(v) = \text{const} \ , \tag{3.2}$$

where $W = W(v)$ is the energy density. System (3.1),(3.2) is elliptic provided $\mathrm{d}v/\mathrm{d}\Delta > 0$, and then it predicts a transverse instability of a plane soliton. Since the ray width Δ is typically inversely proportional to the soliton amplitude, the ray equations define a universal criterion for the self-focusing instability of one-parametric solitons in an isotropic medium: *transverse self-focusing of a plane soliton occurs when the soliton velocity decreases with an increase of its amplitude*. In this case, the corresponding rays form a converging cylindrical wave, as shown schematically in Fig. 1(b).

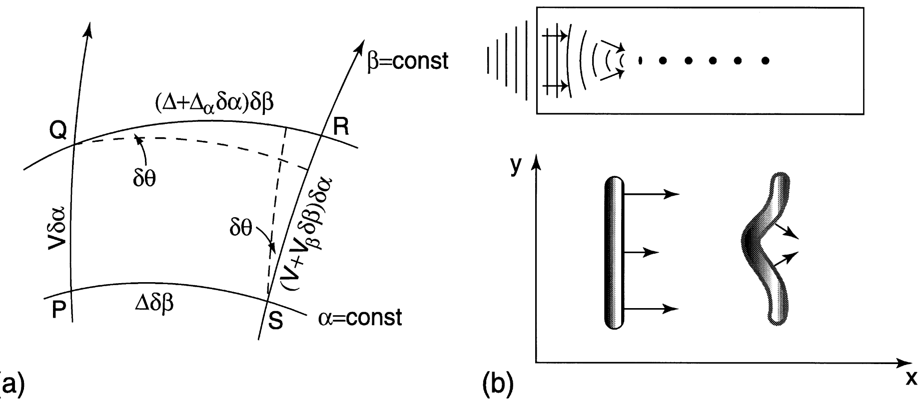


Fig. 1. (a) Coordinate system for the application of the geometric optics approach. (b) Schematic of the soliton self-focusing and the instability mechanism described by the geometric optics.

This simple physical mechanism explains the transverse instability of the *dark NLS solitons* (Kuznetsov and Turitsyn, 1988), for which the dependence of the velocity v on the amplitude a is defined by the relation, $v = \sqrt{\rho^2 - a^2}$ [see Eq. (2.3)]. Similarly, this criterion is valid for the *KdV solitons* if one considers the solitons within the original isotropic problem in a medium with a positive dispersion (Kadomtsev and Petviashvili, 1970).

It is straightforward to generalize Eqs. (3.1) and (3.2) of the geometric optics approach to the case of the solitons modulated with respect to their frequency rather than velocity. As a result, the same criterion of the soliton self-focusing is valid in the case when the soliton frequency ω decreases with the soliton amplitude a. This is valid for the *bright NLS solitons* that have the frequency ω in an effective gap of the linear spectrum band, $\omega = \omega_c - \gamma a^2$, where ω_c is the cutoff frequency of the linear band, and γ is positive. Indeed, the bright solitons are known to be unstable with respect to transverse perturbations (Zakharov and Rubenchik, 1973).

The geometric optics method is not applicable directly to certain nonlinear models, e.g. those described by Eqs. (2.1) and (2.6), where the dependences $\omega = \omega(a)$ and $v = v(a)$ have already been renormalized. In addition, Eqs. (3.1) and (3.2) are limited by the wave propagation in an isotropic medium and by the case of one-parametric solitons. For instance, the hyperbolic NLS equation is beyond the applicability of the ray method because (i) this model is inherently anisotropic, and (ii) the symmetry-breaking instability of bright solitons leads to a self-consistent coupled dynamics of both translational and oscillatory degrees of freedom of the bright NLS soliton.

An alternative and mathematically more accurate method is based on the derivation of the Whitham modulation equations for the soliton parameters (Whitham, 1974). The Whitham equations describe both longitudinal and transverse modulations of a quasi-plane nonlinear wave, and they can be derived for both periodic and localized waves (see, e.g., examples of the application of this method in the book by Infeld and Rowlands (1990)). If a resulting system of the Whitham equations is elliptic, the solitary waves are unstable and their evolution leads to the self-focusing dynamics. In a particular case of transverse modulations, the Whitham modulation theory reduces to an averaged Lagrangian method which we discuss below in Section 4.2.

3.2. Linear eigenvalue problems

Another basic approach for analyzing the self-focusing soliton instabilities is based on the analysis of a linear eigenvalue problem that is obtained by linearizing the equations of motion near the exact solitary wave solution. Because a steady-state solution for a solitary wave depends only on a few parameters (*soliton coordinates and phases*), one can separate variables and reduce the analysis of the corresponding linear problem to the analysis of an eigenvalue spectrum of a certain linear operator.

For the *bright NLS solitons*, linear perturbations are considered in the form,

$$\delta\psi = \psi(x,y,t) - \Phi_b(x;\omega)\,\mathrm{e}^{\mathrm{i}\omega t} = [u(x) + \mathrm{i}w(x)]\,\mathrm{e}^{\mathrm{i}\omega t + \Gamma t + \mathrm{i}py}\ ,$$

where p is the transverse wave number, Γ is an eigenvalue that determines the instability growth rate, $\Phi_b(x;\omega)$ is given by Eq. (2.2), and the real functions $u(x)$ and $w(x)$ satisfy the linear eigenvalue problem following from Eq. (2.1) at $\sigma_\mathrm{n} = +1$,

$$(\hat{\mathscr{L}}_1 + \sigma_\mathrm{d} p^2)u = -\Gamma w, \quad (\hat{\mathscr{L}}_0 + \sigma_\mathrm{d} p^2)w = \Gamma u\ , \tag{3.3}$$

where

$$\hat{\mathscr{L}}_0 = -\partial_x^2 + \omega - 2\Phi_\mathrm{b}^2, \quad \hat{\mathscr{L}}_1 = \hat{\mathscr{L}}_0 - 4\Phi_b^2\ .$$

For the *dark NLS solitons*, the linear eigenvalue problem follows from Eq. (2.1) at $\sigma_\mathrm{n} = -1$ and $\sigma_\mathrm{d} = +1$ after substituting,

$$\delta\psi = \psi(x,y,t) - \Phi_\mathrm{d}(\xi;v)\,\mathrm{e}^{-2\mathrm{i}\rho^2 t} = \phi(\xi)\,\mathrm{e}^{-2\mathrm{i}\rho^2 t + \Gamma t + \mathrm{i}py}\ ,$$

where $\xi = x - 2vt$, $\Phi_\mathrm{d}(\xi;v)$ is given by Eq. (2.3), and $\phi(\xi)$ satisfies the linear eigenvalue equation,

$$\hat{\mathscr{L}}\phi + p^2\phi = \mathrm{i}\Gamma\phi\ , \tag{3.4}$$

where

$$\hat{\mathscr{L}} = -\partial_\xi^2 + 2\mathrm{i}v\partial_\xi - 2(\rho^2 - 2|\Phi_\mathrm{d}|^2) + 2\Phi_\mathrm{d}^2(^*)\ .$$

For the *KdV solitons*, the linear eigenvalue problem follows from the KP equation (2.6) after the substitution,

$$\delta u = u(x,y,t) - U(\xi,v) = W(\xi)\,\mathrm{e}^{\Gamma t + \mathrm{i}py}\ ,$$

where $\xi = x - vt$, $U(\xi;v)$ is given by Eq. (2.7), and $W = W(\xi)$ satisfies the problem

$$(\hat{\mathscr{L}}W)_{\xi\xi} - 4p^2 W = 4\Gamma W_\xi\ , \tag{3.5}$$

with

$$\hat{\mathscr{L}} = -\partial_\xi^2 + 4v - 12U\ .$$

Self-focusing instability of solitary waves occurs if the corresponding linear eigenvalue problem possesses at least one localized eigenmode with an eigenvalue $\Gamma = \Gamma(p)$ having *a positive real part* for $p > 0$. Additionally, if a soliton is stable against longitudinal perturbations preserving its symmetry, there exist no unstable eigenvalues at $p = 0$, i.e. $\Gamma(0) = 0$.

In most of the cases, no exact solutions can be found for linear eigenvalue problems of this type. However, there exist several analytical methods to approximate the unstable eigenvalue $\Gamma = \Gamma(p)$ and to derive the instability threshold condition. These methods are based on the asymptotic expansions for either small wave numbers ($p \to 0$) or finite wave numbers near a critical (cut-off) value ($p \to p_c$). We define these asymptotic approximations as *long-scale* and *short-scale* expansions, respectively.

The long-scale asymptotic method is based on the Taylor expansions of the instability eigenmode for small values of the wave number p. Such a technique is sometimes called the *p-expansion method*, and it may often detect the existence of self-focusing soliton instability since the instability typically occurs for large spatial wavelengths (i.e. small p) of transverse modulations. The method is originated from the first papers (Kadomtsev and Petviashvili, 1970; Zakharov and Rubenchik, 1973), and it is commonly used nowadays (see, e.g., Infeld and Rowlands, 1990). Using the discrete-spectrum modes of linear eigenvalue problem at $p = 0$ and $\Gamma = 0$ usually known in an explicit analytical form (the so-called *neutral modes*), one can construct the Taylor expansion of the eigenfunctions and eigenvalues of the corresponding linear problem and derive the first terms of the asymptotic expansions. These first terms define the spatial symmetry of perturbations leading to the transverse instability, and also provide an approximate expression for the instability growth rate $\Gamma(p)$ for small p. Below, we present the neutral modes which generate the corresponding long-scale expansions, and the asymptotic approximation for $\Gamma(p)$ derived for the basic soliton equations.

(i) *Bright NLS solitons in the elliptic problem* (Zakharov and Rubenchik, 1973; Kuznetsov et al., 1986)

$$u = 0, \quad w = \Phi_b(x;\omega), \quad \Gamma^2(p) = 4\omega p^2 - \frac{4}{3}\left(1 + \frac{\pi^2}{3}\right)p^4 + \mathrm{O}(p^6)\ ; \tag{3.6}$$

(ii) *Bright NLS solitons in the hyperbolic problem* (Zakharov and Rubenchik, 1973; Kuznetsov et al., 1986)

$$u = [\Phi_b(x;\omega)]_x, \quad w = 0, \quad \Gamma^2(p) = \frac{4}{3}\omega p^2 - \frac{4}{9}\left(\frac{\pi^2}{3} - 1\right)p^4 + \mathrm{O}(p^6)\ ; \tag{3.7}$$

(iii) *KdV solitons* (Shrira and Pesenson, 1983; Pesenson, 1991)

$$W = U_\xi(\xi;v), \quad \Gamma^2(p) = \frac{4}{3}vp^2 - \frac{4}{3\sqrt{v}}p^2\Gamma + \mathrm{O}(p^4)\ ; \tag{3.8}$$

(iv) *Dark NLS solitons* (Kuznetsov and Turitsyn, 1988)

$$\phi = [\Phi_d(\xi;v)]_\xi, \quad \Gamma^2(p) = \frac{4}{3}(\rho^2 - v^2)p^2 - \frac{2(\rho^2 + v^2)}{3\rho\sqrt{\rho^2 - v^2}}p^2\Gamma + \mathrm{O}(p^4)\ . \tag{3.9}$$

We would like to point out that numerical coefficients in Eqs. (3.6) and (3.7) are slightly different from those which can be found in the original papers (Zakharov and Rubenchik, 1973; Kuznetsov et al., 1986) because of misprints.

The short-scale asymptotic method is based on the Taylor expansion with respect to another small parameter $(p_c - p)$, where p_c is a critical (or cutoff) value of p where Γ vanishes, i.e. $\Gamma(p_c) = 0$. This method uses the expressions for the eigenmodes of the linear eigenvalue problems at $p = p_c$ and $\Gamma = 0$. The eigenmodes can be often found by a direct analysis of the linearized equations. Below, we present the critical eigenmodes, eigenvalues, as well as the asymptotic approximations for the growth rate $\Gamma(p)$ in the limit $p \to p_c$.

(i) *Bright NLS solitons in the elliptic problem* (Janssen and Rasmussen, 1983; Kuznetsov et al., 1983)

$$u = \mathrm{sech}^2(\sqrt{\omega} x), \quad w = 0, \quad p_c = \sqrt{3\omega}\ ,$$

$$\Gamma^2(p) = \frac{24\omega p_c}{(\pi^2 - 6)}(p_c - p) + \mathrm{O}((p_c - p)^2)\ ; \tag{3.10}$$

(ii) *Bright NLS solitons in the hyperbolic problem* (Pelinovsky, 2000)

$$u = 0, \quad w = \tanh(\sqrt{\omega} x), \quad p_c = \sqrt{\omega}\ ,$$

$$\Gamma^2(p) = \frac{16\omega p_c}{3\pi^2}\sqrt{p_c^2 - p^2} + \mathrm{O}(p_c - p)\ . \tag{3.11}$$

We notice that in this case the critical eigenfunction is delocalized.

(iii) *KdV solitons* (Gorshkov and Pelinovsky, 1995a, 1995b)

$$W = \mathrm{sech}(\sqrt{v}\xi) - 2\,\mathrm{sech}^3(\sqrt{v}\xi), \quad p_c = \frac{\sqrt{3}}{2}v\ ,$$

$$\Gamma^2(p) = \frac{4}{3}p_c(p_c - p) + \mathrm{O}((p_c - p)^2)\ ; \tag{3.12}$$

(iv) *Dark NLS solitons* (Pelinovsky et al., 1995)

$$\phi = -\frac{3v\sinh(k\xi)}{2k\cosh^2(k\xi)} + \mathrm{i}\frac{(p_c^2 + 3k^2)}{4k^2\cosh(k\xi)},$$

$$p_c = [-(\rho^2 + v^2) + 2\sqrt{v^4 - v^2\rho^2 + \rho^4}]^{1/2}\ ,$$

$$\Gamma^2(p) = \frac{p_c}{\beta(k)}(p_c - p) + \mathrm{O}((p_c - p)^2)\,. \tag{3.13}$$

In the latter case, the dependence $\beta = \beta(k)$ was found numerically [see Fig. 2(b) in Pelinovsky et al., 1995] within the domain $1.88 \le \beta/k^2 \le 3.00$, the lower limit is for $k = 1$ ($v = 0$) and the upper limit is for $k \to 0$ ($v \to \rho$).

A complete spectrum of unstable eigenvalues can be found only in special cases, e.g. if the linear eigenvalue problem is integrable. For example, this is the case of the transverse instabilities of the KdV solitons because the KP equation (2.6) is known to be integrable by means of the inverse scattering transform (see, e.g., Ablowitz and Segur, 1981). Using the inverse scattering technique, Zakharov (1975) constructed the spectrum of the transverse instability of the KdV solitons (see also Alexander et al., 1997). Indeed, one can verify that the eigenfunction,

$$W = \frac{\partial^2}{\partial\xi^2}[\mathrm{e}^{\kappa\xi}\,\mathrm{sech}(\sqrt{v}\xi)], \quad \kappa = \left(v - \frac{2}{\sqrt{3}}p\right)^{1/2}\ . \tag{3.14}$$

solves Eq. (3.5) for

$$\Gamma(p) = \frac{2}{\sqrt{3}} p \left(v - \frac{2}{\sqrt{3}} p \right)^{1/2} . \tag{3.15}$$

The expansion of Eq. (3.15) in the limits $p \to 0$ ($\kappa \to \sqrt{v}$) and $p \to \sqrt{3}v/2$ ($\kappa \to 0$) coincide with the results of the asymptotic analysis given by Eqs. (3.8) and (3.12).

3.3. Analogy with modulational instability

Modulational instability of a monochromatic nonlinear wave occurs via the four-wave interaction between the wave of the main frequency and generated wave satellites which form a certain resonant configuration (see Infeld and Rowlands, 1990, and references therein). A possibility of such a resonant four-wave mixing follows from the structure of the dispersion relation of linear waves, the dependence of the wave frequency $\omega(k)$ on the wave number k. The linear dispersion relation for the NLS equation (1.1) is $\omega(k) = k^2$, and such a frequency dependence allows resonant four-wave interactions according to the following frequency mixing rule:

$$2\omega(2k) = \omega(k + \delta k) + \omega(k - \delta k) , \tag{3.16}$$

where $\delta k = \sqrt{3}k$. Such a resonant wave-mixing mechanism explains the appearence of modulational instability of a fundamental wave with respect to the parametric excitations of two wave satellites. The instability follows from the analysis of the linearized version of the NLS equation (1.1). To show this, we present the linearized solution in the form,

$$\varphi = [\rho + (\varepsilon_u + i\varepsilon_v)e^{ipx + \Gamma t} + (\varepsilon_u^* + i\varepsilon_v^*)e^{-ipx + \Gamma^* t}] e^{ikx - ik^2 t + i(r+1)\rho^{2r} t} , \tag{3.17}$$

where ε_u and ε_v are coupled through the relation, by the expression

$$\varepsilon_v/\varepsilon_u = (\Gamma + 2ikp)/p^2 . \tag{3.18}$$

Then, the instability growth rate $\Gamma = \Gamma(p)$ is defined by a solution of a quadratic equation,

$$(\Gamma + 2ikp)^2 = 2r(r+1)\rho^{2r} p^2 - p^4 . \tag{3.19}$$

A simple analysis of Eq. (3.19) indicates that the c.w. solution, $\varphi = \rho e^{i(r+1)\rho^{2r} t}$, is unstable against modulations with the wave numbers p selected within the band $0 < p < p_c$, where

$$p_c = [2r(r+1)\rho^{2r}]^{1/2} .$$

Nonlinear dynamics of modulational instability of a c.w. solution can be also analyzed in the limit of long-wave modulations, i.e. when both p and Γ are small. To do so, we use the well-known presentation of the complex field in the fluid-dynamics form (see, e.g., Spiegel, 1980),

$$\varphi = [q(X, T)]^{1/2} e^{i\theta(X,T)/\varepsilon} , \tag{3.20}$$

where $X = \varepsilon x$, $T = \varepsilon t$, and $\varepsilon \ll 1$. Then, the NLS equation reduces to a system of two coupled equations for real θ and q,

$$\begin{aligned} &q_T + 2(q\theta_X)_X = 0 , \\ &\theta_T + \theta_X^2 - (r+1)q^r - \varepsilon^2 \left(\frac{q_{XX}}{2q} - \frac{q_X^2}{4q^2} \right) = 0 , \end{aligned} \tag{3.21}$$

Coupled equations (3.21) possess a c.w. solution, $q = q_0$ and $\theta = (r+1)q_0^r t + \theta_0$, where q_0 and θ_0 are constants. Linear expansion around the c.w. solution produces the characteristic equation (3.19) with $q_0 = \rho^2$. In the limit $\varepsilon \to 0$, system (3.21) transforms into the gas-dynamics equations (Trubnikov and Zhdanov, 1987) which describes a dispersionless limit of the NLS equation (Kamchatnov, 1997). Equivalently, the gas dynamics equations coincide with the geometric optics approximation for the wave propagation in nonlinear diffractive media, where the first equation gives the energy balance, while the second equation is a kinematic relation for the wave phase. The modulational instability of a c.w. background can be then studied within the coupled equations (3.21) in the limit of small but finite values of ε, by applying the Whitham modulation theory (see, for a review, Kamchatnov, 1997). However, the applicability of the Whitham theory is limited by integrable systems because it is based on exact periodic solutions.

It is a purpose of our survey to show that the modulation equations which describe the development of the soliton self-focusing instability are similar to those of the fluid dynamics form (3.21) of the NLS equation. Moreover, these modulation equations can be investigated systematically by means of the regular asymptotic expansion methods. The latter methods use only a balance between the amplitude and the spatial scale of the soliton modulations and do not rely upon the integrability properties of the modulation equations. Thus, we expect that the methods described here are more general than the Whitham theory, and they can be applied to much broader class of problems related to the symmetry-breaking instabilities of nonlinear waves.

4. Equations for soliton parameters

The analytical approaches described above are based on two different approximations imposed on the soliton transverse modulations. The geometric optics approach describes the evolution of strongly nonlinear long-wavelength modulations in the dispersion less limit, whereas the linear eigenvalue problem allows to describe the evolution of perturbations of all scales but within a small amplitude (linear) approximation. Depending on a type of balance between linear and nonlinear effects in the soliton dynamics, different analytical methods have been employed in the literature for deriving the modulation equations describing nonlinear regimes of the soliton self-focusing. In this section, we review some of those methods and also discuss their applicability limits.

The basic assumptions we use here are the following: (i) a planar soliton is stable against the symmetry-preserving perturbations, and (ii) the instability occurs when the wavelength of the soliton transverse modulations is much larger than the soliton width whereas the amplitude of the soliton modulations may be not small. The modulation equations are derived below by means of direct asymptotic expansions (Section 4.1) or, equivalently, by an averaged Lagrangian method (Section 4.2). These equations generate, in the leading order, ill-posed (elliptic) equations for the dynamics of an equivalent (unstable) gas which displays the development of singularities in finite time (Section 4.3). The gas-dynamics equations represent the dispersionless limit of the NLS equation written in the hydrodynamical form (3.21) (see Section 4.4). We can improve the applicability of the asymptotic equations by extending the perturbation theory into higher orders to include dispersive and/or dissipative effects (see Section 4.5). The bounded scenario of the transverse self-focusing can be then described within the extended models (Section 4.6). For integrable evolution equations, the soliton self-focusing can be studied in the framework of exact analytical solutions (Section 4.7).

4.1. Direct asymptotic expansions

We consider the NLS equation (2.1) and the KP equation (2.6) in the asymptotic limit of long transverse modulations, i.e. we write the fields as $\psi = \psi(x, Y, t)$ and $u = u(x, Y, t)$, where $Y = \varepsilon y$ in the stretched coordinate and $\varepsilon \ll 1$. Then, those equations transform to the perturbed NLS and KdV equations, respectively,

$$\mathrm{i}\psi_t + \psi_{xx} + 2\sigma_{\mathrm{n}}|\psi|^2\psi = \varepsilon^2\sigma_{\mathrm{d}}\psi_{YY} , \tag{4.1}$$

$$(4u_t + 12uu_x + u_{xxx})_x = 4\varepsilon^2 u_{YY} . \tag{4.2}$$

The soliton dynamics induced by such effective perturbations can be analyzed in the framework of a regular soliton perturbation theory (Kivshar and Malomed, 1989; Pelinovsky and Grimshaw, 1997). This theory implies that a planar soliton evolves *adiabatically* under the action of long-scale transverse perturbations, i.e. it is locally close to the profile of a planar solitary wave solution but with the parameters varying slowly with Y and also depending on slow time $T = \varepsilon t$. Then, the soliton perturbation theory allows us to derive, in a systematic way, a system of modulation equations for the soliton parameters. We show below how to derive those equations for several important physical cases.

4.1.1. Bright NLS solitons in the elliptic problem

The transverse self-focusing of a bright soliton in the elliptic problem is generated by the phase and frequency modulations [see Eq. (3.6)]. Therefore, we neglect the parameters v and s in Eq. (2.2) and assume the following asymptotic series:

$$\psi(x,y,t) = \left(\Phi_b(x;\omega) + \varepsilon\phi_1(x,t;Y,T) + \sum_{n=2}^{\infty}\varepsilon^n\phi_n(x,t;Y,T)\right)\mathrm{e}^{\mathrm{i}\theta/\varepsilon} , \tag{4.3}$$

where $\Phi_b(x;\omega)$ is the soliton profile (2.2), the parameters are some functions of slow variables, i.e. $\omega = \omega(Y,T)$ and $\theta = \theta(Y,T)$, and $\phi_1(x,t;Y,T)$ is a first-order correction to the soliton shape. Substituting Eq. (4.3) into Eq. (4.1) for $\sigma_{\mathrm{n}} = +1$ and $\sigma_{\mathrm{d}} = +1$ and neglecting the terms ϕ_n for $n \geq 2$, we find the linear equation for the function $\phi_1 = u_1 + \mathrm{i}w_1$,

$$\begin{aligned} w_{1t} + \hat{\mathscr{L}}_1 u_1 = H_1 &\equiv -\varepsilon^{-1}(\theta_T + \theta_Y^2 - \omega)\Phi_b + \varepsilon\left(\frac{\partial\Phi_b}{\partial\omega}\omega_{YY} + \frac{\partial^2\Phi_b}{\partial\omega^2}\omega_Y^2\right) , \\ -u_{1t} + \hat{\mathscr{L}}_0 = H_2 &\equiv \frac{\partial\Phi_b}{\partial\omega}(\omega_T + 2\omega_Y\theta_Y) + \theta_{YY}\Phi_b , \end{aligned} \tag{4.4}$$

where $\hat{\mathscr{L}}_0$ and $\hat{\mathscr{L}}_1$ are defined after Eq. (3.3). It is clearly seen from Eq. (4.4) that the asymptotic balance occurs when the phase factor $(\theta_T + \theta_Y^2 - \omega)$ becomes of order of $\mathrm{O}(\varepsilon^2)$ and the terms of order of $\mathrm{O}(\varepsilon)$ should be moved from Eq. (4.4) into the next-order equation for ϕ_2. One of the typical approximation is however to keep all terms into the same equation (4.4) and proceed with a solution. A solution can be found by expanding u_1 and w_1 through a complete set of the continuous spectrum eigenfunctions (Kaup, 1990). The discrete and associated eigenfunctions are to be removed from the expansion by means of the orthogonality conditions,

$$\int_{-\infty}^{\infty} \Phi_b \; H_2 \, dx = 0, \quad \int_{-\infty}^{\infty} \frac{\partial \Phi_b}{\partial \omega} H_1 \, dx = 0 \; . \tag{4.5}$$

It can be shown that the first condition gives a power balance for a transversely modulated bright soliton, where the power is defined as $N = \int_{-\infty}^{\infty} |\psi|^2 \, dx$, and the second equation is a condition that the correction term ϕ_1 does not change the value of the soliton energy $N_b(\omega)$, i.e. $N = N_b(\omega) + O(\varepsilon^2)$. The latter condition can be omitted in the soliton perturbation theory in one dimension (Pelinovsky and Grimshaw, 1997) because it just renormalizes the energy conservation equation into the next asymptotic order. However, the second equation (4.5) has its own meaning for the soliton dynamics in two dimensions as the kinematic relations (3.1) within the geometric optics approach. Evaluating integrals in Eqs. (4.5), we derive a system of modulation equations for the parameters of a bright NLS soliton,

$$\begin{aligned} &\omega_T + 2\omega_Y \theta_Y + 4\omega\theta_{YY} = 0 \; , \\ &\theta_T + \theta_Y^2 - \omega + \varepsilon^2 \mu \left(3\frac{\omega_Y^2}{\omega^2} - 4\frac{\omega_{YY}}{\omega} \right) = 0 \; , \end{aligned} \tag{4.6}$$

where

$$\mu = \frac{1}{12}\left(1 + \frac{\pi^2}{12}\right) .$$

Eqs. (4.6), without the terms of order of $O(\varepsilon^2)$ represent the energy conservation law and the eikonal equation of the geometric optics method. The additional terms of the order of $O(\varepsilon^2)$ describe the effect of dispersion on the soliton self-focusing.

4.1.2. Bright NLS solitons in the hyperbolic problem

In the hyperbolic problem, the self-focusing instability of a bright NLS soliton is induced via the coordinate and velocity modulations [see Eq. (3.7)] coupled to the phase and frequency modulations. As a result, *a variation of all four parameters* of a bright NLS soliton (2.2) should be taken into account in the asymptotic expansion,

$$\psi(x, y, t) = \left(\phi_0(\xi; \omega) + \varepsilon\phi_1(\xi, t; Y, T) + \sum_{n=2}^{\infty} \varepsilon^n \phi_n(\xi, t; Y, T) \right) e^{iv\xi + i\theta/\varepsilon} \; , \tag{4.7}$$

where $\phi_0 = \sqrt{1 - 4s_Y^2}\,\Phi_b(\xi; \omega)$, $\xi = x - 2s/\varepsilon$, and all parameters (v, s, ω, θ) are assumed to be some functions of slow variables Y and T. The amplitude factor $\sqrt{1 - 4s_Y^2}$ appears due to a curvature of the soliton front induced by the coordinate modulations. We investigate here only the leading order of the modulation equations. To find these equations, we derive from Eq. (4.1) at $\sigma_n = +1$ and $\sigma_d = -1$ the system of equations for the function $\phi_1 = u_1 + iw_1$,

$$\begin{aligned} w_{1t} + (1 - 4s_Y^2)\hat{\mathscr{L}}_1 u_1 &= -(v_T - 2v_Y\theta_Y + 4vv_Y s_Y)\xi\phi_0 + 2s_{YY}\phi_{0\xi} + 4s_Y\phi_{0\xi Y} \; , \\ -u_{1t} + (1 - 4s_Y^2)\hat{\mathscr{L}}_0 &= (2vs_{YY} + 4v_Y s_Y - \theta_{YY})\phi_0 + \phi_{0T} + 4v_Y s_Y \xi\phi_{0\xi} \\ &\quad - 2\theta_Y\phi_{0Y} + 4vs_Y\phi_{0Y} \; . \end{aligned} \tag{4.8}$$

Solvability of this system is determined by four orthogonality conditions for the neutral and associated modes of the discrete spectrum. As a result of applying those conditions, we obtain the following system of modulation equations:

$$\begin{aligned}
&v_T - 2v_Y\theta_Y + 2\omega_Y s_Y + 4vv_Y s_Y + \frac{4(1-12s_Y^2)}{3\,(1-4s_Y^2)}\omega s_{YY} = 0 \; ,\\
&s_T - v(1-4s_Y^2) - 2s_Y\theta_Y = 0 \; ,\\
&\omega_T - 2\omega_Y\theta_Y + \; + 4v\omega_Y s_Y - 8\omega v_Y s_Y + 4\frac{(1+4s_Y^2)}{(1-4s_Y^2)}\omega(2vs_{YY} - \theta_{YY}) = 0 \; ,\\
&\theta_T - \theta_Y^2 - (\omega + v^2)(1-4s_Y^2) = 0 \; .
\end{aligned} \tag{4.9}$$

At $v = s = 0$, this system reduces to the modulation equations (4.6) after changing $\partial_Y \to \mathrm{i}\partial_Y$ and neglecting the terms of order of $O(\varepsilon^2)$. On the other hand, we are not able to decouple the evolution of the parameters (v, s) from that of the parameters (ω, θ), and therefore the self-focusing dynamics of a bright NLS soliton in the hyperbolic problem involves a coupling between all soliton degrees of freedom.

4.1.3. KdV solitons

Transverse self-focusing of a KdV soliton is an effect of the coordinate and velocity modulations [see Eq. (3.8)]. Therefore, we are looking for the asymptotic expansion of Eq. (4.2) in the form,

$$u = U(\xi; v) + \varepsilon u_1(\xi, t; Y, T) + \sum_{n=2}^{\infty} \varepsilon^n u_n(\xi, t; Y, T) \; , \tag{4.10}$$

where $\xi = x - s/\varepsilon$, $U(\xi; v)$ is the soliton profile (2.7), and the soliton parameters depend on slow variables, i.e. $v = v(Y, T)$ and $s = s(Y, T)$. The first-order correction u_1 to the soliton shape U satisfies the linear equation,

$$4u_{1t} - (\hat{\mathscr{L}}u_1)_\xi = H_1 \equiv 4\varepsilon^{-1}(s_T + s_Y^2 - v)U_\xi - 4(v_T + 2s_Y v_Y)\frac{\partial U}{\partial v} - 4s_{YY}U \; ,$$

where $\hat{\mathscr{L}}$ is given by Eq. (3.5), and the terms of order of $O(\varepsilon)$ are neglected. Solutions to this equation can be presented through the continuous-spectrum eigenfunctions subject to the orthogonality conditions for discrete and associate eigenmodes,

$$\int_{-\infty}^{\infty} UH_1 \,\mathrm{d}\xi = 0, \quad \int_{-\infty}^{\infty} \frac{\partial U}{\partial v}\left(\int_0^{\xi} H_1 \,\mathrm{d}\xi - 4va \right)\mathrm{d}\xi = 0 \; . \tag{4.11}$$

Here $a = a(Y, T)$ is an integration constant with respect to the variable ξ. The first condition reproduces a balance equation for the momentum of a transversely modulated KdV soliton, i.e. $P = \int_{-\infty}^{\infty} u^2 \,\mathrm{d}x$, while the second one is equivalent to the condition that the first-order correction term u_1 does not change the value of the soliton momentum $P_0(v)$, i.e. $P = P_0(v) + O(\varepsilon^2)$. Using these conditions, we derive a system of modulation equations for the parameters of a KdV soliton,

$$\begin{aligned}
&v_T + 2v_Y s_Y + \tfrac{4}{3}v s_{YY} = 0 \; ,\\
&s_T + s_Y^2 - v - \varepsilon a = 0 \; .
\end{aligned} \tag{4.12}$$

System (4.12) is not closed because the parameter a is not specified so far. The parameter a is associated with a radiation field generated by a soliton. As a consequence, the term of order of $O(\varepsilon)$

included into Eqs. (4.12) describes dissipative effects in the soliton dynamics induced by radiation. The dispersive effects, similar to those of Eqs. (4.6), have the order of $O(\varepsilon^2)$, and therefore they are negligible in comparison with dissipative effects. In the limit $\varepsilon \to 0$, system (4.12) reproduces the momentum conservation equation and the eikonal equation of the geometric optics approach.

4.1.4. Dark NLS solitons

The scenario of the self-focusing dynamics of dark solitons is similar to that of the KdV solitons. The corresponding analysis can be developed for Eq. (4.1) for $\sigma_n = -1$ and $\sigma_d = +1$. We reproduce here only main steps of the corresponding asymptotic analysis. First, we look for a solution for a perturbed soliton in the form

$$\psi(x,y,t) = \left(\Phi_d(\eta; v) + \varepsilon\phi_1(\eta, t; Y, T) + \sum_{n=2}^{\infty} \varepsilon^n \phi_n(\eta, t; Y, T) \right) e^{-2i\rho^2 t + i\theta} , \tag{4.13}$$

where $\Phi_d(\eta; v)$ is given by Eq. (2.3), η is a streched coordinate that describes a curved soliton front,

$$\eta = \frac{x - 2s/\varepsilon}{\sqrt{1 + 4s_Y^2}} ,$$

and the parameter θ is associated with the radiation emitted by the dark soliton. The first-order correction ϕ_1 satisfies the linear inhomogeneous equation,

$$\begin{aligned} \hat{\mathscr{L}}\phi_1 = & -2i\varepsilon^{-1}\left(\frac{s_T}{\sqrt{1+4s_Y^2}} - v \right)\Phi_{d\eta} + i\Phi_{dT} - \theta_T \Phi_d - \frac{4is_Y s_{YT}}{(1+4s_Y^2)}\eta\Phi_{d\eta} \\ & + \frac{16 s_Y^2 s_{YY}}{(1+4s_Y^2)^{3/2}}\eta\Phi_{d\eta\eta} - \frac{2(1-4s_Y^2)}{(1+4s_Y^2)^{3/2}} s_{YY}\Phi_{d\eta} - \frac{4s_Y}{\sqrt{1+4s_Y^2}}(\Phi_{dY\eta} + i\theta_Y \Phi_{d\eta}) , \end{aligned} \tag{4.14}$$

where the operator $\hat{\mathscr{L}}$ is defined above, see Eq. (3.4). As the result, the slowly varying parameters of a dark soliton, $v = v(Y,T)$ and $s = s(Y,T)$, satisfy the modulation equations following from Eq. (4.14),

$$\begin{aligned} & v_T + \frac{4(\rho^2 - v^2)s_{YY}}{3(1+4s_Y^2)^{3/2}} - \frac{4vv_Y s_Y}{\sqrt{1+4s_Y^2}} = 0 , \\ & v - \frac{s_T}{\sqrt{1+4s_Y^2}} = 0 . \end{aligned} \tag{4.15}$$

In the limit $v \to -\rho + \frac{1}{2}\tilde{v}$, $s \to -\rho T + \frac{1}{2}\tilde{s}$, where $|\tilde{v}|, |\tilde{s}| \ll 1$ and $\rho = 1$, these equations coincide with Eqs. (4.12) for a KdV soliton, provided the terms of order of $O(\varepsilon)$ are neglected.

4.2. Averaged Lagrangian method

An alternative and rather simple method for deriving the modulation equations in the theory of the soliton transverse self-focusing is to apply the average Lagrangian method also known as *a variational approach*. Such a method is a particular case of the general Whitham modulation theory (Whitham, 1974) when the soliton modulations are assumed to be transverse with respect to a planar soliton. The average Lagrangian method is based on a variational problem equivalent to the nonlinear evolution equation, $\delta S = 0$, where S is the action expressed through the Lagrangian L, $S = \int_0^t \mathrm{d}t \iint_{-\infty}^{\infty} L\,\mathrm{d}x\,\mathrm{d}y$. A trial function is usually chosen in the form of a steady-state soliton but with the parameters slowly varying with respect to the transverse coordinates and time. Then, a variation of the soliton parameters within the averaged variational problem reproduces the leading order of the modulation equations for the soliton self-focusing (see Makhankov, 1978; Trubnikov and Zhdanov, 1987). However, we show below that the straightforward application of the averaged Lagrangian method may meet some difficulties associated with the appearance of diverging integrals responsible for the radiation emitted by solitary waves. To describe those effects, a proper trial function should include a nonlocalized radiation component.

4.2.1. Bright NLS solitons in the elliptic problem

Lagrangian for the NLS equation (2.1) is defined as (for $\sigma_{\mathrm{n}} = \sigma_{\mathrm{d}} = +1$),

$$L = \frac{\mathrm{i}}{2}(\psi^*\psi_t - \psi\psi_t^*) - |\psi_x|^2 - |\psi_y|^2 + |\psi|^4 \ . \tag{4.16}$$

As a trial function, we take a bright NLS soliton, $\psi = \Phi_b(x;\omega)\mathrm{e}^{\mathrm{i}\theta/\varepsilon}$, with the slowly varying parameters $\omega = \omega(Y,T)$ and $\theta = \theta(Y,T)$. Integrating L with respect to x, we find the averaged Lagrangian in terms of the soliton parameters ω and θ,

$$\langle L\rangle = \frac{1}{2}\int_{-\infty}^{\infty} L\,\mathrm{d}x = -\sqrt{\omega}(\theta_T + \theta_Y^2) + \frac{1}{3}\omega^{3/2} - \varepsilon^2\mu\frac{\omega_Y^2}{\omega^{3/2}} \ , \tag{4.17}$$

where μ is the same as in Eqs. (4.6). The variation of $\langle L\rangle$ with respect to θ yields the energy conservation law for the system (4.6), whereas the variation with respect to ω generates an eikonal equation. The soliton modulation equations were first obtained with the help of the averaged Lagrangian method by Makhankov (1978).

4.2.2. KdV solitons

Lagrangian for the KP equation (2.6) is defined as

$$L = w_t w_x - \tfrac{1}{4}w_{xx}^2 - w_y^2 + w_x^3 \ , \tag{4.18}$$

where $u = w_x$. As a trial function, we consider a KdV soliton, $u = U(x - s/\varepsilon; v)$, where $s = s(Y,T)$ and $v = v(Y,T)$ are slowly varying parameters, so that $w = \sqrt{v}\tanh[\sqrt{v}(x - s/\varepsilon)]$. Integrating L with respect to x, we find the averaged Lagrangian in the form,

$$\langle L\rangle = \int_{-\infty}^{\infty} L\,\mathrm{d}x = -\frac{4}{3}v^{3/2}(s_T + s_Y^2) + \frac{4}{5}v^{5/2} + \varepsilon^2\nu\frac{v_Y^2}{v^{3/2}} - \varepsilon^2\frac{v_Y^2}{4v}\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}x \ , \tag{4.19}$$

where

$$\nu = \frac{1}{6}\left(1 - \frac{\pi^2}{6}\right) .$$

The last term in Eq. (4.19) is diverging, and it is of order of $\mathrm{O}(\varepsilon^2)$. The appearance of such a secular term indicates the necessity to include a nonlocalized correction into a trial function for applying the averaged Lagrangian method. Thus, the self-focusing dynamics of a KdV soliton is accompanied by a strong interaction between the soliton and the radiation it induces. This fact was overlooked by Katyshev and Makhankov (1976; see also Makhankov, 1978) where the modulation equations for a KdV soliton were analyzed neglecting the diverging term of the averaged Lagrangian (4.19). As a result, the modulation equations led to wrong predictions, see discussions in Laedke and Spatschek (1979). As a matter of fact, the averaged Lagrangian (4.19) for a KdV soliton is only applicable in neglecting the terms of order of $O(\varepsilon^2)$ when this method reproduces the equations of the geometric optics approach. In some other (rather exotic) problems when the self-focusing dynamics of a long-wave soliton is not associated with the generation of a radiation field up to the order of $O(\varepsilon^2)$, the averaged Lagrangian method is as much effective as in the case of a bright soliton of the NLS equation. For example, this is the case of the soliton self-focusing described by the Benjamin–Ono equation (see Pelinovsky and Shrira, 1995).

4.3. *Gas dynamics equations*

A consistent asymptotic analysis requires neglecting the terms of order of $O(\varepsilon)$ or $O(\varepsilon^2)$ in the soliton modulation equations. Such an approximation corresponds to the dispersion-less limit of nonlinear evolution equations (Kamchatnov, 1997) when the results in the leading order of the asymptotic expansions coincides with the gas-dynamics equations. However, the self-focusing phenomenon is equivalent to the dynamics of an *unstable* gaseous medium (Trubnikov and Zhdanov, 1987) rather than to a standard evolution of a quasi-linear system described by characteristics (Whitham, 1974). In particular, the initial-value problem is elliptic (i.e., ill-posed) so that the characteristic velocities are complex. As a result, the Riemann method cannot be applied in such an unstable case. Instead, an analytical technique based on the so-called hodograph transformation allows us to construct exact analytical solutions to the gas-dynamics equations (Trubnikov and Zhdanov, 1987). These exact solutions describe the formation of singularities in the dispersion-less elliptic initial-value problem. Within the original problem, a singularity of the gas dynamics equations resembles an initial stage of a growth of a self-focusing spike along the front of a planar soliton.

Following Trubnikov and Zhdanov (1987), we reproduce here the analytical solutions for the gas-dynamics equations (4.6) and (4.12) in the limit $\varepsilon \to 0$ that describes the self-focusing dynamics of a planar soliton under the action of a periodic transverse modulation.

4.3.1. *Bright NLS solitons in the elliptic problem*

The exact analytical solution to Eqs. (4.6) in the limit $\varepsilon \to 0$ is obtained after the transformation $\omega = r^2$ and $\theta_Y = z$, where

$$r = r(\xi,\eta) = \frac{\sinh \xi}{(\cosh \xi - \cos \eta)}, \quad z = z(\xi,\eta) = -\frac{\sin \eta}{(\cosh \xi - \cos \eta)}, \tag{4.20}$$

and the variables ξ and η are implicitly related to T and Y as follows:

$$\Gamma T = \frac{\xi(\cos \eta - \cosh \xi)}{\sinh \xi}, \quad pY = \eta + \frac{\xi \sin \eta}{\sinh \xi}. \tag{4.21}$$

Here Γ is the instability growth rate in the dispersionless limit, i.e. $\Gamma = 2p$ [see Eq. (3.6)]. The parameters (ω, θ) of a bright soliton reproduce, for large negative time, a planar bright soliton with the boundary condition $\omega \to 1$ as $T \to -\infty$ (see Trubnikov and Zhdanov, 1987). The analytical solution $\omega = \omega(Y, T)$ expressed through Eq. (4.21) is shown in Fig. 2(a).It has the form of a bubble which becomes tighter and higher as $T \to 0$.

4.3.2. KdV solitons

Exact analytical solution to Eq. (4.12) in the limit $\varepsilon \to 0$ is obtained by means of the transformation $v = r^2$ and $s_Y = \sqrt{3}z$, where $r = r(\xi, \eta)$ and $z = z(\xi, \eta)$ are the same as in Eq. (4.20) while ξ and η are defined this time by the expressions

$$\Gamma T = -\frac{(\xi \coth \xi - 1)(\cosh \xi - \cos \eta)^2}{\sinh^2 \xi}\,, \qquad pY = \eta + \frac{\sin \eta}{\sinh \xi}\left(\frac{3(\xi \coth \xi - 1)(\cosh \xi - \cos \eta)}{\sinh \xi} - \xi\right), \tag{4.22}$$

where $\Gamma = (2/\sqrt{3p})$ [see Eq. (3.8)]. The analytical solution $v = v(Y, T)$ is shown in Fig. 2(b). It displays the development of a singularity similar to Fig. 2(a).

Thus, the gas-dynamics equations exhibit the singular solutions for the transverse soliton self-focusing. The principal further problem is to predict whether the singularities of the gas-dynamics equations still persist within the original evolution problem, or they are removed by taking into account a weak dispersion or dissipation. For example, the development of modulational instability described by the cubic NLS equation in one dimension displays a bounded scenario ("recurrence") (Yuen and Lake, 1986). Also, the characteristic features of the soliton self-focusing might be different for the nonlinear regime of the soliton dynamics. The

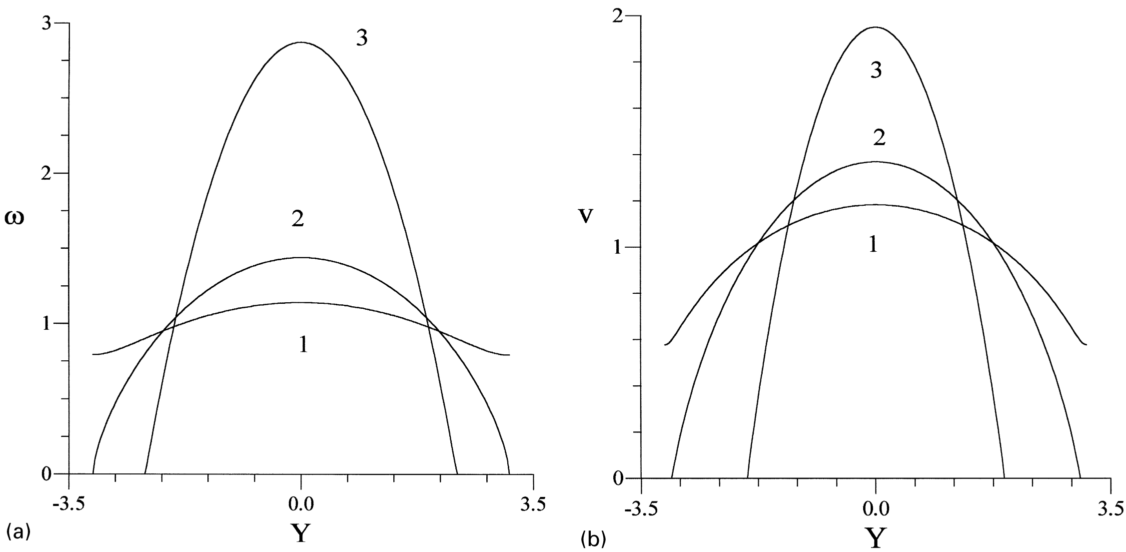


Fig. 2. (a) The analytical solution (4.21) describing the transverse self-focusing of a bright NLS soliton within the gas dynamics equations: $1 - T = -1.6$, $2 - T = -1.0$, and $3 - T = -0.4$. (b) The analytical solution (4.22) describing the transverse self-focusing of a KdV soliton for the same times as above.

aforementioned problems can be solved only within the modulation equations modified by dispersive or dissipative corrections.

4.4. Reduction to the NLS equation

The gas-dynamics equations are similar to the hydrodynamics form (3.21) of the NLS equation. Therefore, some predictions about the long-term dynamics of the soliton transverse instability may be extracted by deriving the effective power p of a nonlinear term in the gas-dynamics equations. In some exceptional cases, further remarkable reductions to the NLS equation (1.1) may also take place.

4.4.1. Bright NLS solitons in the elliptic problem

It was first realized by Degtyarev et al. (1975) that the truncated modulation equations (4.6) are nothing but the hydrodynamics form (3.21) of the quintic NLS equation (1.1) for $r = 2$. Indeed, the corresponding transformation is given by

$$\varphi(x,t) = \left(\frac{\omega}{3}\right)^{1/4} \exp\left(\frac{i\theta}{4\sqrt{\mu\varepsilon}}\right),$$

where the new variables are

$$x = \frac{Y}{4\sqrt{\mu\varepsilon}}, \quad t = \frac{T}{4\sqrt{\mu\varepsilon}} .$$

Such a reduction leads to *several important consequences*. First of all, a steady-state periodic solution to Eq. (1.1) resembles a steady-state transversely modulated soliton in two dimensions, while the soliton solution to Eq. (1.1) approximates a two-dimensional soliton in Eq. (2.1). Second, it is well-known that the steady-state solutions are weakly unstable for the quintic NLS equation and possess singularities associated with the so-called *critical collapse* (Rasmussen and Rypdal, 1986). The latter feature enables us to predict that the self-focusing of a planar bright soliton within the elliptic NLS equation leads to the formation of two-dimensional localized modes along the soliton front which then collapse according to the critical NLS equation. A self-similar form of each individual localized mode is described by an asymptotic solution of Eq. (1.1) at $r = 2$ (Fibich and Papanicolaou, 1998; Pelinovsky, 1998). We express this solution in terms of the parameters of a bright NLS soliton,

$$\omega \sim 3\Omega\,\mathrm{sech}^2\left(\frac{\sqrt{\Omega}}{2\mu\varepsilon} Y\right), \quad \theta \sim \int_0^T \Omega(T')\,\mathrm{d}T' , \tag{4.23}$$

where the scaling law for the singularity formation modified by the radiation-induced factor is the following:

$$\Omega(T) \to \frac{\log|\log(T_0 - T)|}{(T_0 - T)} \quad \text{as } T \to T_0$$

(see also Fraiman, 1985; Malkin, 1990). It is well-known that the two-dimensional elliptic NLS equation (2.1) for $\sigma_n = \sigma_d = +1$ also possesses the critical collapse dynamics. Thus, the reduced modulation equations (4.6) preserve this principal property of the model (2.1). We exploit this analogy further in Section 7.1.1, where some generalizations of the NLS equation are considered.

4.4.2. KdV solitons

Exact reduction of the modulation equations (4.12) to a NLS model cannot be verified because these equations have dissipative rather than dispersive perturbative terms. However, in the leading order, the modulation equations (4.12) coincide with the dispersionless limit ($\varepsilon \to 0$) of the NLS equation (1.1) for $r = 2/3$. The corresponding transformation is given by the relation

$$\varphi(x,t) = \left(\frac{3}{5}v\right)^{3/4} \exp\left(\frac{i\lambda s}{\varepsilon}\right) ,$$

where $x = \lambda Y/\varepsilon$, $t = \lambda T/\varepsilon$, and λ is arbitrary. As is well-known, the NLS equation (1.1) with $r = 2/3$ is *subcritical* (Rasmussen and Rypdal, 1986), i.e. it does not display any singularity dynamics or any instability of steady-state localized solution. Therefore, we can expect that the self-focusing of a KdV soliton is suppressed by the nonlinear and dissipative effects and displays a bounded scenario of the soliton evolution. However, this hypothesis can only be checked within the extended modulation equations.

4.5. Higher-order perturbation theory

Here we extend the asymptotic analysis to include higher-order approximations for the cases of the bright NLS and KdV solitons. To do so, we construct a self-consistent solution to the linear equation of the first-order perturbation theory. This perturbation term determines distortions of the soliton shape due to the self-focusing dynamics as well as the structure of radiation emitted outside the soliton core. Depending on the type of the radiation emitted, the modulation equations include either dispersive or dissipative terms. We show that a balance between the dispersive and dissipative effects depends on properties of a nonlinear system under consideration.

4.5.1. Bright NLS solitons in the elliptic problem

For a systematic asymptotic procedure we set

$$\begin{aligned} &\omega_T + 2\omega_Y\theta_Y + 4\omega\theta_{YY} = \varepsilon^2\bar{\omega}_T \ , \\ &\theta_T + \theta_Y^2 - \omega = \varepsilon^2\bar{\theta}_T \ . \end{aligned} \tag{4.24}$$

Then, neglecting in Eq. (4.4) the terms of order of $O(\varepsilon)$, we find the first-order perturbation term induced due to the soliton adiabatic dynamics, $\phi_1 = u_1(x;Y,T) + iw_1(x;Y,T)$, where $u_1 = 0$ and

$$w_1 = \tfrac{1}{2}\theta_{YY}x^2\Phi_b(x;\omega) \ . \tag{4.25}$$

Generally speaking, a solution of an initial-value problem associated with Eq. (4.4) is decomposed through the wave packets of the continuous spectrum which may evolve also at the fast time scale t. Here we have supposed that the induced wave packets are self-consistent with the adiabatic self-focusing dynamics of a solitary wave. The first-order perturbation (4.25) to the soliton shape (2.2) represents a quadratically growing chirp of a complex phase of ψ [see Eq. (4.3)]. The phase chirp is responsible for radiation emitted by a bright NLS soliton but such a radiation is exponentially small in ε (Pelinovsky, 1998). Therefore, this effect is negligible for the dynamics of a self-focusing bright soliton. Then, one can proceed to higher orders where the linear equations for $\phi_2 = u_2(x; Y, T)$ and $\phi_3 = \mathrm{i}w_3(x; Y, T)$ are defined as follows:

$$\begin{aligned} -\hat{\mathscr{L}}_1 u_2 &= \bar{\theta}_T \Phi_b + w_{1T} + 2\theta_Y w_{1Y} + \theta_{YY} w_1 - \Phi_{bYY} - 2\Phi_b w_1^2 , \\ -\hat{\mathscr{L}}_0 w_3 &= -\bar{\omega}_T \frac{\partial \Phi_b}{\partial \omega} + \bar{\theta}_T w_1 - u_{2T} - 2\theta_Y u_{2Y} - \theta_{YY} u_2 - w_{1YY} - 4\Phi_b w_1 u_2 - 2w_1^3 , \end{aligned} \tag{4.26}$$

where $\hat{\mathscr{L}}_0$ and $\hat{\mathscr{L}}_1$ are the same as in Eqs. (3.3). Using the orthogonality conditions (4.5) and Eqs. (4.24), we extend the modulation equations for the soliton self-focusing to include the second-order effects. After a simple algebra, the generalized modulation equations for the soliton parameters can be written in the form,

$$\begin{aligned} &\omega_T + 2\omega_Y \theta_Y + 4\omega\theta_{YY} + \varepsilon^2 \left[\mu_2 \left(\theta_{YYYY} - \frac{\omega_Y \theta_{YYY}}{2\omega} \right) + \mu_4 \left(\frac{\theta_{YY}^3}{\omega} + \frac{\omega_{YY}\theta_{YY}}{2\omega} \right) \right] = 0 , \\ &\theta_T + \theta_Y^2 - \omega + \varepsilon^2 \left(3\mu \frac{\omega_Y^2}{\omega^2} - \frac{(16\mu + \mu_2)}{4\omega} \omega_{YY} - \frac{\mu_4}{4\omega} \theta_{YY}^2 \right) = 0 , \end{aligned} \tag{4.27}$$

where μ is the same as in Eqs. (4.6), $\mu_2 = (\pi^2/6)$, and $\mu_4 = (7\pi^4/120)$. The $O(\varepsilon^2)$ terms describe completely the dispersive effects of the soliton dynamics while the dissipative effects are negligible (exponentially small) in terms of ε. The modulation equations (4.27) represent an extended version of the modulation equations (4.6) of the first-order asymptotic theory.

4.5.2. KdV solitons

We apply the leading-order system (4.12) and find the explicit form of the first-order perturbation term $u_1 = u_1(\xi; Y, T)$, where

$$u_1 = \frac{s_{YY}}{6\sqrt{v}} \frac{\partial^2}{\partial \xi^2} [\xi^2 \tanh(\sqrt{v}\xi)] + a\left(1 - 2\frac{\partial U}{\partial v}\right) . \tag{4.28}$$

It follows from Eq. (4.28) that the perturbation term u_1 is not localized at infinity, i.e. $u_1 \to u^\pm$ as $\xi \to \pm\infty$. This indicates the generation of a radiation tail similar to that which appears in the instability-induced dynamics of a KdV soliton (Pelinovsky and Grimshaw, 1996). Moreover, the second-order perturbation term u_2 also grows at infinity, i.e. $u_2 \sim u_X^\pm \xi$ as $\xi \to \pm\infty$. Proceeding to the second-order approximation, we find the linear equation for $u_2 = u_2(\xi; Y, T)$,

$$(\hat{\mathscr{L}} u_2)_\xi = 4\left(vb + \bar{v}_T \frac{\partial U}{\partial v} + u_{1T} + 2s_Y u_{1Y} + s_{YY} u_1 - a u_{1\xi} - \int_0^\xi U_{YY} \, \mathrm{d}\xi + 3u_1 u_{1\xi} \right) , \tag{4.29}$$

where $\hat{\mathscr{L}}$ is given after Eq. (3.5), $b = b(Y, T)$ is another integration constant, and $\bar{v}_T$ is the extension of v_T to the next order. Then, applying the orthogonality condition (4.11) to Eq. (4.29), we extend the modulation equations for a KdV soliton to the order of $O(\varepsilon)$,

$$\begin{aligned} &v_T + 2v_Y s_Y + \tfrac{4}{3} v s_{YY} + \varepsilon(2vb - \tfrac{2}{3} a s_{YY}) = 0 , \\ &s_T + s_Y^2 - v - \varepsilon a = 0 . \end{aligned} \tag{4.30}$$

Parameters a and b define the profile of the asymptotic series (4.10) extended outside the soliton core, $u = \varepsilon u^\pm(X, Y, T)$, where $X = \varepsilon x$. The boundary conditions for the radiation fields $u^\pm$

calculated at the soliton position, i.e. at $X = s(Y,T)$, are given by the matching conditions,

$$\begin{aligned} u^{\pm}|_{X=s} &= a \pm \frac{s_{YY}}{3\sqrt{v}}\,, \\ u^{\pm}_X|_{X=s} &= b + \frac{1}{v}(a_T + 2s_Y a_Y + s_{YY} a) \mp \frac{1}{v^{3/2}}\left(\frac{1}{9}s_{YY}^2 + \frac{1}{6}v_{YY} - \frac{1}{4}\frac{v_Y^2}{v}\right). \end{aligned} \tag{4.31}$$

Outside the soliton core, the function $u^{\pm}(X,Y,T)$ satisfies the reduced wave equation in two dimensions,

$$u^{\pm}_{TX} = u^{\pm}_{YY}\ . \tag{4.32}$$

The boundary conditions (4.31) supplemented by system (4.30) determine the radiation fields $u^{\pm}$ propagating according to the wave equations (4.32). A proper solution of this (radiation) problem closes the system (4.30) by additional relations between the parameters a and b and the soliton parameters s and v. We solve this problem in a small-amplitude approximation (see Section 5.2.3). In Eqs. (4.30), the terms of order of $O(\varepsilon)$ describe the radiation-induced dissipative effects in the soliton dynamics while the dispersive effects [of order of $O(\varepsilon^2)$] are beyond this approximation.

4.6. Different scenarios of soliton self-focusing

As has been discussed above, the extended modulation equations include either dispersive or dissipative effects in the effective elliptic-type asymptotic problem. The solution of those equation in the form of a (unstable) c.w. mode always corresponds to an unperturbed planar soliton. For example, the modulation equations (4.6) and (4.9) have the c.w. solution, $\omega = \omega_0$ and $\theta = \omega_0 T + \theta_0$, where ω_0 and θ_0 are constants; this solution corresponds to a planar bright NLS soliton. On the other hand, Eqs. (4.12) and (4.15) have the c.w. solution $v = v_0$ and $s = v_0 T + s_0$ corresponding to unperturbed KdV and dark solitons.

Although the gas-dynamics equations always lead to a formation of singularities for elliptically unstable problems, the higher-order dispersive or dissipative effects may suppress an exponential growth of modulations. Depending on the type of higher-order effects, we can distinguish, in general, *four different types* (or *scenarios*) of the instability-driven soliton self-focusing dynamics:

- *wave collapse* or formation of a chain of two-dimensional localized singularities along the front of a planar soliton;
- *monotonic transition* from a planar soliton to a periodic chain of two-dimensional solitons;
- *breakup* of a planar soliton into *localized states* which gradually decay due to the action of the wave dispersion;
- *quasi-recurrence*, i.e. a periodic growth and damping of transverse modulations along the front of a planar soliton.

Unfortunately, the extended modulation equations that describe all such types of the solitons self-focusing dynamics, can be usually solved only numerically. As an exception, some exact solutions for the soliton self-focusing can be found in an explicit analytical form, if an original nonlinear equation is integrable by the inverse scattering transform method (see Section 4.7). Nevertheless, the extended modulations equations for the soliton parameters can be analyzed effectively by employing small-amplitude asymptotic expansions summarized below in Sections 5 and 6. Applying those methods, we are able to demonstrate that the bright NLS solitons of the elliptic NLS equation display the first scenario of the soliton self-focusing, the KdV solitons decay according to the second scenario, and the bright NLS solitons in the hyperbolic NLS equation follow the third scenario of the soliton self-focusing. Dark solitons, depending on their initial parameters, can display either second or fourth scenarios of the soliton instability-induced decay.

4.7. Exact solutions for the soliton self-focusing

Some nonlinear evolution equations describing the soliton self-focusing in two or more dimensions are solvable by means of the inverse scattering transform method (e.g., Ablowitz and Segur, 1981). These so-called *integrable models* possess a rich functional structure of explicit solutions that allow to describe, in some particular cases, both linear and nonlinear regimes of the soliton self-focusing (see, e.g., Zakharov, 1975; Kuznetsov et al., 1984; Pelinovsky and Stepanyants, 1993; Allen and Rowlands, 1997). Here we present exact solutions for the soliton self-focusing obtained in the framework of the KP equation (2.6). Analytical solution of the KP equation can be expressed in a bilinear form,

$$u(x,y,t) = \partial_x^2 \log \tau \ , \tag{4.33}$$

where the function $\tau = \tau(x,y,t)$ possesses a determinant representation (Pelinovsky and Stepanyants, 1993). We discuss here only a scalar reduction of this solution defined by the following expression for τ:

$$\tau = 1 + \int_{-\infty}^{x} |\phi|^2 \, dx \ , \tag{4.34}$$

where the function $\phi = \phi(x,y,t)$ satisfies the system of linear equations,

$$\frac{2}{\sqrt{3}} i\phi_y + \phi_{xx} = 0, \quad \phi_t + \phi_{xxx} = 0 \ .$$

In a particular case when $\phi \sim \exp[\sqrt{v}x + i(\sqrt{3}v/2)y - \sqrt{v^3}t]$, the solution given by Eqs. (4.33) and (4.34) reproduces a single KdV soliton (2.7). To find more general solutions, we select $\phi(x,y,t)$ in the form of two exponentials,

$$\phi = (2\sqrt{v})^{1/2} \exp\left(\sqrt{v}x + \frac{i\sqrt{3}v}{2}y - \sqrt{v^3}t\right) + (2\kappa)^{1/2} a \exp\left(\kappa x + \frac{i\sqrt{3}}{2}\kappa^2 y - \kappa^3 t\right) , \tag{4.35}$$

where v, κ, and a are real parameters. Then, the corresponding exact solution of the KP equation has the form (Zakharov, 1975)

$$\tau(x,y,t) = 1 + e^{2\sqrt{v}(x-vt)} + 4a\frac{(\kappa\sqrt{v})^{1/2}}{(\kappa+\sqrt{v})} e^{(\kappa+\sqrt{v})(x-vt)+\Gamma t} \cos(py) + a^2 e^{2\kappa(x-vt)+2\Gamma t} \ , \tag{4.36}$$

where Γ and p are defined as

$$p = \frac{\sqrt{3}}{2}(v - \kappa^2), \quad \Gamma = \kappa(v - \kappa^2) .$$

In the limit $a \to 0$, a Taylor expansion of Eqs. (4.33) and (4.36) leads to the exact solution (3.14) of the linear eigenvalue problem (3.5) obtained for the KP equation. Exact solution (4.36) for finite values of the parameter a describes a monotonic transition ('splitting') of a planar soliton moving initially with the velocity v into a chain of two-dimensional KP solitons, the so-called *lumps*, propagating with the velocity $v_1 = v + \sqrt{v\kappa} + \kappa^2$, and a complimentary small-amplitude planar KdV soliton moving with a small velocity $v_2 = \kappa^2$.

Numerical simulations (Infeld et al., 1994, 1995) completely confirmed this result as the basic physical mechanism of a decay of a plane KdV soliton in two dimensions. However, in three dimensions the two-dimensional lump solitons are known to be *unstable* and, moreover, the instability of three-dimensional azimuthally symmetric solitons were found to lead to gradual collapse (Kuznetsov et al., 1983; Kuznetsov and Musher, 1986). Recent numerical simulations (Senatorski and Infeld, 1998) confirmed the scenario of the graduate collapse and display steepening, narrowing, and slow disintegration of two-dimensional solitons in higher dimensions.

5. Long-scale approximation

Long-scale small-amplitude asymptotic approximation is based on the assumption that the amplitude of the long-scale transverse perturbation remains small compared to the amplitude of a planar (unperturbed) soliton. This approximation simplifies the extended modulation equations derived in Section 4 and it allows to reduce them to a number of universal asymptotic equations (Section 5.1). The main objective for this reduction is to improve the predictions of the gas dynamics equations by describing a balance between weak nonlinear (e.g. quadratic or cubic) and linear (e.g. dispersive or dissipative) terms of the extended modulation equations (Section 5.2). Then, the asymptotic equations can be employed for constructing analytical solutions describing the development of the transverse instability in the case of periodic modulations of the soliton front (Section 5.3). Extensions of these equations can also be derived for the problems, where some of the coefficients vanish in the small-amplitude limit (Section 5.4).

5.1. Basic asymptotic equations

Two universal asymptotic equations appear within the long-scale small-amplitude approximation depending on a type of a balance between the dispersive and dissipative effects. If the dispersive effects are dominant over the dissipative ones, the small-amplitude approximation reduces the extended modulation equations to *the elliptic Boussinesq equation* (see, e.g., Pelinovsky and Shrira, 1995),

$$\Theta_{TT} + \alpha\Theta_{YY} + \varepsilon^2(\beta\Theta_{YYYY} + \gamma_1\Theta_Y\Theta_{YT} + \gamma_2\Theta_T\Theta_{YY}) + O(\varepsilon^4) = 0 , \tag{5.1}$$

where the coefficients α and β are determined from a linear approximation and they are positive within the instability band (see Section 3.2), and the coefficients γ_1 and γ_2 are responsible for the effect of quadratic nonlinearity. In the opposite case, when the dissipative terms are dominating, the small-amplitude approximation results in *the elliptic Shrira–Pesenson equation* (see Shrira and Pesenson, 1983),

$$S_{TT} + \alpha S_{YY} + \varepsilon(-\beta S_{TYY} + \gamma_1 S_Y S_{YT} + \gamma_2 S_T S_{YY}) + \mathrm{O}(\varepsilon^2) = 0 \ , \tag{5.2}$$

where the coefficients α, β, γ_1, and γ_2 have the same meaning as in Eq. (5.1). The hyperbolic analogue of the governing equation (5.2) was first derived by Shrira and Pesenson (1983) (see also Pesenson, 1991) for describing the weakly nonlinear dispersive waves propagating along a transversely stable planar soliton.

The reduction to a linear problem can be made by substituting Θ, $S \sim e^{\Gamma t + ipY}$, where the dependence $\Gamma = \Gamma(p)$ has the form,

$$\Gamma^2 = \alpha p^2 - \varepsilon^2 \beta p^4 \ , \tag{5.3}$$

for Eqs. (5.1), and

$$\Gamma^2 = \alpha p^2 - \varepsilon \beta p^2 \Gamma \ , \tag{5.4}$$

for Eq. (5.2), respectively. For some problems, the coefficients γ_1 and γ_2 in Eqs. (5.1) and (5.2) may vanish. For example, this occurs for bright solitons in the hyperbolic NLS equation and for dark solitons in the limit of zero velocities (the so-called 'black solitons'). Then, the asymptotic equations should include higher-order (namely, cubic) nonlinear terms (see Section 5.4).

5.2. Derivation

Equations for the long-scale modulations have constant-background solutions corresponding to an unperturbed planar soliton. Then, the purpose of the small-amplitude asymptotic expansion technique is to derive a nontrivial evolution equation governing the evolution of small-amplitude modulations of the background solution. Such evolution equations are valid for certain time intervals when the amplitude of the perturbation mode becomes comparable with the background amplitude. Since the instability leads to a growth of the amplitude, the applicability of the small-amplitude evolution equations is generally limited by a certain time interval. Nevertheless, we show that the results obtained in the small-amplitude approximation give a surprising good agreement with the results that can be obtained by some other methods for analyzing modulational and self-focusing instabilities.

5.2.1. Modulational instability

We start with the hydrodynamical form (3.21) of the NLS equation (1.1) and set the asymptotic scaling,

$$\theta = (r+1) q_0^r T + \varepsilon^2 \Theta(X, T) \ .$$

Then, the kinematic equation of system (3.21) defines the asymptotic expansion for $q(X,T)$,

$$q = q_0 + \frac{\varepsilon^2}{r(r+1)q_0^{r-1}}\Theta_T + \varepsilon^4\left[\frac{1}{r(r+1)q_0^{r-1}}\Theta_X^2 - \frac{r-1}{2r^2(r+1)^2q_0^{2r-1}}\Theta_T^2 - \frac{1}{2r^2(r+1)^2q_0^{2r-1}}\Theta_{TXX}\right] + \mathrm{O}(\varepsilon^6)\,. \quad (5.5)$$

This expansion describes small-amplitude long-scale perturbations propagating along the constant background $q = q_0$. It follows from the first equation of system (3.21) that the function $\Theta = \Theta(X,T)$ satisfies the elliptic Boussinesq equation (5.1) (written for the variables X and T) with the coefficients

$$\alpha = 2r(r+1)q_0^r, \quad \beta = 1, \quad \gamma_1 = 4, \quad \gamma_2 = 2r\,. \quad (5.6)$$

Linear dispersion relation (5.3) with these coefficients corresponds to the linear relation (3.19) for $k = 0$ and $q_0 = \rho^2$.

5.2.2. Bright NLS solitons in the elliptic problem

We impose the asymptotic expansion for the extended modulation equations (4.27),

$$\begin{aligned}\theta &= \omega_0 T + \varepsilon^2\Theta(Y,T)\,,\\ \omega &= \omega_0 + \varepsilon^2\Theta_T + \varepsilon^4\left(\Theta_Y^2 - \frac{(16\mu + \mu_2)}{4\omega_0}\Theta_{TYY}\right) + \mathrm{O}(\varepsilon^6)\,.\end{aligned} \quad (5.7)$$

This expansion describes small-amplitude long-scale disturbances at a planar soliton background with the propagation constant $\omega = \omega_0$. It follows from Eqs. (4.27) and (5.7) that the function $\Theta(Y,T)$ satisfies asymptotically the elliptic Boussinesq equation (5.1) with the coefficients

$$\alpha = 4\omega_0, \quad \beta = \frac{4}{3}\left(1 + \frac{\pi^2}{3}\right), \quad \gamma_1 = 4, \quad \gamma_2 = 4\,. \quad (5.8)$$

The linear part (5.3) reproduces the dispersion relation (3.6). The instability region is limited by a critical wave number $p = p_c = \sqrt{3\omega_0}$. Expansion (3.6) however gives an approximation for p_c beyond the applicability of this expansion, i.e. $\varepsilon p_c = 3\sqrt{\omega_0}/\sqrt{3+\pi^2} \approx 0.836\sqrt{\omega_0}$. Nevertheless, the existence of the instability band cut off is described by the dispersive term of the Boussinesq equation (5.1), and therefore it provides an improved version of the elliptic gas dynamics equations, where the instability band is not bounded.

We mention that the same small-amplitude approximation can be applied to the truncated version of the modulation equations (4.6) and it also results in the elliptic Boussinesq equation (5.1) but with the numerical constant β replaced by 16μ.

5.2.3. KdV solitons

We start with the extended modulation equations (4.30) for a KdV soliton and the associated radiation problem (4.31) and (4.32) and impose the asymptotic expansion, $s = v_0 T + \varepsilon S(Y,T)$, $a = \varepsilon A(Y,T)$, $b = \varepsilon B(Y,T)$, $u^{\pm} = \varepsilon U^{\pm}(X,Y,T)$, and $v = v_0 + \varepsilon S_T + \varepsilon^2(S_Y^2 - A) + \mathrm{O}(\varepsilon^3)$. For small-amplitude expansions, system (4.30) reduces to a single equation,

$$S_{TT} + \frac{4v_0}{3}S_{YY} + \varepsilon\left(4S_Y S_{YT} + \frac{4}{3}S_T S_{YY} - A_T + 2v_0 B\right) + \mathrm{O}(\varepsilon^2)\ . \tag{5.9}$$

In order to connect the parameters A and B with the parameter S, we solve the problem for the radiation field. In the reference frame propagating with the soliton velocity $v = v_0$ wave equation (4.32) is rewritten as

$$U^{\pm}_{TX}|_{X=v_0 T} = v_0 U^{\pm}_{XX}|_{X=v_0 T} + U^{\pm}_{YY}|_{X=v_0 T}\ . \tag{5.10}$$

Now we connect the Y-derivatives with the T-derivatives using the leading order of modulation equation (5.9), i.e.

$$U^{\pm}_{YY}|_{X=v_0 T} = -\frac{3}{4v_0}U^{\pm}_{TT}\bigg|_{X=v_0 T}\ .$$

Then, two characteristic velocities for the radiative fields can be found from Eq. (5.10), i.e. $U^{\pm} = U^{\pm}(X - v_0 T - \lambda_{\pm} T)$, where $\lambda_+ = 2v_0$ and $\lambda_- = -(2v_0/3)$. Thus, we come to the conclusion that the radiation field U^+ in front of a KdV soliton propagates with the velocity equal to a triple soliton velocity in a laboratory reference frame, while the radiation field U^- behind the soliton moves with the third of the soliton velocity. This explicit solution of the wave emission problem results in the differential relations for the boundary values (4.31),

$$U^{\pm}_{T}|_{X=v_0 T} = -\lambda_{\pm} U^{\pm}_{X}|_{X=v_0 T}\ . \tag{5.11}$$

Substituting Eq. (4.31) taken into the small-amplitude approximation into Eqs. (5.11), we define the parameters A and B in terms of S,

$$A = \frac{1}{3\sqrt{v_0}}S_{YY}, \quad B = -\frac{1}{2\sqrt{v_0^3}}S_{YYT}\ . \tag{5.12}$$

It follows from Eqs. (4.31) and (5.12) that the radiation field behind the KdV soliton is not excited along the characteristics λ_-, i.e. $U^- \equiv 0$, while the radiation field in front of the KdV soliton is generated, and it can be determined from the boundary condition,

$$U^+|_{X=v_0 T} = \frac{2}{3\sqrt{v_0}}S_{YY}\ . \tag{5.13}$$

We notice that the radiation induced due to the long-scale soliton self-focusing differs from a typical adiabatic evolution of a KdV soliton, when the radiation is emitted behind the KdV soliton (Pelinovsky and Grimshaw, 1996). The relations (5.12) enable us to close the asymptotic equation (5.9) for the function $S(Y,T)$ and reduce it to the elliptic Shrira–Pesenson equation (5.2) with the coefficients

$$\alpha = \frac{4v_0}{3}, \quad \beta = \frac{4}{3\sqrt{v_0}}, \quad \gamma_1 = 4, \quad \gamma_2 = \frac{4}{3}\ . \tag{5.14}$$

The linear part (5.4) reproduces the result of the linear stability analysis (3.8). If we use the approximation $\Gamma \approx 2\sqrt{v_0}p/\sqrt{3}$ to simplify the last term in Eq. (3.8), then this expansion coincides with the exact result (3.15). The instability band is limited by the transverse wave numbers, $p < p_c$, where $\varepsilon p_c = \sqrt{3} v_0/2$.

5.2.4. Dark NLS solitons

In small-amplitude limit, a dark NLS soliton transforms into a KdV soliton. As a result, the elliptic Shrira–Pesenson equation (5.2) can be derived for a dark soliton of a finite amplitude. In order to derive this equation, we use the asymptotic representation (4.13) together with the small-amplitude expansion, $s = v_0 T + \varepsilon S(Y,T)$, $\theta = \varepsilon \Theta(Y,T)$, and $v = v_0 + \varepsilon S_T - 2\varepsilon^2 v_0 S_Y^2 + \mathrm{O}(\varepsilon^3)$. The first-order correction $\phi_1(v; Y, T)$ can be found from Eq. (4.14) within the small-amplitude approximation,

$$\phi_1 = Q\left(\mathrm{i}\xi\Phi_d - \frac{\partial\Phi_d}{\partial v} + v_0\frac{\partial\Phi_d}{\partial\rho^2}\right) - \frac{\Theta_T}{4k}\left(\tanh(k\xi) + \frac{k\xi}{\cosh^2(k\xi)}\right) - 2S_Y^2\frac{k^2\xi}{\cosh^2(k\xi)}$$
$$+ S_{YY}\left(-\frac{2}{3}v_0\xi\frac{\partial\Phi_d}{\partial v} - \frac{1}{3}v_0^2\frac{\xi^2}{\cosh^2(k\xi)} + \frac{k^2 - v_0^2}{3kv_0}\mathrm{i}\tanh(k\xi)\right),$$

where $\xi = x - 2v_0 t - 2S(Y,T)$, $k = (\rho^2 - v_0^2)^{1/2}$, and $Q = Q(Y,T)$ is a parameter associated with radiation. The analysis shows (see Pelinovsky et al., 1995a, for details) that $S = S(Y,T)$ satisfies the extended evolution equation,

$$S_{TT} + \frac{4}{3}k^2 S_{YY} + \varepsilon\left[-8v_0 S_Y S_{YT} - \frac{8v_0}{3}S_T S_{YY} + v_0\Theta_{YY} - \frac{1}{2}Q_T\right] + \mathrm{O}(\varepsilon^2)\,, \tag{5.15}$$

where Θ and Q are related to S through a solution of the radiation problem. We notice that the first linear and quadratic terms of Eq. (5.15) can also be obtained from Eq. (4.15) within the small-amplitude approximation. The radiation field, $\psi \to \psi^{\pm}(X,Y,T)$ as $\xi \to \pm\infty$ and $X = \varepsilon x \sim \mathrm{O}(1)$, can be found in the asymptotic form (2.5), where the boundary conditions for $u^{\pm}$ and $R^{\pm}$ are defined at the soliton $X = 2v_0 T$,

$$\begin{aligned} u^{\pm}|_{X=2v_0T} &= -\frac{1}{\rho}\left(v_0 Q - \frac{1}{2}\Theta_T \pm \frac{2(k^2 - v^2)}{3k}S_{YY}\right), \\ R_X^{\pm}|_{X=2v_0T} &= Q \mp \frac{2v}{3k}S_{YY}\,. \end{aligned} \tag{5.16}$$

Outside the soliton, the radiation field $u^{\pm} = u^{\pm}(X,Y,T)$, where $X = \varepsilon x$, satisfies the scalar wave equations,

$$u_{TT}^{\pm} - 4\rho^2(u_{XX}^{\pm} + u_{YY}^{\pm}) = 0\,. \tag{5.17}$$

Using the analysis similar to the case of KdV solitons, we find from Eq. (5.17) that the radiation fields are generated along the characteristic directions $u^{\pm} = u^{\pm}(X - 2v_{\pm}T, Y)$, where

$$v_{\pm} = \frac{\pm 2\rho(\rho^2 - v_0^2) + 3\rho^2 v_0}{4\rho^2 - v_0^2}\,, \tag{5.18}$$

Together with Eqs. (5.16), this relation allows us to find the parameters Θ and Q as follows:

$$\Theta = -\frac{v_0}{\rho k} S_T, \quad Q = \frac{2(\rho^2 + k^2)}{3\rho k} S_{YY} . \tag{5.19}$$

As a result, we derive from Eq. (5.15) the elliptic Shrira–Pesenson equation (5.2) with the coefficients,

$$\alpha = \frac{4}{3}(\rho^2 - v_0^2), \quad \beta = \frac{2(\rho^2 + v_0^2)}{3\rho\sqrt{\rho^2 - v_0^2}}, \quad \gamma_1 = -8v_0, \quad \gamma_2 = -\frac{8}{3}v_0 . \tag{5.20}$$

The linear part (5.4) corresponds to the relation (3.9). In addition, we find from Eqs. (5.16) and (5.19) the boundary conditions for the radiation fields,

$$u^{\pm}|_{X = 2v_0 T} = \frac{2[-\rho v_0 \pm (2v_0^2 - \rho^2)]}{3\rho\sqrt{\rho^2 - v_0^2}} S_{YY} . \tag{5.21}$$

Thus, in contrast to the evolution of an unstable KdV soliton, the radiation fields are excited in both directions. In the limit $v_0 \to -\rho + \frac{1}{2}\tilde{v}_0$ and $S \to \frac{1}{2}\tilde{S}$, results coincide with those for a KdV soliton at $\rho = 1$, so that the field u^- vanishes.

5.3. Analysis

Asymptotic reduction either to the elliptic Boussinesq equation (5.1) or to the Shrira–Pesenson equation (5.2) helps us to construct explicit asymptotic solutions for describing the transverse soliton self-focusing. We mention that the Boussinesq equation (5.1) can be transformed, within the same asymptotic approximation, to the integrable Boussinesq equation, where the exact solutions can be found by regular methods (see Pelinovsky and Shrira, 1995, and references therein). However, there exists a universal method that allows to construct these asymptotic solutions. This method does not use the properties of integrability but, instead, reduce the elliptic equations to two partial ("one-wave") equations by means of the Laplace (complex) coordinates. We apply this method to both Eqs. (5.1) and (5.2) as well as to their extentions.

5.3.1. The Boussinesq equation

The Laplace coordinates are defined by the relations,

$$z = T + i\alpha^{-1/2} Y, \quad \bar{z} = T - i\alpha^{-1/2} Y . \tag{5.22}$$

Then, the function $\Theta = \Theta(Y, T)$ can be expanded into an asymptotic series,

$$\Theta = \Upsilon(z, \tau) + \bar{\Upsilon}(\bar{z}, \tau) + \varepsilon^2 \Theta_2(z, \bar{z}, \tau) + O(\varepsilon^4) , \tag{5.23}$$

where $\tau = \varepsilon^2 T$ and $\bar{\Upsilon}$ is a complex conjugate to Υ. The straightforward asymptotic analysis reduces Eq. (5.1) to the KdV equation for the function $\Upsilon(z, \tau)$,

$$\Upsilon_{z\tau} - \frac{(\gamma_1 + \gamma_2)}{2\alpha} \Upsilon_z \Upsilon_{zz} + \frac{\beta}{2\alpha^2} \Upsilon_{zzzz} = 0 . \tag{5.24}$$

The complex conjugated function $\bar{\Upsilon}$ satisfies the same equation (5.24), while the correction term Θ_2 can be explicitly found as the following:

$$\Theta_2 = \frac{(\gamma_2 - \gamma_1)}{4\alpha}(\Upsilon\bar{\Upsilon}_z + \Upsilon_z\bar{\Upsilon}) . \tag{5.25}$$

The KdV equation (5.24) has a family of soliton solutions (Ablowitz and Segur, 1981). In particular, a one-soliton solution for $\Upsilon(z,\tau)$ reproduces, through Eq. (5.23), an asymptotic solution describing the transverse soliton self-focusing under the action of a periodic perturbation,

$$\Theta = -\frac{12\beta p}{\sqrt{\alpha}(\gamma_1 + \gamma_2)} \frac{\sinh(\Gamma T)}{[\cosh(\Gamma T) - \cos(pY)]} , \tag{5.26}$$

where Γ is the linear instability growth rate (5.3). In application to the *bright NLS solitons in the elliptic problem*, the asymptotic solution (5.26) describes the appearance of a periodic chain of large-amplitude (singular) localized modes along the planar soliton front. The corresponding analytical solution $\omega = \omega(Y,T)$ is expressed through Eqs. (5.7) and (5.26) and it is shown in Fig. 3(a).The variable $\psi(x,y,t)$ [see Eq. (4.3)] for the elliptic NLS equation is displayed in Fig. 3(b).

Thus, the asymptotic solution is singular as $T \to 0$ at the center of a self-focusing region. This singular structure differs quantatively from that predicted by the elliptic gas dynamics equations [see Eq. (4.21) or Fig. 2(a)] because the latter equations are limited by a shorter time scale. The formation of collapsing two-dimensional modes along the front of a planar bright soliton was numerically confirmed for the elliptic NLS equation (Degtyarev et al., 1975; Litvak et al., 1991).

5.3.2. The Shrira–Pesenson equation

We introduce the same Laplace coordinates z and $\bar{z}$ as in Eq. (5.22) and expand $S = S(Y,T)$ into the asymptotic series,

$$S = \Upsilon(z,\tau) + \bar{\Upsilon}(\bar{z},\tau) + \varepsilon S_1(z,\bar{z},\tau) + O(\varepsilon^2) , \tag{5.27}$$

where $\tau = \varepsilon T$ and $\bar{\Upsilon}$ is a complex conjugate to Υ. The straightforward asymptotic analysis reduces Eq. (5.2) to the Burgers equation for the function $\Upsilon(z,\tau)$,

$$\Upsilon_{z\tau} - \frac{(\gamma_1 + \gamma_2)}{2\alpha}\Upsilon_z\Upsilon_{zz} + \frac{\beta}{2\alpha}\Upsilon_{zzz} = 0 . \tag{5.28}$$

The complex conjugate function $\bar{\Upsilon}$ satisfies the same equation (5.28), while the correction term S_1 can be found through Υ as follows:

$$S_1 = \frac{(\gamma_2 - \gamma_1)}{4\alpha}(\Upsilon\bar{\Upsilon}_z + \Upsilon_z\bar{\Upsilon}) . \tag{5.29}$$

We mention that the reduction of the KP equation (2.6) to the Burgers equation (5.28) was first presented by Shrira and Pesenson (1983) for describing the propagation of finite-amplitude modulations along the transversely stable planar KdV soliton in a medium with a negative dispersion (see also Pesenson, 1991). The corresponding KP equation differs from Eq. (2.6) by the negative sign in front of the y-derivative term. In the latter problem, the small-amplitude asymptotic approach allows to describe a shock wave propagating along the stable soliton. This

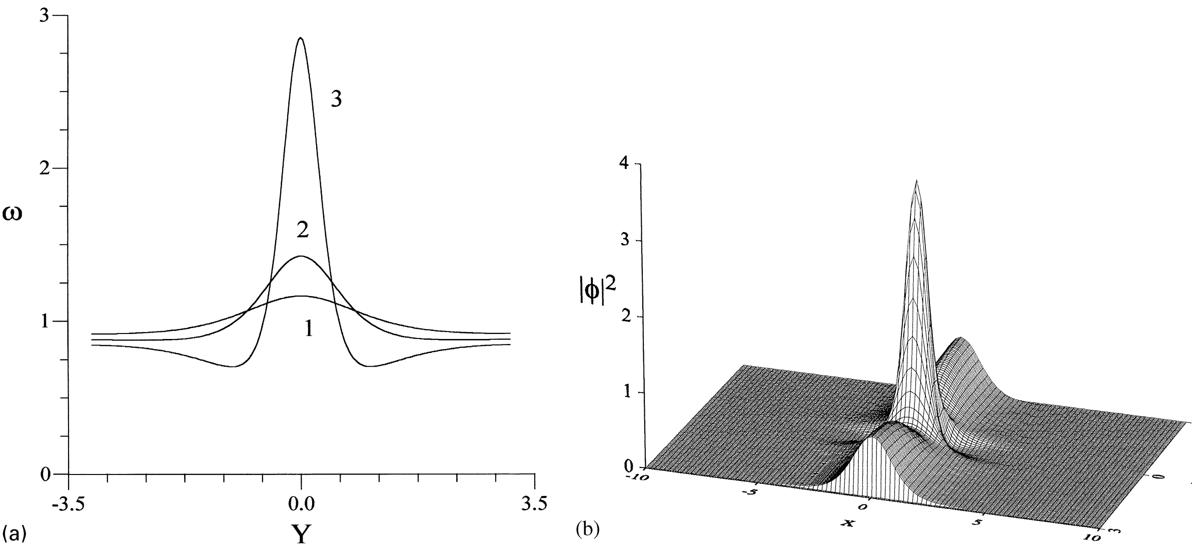


Fig. 3. (a) The analytical solution (5.26) describing the transverse self-focusing of a bright NLS soliton in the elliptic problem within the long-scale analysis: $\varepsilon = 0.2$ and $1 - T = -0.9$, $2 - T = -0.6$, and $3 - T = -0.3$. (b) The corresponding profile of $|\psi(x,y)|^2$ at $T = -0.2$.

phenomenon is typical for nonlinear dynamics of stable solitons in the negative-dispersion medium (Zakharov, 1986; Anders, 1995) and it leads to the formation of resonant soliton triads (see also Infeld and Rowlands, 1990, and references therein).

In the problems of the transverse soliton self-focusing, the Burgers equation is written in terms of the complex Laplace coordinates z. However, one can still proceed with constructing an asymptotic solution for the soliton self-focusing under the action of a periodic perturbation. To do this, we find a one-soliton solution of Eq. (5.28) for $\Upsilon(z,\tau)$ which reproduces, with the help of Eq. (5.27), an asymptotic solution of Eq. (5.2),

$$S = -\frac{2\beta}{(\gamma_1 + \gamma_2)}\{\Gamma T + \log[\cosh(\Gamma T) - \cos(pY)]\} , \tag{5.30}$$

where Γ is the instability growth rate (5.4). In application to the *KdV solitons*, the asymptotic solution (5.30) describes a chain of large-amplitude (nonsingular) localized mode growing along the front of a planar soliton. The asymptotic solution for $v = v(Y,T)$ follows from Eqs. (5.27) and (5.30) and it is shown in Fig. 4(a). In addition, we find from Eq. (5.13) the radiation field $U^+ = U^+(X - 3v_0, Y)$ generated in front of the self-focusing soliton,

$$U^+ = k^2\frac{1 - \cos(pY)\cosh[k(X - 3v_0 T)]}{[\cosh(k(X - 3v_0 T)) - \cos(pY)]^2} \tag{5.31}$$

where $k = p/\sqrt{3}$. The corresponding function u [see Eq. (4.10)] satisfying the KP equation (2.6) is presented in Fig. 4(b).

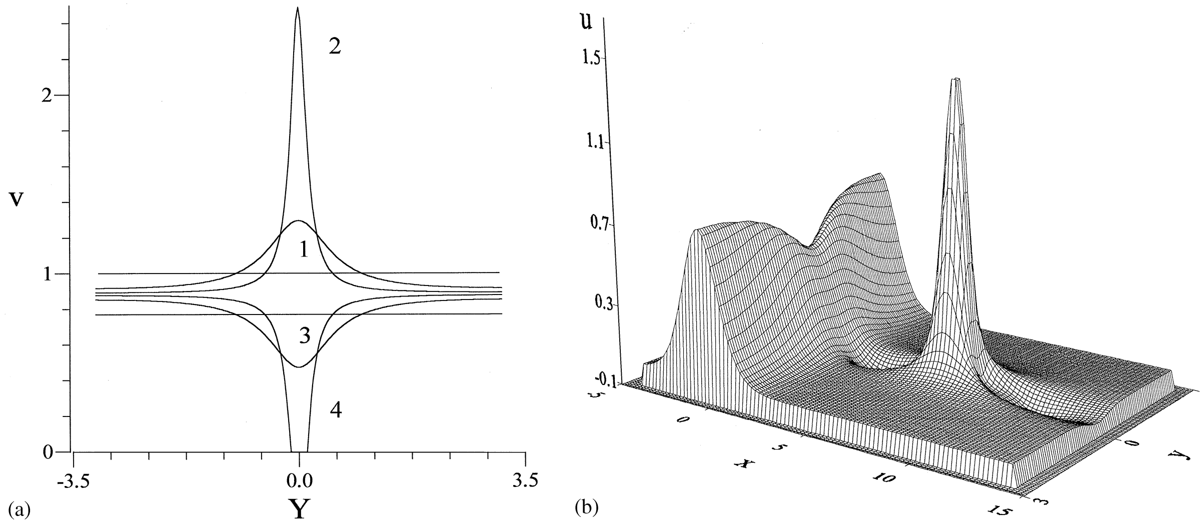


Fig. 4. The analytical solution (5.30) describing the transverse self-focusing of a KdV soliton within the long-scale analysis: (a) $\varepsilon = 0.2$ and $1 - T = -0.5$, $2 - T = -0.125$, $3 - T = 0.125$, and $4 - T = 0.5$. (b) The corresponding profile of $u(x, y)$ at $T = 0.75$.

It is remarkable that the spatial structure defened by Eq. (5.31) is identical to a chain of two-dimensional solitons in the asymptotic limit of a long period of transverse modulation. Still a singularity is presented at the centers of the self-focusing, this *feature is intrinstical* for the long-wave small-amplitude expansions. In spite of the presence of singularities, the asymptotic solution shown in Fig. 4(b) clearly corresponds to a decay of a planar soliton with velocity $v = v_0$ into a chain of two-dimensional solitons propagating with the velocity $v_1 = 3v_0$ and *a residuent* planar soliton moving with the velocity $v_2 = v_0 - \varepsilon\Gamma$. This asymptotic solution agrees, in the limit $p \to 0$ ($\kappa \to \sqrt{v}$), with exact solution (4.36). Thus, in the long-scale asymptotic approximation, we have improved drastically the predictions of the gas dynamics equations [see Eq. (4.22) and Fig. 2(b)].

In application to the *dark NLS solitons*, the asymptotic solution (5.30) describes the self-focusing instability of a finite-amplitude dark soliton. If the dark soliton has the velocity v_0 within the interval $-\rho < v_0 < -\frac{1}{2}\rho$, then the radiation fields $u^{\pm}$ describe chains of two-dimensional dark solitons. The corresponding solution for the function $\psi = \psi(x, y, t)$ [see Eq. (4.13)] is shown in Fig. 5(a). The scenario of the splitting of a dark soliton into a residuent dark soliton with generation of two chains of two-dimensional dark solitons agree with numerical simulations of the NLS equation (Pelinovsky et al., 1995).

For $v_0 = -\frac{1}{2}\rho$ the radiation field u^+ vanishes because the generation of this field implies $v_+ = 0$ [see Eq. (5.18)]. For $-\frac{1}{2}\rho < v_0 < 0$ the field u^+ acquires an "opposite" polarity to the field u^- [see Eq. (5.21)]. This corresponds to a π-phase shift between two-dimensional dark solitons generated in the opposite directions of the radiation fields. When a dark soliton transforms into a stationary black soliton, i.e. v_0 tends to zero, the coefficients γ_1 and γ_2 for the quadratic nonlinear terms in Eq. (5.2) vanish. In this case, the self-focusing dynamics becomes symmetric in space [see Figs. 5(b) and (c)], and it is governed by an effective equation with cubic nonlinear terms (see Section 5.4).

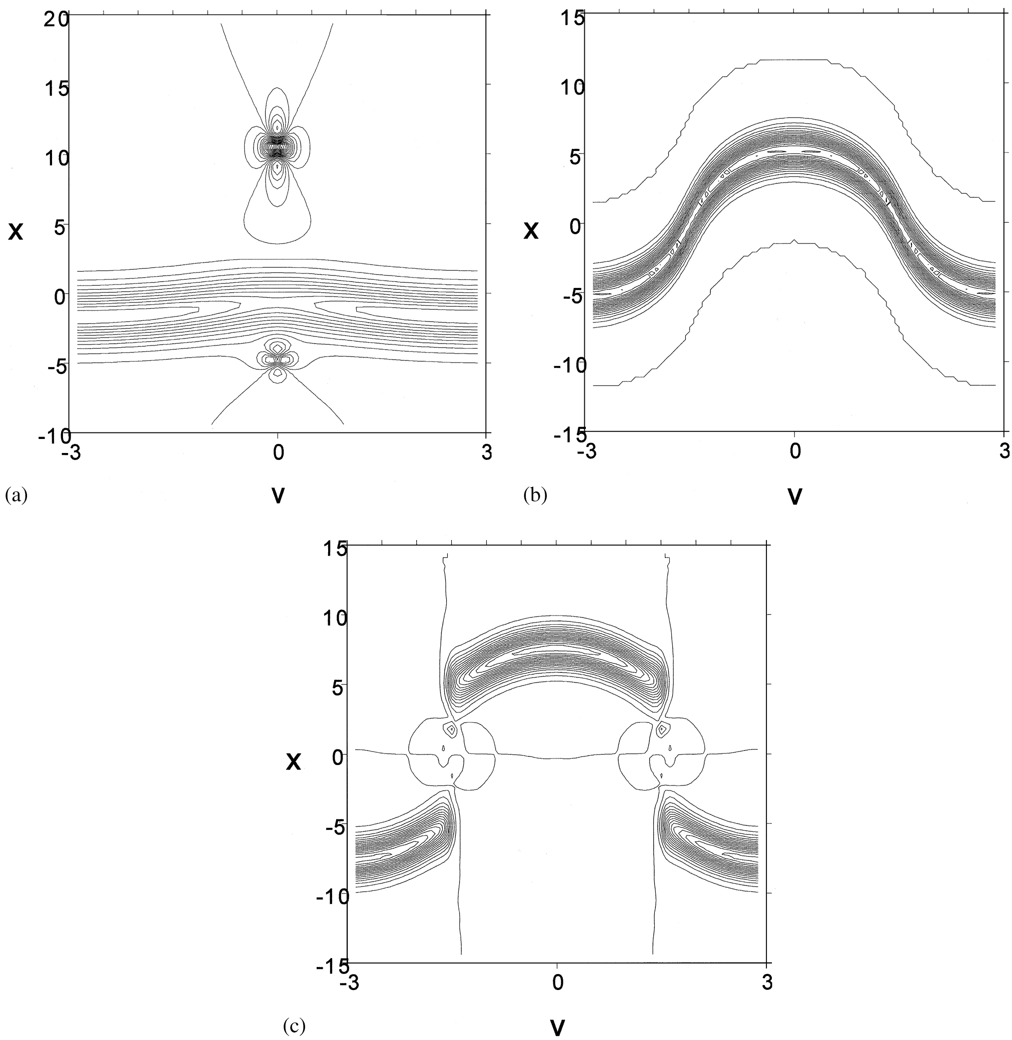


Fig. 5. (a) The level lines of $|\psi(x,y)|^2$ for the transverse self-focusing of a dark NLS soliton within the long-scale analysis: $\varepsilon = 0.1$, $v_0 = -0.75$ and $T = 1.5$. (b)–(c) The level lines of $|\psi(x,y)|^2$ for the transverse self-focusing of a black NLS soliton within the long-scale analysis: $\varepsilon = 0.1$, $v_0 = 0.$, and (b) $T = -0.2$ and (c) $T = 0.2$.

5.3.3. Necessary criteria for collapse

Here we discuss the necessary condition for collapse, that can be obtained in the framework of the extended modulation equations. It follows from Eqs. (5.23) and (5.25) for the particular solution (5.26) that the correction Θ_2 has the same sign of infinite singularity as the leading term $\Theta_0 = \Upsilon + \bar{\Upsilon}$ provided the following condition helds:

$$\delta\gamma = (\gamma_1 - \gamma_2)/(\gamma_1 + \gamma_2) < 0 . \tag{5.32}$$

In the opposite case, i.e. when $\delta\gamma > 0$, the correction Θ_2 has the opposite sign of a singularity and, being taking together with the main term, it can prevent the singularity to be developed in the latter case. Other words, summation of the (singular) asymptotic series (5.23) may result in a regular solution, while no regular solution is expected for $\delta\gamma < 0$. We conclude therefore that, in the extended modulation equations, collapse takes place if the small-amplitude asymptotic expansion satisfy $\delta\gamma < 0$. Similar conclusion follows also for the Shrira–Pesenson equation by analyzing Eqs. (5.27) and (5.29).

In application to the *modulational instability of the NLS equation* (*1.1*), we find from Eq. (5.32) that the collapse is supported by the power nonlinearity with $r > 2$. In the opposite case, i.e. when $r < 2$, the bounded scenarios of the soliton self-focusing are likely to happen. This conclusion agrees with the conventional classification of the NLS equations into *supercritical* and *subcritical* cases (Rasmussen and Rypdal, 1986). For instance, the modulational instability in the integrable NLS equation ($r = 1$) results in long-lived periodic oscillations discussed in Section 4.6 (Yajima, 1983; Ma, 1984).

In application to the *KdV and dark NLS solitons*, we find from Eqs. (5.14) and (5.20) that the condition $\delta\gamma > 0$ is satisfied. As a result, the instability-induced dynamics of those solitons displays a bounded scenario of the soliton self-focusing, i.e. a monotonic transition to a chain of two-dimensional solitons.

Finally, the *bright NLS solitons in the elliptic problem* satisfy the condition $\delta\gamma = 0$, i.e. $\gamma_1 = \gamma_2$ (see Eq. (5.8)). This is a marginal case between unbounded and bounded soliton dynamics, and it still possesses *critical collapse* (Rasmussen and Rypdal, 1986).

5.4. Extensions

In a number of nonlinear problems, the small-amplitude evolution equations (5.1) and (5.2) turn out to be inconsistent because either linear or nonlinear coefficients vanish. In these special cases, (5.1) and (5.2) should include higher-order corrections. Here we discuss two such cases which include dark solitons of near-zero velocity and bright solitons in the hyperbolic NLS equation.

5.4.1. 'Almost Black' NLS solitons

When v_0 approaches zero, the coefficients γ_1 and γ_2 vanish and one should reconsider the small-amplitude asymptotic expansion to include the higher-order (cubic) nonlinear terms into the asymptotic balance with the linear (dissipative) term in Eq. (5.2). This can be done by a simple scaling $s = \sqrt{\varepsilon} S(Y,T)$ and the extention of v following from Eqs. (4.15), $v = \varepsilon^{1/2} S_T - 2\varepsilon^{3/2} S_T S_Y^2 + O(\varepsilon^{5/2})$. Then, it can be shown that the function $S = S(Y,T)$ satisfies *the modified elliptic Shrira–Pesenson equation*,

$$S_{TT} + \frac{4\rho^2}{3} S_{YY} + \varepsilon\left(-\frac{2}{3} S_{TYY} - 8 S_T S_Y S_{TY} - \frac{4}{3} S_T^2 S_{YY} - \frac{16\rho^2}{3} S_Y^2 S_{YY} \right) + O(\varepsilon^2) = 0 . \tag{5.33}$$

A standard reduction based on the Laplace coordinates, z and $\bar{z}$ given by Eq. (5.22) and the asymptotic expansion $S = \Upsilon(z,\tau) + \bar{\Upsilon}(\bar{z},\tau) + \varepsilon S_1(z,\bar{z},\tau) + O(\varepsilon^2)$, where $\tau = \varepsilon T$, leads to *the modified Burgers equation* for the function Υ,

$$\Upsilon_{z\tau} + \frac{(\delta_3 - (\delta_1 + \delta_2)\alpha)}{2\alpha^2} \Upsilon_z^2 \Upsilon_{zz} + \frac{\beta}{2\alpha} \Upsilon_{zzz} = 0 . \tag{5.34}$$

Then, the asymptotic solution can be obtained in the form,

$$S = \left(\frac{3\beta\sqrt{\alpha}}{[\delta_3 - (\delta_1 + \delta_2)\alpha]p} \right)^{1/2} \log |F|^2 , \tag{5.35}$$

where

$$F = \mathrm{e}^{\Gamma T + \mathrm{i}pY} + (1 + \mathrm{e}^{2(\Gamma T + \mathrm{i}pY)})^{1/2} ,$$

and Γ is the linear instability growth rate (5.4). This solution describes the transverse instability of a black soliton under the action of a periodic perturbation. The soliton instability develops symmetrically in space, and it results in a break-up of the planar soliton at the places of location of two-dimensional vortices, the process is accompanied by small radiation propagating with the velocities $v^{\pm} = \pm \frac{1}{2}\rho$ [see Figs. 5(b) and (c)]. This scenario agrees with the results of numerical simulations of the vortex generation in the defocusing NLS equation (McDonald et al., 1993; Law and Swartzlander, 1993; Pelinovsky et al., 1995).

5.4.2. Bright NLS solitons in the hyperbolic problem

The system of modulation equations (4.9) derived for the soliton self-focusing in the framework of the hyperbolic NLS equation can be reduced, in the small-amplitude approximation, to a system of coupled equations. The scaling $s = \varepsilon S(Y, T)$ and $\theta = \omega_0 T + \varepsilon^2 \Theta(Y, T)$ results in the asymptotic expansions for v and ω,

$$v = \varepsilon S_T + \varepsilon^3(4S_T S_Y^2 - 2S_Y \Theta_Y) + \mathrm{O}(\varepsilon^5) ,$$

$$\omega = \omega_0 + \varepsilon^2(\Theta_T + 4\omega_0 S_Y^2 - S_T^2) + \mathrm{O}(\varepsilon^4) .$$

Then, the coupled system for $S = S(Y, T)$ and $\Theta = \Theta(Y, T)$ follows from Eqs. (4.9),

$$\begin{aligned} &S_{TT} + \frac{4}{3}\omega_0 S_{YY} + \varepsilon^2 \left[\frac{4}{9}\left(\frac{\pi^2}{3} - 1\right) S_{YYYY} + \frac{16}{3}\omega_0 S_Y^2 S_{YY} - \frac{4}{3} S_T^2 S_{YY} \right. \\ &\quad \left. + 8 S_T S_Y S_{TY} - 4 S_{YT}\Theta_Y + \frac{4}{3} S_{YY}\Theta_T \right] + \mathrm{O}(\varepsilon^4) = 0 , \\ &\Theta_{TT} - 4\omega_0 \Theta_{YY} + 8\omega_0 S_T S_{YY} - 2 S_T S_{TT} + \mathrm{O}(\varepsilon^2) = 0 . \end{aligned} \tag{5.36}$$

Here we have included the linear dispersion term from Eq. (3.7), which can be obtained by means of a direct asymptotic analysis.

The important property of the coupled equations (5.37) is that the reduction to the Laplace coordinates fails since the nonlinear (cubic) term vanishes. Moreover, we can show that any power nonlinearity cannot stabilize the growth of linear perturbations because the nonlinearity vanishes. Indeed, the system of modulation equations (4.9) possess an explicit (complex-valued) solution,

$$S_Y = u(Y - c_0 T), \quad \Theta_Y = -\frac{\mathrm{i}\sqrt{\omega_0}}{\sqrt{3}}(\sqrt{1 - u^2} - 1) ,$$

$$v = \frac{2\mathrm{i}\sqrt{\omega_0}}{\sqrt{3}} \frac{u}{\sqrt{1 - 4u^2}}, \quad \omega = \frac{\omega_0}{(1 - 4u^2)} ,$$

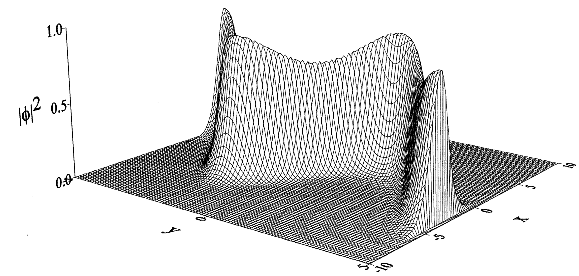


Fig. 6. The level lines of $|\psi(x, y)|^2$ for the transverse self-focusing of a bright NLS soliton in the hyperbolic problem within the long-scale analysis.

where c_0 and ω_0 are constants and $u = u(Y)$. The complex-valued parameter c depends usually on u and this provides an effective nonlinearity in the evolution equations of a "single-wave" approximation. Here $c = c_0 = \text{const}$ and the nonlinearity vanishes identically for any power. As a result, the development of self-focusing instability of a planar soliton cannot be stabilized in the hyperbolic NLS equation and this leads eventually to a break-up of the soliton. The corresponding solution is shown in Fig. 6. This scenario corresponds to numerical simulations reported earlier by Pereira et al. (1978; see Fig. 4). Since two-dimensional solitons do not exist in the hyperbolic NLS equation, each individual part of a planar soliton spreads out and gradually decays due to dispersion.

6. Short-scale approximation

Asymptotic equations obtained in the long-scale small-amplitude approximation discussed in Section 5 still possess singularities within the validity of the asymptotic expansion technique. This occurs even in the problems where the exponential growth of the soliton amplitude due to the transverse self-focusing is bounded by nonlinear effects leading to the formation of a chain of *non-singular two-dimensional solitons*. The reason for such singularities of the asymptotic expansions can be explained by the fact that, within the long-scale asymptotic approximation, two-dimensional solitons formed due to the development of the transverse instability have large amplitudes compared to the amplitude of the initially unstable planar soliton. Therefore, although the long-scale expansion method is usually very simple for the asymptotic analysis, the applicability of the asymptotic results is limited by the temporal and spatial domains where such asymptotic singularities appear.

The short-scale approximation described in this section is based on the asymptotic analysis near the cutoff of the instability band. Under the action of short-scale transverse modulations, two-dimensional solitons formed in result of the development of instability have the amplitudes compared with the amplitude of an initially unstable planar soliton. If the soliton self-focusing is bounded by nonlinear effects (no collapse), the asymptotic technique is free of singularities and it allows to describe, in a self-consistent manner, the nonlinear regime of the soliton self-focusing. A disadvantage of this approach is that, in most of the cases, the cutoff wave number of the instability band and the corresponding linear eigenmode cannot be found analytically, so that the whole asymptotic scheme may be developed only formally with the subsequent numerical calculations of the parameters and eigenfunctions. Here we discuss the basic asymptotic equation of the short-scale approximation (Section 6.1) and also present a derivation of this equation for the important examples (Section 6.2).

6.1. Basic asymptotic equations

From the physical point of view, the long-scale soliton transverse instability corresponds to either phase or coordinate modulations which destabilize the propagation of a planar soliton. In contrast, the short-scale instability is associated with the instability of the amplitude modulations along the front of a planar soliton. Such an origin of the instability is supported by the existence and bifurcations of the transversely periodic solitary-wave structures that occur near the cutoff wave number (Laedke et al., 1986). Therefore, although the existence of the cutoff of the instability band can be predicted within the long-scale asymptotic equations (5.1) and (5.2), the rigorous asymptotic analysis should be developed on the basis of direct asymptotic expansions of nonlinear evolution equations near $p = p_c$.

We assume that a perturbation applied to a planar soliton is nearly periodic along the transverse coordinate, i.e. it can be written in the form,

$$\delta u(x,y,t) \sim [a(Y,T)e^{ip_c y} + a^*(Y,T)e^{-ip_c y}]\, U(x) , \tag{6.1}$$

where p_c is the critical transverse wave number for the instability cutoff, $U(x)$ is the corresponding eigenmode of the linear eigenvalue problem. A slowly varying complex amplitude $a = a(Y,T)$ depends on a slow time $T = \varepsilon t$ and the strechted transverse coordinate $Y = \varepsilon^2 y$ [notice that a different streched coordinate, $Y = \varepsilon y$, has been used in the expansions of Sections 4 and 5]. Perturbation δu in the form (6.1) describes, in the leading order, the amplitude modulations along the front of a planar soliton. The main target of the asymptotic analysis is to derive a governing equation for the amplitude $a(Y,T)$. As a matter of fact, the universal equation that appears in the short-scale small-amplitude asymptotic expansions is the unstable NLS equation (Wadati et al., 1991),

$$-ip_c a_Y + \beta a_{TT} + \gamma |a|^2 a = 0 , \tag{6.2}$$

where the coefficient β is determined from the linear analysis, and the coefficient γ describes the nonlinear effects. According to the linear analysis, the instability appears for longer transverse perturbations, i.e. for $p < p_c$, this condition specifies β to be positive. Indeed, for linear perturbations of the form $a \sim e^{\Gamma T + i\Delta_p Y}$, we obtain

$$\Gamma^2 = -\beta^{-1} p_c \Delta_p , \tag{6.3}$$

where Γ is the instability growth rate ($\Gamma > 0$) while Δ_p is the deviation of the transverse wave number p from p_c, i.e. $p = p_c + \varepsilon^2 \Delta_p$ so that $\Delta_p < 0$ in the instability domain. Within the unstable NLS equation (6.2), the instability domain is not bounded, and therefore the initial-value problem is ill-posed. However, this model still can provide accurate results for the transverse self-focusing under the action of periodic or multi-periodic perturbations (see Janssen and Rasmussen, 1983; Gorshkov and Pelinovsky, 1995a).

We should also mention that an extension of the underlying model (6.2) to include higher-order terms of the asymptotic expansions may bound the instability band making the initial-value problem to be well-posed, similar to the case of the extended long-scale modulation equations described below in Sections 4 and 5. However, for the self-consistent predictions of the instability-induced soliton dynamics, it is usually sufficient to analyze only the unstable NLS equation (6.2) or its analogues.

The coefficient γ in Eq. (6.2) determines whether the nonlinear dynamics of the transverse instability remains bounded or not. If $\gamma < 0$, Eq. (6.2) describes an unbounded scenario of the nonlinear dynamics, i.e. collapse may occur as a result of the self-focusing process. If $\gamma > 0$, long-lived bounded oscillations ('quasi-recurrence') is the most typical scenario for, at least, intermediate regimes of the nonlinear dynamics. At last, the case $\gamma = 0$ is special, and the unstable NLS equation (6.2) should be modified by higher-order nonlinear terms. For example, this situation occurs in integrable models where the transverse self-focusing displays a monotonic (resonant) transition from a planar soliton to a transversely modulated soliton-like structure. Radiative effects, which are usually beyond the leading-order approximation described by the unstable NLS equation (6.2), should be also taken into account for those special (resonant) cases.

6.2. Derivation

Since the asymptotic analysis is straightforward, we present here only the main steps of the derivation and the final results obtained by the short-scale expansion technique referring to the original papers where more details can be found. We consider the basic soliton equations and construct the asymptotic solutions for different scenarios of the instability-induced evolution of a planar soliton under the action of periodic short-scale transverse modulations.

6.2.1. Bright NLS solitons in the elliptic problem

We start from the following asymptotic expansion for the NLS equation (2.1) for $\sigma_{\mathrm{n}} = \sigma_{\mathrm{d}} = +1$ [cf. Eqs. (4.3) and (5.7)],

$$\psi = \left[\Phi_b(x;\omega) + \varepsilon\phi_1(x,y;Y,T) + \sum_{n=2}^{\infty} \varepsilon^n \phi_n(x,y;Y,T) \right] \mathrm{e}^{\mathrm{i}(t+\varepsilon\theta)} , \tag{6.4}$$

where $\omega = 1 + \varepsilon^2\theta_T + \mathrm{O}(\varepsilon^4)$, $\Phi_b(x;\omega)$ is given by Eq. (2.2), and the first-order perturbation $\phi_1(x,y;Y,T)$ is selected according to the structure of the linear eigenmode (3.10),

$$\phi_1 = [a(Y,T)\mathrm{e}^{\mathrm{i}p_c y} + a^*(Y,T)\mathrm{e}^{-\mathrm{i}p_c y}]\,\mathrm{sech}^2 x , \tag{6.5}$$

where the cutoff wave number is $p_{\mathrm{c}} = \sqrt{3}$. Such a perturbation describes nearly periodic amplitude modulations along a planar bright soliton. Self-consistent phase modulations are introduced through the slowly varying function $\theta = \theta(Y,T)$. The short-scale asymptotic analysis was considered by Janssen and Rasmussen (1983) where the first terms of the asymptotic series (6.4) were found for a particular case of periodic modulations, $a = A(T)\mathrm{e}^{\mathrm{i}\Delta_p Y}$,

$$\phi_2 = |a|^2 u_0(x) + [\mathrm{i}a_T \mathrm{e}^{\mathrm{i}p_c y} + \mathrm{i}a_T^* \mathrm{e}^{-\mathrm{i}p_c y}] w_1(x) + [a^2 \mathrm{e}^{2\mathrm{i}p_c y} + a^{*2}\mathrm{e}^{-2\mathrm{i}p_c y}] u_2(x) , \tag{6.6}$$

where $u_0 = -4\,\mathrm{sech}\,x + 2\,\mathrm{sech}^3 x$, while the real functions $w_1(x)$ and $u_2(x)$ can be expressed through the Legendre functions (see Appendix B in the original paper by Janssen and Rasmussen, 1983). Furthermore, the parameter $\theta(Y,T)$ is related to the amplitude $a(Y,T)$ through the relation

$$\theta_T = 8|a|^2 ,$$

and the modulation amplitude $a(Y,T)$ satisfies the unstable NLS equation (6.2) with the coefficients,

$$\beta = \frac{1}{4}\left(\frac{\pi^2}{6} - 1\right), \quad \gamma = -\left(\frac{108}{35} + \frac{27}{2} I\right),$$

where I is given by Eq. (B6) in the paper by Janssen and Rasmussen (1983). It was found numerically that $I \approx 0.164$ and therefore $\gamma \approx -5.300$.

The unstable NLS equation (6.2) describes an unbounded scenario of a singularity formation for the amplitude of a modulated bright NLS soliton. Fig. 7(a) shows the corresponding phase plane of the dynamical system (6.2) for the periodic perturbation $a = A(T)e^{i\Delta_p Y}$. A particular analytical solution for a separatrix trajectory represents a nonlinear regime of the soliton self-focusing,

$$A(T) = \left(\frac{2\beta}{|\gamma|}\right)^{1/2} \Gamma\,\mathrm{cosech}(\Gamma T) , \tag{6.7}$$

where Γ is defined by Eq. (6.3). This asymptotic solution, rewritten for the original field ψ with the help of Eqs. (6.4) and (6.7), is presented in Fig. 7(b). The unbounded dynamics of the soliton self-focusing is associated with the development of a two-dimensional collapse known to occur for the elliptic NLS equation (Janssen and Rasmussen, 1983).

6.2.2. KdV solitons

We analyze the short-scale transverse modulations of the KdV soliton in the framework of the KP equation (2.6). We start from the following asymptotic expansion [cf. Eqs. (4.10) and (5.9)],

$$u = U(\xi; v) + \varepsilon u_1(\xi, y; Y, T) + \sum_{n=2}^{\infty} \varepsilon^n u_n(\xi, y; Y, T) , \tag{6.8}$$

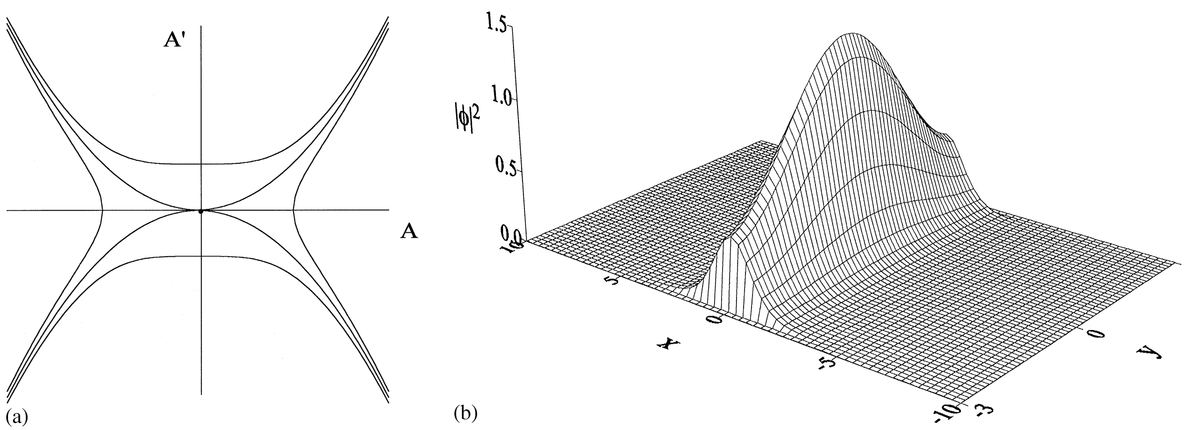


Fig. 7. (a) The phase plane corresponding to Eq. (6.2) for a bright NLS soliton under a periodic transverse perturbation. (b) The profile of $|\psi(x,y)|^2$ within the short-scale analysis with $p = 1$, $\varepsilon = 1$, and $T = -0.8$.

where $\xi = x - t - \varepsilon s$, $v = 1 + \varepsilon^2 s_T + O(\varepsilon^4)$, $U(\xi; v)$ is the profile of a KdV soliton (2.7), and the first-order perturbation $u_1(\xi, y; Y, T)$ is specified by the linear eigenmode,

$$u_1 = [a(Y,T)e^{ip_c y} + a^*(Y,T)e^{-ip_c y}](\operatorname{sech}\xi - 2\operatorname{sech}^3\xi) , \tag{6.9}$$

where $p_c = \sqrt{3}/2$. Perturbation of this form describes nearly periodic amplitude modulations along a planar KdV soliton. The amplitude modulation induces also the coordinate modulation described by the parameter $s = s(Y, T)$. The second-order term of the asymptotic expansions was found explicitly by Gorshkov and Pelinovsky (1995a), and it can be written in the following form:

$$u_2 = |a|^2 w_0(\xi) + [a_T e^{ip_c y} + a_T^* e^{-ip_c y}] w_1(\xi) + [a^2 e^{2ip_c y} + a^{*2} e^{-2ip_c y}] w_2(\xi) , \tag{6.10}$$

where $w_0(\xi) = 3\xi \tanh\xi \operatorname{sech}^2\xi - 7\operatorname{sech}^2\xi + 6\operatorname{sech}^4\xi$, $w_1(\xi) = (\partial^2/\partial\xi^2)(\xi \operatorname{sech}\xi)$, $w_2(\xi) = -2\operatorname{sech}^2\xi + 3\operatorname{sech}^4\xi$. The coordinate $s(Y, T)$ is related to the amplitude $a(y, T)$ as $s_T = |a|^2$, and the amplitude $a(Y, T)$ satisfies Eq. (6.2) with $\beta = 3/4$ and $\gamma = 0$. Thus, the asymptotic expansion done in the framework of the integrable KP equation corresponds to a critical soliton self-focusing, and therefore it should include higher-order nonlinear terms. Such a modification of the asymptotic procedure was developed by Gorshkov and Pelinovsky (1995a, 1995b) who derived the unstable Eckhaus equation (see, e.g., Calogero and Eckhaus, 1987) for the amplitude $a = a(Y, T)$ rescaled to the order of $\sqrt{\varepsilon}$,

$$-\frac{4}{3} i p_c a_Y + a_{TT} + |a|^4 a + 2a(|a|^2)_T = 0 . \tag{6.11}$$

The last term in Eq. (6.11) describes an effective dissipation due to the emission of radiation behind the KdV soliton. Within the asymptotic analysis (see Gorshkov and Pelinovsky, 1995a), the generation of a radiation field behind the soliton, $u = \varepsilon^2 u^-$, is defined by the boundary condition at the soliton front,

$$u^-|_{X=T} = 2(|a|^2)_T . \tag{6.12}$$

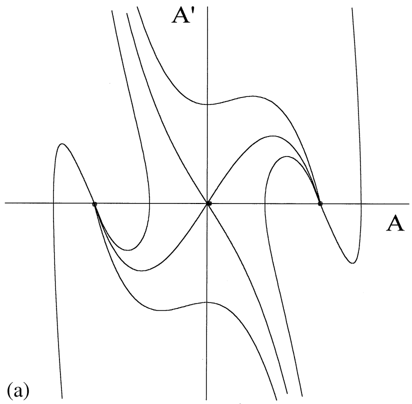
In the framework of the short-scale approximation, the generation of the radiation field in front of the KdV soliton does not occur.

Model (6.11), derived by means of an extended short-scale asymptotic analysis, predicts a monotonic transition of an unstable planar KdV soliton to a steady-state transversely modulated soliton structure. Fig. 8(a) presents the phase plane of the dynamical system (6.11) for the periodic modulations in the form $a = A(T)e^{i\Delta_p Y}$. A particular solution for the separatrix trajectory can be found analytically,

$$A(T) = \left[\frac{\Gamma}{2} e^{\Gamma T} \operatorname{sech}(\Gamma T)\right]^{1/2} , \tag{6.13}$$

where Γ is defined by Eq. (6.3) for $\beta = 3/4$. A monotonic transition from a planar soliton to a transversely modulated structure described by the solution (6.13) is accompanied by the generation of a radiation field u^- behind the solitary wave,

$$u^- = \Gamma^2 \operatorname{sech}^2(\Gamma X) . \tag{6.14}$$



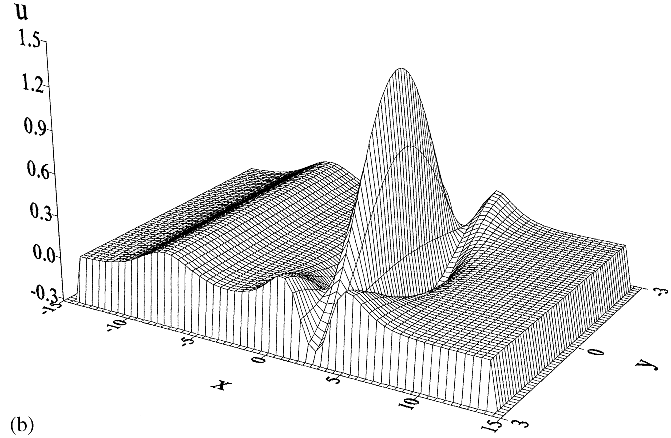


Fig. 8. (a) The phase plane corresponding to Eq. (6.2) for a KdV soliton under a periodic transverse perturbation. (b) The profile of $u(x, y)$ within the short-scale analysis with $p = 0.7$, $\varepsilon = 1$, and $T = 7.5$.

Radiation field (6.14) has a structure of a planar KdV soliton (2.7) of smaller amplitude and velocity. Thus, within the short-scale asymptotic analysis, the analytical solution describes a splitting of an initially unstable planar KdV soliton moving with the velocity $v_0 = 1$ into a chain of two-dimensional KP solitons moving with the velocity $v_1 = 1 + \varepsilon^2 \Gamma$ and a complimentary planar KdV soliton moving with a small velocity $v_2 = \varepsilon^2 \Gamma^2$ remaining behind the modulated soliton chain. The asymptotic solution for the field u [see Eqs. (6.8) and (6.14)] is shown in Fig. 8(b). The asymptotic solution (6.13) and(6.14) agrees with the exact solution (4.36) calculated in the limit $p \to p_c$ ($\kappa \to 0$).

6.2.3. Dark NLS solitons

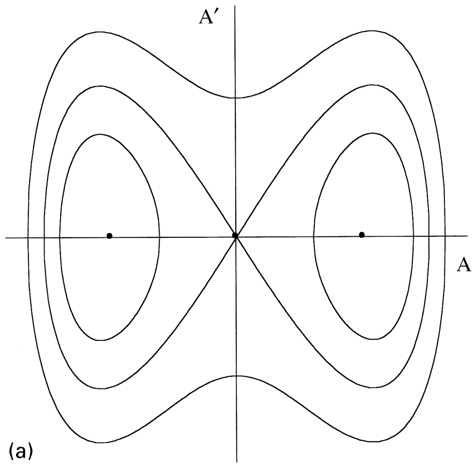
The analysis of short-scale self-focusing of dark solitons is similar to that for KdV solitons. In particular, the short-scale asymptotic expansion has the form [cf. Eq. (4.13)],

$$\psi = \left[\Phi_d(\xi; v) + \varepsilon \phi_1(\xi, y; Y, T) + \sum_{n=2}^{\infty} \varepsilon^n \phi_n(\xi, y; Y, T) \right] e^{-2i\rho^2 t} , \tag{6.15}$$

where $\xi = x - 2vt - \varepsilon s$, $\Phi_d(\xi; v)$ is defined by the profile of a dark soliton (2.3), and the first-order perturbation $\phi_1(\xi, y; Y, T)$ is given by

$$\phi_1 = [a(Y,T)e^{ip_c y} + a^*(Y,T)e^{-ip_c y}] \left\{ -\frac{3v \sinh(k\xi)}{2k \cosh^2(k\xi)} + i\frac{(p_c^2 + 3k^2)}{4k^2 \cosh(k\xi)} \right\} , \tag{6.16}$$

where $k = \sqrt{\rho^2 - v^2}$ and $p_c = [-(\rho^2 + v^2) + 2\sqrt{v^4 - v^2\rho^2 + \rho^4}]^{1/2}$. The instability dynamics is described again by the effective NLS equation (6.2) with the coefficients calculated numerically (Pelinovsky et al., 1995). It was found that the coefficient $\gamma = \gamma(k)$ is positive and $\gamma \to 0$ as $k \to 0$ (the limit of a KdV soliton). The radiation fields $u^{\pm}$ propagate to the right and to the left with the sound speed $\pm 2\rho$. Their profiles are generated by certain boundary conditions (see Pelinovsky et al., 1995) similar to Eq. (6.12). Thus, the short-scale self-focusing of a dark soliton displays the bounded



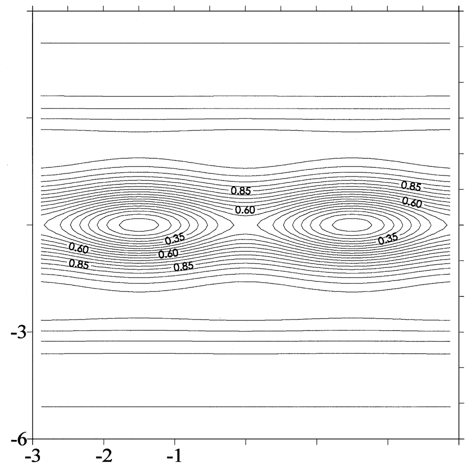


Fig. 9. (a) The phase plane corresponding to Eq. (6.2) for a black NLS soliton under a periodic transverse perturbation. (b) The level lines of $|\psi(x,y)|^2$ within the short-scale analysis with $p = 0.5$, $\varepsilon = 1$, and $T = 1.5$.

scenario of long-lived oscillations of a modulated dark soliton. Fig. 9(a) displays the corresponding phase plane of the dynamical system (6.2) for the periodic perturbation $a = A(T)e^{i\Delta_p Y}$. A particular analytical solution for the separatrix trajectory is given by the explicit result,

$$A(T) = \left(\frac{2\beta}{\gamma}\right)^{1/2} \Gamma \operatorname{sech}(\Gamma T) , \tag{6.17}$$

where Γ is the same as in Eq. (6.3). The asymptotic solution for the field ψ [see Eqs. (6.15) and (6.17)] is presented in Fig. 9(b). The analytical asymptotic results agree with numerical simulations of the defocusing NLS equation which revealed formation of a train of vortex pairs from a planar soliton (McDonald et al., 1993; Pelinovsky et al., 1995) and long-lived periodic oscillations (Pelinovsky et al., 1995). As $k \to 0$, one can modify the asymptotic analysis and derive the mixed unstable NLS and Eckhaus equation that describes a transformation of a small-amplitude planar soliton into a chain of two-dimensional solitons accompanied by some intermediate oscillations (see Pelinovsky et al., 1995).

7. Some other models

In the previous sections, we have presented an overview of different approaches for analyzing the soliton self-focusing phenomena on the basis of a few fundamental nonlinear models. Different varieties of soliton-bearing models still appear in physical problems of different physical context, and they bring many novel modifications of the classical methods. Here we review a few more examples, which we classify into several groups, namely the NLS-type models (Section 7.1), the KdV-type models (Section 7.2), and the kink-type models (Section 7.3).

7.1. NLS-type models

There exist several different modifications of the NLS equation (2.1). Some of them have been already mentioned in Section 2.2 in the discussions of realistic physical models employed to describe the suppression of wave collapse. Here we analyze more examples of this type.

7.1.1. Power-law nonlinearities

A balance between power-law nonlinearity and wave diffraction can be described by the generalized NLS equation in D dimensions,

$$i\psi_t + \nabla_D^2 \psi + (r+1)|\psi|^{2r}\psi = 0 , \tag{7.1}$$

where ∇_D^2 is the Laplace operator in D dimensions, and r is the power of nonlinearity. It is well-known that the D-dimensional soliton of the model (7.1) is stable provided $D < 2/r$, and unstable otherwise. In the latter case Eq. (7.1) displays collapse of localized solitons (Rasmussen and Rypdal, 1986). The c.w. background is *modulationally unstable* within the generalized NLS equation (7.1). Being related to the analysis of the soliton stability, the modulational instability of the c.w. background develops into singularities for the case $D > 2/r$ while, in the opposite case, it results in long-lived periodic oscillations between the c.w. and modulated states. Stable D-dimensional solitons can be formed at later stages of the instability-induced dynamics. The case $r = 2/D$ corresponds to a weak instability of a soliton which results in a critical collapse of slowly growing perturbations (see, e.g., Rasmussen and Rypdal, 1986). Here we discuss how these well-known properties of the generalized NLS equation (7.1) are related to the symmetry-breaking instabilities and the transverse self-focusing of planar solitons.

It was shown in Section 4.4 that the modulation equations for transverse perturbations of a one-dimensional soliton of Eq. (2.1) reduce to the one-dimensional NLS equation (1.1) for $r = 2$. Here we generalize this result and show that the modulation equations for the transverse perturbations of a d-dimensional ($d < D$) soliton of Eq. (7.1) with the parameters D and r reduce to the same NLS equation (7.1) but with the different parameters, $\hat{D}$ and $\hat{r}$, where

$$\hat{D} = D - d, \quad \hat{r} = \frac{2r}{2 - rd} . \tag{7.2}$$

The generalized NLS equation (7.1) follows from the Lagrange function,

$$L = \frac{i}{2}(\psi^* \psi_t - \psi \psi_t^*) - |\nabla_\parallel \psi|^2 - |\nabla_\perp \psi|^2 + |\psi|^{2r+2} , \tag{7.3}$$

where the gradient vector $\nabla_\parallel$ includes d dimensions *parallel* to a planar soliton, while the vector $\nabla_\perp$ includes $\hat{D} = D - d$ dimensions *transverse* to the soliton. The planar soliton solution can be written in the form,

$$\psi = \omega^{1/2r} f(\boldsymbol{X}_\parallel) e^{i\theta/\varepsilon} , \tag{7.4}$$

where $\boldsymbol{X}_\parallel = \sqrt{\omega}\boldsymbol{x}_\parallel$, $\theta/\varepsilon = \omega t$, and the function f satisfies the equation for a steady-state normalized envelope, $\nabla_\parallel^2 f - f + (r+1) f^{2r+1} = 0$. The transverse modulation of a planar soliton can be described by Eq. (7.4) with the varying soliton parameters, $\omega = \omega(\boldsymbol{X}_\perp, T)$ and $\theta = \theta(\boldsymbol{X}_\perp, T)$, where

$\boldsymbol{X}_\perp = \varepsilon \boldsymbol{x}_\perp$, $T = \varepsilon t$, and $\varepsilon \ll 1$. Integrating the function (7.3) with respect to $\boldsymbol{x}_\parallel$, we find the following averaged Lagrangian (see also Kuznetsov et al., 1986; Trubnikov and Zhdanov, 1987),

$$\langle L \rangle = \frac{\int_{-\infty}^{\infty} L \, \mathrm{d}\boldsymbol{x}_\parallel}{\int_{-\infty}^{\infty} f^2 \, \mathrm{d}\boldsymbol{X}_\parallel} \tag{7.5}$$

in the form

$$\langle L \rangle = -\omega^{(2-rd)/2r}[\theta_T + (\nabla_\perp \theta)^2] + \frac{2-rd}{2(r+1)-rd}\omega^{(2(r+1)-rd)/2r} - \varepsilon^2 \mu \omega^{(2(1-2r)-rd)/2r}(\nabla_\perp \omega)^2 ,$$

where

$$\mu = \frac{\int_{-\infty}^{\infty} (\frac{1}{r} f + \boldsymbol{X}_\parallel \cdot \nabla_\parallel f)^2 \, \mathrm{d}\boldsymbol{X}_\parallel}{4\int_{-\infty}^{\infty} f^2 \, \mathrm{d}\boldsymbol{X}_\parallel} > 0 .$$

For $r = 1$ and $d = 1$ this expression reduces to Eq. (4.17). The modulation equations can be obtained from Eq. (7.5) by varying $\langle L \rangle$ with respect to the parameters ω and θ,

$$\begin{aligned} &(\omega^{(2-rd)/2r})_T + 2\nabla_\perp(\omega^{(2-rd)/2r}\nabla_\perp \theta) = 0 , \\ &\theta_T + (\nabla_\perp \theta)^2 - \omega - \varepsilon^2 \mu \left[\frac{4r}{2-rd} \frac{\nabla_\perp^2 \omega}{\omega} - \frac{2(1-2r)-rd}{2-rd} \frac{(\nabla_\perp \omega)^2}{\omega^2} \right] = 0 , \end{aligned} \tag{7.6}$$

The new function $\hat{\psi}(\eta, \tau) = c\omega^{(2-rd)/4r} e^{i\theta/\lambda}$ satisfies Eq. (7.1) written for the variables $\eta = \boldsymbol{X}_\perp/\lambda$ and $\tau = T/\lambda$ with the parameters $\hat{D}$ and $\hat{r}$ given by Eqs. (7.2). The constants c and λ are defined as

$$c = \left[\frac{2-rd}{2(r+1)-rd} \right]^{(2-rd)/4r} , \quad \lambda = \frac{4r}{(2-rd)}\sqrt{\mu\varepsilon} .$$

For $r < 2/d$, the modulation equations (7.6) are elliptic, and this property indicates immediately the existence of a self-focusing instability of planar (d-dimensional) solitons. The scenario of the development of the self-focusing instability depends on the ratio between $\hat{D}$ and $2/\hat{r}$, according to the scheme discussed above. It follows from Eq. (7.2) that the critical NLS equation, with respect to the modulational instability of a c.w. background, i.e. $D = 2/r$, remains the critical NLS equation with respect to the self-focusing instability of a planar soliton, i.e. $\hat{D} = 2/\hat{r}$. In addition, the subcritical ($D < 2/r$) and the supercritical ($D > 2/r$) cases of Eq. (7.1) transform to the corresponding cases of Eqs. (7.6). This implies that the long-term instability-induced dynamics of a modulated soliton should display *a collapse scenario*, for $\hat{D} \geq 2/\hat{r}$, and *a quasi-recurrence scenario*, for $\hat{D} < 2/\hat{r}$. Thus, we come to the conclusion that there exists one-to-one correspondence between modulational instability of continuous waves and self-focusing instability of solitons in nonlinear dispersive/diffractive media.

For $r \geq 2/d$, the modulation equations (7.6) are hyperbolic, i.e. they fail to predict the self-focusing instability of a planar (d-dimensional) soliton. This is explained by the appearance of longitudinal instabilities of planar solitons within the generalized NLS equation (7.1). If a planar soliton is unstable against the symmetry-preserving (longitudinal) perturbations, none of the methods discussed in this survey can help to describe the instability (see, e.g., discussions in Pelinovsky and Grimshaw, 1997).

7.1.2. Parametric quadratic solitons

For three decades optical self-trapped beams (or spatial solitons) confined in the transverse plane were commonly believed to be a prerogative of media with cubic-like nonlinearities or their generalizations. However, about 25 yr ago, Karamzin and Sukhorukov (1974) predicted a possibility to achieve beam the self-trapping in an optical medium with a quadratic nonlinear response in the form of mutually interacting and coupled beams of the fundamental and second-harmonic fields, the so-called *parametric quadratic solitons*. The field of quadratic solitons has acquired importance only recently (e.g., Buryak and Kivshar, 1994, 1995; Torner et al., 1996; He et al., 1996; see also the review papers by Stegeman et al., 1996, and Kivshar, 1998b) being also actively stimulated by experiment. Since the parametric solitons are, strictly speaking, solitary waves, i.e. localized solutions of nonintegrable equations, a crucial issue is their stability, including the symmetry-breaking instabilities.

To investigate the symmetry-breaking instabilities of quadratic solitons, we consider a nearly phase-matched interaction of two beams in a quadratic medium governed by two coupled equations for the normalized field envelopes u_1, at the fundamental frequency ω, and u_2, at the second-harmonic frequency 2ω (see, e.g., De Rossi et al., 1997a,b)

$$\begin{aligned} &\mathrm{i}u_{1z} + \frac{1}{2}\nabla_\perp^2 u_1 - \frac{\gamma_1}{2}u_{1tt} + u_2 u_1^* = 0\ , \\ &\mathrm{i}u_{2z} + \frac{1}{2\sigma}\nabla_\perp^2 u_2 - \frac{\gamma_2}{2}u_{2tt} + \delta k u_2 + \frac{u_1^2}{2} = 0\ , \end{aligned} \tag{7.7}$$

where $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$, is the transverse Laplacian, z is the propagation distance, the normalized time t is in the reference frame travelling at the common group velocity, $\delta k \equiv \Delta k z_d = (k_2 - 2k_1)k_1 r_0^2$ is the phase-mismatch, and $\sigma \equiv k_2/k_1$. Here k_2 and k_1 are the wave numbers at the corresponding frequencies and r_0 is the characteristic beam width.

Similar to the NLS solitons, two-wave quadratic solitons of a plane geometry display the symmetry-breaking instabilities in a planar geometry (De Rossi et al., 1997a,b; Skryabin and Firth, 1998b) or in higher dimensions (Skryabin and Firth, 1998b). Importantly, it was proven for different cases (Kanashov and Rubenchik, 1981; Bergé et al., 1995; Turitsyn, 1995) that the equations for quadratic solitons possess *no collapse dynamics* for localized solutions. Therefore, as was shown in the recent studies, the development of the instability leads either to the formation of a train of higher-dimensional solitons (De Rossi et al., 1997b), similar to the elliptic NLS equation, or to the complete disintegration and radiative decay of a plane soliton (De Rossi et al., 1997a), similar to the case of the hyperbolic NLS equation.

Because the model (7.7) is not integrable, explicit results can be obtained when the soliton profile is known in a closed analytical form (see, e.g., Buryak and Kivshar, 1995). The full analysis of the linear eigenvalue problem was performed numerically for both the cases (DeRossi et al., 1995a,b).

Once established that the plane solitons are unstable, a crucial issue is their long-range evolutions. For the transverse instability, whenever the eigenfunction profiles follow those of the bell-shaped soliton, the dynamics of the instability process should show no significant changes along the trapping dimension and remain essentially one dimensional. However, in the model (7.7) the problem of long-range evolutions of plane solitons is complicated by a large number of effective frequency modes (at least two carriers and two pairs of sidebands). De Rossi et al., 1995a,b; see also Baboiu and Stegeman, 1998) investigated a nonlinear stage of the soliton symmetry-breaking instability by integrating numerically Eqs. (7.7) with slightly modulated initial front corresponded to one of the instability eigenmodes. A typical result for the focusing case (De Rossi et al., 1997b) is that a stripe breaks up into a periodical sequence of spots forming a lattice of trapped waves, which are naturally expected to be the (2 + 1)-dimensional solitons, the so-called 'neck instability' (see Fig. 10, left column).

In the problem of temporal or modulational instability, the temporal break-up of guided modes in waveguides leads to spatio-temporal trapping for the case of the anomalous dispersion with no qualitative changes in comparison with the transverse instability. Conversely, in the normal dispersion regime no spatial analogy exists. The unstable modes are anti-symmetric and they lead to spatio-temporal wave breaking with characteristic snake-like shapes (De Rossi et al., 1997a), followed by the radiative decay of the soliton (see Fig. 10, right column).

Thus, parametric solitons undergo symmetry-breaking instabilities in the way quite similar to the cases discussed above in the framework of the hyperbolic and elliptic NLS equations with cubic nonlinearity. The resulting collapse-free soliton dynamics can be compared with the effect produced by a saturable nonlinearity in the NLS models, in particular, the transverse break-up of plane solitons leads to the soliton buncing (via the neck instability), and oscillations around a lattice of stable higher-dimensional solitons.

Recently, Skryabin and Firth (1998c) analyzed also the symmetry-breaking instabilities in a more general case of *nondegenerate three-wave mixing* which describes a phase matched interaction between three waves satisfying the resonant condition, $\omega_3 = \omega_1 + \omega_2$, so that the model (7.7)

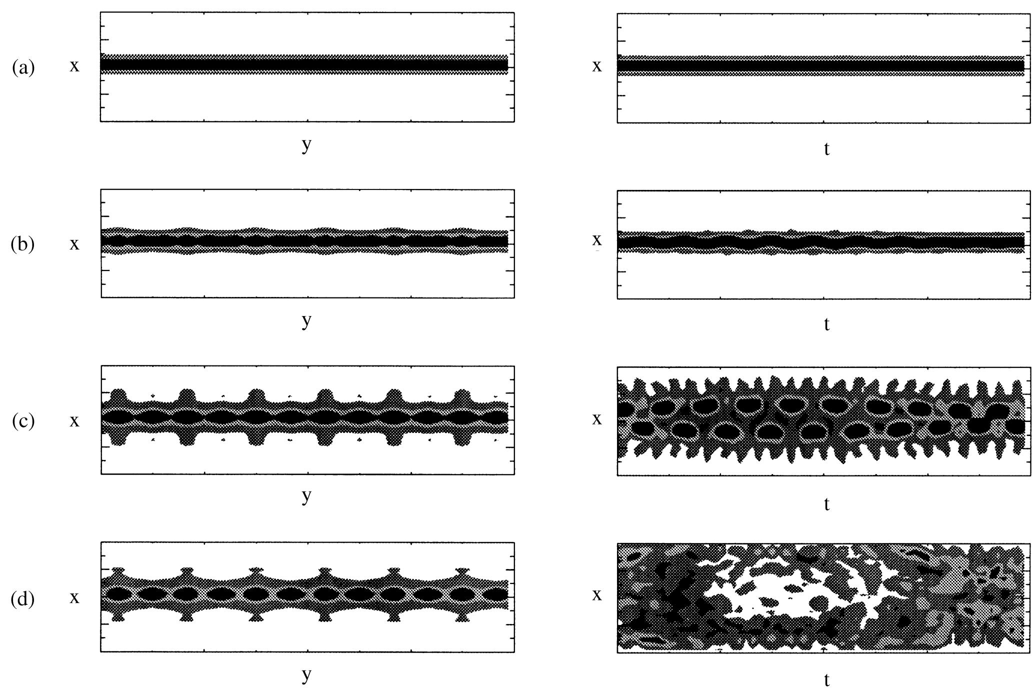


Fig. 10. Self-focusing modulational instability of two-wave parametric quadratic solitons in the cases of (a) normal and (b) anomalous dispersion. In the latter case, the stripe decays into a periodic train of solitary waves stable in higher dimension.

is a particular case. These authors found *a new branch of unstable eigenvalues* corresponding to a two-parameter phase symmetry of the three-wave solitons which appears due to the second invariant of the equations associated with the conservation of the power unbalance of the different frequencies fields. This new branch was shown to have a dramatic effect on the soliton instability in the case of *the normal dispersion*. Indeed, this new branch of the symmetry preserving unstable eigenvalues corresponds to *a neck instability* which, under certain conditions, may become dominant even in a normal dispersion case when the corresponding NLS model displays only the snake-like soliton instabilities. The similar results have been very recently obtained for two incoherently coupled NLS equations (Skryabin and Firth, 1999).

7.1.3. The Davey–Stewartson equation

There are known several physical situations when a resonant coupling occurs between high- and low-frequency waves. In one-dimensional systems, this effect is described by the so-called Zakharov model which, with the same accuracy, reduces to an effective NLS equation. For multi-dimensional case, such a simple reduction is no longer valid, leading to a new model described by the Davey–Stewartson (DS) equation. As one of the possible examples of the corresponding physical system, we mention the evolution of weakly nonlinear gravity-capillary waves at a free surface where a fundamental wave is coupled to an induced mean field (Ablowitz and Segur, 1981; Craig et al., 1997). If the fluid is deep, the governing model reduces either to the hyperbolic or to the elliptic NLS equation. In the opposite limit of a shallow fluid, the equations can be reduced to the DS equation,

$$\begin{aligned} &\mathrm{i}\psi_t + \sigma\psi_{xx} + \psi_{yy} + 2\psi(n - |\psi|^2) = 0 , \\ &n_{xx} - \sigma n_{yy} - 2(|\psi|^2)_{xx} = 0 . \end{aligned} \tag{7.8}$$

Here ψ is the amplitude of a wave packet, n is a self-consistent mean flow, and $\sigma = \pm 1$. The case $\sigma = +1$ occurs for weak capillary effects, and it is referred to as the *DS-I equation*. The case $\sigma = -1$ occurs for pure gravity waves and it is referred to as the *DS-II equation*. Both the cases are known to be integrable by means of the inverse scattering technique. Soliton solutions (bright solitons) are stable with respect to transverse perturbations for $\sigma = -1$ and unstable for $\sigma = +1$ (Ablowitz and Segur, 1979). Exact solutions describing the development of the soliton self-focusing instability were constructed for the DS-I equation by Pelinovsky (1994), who considered both dark and bright solitons of Eqs. (7.8) for $\sigma = +1$. However, it can be shown that the nonvanishing c.w. background is also unstable in this model, therefore, dark solitons do not survive at wave background due to the background instability. Here we reproduce the exact solutions for bright solitons of Eq. (7.8) at $\sigma = +1$.

The DS-I equation has an explicit bilinear representation,

$$\psi = \frac{\tau^+}{\tau}, \quad |\psi|^2 = (\partial_x^2 - \partial_y^2)\log\tau, \quad n = 2\partial_x^2 \log\tau . \tag{7.9}$$

The bilinear functions $\tau(x, y, t)$ and $\tau^+(x, y, t)$ can be specified in the following particular representation (Pelinovsky, 1994):

$$\tau^+ = 2\phi\chi^*, \qquad \tau = 1 + \int_{-\infty}^{x} |\phi|^2\,\mathrm{d}x \int_{-\infty}^{x} |\chi|^2\,\mathrm{d}x , \tag{7.10}$$

where the functions ϕ and χ satisfy the system of linear equations,

$$\phi_x + \phi_y = 0, \quad \chi_x - \chi_y = 0, \quad \mathrm{i}\phi_t + \phi_{xx} = 0, \quad -\mathrm{i}\chi_t + \chi_{xx} = 0 \ . \tag{7.11}$$

We consider the plane-wave solution to Eqs. (7.11),

$$\begin{aligned} \phi &= c_1 \exp[p(x-y) + 2\mathrm{i}p^2 t] + c_2 \exp[\kappa(x-y) + 2\mathrm{i}\kappa^2 t] \ , \\ \chi &= c_1 \exp[p(x+y) - 2\mathrm{i}p^2 t] + c_3 \exp[\kappa(x+y) - 2\mathrm{i}\kappa^2 t] \ , \end{aligned} \tag{7.12}$$

where $\kappa = p - \mathrm{i}\mu$, $c_1 = (2p)^{1/2}$, $c_2 = (2p)^{1/2}a$, $c_3 = (2p)^{1/2}b$, and a, b, and μ are real parameters. Taking, for simplicity, the case $b = 0$, we can write the following exact solution to Eq. (7.8),

$$\psi(x,y,t) = \frac{4p\mathrm{e}^{2px + 4\mathrm{i}p^2 t}(1 + \mathrm{e}^{-\mathrm{i}\mu(x-y+2\mu t) + \Gamma t})}{1 + \mathrm{e}^{4px}\left(1 + \dfrac{2p}{2p - \mathrm{i}\mu}\mathrm{e}^{-\mathrm{i}\mu(x-y+2\mu t)+\Gamma t} + \dfrac{2p}{2p + \mathrm{i}\mu}\mathrm{e}^{\mathrm{i}\mu(x-y+2\mu t)+\Gamma t} + \mathrm{e}^{2\Gamma t}\right)} \ , \tag{7.13}$$

where $\Gamma = 4p\mu$. If $\Gamma > 0$, this solution describes, for $t \to -\infty$, a planar soliton perturbed by a transverse periodic perturbation that exponentially grows in time. The instability is induced by asymmetric (translational) eigenfunctions in the long-scale limit ($\mu \to \infty$), as in the case of bright solitons of the hyperbolic NLS equation. The instability domain is not bounded from above, i.e. $\Gamma \to \infty$ as $\mu \to \infty$. This unusual feature of the soliton self-focusing is explained by the profile of $n(x,y,t)$. It is clear from Eqs. (7.9), (7.10), and (7.12) that the self-focusing of a bright soliton is driven by a transverse periodic modulation of the self-consistent mean flow *n nonlocalized* in the direction of the soliton as $x \to +\infty$. This perturbation in the mean flow pumps the fundamental wave and induces the soliton transverse instability.

At a nonlinear stage of the instability development, the growth of transverse perturbations is stabilized, and it alternates with damping. As a result, the bright soliton returns to its unperturbed planar shape but it acquires a drift velocity, $v = -4\mu$. In the framework of this exact solution, the energy of a soliton is not changed and radiation is not generated. In the symmetric case, $a = b$, the exact solution for the soliton self-focusing in the DS-I equation is presented in Figs. 11(a)–(c).

7.1.4. Discrete NLS equations

All the models analyzed above describe nonlinear waves and instabilities in the continuous systems. However, in the solid state physics, continuous models often appear as a limiting case of more general, *discrete physical models* where the lattice spacing is a fundamental physical parameter. Discreteness introduces a number of new features in the system dynamics, in particular, it modifies the conditions for modulational instability of plane waves (Kivshar and Peyrard, 1992). A simplest model to demonstrate some basic features introduced by discreteness is a model of an array of optical fibers (or planar optical waveguides) coupled by a weak overlapping of the guided wave fields excited in each core of the waveguide array (Christodoulides and Joseph, 1988; Kivshar, 1993; Aceves et al., 1994b),

$$\mathrm{i}\partial_t \psi_n + \partial_x^2 \psi_n + K(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + 2|\psi_n|^2 \psi_n = 0 \ , \tag{7.14}$$

where the variable ψ_n stands for an envelope of the average electric field in the fiber with the number n, and the variables t and x have a reverse meaning in the fiber optics, t is the coordinate along the fiber in the reference frame moving with the group velocity and x is the retarded time. If we neglect

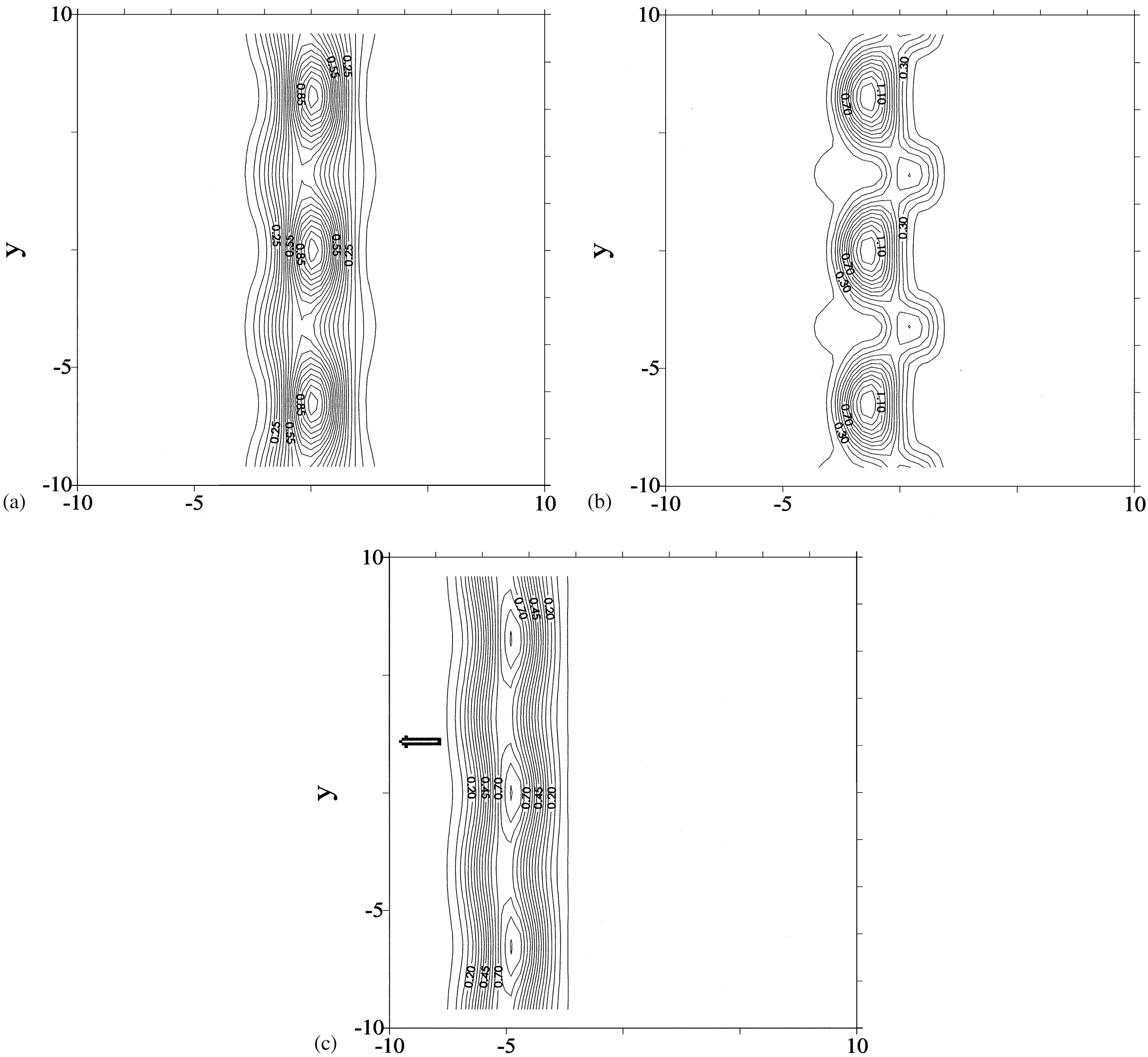


Fig. 11. (a)–(c) Self-focusing instability in the DS-I system described by the exact analytical solutions.

the coupling, i.e. $K = 0$, Eq. (7.14) becomes a standard (1 + 1)-dimensional NLS equation, and therefore the coupling introduces the second dimension described by the discrete variable n.

We follow Aceves et al. (1994b) and study transverse stability of the plane solution $\psi_n(x,t) = \Phi_s(x)e^{i\omega t}$, where $\Phi_s(x) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}x)$, to the perturbation in the discrete lattice varying with n. To do so, we write as usual $\psi_n(x,t) = [\Phi_s(x) + \delta\psi_n(x,t)]e^{i\omega t}$, where this time the small perturbation is selected in the form, $\delta\psi_n(x,t) = \delta\psi(x,t)\cos(Qn)$. Linear eigenvalue problem for $\delta\psi$ is standard, and it is known to possess unstable eigenmodes for the interval $|Q| < Q_c$, where Q_c is defined differently for the discrete problem,

$$4K\sin^2(Q_c/2) = 3\omega\ , \tag{7.15}$$

which, as expected, transforms into the well-known result of the continuum limit for $Q \ll 1$.

Aceves et al. (1994b) analyzed numerically the scenario of a decay of a plane soliton in the case of the transverse instability in such a partially discretized lattice. In a sharp contrast with the (2 + 1) dimensional NLS equation displaying the collapsing dynamics, in the discrete model Eq. (7.14) localized states are formed after the early stage of the instability development and exponential growth of the amplitude. Such localized modes are the analog of multi-dimensional solitons localized in both time (continuous variable) and space (discrete variable) (see e.g., Aceves et al., 1994a; and also Pouget et al., 1993, for the similar localized modes in a discrete two-dimensional lattice). The collapse mechanism can be associated with the initial stage of the evolution, whereas any singular dynamics is suppressed by discreteness (Aceves et al., 1995).

Thus, the lattice discreteness in the transverse dimension prevents collapse and it allows to develop a sequence of spatially localized states as a result of the transverse instability of a plane soliton acting, in some sense, as an effective nonlinearity saturation.

Some further results in the analysis of transverse instabilities of solitons in discrete lattices and waveguide arrays were obtained by Darmanyan et al. (1997), who considered transverse instability of envelope solitons in the model (7.14) but with an arbitrary wave number of the carrier wave. Unlike the case of the continuous NLS equation where the instability growth rate does not depend on the carrier wave number, this is not so even for modulational instability in discrete lattices where the carrier wave modifies the effective dispersion and it can lead to stabilization of plane waves (Kivshar and Peyrard, 1992).

Considering the stability of the NLS solitons in the model (7.14) with a moving carrier wave, i.e.

$$\psi_n(x,t) = \Phi_s(x - vt)e^{i(\omega t - qn - \kappa x)} ,$$

Darmanyan et al. (1997) obtained the following result for the cut-off wave number Q_c of the instability band [cf. Eq. (7.15)]

$$4K \cos q \sin^2(Q_c/2) = 3\omega , \tag{7.16}$$

which is valid for $\cos q > 0$ and therefore it generalizes the result (7.15). Similarly, for $\cos q > 0$ the plane soliton forms a train of stable localized modes of higher dimensions (the soliton bunching effect), as in the case of the elliptic NLS equation with a saturable nonlinearity.

For the case $\cos q < 0$ the analysis is more involved, and effectively this case coresponds to the hyperbolic NLS equation, and it can be analyzed by an asymptotic technique. In that case, the maximum growth rate is reached near the edge of the Brillouin zone, and asymmetric instability leads to a bending of the initially plane soliton. Thus, the carrier wave number q allows to vary the effective dispersion in the lattice and to transform the scenario of the soliton instability from the elliptic case to the hyperbolic one.

Recently, Relke (1998) extended this kind of analysis to the case of a waveguide array with a periodically varying coupling constant in the array, i.e. for $K \to K_n$. In a contrast to continuous models, in the mixed continuous-discrete NLS model (7.14) the asymptotic analysis of the soliton stability can be performed in the approximation of *weak coupling* but for arbitrary values of the perturbation wave numbers. For a periodic variation of the coupling constant, $K_n = K + \Delta K \cos[\pi(n + \frac{1}{2})]$, Relke (1998) revealed the existence of two types of unstable eigenmodes, *optical* and *acoustic* ones. This analogy comes from the fact that a discrete array of waveguides coupled by a periodically varying constant K_n form an effective *diatomic lattice* with the spectrum consisting of two branches separated by a gap (see e.g., Kivshar and Flytzanis, 1992).

The existence of two branches of unstable eigenvalues open a new mechanism of the instability scenario when at the initial stage of the instability-driven evolution the energy gets redistributed between adjacent solitons in the array, and a kind of doubling of a spacial period occurs (Relke, 1998).

7.2. KdV-type models

The integrable KP equation (2.6) is an example of a higher-dimensional model that describes *long-wave* transverse modulations of a KdV soliton. Here we discuss a few more examples of this type.

7.2.1. The Zakharov–Kuznetsov equation

The Zakharov–Kuznetsov (ZK) equation is another, alternative version of a nonlinear model describing two-dimensional modulations of a KdV soliton. It was first derived for describing the evolution of the ion density in strongly magnetized ion-acoustic plasmas (Zakharov and Kuznetsov, 1974). If a magnetic field is directed along the axis x, the ZK equation in renormalized variables takes the form,

$$u_t + uu_x + \nabla^2 u_x = 0 \ . \tag{7.17}$$

The ZK equation appears as a generalization of the KdV equation to two spatial dimensions but, unlike the KP equation, it is not integrable by the inverse scattering transform method (see, e.g., Shivamoggi et al., 1993).

Instability of a plane KdV soliton to transverse dimensions, in the framework of the model (7.17), was extensively investigated analytically and numerically (e.g., Laedke and Spatschek, 1982; Laedke et al., 1986; Infeld and Frycz, 1987; Frycz and Infeld, 1989a, 1989b; Allen and Rowlands, 1993, 1995; Infeld, 1985; Infeld et al., 1995; Bettinson and Rowlands, 1998a,b). To analyze the soliton instabilities, Laedke et al. (1986; see also Laedke and Spatschek, 1982) developed a modification of the short-scale small-amplitude asymptotic analysis. Instead of the unstable NLS equation (6.2), for the ZK equation (7.17) they derived a first-order evolution equation,

$$-\,\mathrm{i}p_c a_Y + \beta a_T + \gamma |a|^2 a = 0 \ , \tag{7.18}$$

where the coefficients β and γ are positive. In contrast to the case of the unstable NLS equation, this short-scale asymptotic model describes a monotonic transition from a planar soliton to a periodic train of two-dimensional solitons generated along the soliton front (Laedke et al., 1986). Such a scenario of the soliton decay was confirmed numerically by Frycz and Infeld (1989b) and Frycz et al. (1992). Later, Allen and Rowlands (1993) suggested an extension of the multi-scale perturbation approach and also obtained the maximum growth rate for all k in the form of a two-point Padé approximant (see also the case of obliquely propagating plane solitons discussed by Allen and Rowlands (1995)).

7.2.2. The Shrira and Benjamin–Ono equations

Large-amplitude localized structures were experimentally observed in boundary layers generated by subsurface shear flows (Kachanov et al., 1993). An analytical description of this phenomenon was developed by Shrira (1989) who derived a two-dimensional model generalizing the well-known Benjamin–Ono (BO) equation. If a shear flow is directed along the x-axis, the Shrira model is described by the equation,

$$u_t + uu_x + Q\{u_x\} = 0 , \tag{7.19}$$

where $Q\{u\}$ is the Cauchy–Hadamard integral transform,

$$Q\{u\} = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{u(x', y', t)\, dx'\, dy'}{[(x-x')^2 + (y-y')^2]^{3/2}} .$$

Linear transverse instability of (1 + 1) dimensional solitons of the model (7.19) was predicted and analyzed by Pelinovsky and Stepanyants (1994) (see also D'yachenko and Kuznetsov, 1994). Numerical simulations indicated the formation of *singular two-dimensional structures* in Eq. (7.19) (D'yachenko and Kuznetsov, 1995). This conclusion was confirmed by Pelinovsky and Shrira (1995) who constructed approximate analytical solutions describing a singular localized mode. Eq. (7.19) can be simplified under the assumption $|u_{yy}| \ll |u_{xx}|$, and then it takes the form of a two-dimensional BO equation,

$$u_t + uu_x + H\{u_{xx}\} + \tfrac{1}{2}H\{u_{yy}\} = 0 , \tag{7.20}$$

where $H\{u\}$ is the Hilbert integral transform,

$$H\{u\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(x', y, t)\, dx'}{x' - x} .$$

Transverse modulations of the BO soliton,

$$u(x) = \frac{2v}{[1 + v^2(x-s)^2]} ,$$

can be studied by the averaged Largangian method outlined in Section 4.2, which generates the modulation equations for the soliton parameters,

$$\begin{aligned} &v_T + \tfrac{1}{2}(v^2 s_Y)_Y = 0 , \\ &s_T - v + \frac{1}{2}\varepsilon^2\left(v s_Y^2 + \frac{v_Y^2}{v^3} - \frac{v_{YY}}{v^2} \right) = 0 . \end{aligned} \tag{7.21}$$

A small-amplitude limit of these equations corresponds to the elliptic Boussinesq equation for which the asymptotic solutions for the transverse soliton self-focusing was obtained by Pelinovsky and Shrira (1995). Furthermore, there exists an exact transformation for the modulation equations (7.21),

$$v = \left(\frac{T_0}{T_0 - T}\right)^{1/3} W(\eta), \quad \eta = \left(\frac{T_0}{T_0 - T}\right)^{1/3} Y ,$$

$$s = \frac{3}{2}T_0\left[1 - \left(\frac{T_0 - T}{T_0}\right)^{2/3}\right] - \frac{1}{3T_0}\int_0^{\eta} \frac{z\, dz}{W(z)} ,$$

where

$$W(\eta) = \frac{p^2}{1 - \sqrt{1 - p^2}\cos(p\eta)} ,$$

p is the transverse wave number for a periodic modulation, T_0 is the time of blowup of the asymptotic solution. This solution resembles the formation of singular localized modes along the front of a planar soliton due to the development of a transverse instability. Basically the same phenomenon was observed numerically by D'yachenko and Kuznetsov (1995).

7.3. Kinks in the Cahn–Hilliard equation

A wide class of the stability problems can be formulated for solitary wave solutions of *dissipative models*. In such models, the most common localized solution is a kink (or an interface solution) that connects two equilibrium states of the system. Generally speaking, the asymptotic methods described above for conservative models are well applicable for the stability analysis of localized solutions of dissipative models. To give a specific example and also point out some new features of dissipative models, here we mention some results for the transverse stability of the (1 + 1) dimensional kink solution in the Cahn–Hilliard equation (Cahn and Hilliard, 1958) with general nonlinearity,

$$u_t = \nabla^2\left(\frac{\mathrm{d}F}{\mathrm{d}u} - \nabla^2 u\right) , \tag{7.22}$$

where $F(u)$ is some general nonlinear free energy. Eq. (7.22) admits a stationary (1 + 1) dimensional kink solutions $u_k(x)$ connecting two equilibrium states $u = u_1$ and $u = u_2$, i.e. it is assumed that $F(u_1) = F(u_2) = 0$, $F'(u_1) = F'(u_2) = 0$, $u_1 \leq u_k(x) \leq u_2$.

Eq. (7.22) was derived from classical thermodynamic considerations of the interdiffusion of two components A and B, to describe the phase transition induced rapidly decreasing the temperature from some $T_1 > T_c$ to some $T_2 < T_c$ (for an overview of the physics, see Novick-Cohen and Segel, 1984).

In a particular case, the free energy $F(u)$ can be taken in the form,

$$F(u) = \tfrac{1}{4}(1 - u^2)^2 , \tag{7.23}$$

and then the Cahn–Hilliard equation becomes, $u_t = \nabla^2(u^3 - u - \nabla^2 u)$, and its kink solution can be found in an explicit form, $u_k(x) = \tanh(x/\sqrt{2})$.

To analyze stability of the (1 + 1) dimensional kinks $u_k(x)$ to small perpendicular perturbations of the wave number k, we follow Bettinson and Rowlands (1996a), and write a solution of Eq. (7.22) in the form,

$$u(\boldsymbol{r}, t) = u_k(x) + \varepsilon\delta u(x)\mathrm{e}^{\mathrm{i}(k_y y + k_z z)}\mathrm{e}^{\gamma t} ,$$

where $u_k(x)$ is a kink solution of the equation $(u_k')^2 = 2F(u_k)$, and $\delta u(x)$ stands for a perturbation amplitude. The linear eigenvalue problem for $\delta u(x)$ can be analyzed by means of the asymptotic expansions for small and large $k \equiv \sqrt{k_y^2 + k_z^2}$. For small k, the asymptotic result for the growth rate

$\gamma(k)$ is (Bettinson and Rowlands, 1996a)

$$\gamma(k) = -\gamma_3 k^3 - \gamma_4 k^4 , \tag{7.24}$$

where the positive coefficients γ_3 and γ_4 are defined by the specific form of $F(u)$, e.g. for the potential (7.23) they are $\gamma_3 = \sqrt{2}/3$ and $\gamma_4 = 11/18$. In the lowest order, the result (7.24) agrees with that obtained by a variational method (Shinozaki and Oono, 1993).

In the case of large k, the asymptotic expansion in k^{-1} yields: $\gamma/k^3 = -1 - \gamma_c/k^2 + \cdots$, where, e.g., $\gamma_c \approx 0.303$ for the potential (7.23). Bettinson and Rowlands (1996a) suggested a general Padé approximation to describe the full dispersion relation $\gamma(k)$ for all k, which provides a reasonably good agreement (within 1.3% of the exact results) with numerical results and particular cases. This analysis shows that the kink solution of the dissipative model (7.22) is *stable to transverse perturbations for all k*.

Eq. (7.22) admits also both *cylindrical* and *spherically symmetric* stationary kink solutions. As has been shown by Bettinson and Rowlands (1996b) for the case of large radius R, the cylindrically symmetric kink solution is stable to perturbations involving angular variation, but is unstable to a general perturbation. In contrast, the spherically symmetric kink solution is stable for all small perturbations. This suggests that the unstable cylindrically symmetric solution may decay into spherically symmetric states similar to the cases discussed above for some conservative models.

It is interesting to note that a discrete version of the Cahn–Hilliard equation, which can also have some physical applications, shows new features for the kink stability (see Bettinson and Rowlands, 1998a,b). Analytical kink solution of a discrete model, $u_n = \tanh\beta \tanh(n\beta + s)$, can be found for some special form of $F(u)$, so that the asymptotic analysis for small and large k can be developed, similar to the continuum case. As a result of that analysis, a kink is stable to transverse perturbation in the discrete model as well, however, the growth rate vanishes at the edges of the Brollouin zone, $k = 2\pi p$, p is integer.

8. Experimental observations

8.1. Self-focusing and bright solitons

Theoretical predictions of self-focusing of light in an optical medium with nonlinear refractive index (Askar'yan, 1962; Chiao et al., 1964; Talanov, 1964) were followed by experimental evidence of this phenomenon in different optical materials, e.g. glasses, Raman-active liquids, gas vapors, etc. In particular, Pilipetskii and Rustamov (1965) reported the generation of one-, two- and three-filaments due to self-focusing of a laser beam in different organic liquids. Later, Garmire et al. (1966) reported a direct observation of the evolution of beam trapping in CS_2 in the simplest cylindrical mode. They found that the threshold, trapping length, nonlinearity-induced increase in the refractive index in the trapped region, and beam profile are consistent with theoretical predictions, and the steady-state input beam of circular symmetry asymptotically collapses to a bright filament as small as 50 μm. As a matter of fact, this was one of the first experimental manifestations of the phenomenon which we now call *spatial optical soliton*.

Because ruby-laser beams used in the experiments have intensities far above threshold for self-trapping in CS_2 (25 ± 5 kW), Garmire et al. (1966) also observed the formation of rings around the self-focused spots and the development of many filaments from an apparently homogeneous beam about 1 mm in diameter and considerably above the threshold power. The former effect can be associated with the existence of a set of higher-order circularly symmetric steady-state modes (Yankauskas, 1966; Haus, 1966), whereas the latter effect is a direct manifestation of the transverse beam instability, spatial modulational instability of a broad beam.

Steady-state self-focusing and self-trapping was observed for several other media, including potassium vapor (e.g., Grischkowsky, 1970), sodium vapor (Bjorkholm and Ashkin, 1974), etc. Additionally, detailed studies of a spatial breakup of a broad optical beam due to self-focusing was reported by Campillo et al. (1973, 1974) who used a 50-cm cell of CS_2 to study self-focusing and observed that radially symmetric ring patterns created by circular apertures breakup into focal spots having azimuthal symmetry and regular spacing. This kind of effect can be associated with the transverse modulational instability of quasi-plane bright rings created by the input beam, and the number of the bright spots and critical powers are in a good qualitative agreement with the simple theory of transverse instabilities, as was discussed in detail later by Campillo et al. (1974).

A number of similar experiments were performed later for different types of nonlinear media, including artificial Kerr media made from liquid suspensions of submicrometer particles (e.g., Ashkin et al., 1982), where the smallest-diameter self-trapped filaments ($\sim 2\,\mu$m) were observed.

Similar experiments were recently done for *vortex rings*, i.e. bright rings with a nonzero angular momentum created by passing the laser beam through a diffracting phase mask and then propagating it in a nonlinear medium (a 20-cm cell with rubidium vapor) (Tikhonenko et al., 1995, 1996b) and also for a quadratically nonlinear medium (KTP crystal) (Petrov et al., 1998). An angular momentum introduced in the input beam, strongly affected the dynamics of bright spots (in fact, spatial solitons) created by the transverse instability of the rings, so that they can attract and repel each other, or even fuse together.

Formation of a variety of different patterns of spots (bright spatial solitons) was investigated by Grantham et al. (1991) in a sodium vapor. They varied the input beam power from 30 to 460 mW and observed spatial bifurcation sequences due to spatial instabilities seeded by intentionally introduced aberrations. They used the structure of the instability gain curve for an input-wavefront-encoding feedback to accelerate particular unstable wave vectors, and observed complicated spatial bifurcations as a function of intensity or detuning, with "*… complexity and beauty rivaling that of a kaleidoscope*" (Grantham et al., 1991).

The analysis of self-focusing based on the spatial $(2+1)$ dimensional NLS equation and associated with the spatial instabilities, bifurcations, and formation of spatial solitons is valid for both c.w. beams and long pulses. In contrast, short pulses undergoing self-focusing do not collapse to wavelength dimensions. A number of experimental results (e.g., Strickland and Corkum, 1991) demonstrated the resistance of short pulses (~ 50 fs) to self-focusing. In spite of the fact that these process can be modelled by the hyperbolic NLS equation with normal group-velocity dispersion, experimental results (Strickland and Corkum, 1994) indicate that spectral dispersion and other non-slowly-varying are also important to explain different behavior of short pulses.

A detailed experimental investigation of the self-focusing dynamics of a femtosecond pulse in a normally dispersive (glass) medium was recently reported by Ranka et al. (1996) who observed one of the main effects predicted by the theory based on the hyperbolic NLS equation, i.e. the splitting of a short pulse (85–90 fs) into two pulses for the power above the threshold value, $P > P_{cr} \approx 3\,\mathrm{MW}$, and even additional splittings, for higher powers ($P \approx 4.8\,\mathrm{MW}$). Moreover, Ranka et al. (1996) noticed that above the threshold power for pulse splitting, the pulse spectrum undergoes significant broadening which eventually develops, at higher powers, into *supercontinuum generation* (SCG) or while-light generation first observed by Alfano and Shapiro (1970). Such a spectrum broadening confirms a hypothesis that SCG is a result of the nonlinear dynamics of self-focusing in which the temporal and spatial degrees of freedom are coupled. However, the corresponding model describing both these phenomena, i.e. the pulse splitting and SCG, should be not based on the slowly varying envelope approximation. More recently, Diddams et al. (1998) reported the similar effects for the propagation of intense fs pulses in fused silica. Frequency-resolved optical gating was used to characterize the pulse splitting into subpulses which were found to be not generally symmetric, in accordance with the theoretical predictions based on a three-dimensional NLS equation that includes the Raman effect, linear and nonlinear shock terms, and third-order dispersion (Zozulya et al., 1998).

Theoretical prediction and a number of experimental observations of beam self-trapping in photorefractive media allow to observe spatial solitons in crystals at relatively low input powers. The first observation of two-dimensional spatial solitons was reported by Shih et al. (1995; see also Shih et al., 1996) who used an electric field of 5.8 kV/cm applied to a crystal of strontium barium niobate (SBN) to create an effective self-focusing nonlinearity and trap an optical beam into a filament as small as 9.6 μm at micro-Watt power levels.

Experimental observation of breakup of a quasi-plane bright spatial soliton into a sequence of higher dimensional solitons due to the transverse ('neck'-type) instability was observed by Mamaev et al. (1996c). In the experiments, Mamaev et al. (1996c) used a 10 mW beam from a He–Ne laser ($\lambda = 0.6328\,\mu\mathrm{m}$) to create a highly asymmetric elliptical beam ($15\,\mu\mathrm{m} \times 2\,\mathrm{mm}$) with a controlled waist. The beam was directed into a photorefractive crystal (10×9 mm) of SBN:60, lightly doped with 0.002% by weight Ce. A variable DC voltage was applied along the crystal $\hat{c}$-axis to take advantage of the largest component of the electro-optic tensor of SBN, and to vary an effective self-focusing nonlinearity.

Fig. 12 shows the near-field distributions of the input (a) and outputs [(b) to (f)] beam for different values of the applied voltage, i.e. different values of the nonlinearity. First of all, without the applied voltage, the output beam spreads due to diffraction [Fig. 12(b)]. As the nonlinearity increases, the beam starts to self-focus [Fig. 12(c)] forming a self-trapped channel of light [Fig. 12(d)]. A further increase of nonlinearity leads to the modulational instability, and the self-trapped stripe beam breaks up into a periodic sequence of filaments [Figs. 12(e) and (f)]. No artificial seeding was added to the input beam, and it was argued that the instability developed from the natural level of noise present on the beam and /or in the crystal (Mamaev et al., 1996c).

As a matter of fact, the transverse instability of a plane spatial soliton in photorefractive media is more complicated phenomenon than that in a cubic Kerr medium, due to the applied electric field which makes the problem anisotropic. The equations describing these effects can be written as a system of normalized equations for an optical beam envelope, B, and the normalized electrostatic potential induced by the beam, ϕ, as follows:

$$\begin{aligned} &\mathrm{i}B_z + \tfrac{1}{2}\nabla_\perp^2 B + \phi_x B = 0\ , \\ &\nabla_\perp^2 \phi + \nabla_\perp \ln(1 + |B|^2)\nabla_\perp \phi = \{\ln(1 + |B|^2)\}_x\ , \end{aligned} \tag{8.1}$$

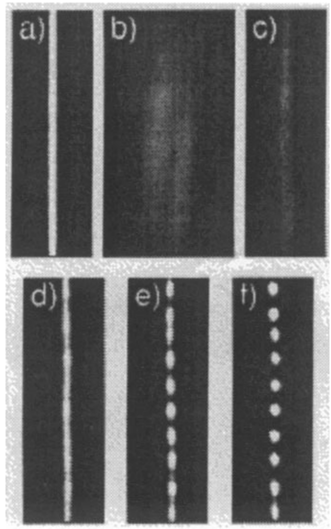


Fig. 12. Experimental observation of a plane-soliton decay in a photorefractive medium: (a) input beam, and (b)–(f) output beams for different values of the applied DC voltage, $V = 0$, 600, 1000, 1560, and 2200 V, respectively (Mamaev et al., 1996c).

where ∇ acts on components x and y perpendicular to the direction of propagation of the beam z (see, e.g., Mamaev et al., 1996c).

Numerical simulations were carried out by Mamaev et al. (1996c) in order to show that the transverse self-focusing of a plane soliton can occur for Eqs. (8.1), similar to the elliptic NLS equation. First of all, a plane soliton is a stationary solution of Eqs. (8.1), which satisfy an equivalent saturable NLS equation because the second equation of Eq. (8.1) can be integrated to yield, $\phi_x = (|B|^2 - B_0^2)/(1 + |B|^2)$. Transverse instability of this stationary solution was analyzed, for both focusing and defocusing cases, in the framework of the complete model (8.1) by Infeld and Lenkowska-Czerwińska (1997), by means of the asymptotic p-expansions for small wave numbers, and also by using a trial sech-type function for the soliton profile.

At last but not least, we would like to mention the most recent observations of the soliton instability due to parametric beam self-focusing in quadratic nonlinear media, via the cascaded interaction between the fundamental wave and its second harmonic under the condition of the phase matched second-harmonic generation process (see, e.g., Stegeman et al., 1996, for a comprehensive review of cascaded nonlinearities). The cascading mechanism of self-focusing allows to observe a number of interesting nonlinear effects in noncentrosymmetric crystals, similar to those already observed in gases and liquids. In particular, Fuerst et al. (1997a; see also Fuerst et al., 1997b) reported the first observation of the transverse modulational instability of a spatial two-wave soliton in a quadratically nonlinear optical medium (see also Section 7.1.2 for discussions of the theoretical results).

In the experiments, the transverse instability was demonstrated for a 1 cm long KTP crystal, similar to that earlier used in the experiments on the beam self-focusing (Torruellas et al., 1995). The 35 ps pulses were generated by Nd:YAG laser with 10 Hz repetition at $\lambda = 1064$ nm. A variable elliptical beam was created with an adjustable cylindrical telescope, so that the small dimension of

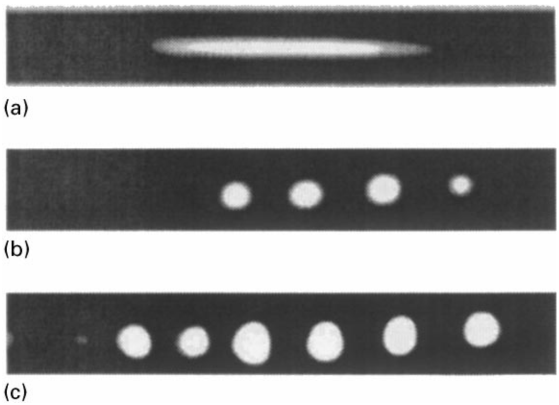


Fig. 13. Break-up of an elliptic beam in a quadratic optical medium: (a) input beam, (b) output beam at 48 GW/cm^2, (c) output beam at 57 GW/cm^2. Shown is the beam of the fundamental frequency but the similar structures appear for the second harmonic beam (Fuerst et al., 1997a).

the beam was about 20 μm, and the larger dimension was varied to be more than in five times larger, at the input face of the crystal. This ratio made reasonable plane soliton since at a 1 cm long crystal the effects of diffraction are minimal with such large waists.

Figs. 13(a)–(c) present the experimental results (Fuerst et al., 1997a) for the evolution of the input elliptic beam with the ratio of its dimension 12:1, shown in Fig. 13(a). Figs. 13(b) and (c) show the beam of the fundamental frequency at the output of the crystal for intensities of 48 and 57 GW/cm^2, respectively. The main effect observed by Fuerst et al. (1997a) is a breakup of the beam into a sequence of well-defined circular spots with the radius 9.5 ± 1.5 μm [see Fig. 13(c)], essentially equal to the value of 10–12 μm obtained for spatial solitons created by 20 μm circular beams (Torruellas et al., 1995). According to the theory, the number of created spatial solitons should depend on the intensity, and indeed the experimental results were found to give a qualitatively good agreement with the results of the simplified theory of modulational instability in quadratic media.

8.2. Dark solitons

Most of the experimental demonstrations of spatial and temporal dark solitons have been recently discussed in the paper by Kivshar and Luther-Davies (1998). For the case of spatial dark solitons, the experiments reported the creation of dark soliton stripes in a $(2+1)$-dimensional geometry. As discussed above, such stripes should be unstable due to transverse modulational instability which leads to stripe breakup and the eventual creation of optical vortex solitons. However, it turns out that this instability was avoided in the early experiments by the use of finite-sized background beams and weak nonlinearity. By increasing nonlinearity, the transverse instability should be observed even with finite sized beams. The first experiments to verify the existence of this transverse instability, and through it the creation of optical vortex solitons, have been performed by Tikhonenko et al. (1996a) using a continuous wave, Ti:sapphire laser and a nonlinear medium comprised of atomic rubidium vapour. Very similar observations, with less evidence of the stripe decay into a sequence of vortex solitons, were performed almost simultaneously by Mamaev et al. (1996a,b) for spatial dark solitons in a photorefractive medium.

In the experiments using rubidium vapour (Tikhonenko et al., 1996a), the laser output was a linearly polarized slightly elliptical Gaussian beam with a wavelength tuned close to the rubidium atom resonance line at 780 nm. A π phase jump was imposed across the beam center using a mask and the resulting beam imaged into the nonlinear medium. The rubidium vapor concentration could be increased up to $10^{13}\,\mathrm{cm}^{-3}$ by changing the cell temperature. Images of the beam at the output of the cell were recorded by a CCD camera and frame capture system. As a matter of fact, a schematic of this experimental arrangement is similar to that used to observe optical vortex solitons (see, e.g., Kivshar et al., 1998, and references therein).

The important step in observing the instability was to resonantly enhance the value of nonlinearity of the medium by tuning the laser frequency close (-0.4 to -1.0 GHz) to the rubidium atom D_2 line and the use of the maximum vapor pressure consistent with tolerable absorption. The power in the beam at the input face of the cell was 240 mW with a $1/e^2$ waist of 0.3 mm. A maximum nonlinear refractive index change of order of 10^{-4} was achieved.

Fig. 14 shows a series of output intensity profiles observed experimentally, with increasing cell temperature and with the detuning fixed at approximately 0.85 GHz. For vanishingly small vapor concentration, the beam underwent linear propagation through the medium. With increasing temperature (i.e., increasing nonlinearity), the output beam developed a vertically uniform dark soliton stripe, as shown in Fig. 14(a). Further increase in the temperature led to the growth of a periodic modulation of the uniformity of the stripe. As the temperature was further increased, the breakup of the stripe began, initially appearing as a growing, 'snake-type' bending [Fig. 14(b)], then as breaking, with field coalescing into dark spots at the inflection points in the bends [Fig. 14(c)]. At the highest nonlinearity the dark spot assumed close to circular symmetry consistent with the predicted formation of a pair of optical vortex solitons [Fig. 14(d)].

Tikhonenko et al. (1996a) also carried out numerical simulations based on the generalized NLS equation including saturation and dissipative effects for comparison with the experimental results. The calculated output intensity distributions showed the same dynamics observed experimentally

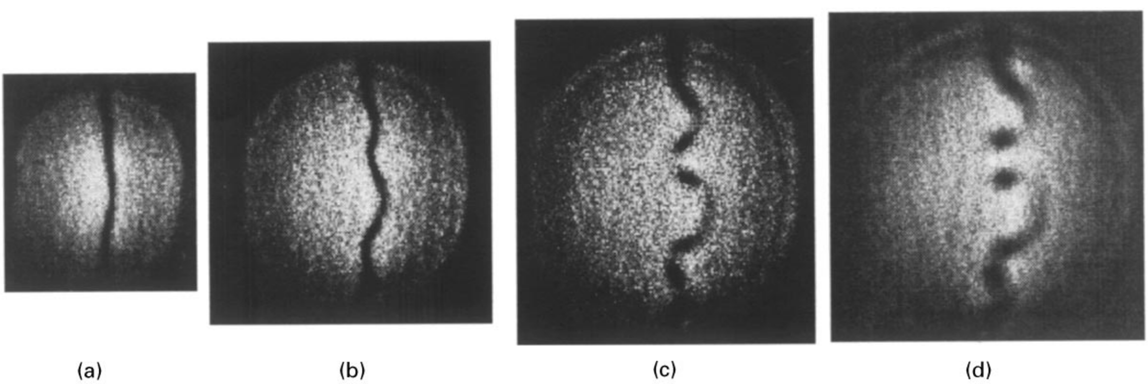


Fig. 14. Out beam intensity profiles demonstrating the instability of a dark soliton stripe as the nonlinearity is increased. The cell temperatures are: (a) 82°C (b) 90°C, (c) 112°C, and (d) 125°C. The power of the beam at the cell input was 240 mW, corresponding to a maximum intensity $\approx 170\,\mathrm{W/cm}^2$ (Tikhonenko et al., 1996a).

with increasing nonlinearity. The beam defocusing, power depletion and instability growth rates seen in experiments appeared to be well approximated by the simulations. However, the period of the transverse perturbation corresponding to the maximum growth rate appeared to be *smaller* than that seen in experiment, by a factor of 1.5–2. This discrepancy is most likely due to (i) the physically complicated nonlinear response of rubidium vapour, which was only approximated by the model used in the simulations; (ii) the difficulty in accurately characterizing the initial field which was found to sensitively affect the simulations. It should be noted that this sensitivity was not observed in the experiments, suggesting that the breakup process may have been partially stabilized by some physical mechanisms not included in the model (e.g., nonlocality in the form of diffusion).

Similar experimental results were reported by Mamaev et al. (1996a). In their experiments a biased photorefractive SBN crystal, irradiated with a 10 mW He–Ne laser beam containing a phase step, was used as the nonlinear medium. Numerical simulations using the generalized NLS equation with a saturable nonlinearity demonstrated a similar breakup of the initial stripe into a set of optical vortex solitons as reported by Tikhonenko et al. (1996a). The effectively nonlinearity could be varied by increasing the bias voltage on the crystal. When zero voltage was applied, the dark stripe spread due to diffraction as did the background beam. As the applied voltage increased the background beam underwent self-defocusing and a dark-stripe soliton was clearly formed. A further increase in the voltage up to 990 V [the maximum voltage reported was 1410 V (Mamaev et al., 1996a) and 2000 V (Mamaev et al., 1996b)] led to the appearance of the snake-like bending of the dark soliton stripe. However, the final state was markedly different from that shown in Fig. 14(d) because it did not present clear evidence of the creation of optical vortex soliton pairs. However, the authors reported the observation of zeroes in the electromagnetic field from interferometric measurements of the output beam with the distances between the zeroes being about 40 μm. This measurement indicates that wave-front dislocations similar to single vortices were being formed.

The experiments, which reported the first observation of the transverse dark-soliton instability, indicate that the scenario of the periodic modulation of a plane soliton is really hard to observe because it is strongly affected by radiative losses and dissipative losses in real physical systems. However, a decay of a plane dark soliton into vortex solitons is readily observed being a fundamental physical phenomenon only slightly affected by dissipation.

9. Concluding remarks

We have discussed above the asymptotic analytical methods for describing linear and nonlinear regimes of the symmetry-breaking instabilities and self-focusing of plane solitons induced by transverse modulations in higher spatial dimensions or self-modulation due to temporal dispersion. The analytical methods are associated with different approximations and approaches, namely

- geometric optics approach,
- linear stability analysis,
- gas dynamics approach (Whitham equations),
- modulation equations for the soliton parameters,

- long-scale and short-scale asymptotic expansions,
- exact solutions for integrable nonlinear equations.

Each of those analytical methods is valid in the corresponding asymptotic domain where analytical solutions can be obtained. We have demonstrated the applications of all of those methods to the analysis of the transverse instability of bright and dark envelope solitons in the cubic NLS equation, and also long-wave solitons of the KdV equation. The asymptotic methods allow to describe, in a self-consistent manner, both weak nonlinear and linear effects in the soliton self-focusing, and they provide a rather informative and adequate picture of the instability-induced soliton dynamics. We believe that these analytical methods will be useful in the analysis of the soliton self-focusing dynamics in other physically important nonlinear models, and also for some other problems we briefly summarize below.

Indeed, in a context of different physics, there exist a number of problems which are closely connected to the topics discussed above. Some of them still require further investigation, and we believe that the analytical methods and physical concepts discussed above may help to achieve a further progress in this research. Below, we mention just a few such problems.

9.1. Random or periodic fluctuations

One of the important physical generalizations of the problems discussed above is the soliton self-focusing and instabilities in the presence of *spatial* (*random or periodic*) *inhomogeneities* or *temporal fluctuations* of the physical parameters. The effect of a periodic spatial modulation of the refractive index in the spatiotemporal pulse evolution was analyzed by a variational aproach (Aceves and De Angelis, 1992), by the virial theorem (Turitsyn, 1993a,b), and by the averaging method (Kivshar and Turitsyn, 1994). More recently, Gaididei and Christiansen (1998) have demonstrated analytically, by means of the virial theorem and the Furutsu–Novikov technique, that delta-correlated random fluctuations 'delay' the threshold power for the pulse self-focusing. This is in agreement with the case of periodic variations where, for the averaged equations, the effect of renormalized nonlinearity was demonstrated (Kivshar and Turitsyn, 1994), suggesting a shift of the instabilily band. The qualitatively similar effects were observed in numerical simulations (see, e.g., Rasmussen et al., 1995; Christiansen et al., 1996a,b).

9.2. Self-focusing of coupled waves

Other class of important physical problems is *self-focusing of coupled waves*. The effect of the wave coupling on modulational instability was first analyzed by Agrawal (1987; see also Agrawal et al., 1989), in the framework of two coupled NLS equations. Agrawal (1987) pointed out that the cross-phase modulation between two waves leads to a number of interesting new effects and, in particular, he predicted that modulational instability of two waves can occur even in the case of normal group-velocity dispersion when a single wave is always stable. For the spatial beams, the effects of the wave interaction have been studied for the mutual focusing of coupled waves (Berkhoer and Zakharov, 1970; McKinstrie and Russell, 1988; Agrawal, 1990b; Bergé, 1998b), and also for the transverse modulational instability of both counterpropagating (Vlasov and Sheinina, 1983; Firth and Paré, 1988; Firth et al., 1990) and copropagating (Agrawal, 1990a; Luther and

McKinstrie, 1990) optical beams, and some of the theoretical predictions were observed experimentally (Grynberg et al., 1988; Gauthier et al., 1990).

In general, the instability growth rate of the coupled waves is larger, and a range of unstable wave numbers is broader, than those of either wave alone. Moreover, waves that are modulationally stable by themselves are often unstable in the other's presence, and this is true for both copropagating and counterpropagating waves. The transverse modulational instability of copropagating waves is always convective.

An overview of transverse effects in other cases, when two or more beams propagate and interact in a nonlinear medium can be found in the paper by Abraham and Firth (1990). A similar kind of transverse instabilities, for both focusing and defocusing media, is expected for two-component (or vector) spatial optical solitons. The simplest problem of this kind is the analysis of the effects of *vectorial* perturbations on *scalar* solitons (see, e.g., Bondeson, 1979). More general analysis of the transverse instability should include the vectorial nature of solitons, and this is still an open direction of research. Recently, Musslimani et al. [1999; see also Skryabin and Firth (1999) for a more general case of incoherently coupled bright solitons] made the first step in this direction and found the long-wave-expansion results for the transverse instability of *vector solitons*, including a special case of dark-bright soliton pairs.

An important class of the problems is associated with the models of interaction between high-frequency and low-frequency fields, such as the Zakharov equation for the nonlinear Langmuir waves (e.g., Kuznetsov et al., 1986, and references therein), the model for multi-dimensional light bullets supported by nonresonant quadratic nonlinearity (Ablowitz et al., 1997), the Iizuka-Kivshar model for the propagation of gap solitons in quadratically nonlinear gratings (Iizuka and Kivshar, 1999), etc. Some preliminary results (e.g., Kuznetsov et al., 1986; Hadžievski and Škorić, 1991) are not sufficient to make a general conclusion about the effect of the wave coupling of the multi-component soliton self-focusing.

At last, Saffman (1998) suggested another form of the coupled NLS equations describing the interaction of copropagating electromagnetic and matter waves, when the optical self-focusing generates dipole forces on the atoms (the original physical concept was suggested by much earlier Klimontovich and Luzgin, 1979). Numerical studies of the coupled equations revealed Saffman (1998) the filamentation due to modulational instability, and the formation of localized waves anticorrelated in the two fields. Even there exists no analysis of the solitary waves and their stability in two spatial dimensions, the numerical results suggest a typical scenario of the soliton self-focusing with the formation of localized solutions stable in higher dimensions.

9.3. Transverse vs. longitudinal instabilities

Several generalized evolution models, such as the saturable NLS equation (Mamaev et al., 1996a,b,c; Tikhonenko et al., 1996a,b) and the model for two- and three-wave mixing in quadratic nonlinear materials (Fuerst et al., 1997a; DeRossi et al., 1997a,b; Skryabin and Firth, 1998b) do not possess properties of the scaling invariance. In such *nonscalingly invariant* models the longitudinal instabilities of solitary waves are known to occur in a certain range of the soliton parameters. Then, the transverse self-focusing may compete with the longitudinal soliton instabilities, the latter may display rather different types of long-term instability-induced soliton dynamics (Pelinovsky et al., 1996a,b). The analytical studies of such a competition between two instabilites and the soliton evolution is still an open problem, although a proper combination of the asymptotic methods developed for the longitudinal instabilities (see, e.g., Pelinovsky and Grimshaw, 1997) with the methods described here are expected to lead to the asymptotic analytical models which will cover the whole class of such complex phenomena.

9.4. Transverse instabilities of nonlinear guided waves

An extensive theoretical literature is devoted to nonlinear stationary waves localized at optical interfaces and in layered dielectric media (see, e.g., Maradudin, 1983, and references therein). Such nonlinear guided waves are localized due to the fact that one or more of the dielectric layers shows either a positive or negative nonlinear optical response to an incident electromagnetic wave, and the trapped surface modes may appear above a certain critical power. These are the so-called *nonlinear guided waves* which can be regarded, in some approximation, as spatial solitons trapped by the interface, and therefore they are expected to be unstable to transverse spatial fluctuations in the dimension parallel to the coupling layers (e.g., Moloney, 1987). As a result of the development of such an instability, localized peaks located within the guided layer or near the interface appear, similar to the decay of plane solitons into two-dimensional solitons (see, e.g., Vysotina et al., 1990). Because a decay of guided waves is affected by an attractive potential created by the interface, the instability scenarious, especially in the case of the bending (snake-type) dynamics of the hyperbolic NLS equation, are expected to be different from those for spatial solitons in homogeneous models. However, an analytical study of this type of problems is still absent.

9.5. Higher-order localized modes

At last, we would like to mention a variety of problems involving the instability and decay of higher-order localized modes and ring-type soliton structures.

Additionally to the radially symmetric (no nodes) localized solution for a self-trapped optical beam, the NLS equation in higher dimensions possesses a discrete number of the so-called *higher-order localized modes* which consist of bright or dark central spots surrounded by one or more rings. Such higher-order nonlinear localized modes have been first suggested as radially symmetric solutions of the $(2+1)$ dimensional cubic NLS equation without (Yankauskas, 1966; Haus, 1966) or with an angular momentum (i.e., 'bright vortex solitons'; see Kruglov and Vlasov, 1985; Kruglov et al., 1992), and then the similar solutions were found in higher dimensions and for other nonlinear models, including the saturable NLS equation (e.g., Edmundson, 1997; and reference therein) and the models of two- and three-wave mixing in a quadratic optical medium (Firth and Skryabin, 1997; Torres et al., 1998; Skryabin and Firth, 1998a).

Stability of the higher-order localized modes is an important problem which was analyzed by many authors for different models. No universal stability criterion is known for this kind of solutions, but it is clear that the instability of the rings (or 'shells', in higher dimensions) is similar to the modulational instability of plane solitons. Indeed, it was shown that all types of higher-order localized modes are *unstable with respect to azimuthally dependent perturbations* and they decay into filaments, i.e. stable bright solitons of the lowest order. The scenario of the decay depends on the angular momentum of the modes. The modes with a bright central spot have no angular momentum, and the rings break up into filaments that move radially to the initial ring, the similar effects were observed experimentally for the transverse modulational instability of broad beams (Campillo et al., 1973,1974). The number of the filaments is defined by the maximum of the instability growth rate (Feit and Fleck, 1988; Soto-Crespo et al., 1991). The decay of the modes with a dark central spot is affected by the angular momentum (Tikhonenko et al., 1995,1996b; Firth and Skryabin, 1997; Petrov et al., 1998).

Recently, Skryabin and Firth (1998a) have analyzed stability and dynamics of higher-order localized modes in the problem of two-wave mixing in a diffractive bulk medium and found another scenario of the decay of these modes. For sufficient negative values of the phase mismatch between the harmonics, rings were found to coalesce with the central peak forming a single oscillating filament. For positive phase mismatch, the standard break-up into filaments was observed. This coalescence scenario is not observed in the saturable NLS models. For a quadratic nonlineaar material [KTP crystal], the first experimental demonstration of the azimuthal self-breaking of intense beams with a vortex phase dislocation into optical spatial solitons was recently reported by Petrov et al. (1998).

Instability of the higher-order localized modes ('light bullets') was investigated numerically in the framework of the three-dimensional saturable NLS equation (Edmundson, 1997). In contrast with the two-dimensional case (e.g., Soto-Crespo et al., 1991), rather than a single structure emerging independent of the weak perturbation, in the three-dimensional case 'the shells' of the higher-order modes decay in a miriad of complicated intermediate patterns attributed to a degeneracy in the number of maximally unstable eigenmodes.

9.6. Ring-like models and ring solitons

A number of interesting physical effects is associated with a ring geometry. Let us bend a plane soliton to form a ring of the length L. Then, if the instability band of a plane soliton is characterized by the maximum wave number $q_{\max}$, we expect that this instability will be suppressed in the ring geometry provided the condition $L < 2\pi/q_{\max}$ (or, for the ring radius R, $q_{\max}R < 1$) holds. This simple observation explains why for the so-called *ring solitary waves*, the transverse instability is either suppressed or completely eliminated. This is true for many types of ring solitons, including bright (Lomdahl et al., 1980; Afanasjev, 1995), and dark (Kivshar and Yang, 1994) ring solitons of the $(2+1)$ dimensional NLS equation, and the solitons of the so-called cylindrical KdV equation (e.g., Maxon and Viecelli, 1974; Ko and Kuehl, 1979; Stepanyants, 1981). However, the ring solitons do not exist as *stationary localized states*, they either expand or contract due to an effective surface tension introduced by the bending into a ring. If the condition $q_{\max}R < 1$ is not fulfield, the ring soliton is expected to decay into a number of filaments, similar to the decay of the higher-order localized modes discussed above. Recently, it has been demonstrated numerically (Soljačić et al., 1998) that a ring of two-dimensional stable bright solitons in the form of the so-called "necklace" beam possesses an unique robustness due to the stabilizing interaction forces between the solitons in the ring. In particular, a necklace beam expands much slower that a quasi-plane soliton in the ring geometry.

9.7. Instabilities in higher dimensions

Many interesting problems related to the soliton self-focusing and symmetry-breaking instabilities appear in higher dimensions. We did not touch this topic at all in the present review paper, however, it is worth mentioning, as the most interesting problem of this kind, the transverse instability of a vortex tube subjected to the action of higher dimensions or dispersion. This problem was first discussed more than 10 yr ago by Jones et al. (Jones et al., 1986) and the most recent analysis has been presented by Kuznetsov and Rasmussen (1995).

Acknowledgements

We are indebted to M.J. Ablowitz, A.V. Buryak, K.A. Gorshkov, A.M. Kamchatnov, V.I. Shrira, Yu.A. Stepanyants, M. Segev and S. Trillo for useful discussions and collaboration. We also thank L. Bergé, G. Fibich, E.A. Kuznetsov, A.A. Nepomnyashchy, J.J. Rasmussen, M. Saffman, and S. Turitsyn for critical reading of different sections of this manuscript, their valuable comments and additional references. The work of Yuri Kivshar has been partially supported by the Australian Photonics Cooperative Research Centre.

References

Ablowitz, M.J., Segur, H., 1979. J. Fluid Mech. 92, 691.
Ablowitz, M.J., Segur, H., 1980. Stud. Appl. Math. 62, 249.
Ablowitz, M.J., Segur, H., 1981. Solitons and the Inverse Scattering Transformation. SIAM, Philadelphia (Chapters 3.8 and 4.3).
Ablowitz, M.J., Biondini, G., Blair, S., 1997. Phys. Lett. A 236, 520.
Abraham, N.B., Firth, W.J., 1990. J. Opt. Soc. Am. B 7, 951.
Aceves, A.B., De Angelis, C., 1992. Opt. Lett. 18, 110.
Aceves, A.B., De Angelis, C., Rubenchik, A.M., Turitsyn, S.K., 1994a. Opt. Lett. 19, 329.
Aceves, A.B., De Angelis, C., Luther, G.G., Rubenchik, A.M., 1994b. Opt. Lett. 19, 1186.
Aceves, A.B., Luther, G.G., De Angelis, C., Rubenchik, A.M., Turitsyn, S.K., 1995. Phys. Rev. Lett. 75, 73.
Afanasjev, V.V., 1995. Phys. Rev. E 53, 3153.
Agrawal, G.P., 1987. Phys. Rev. Lett. 59, 880.
Agrawal, G.P., 1990a. J. Opt. Soc. Am. B 7, 1072.
Agrawal, G.P., 1990b. Phys. Rev. Lett. 64, 2487.
Agrawal, G.P., 1995. Nonlinear Fiber Optics. Academic Press, Boston (Chapters 2 and 6).
Agrawal, G.P., Baldeck, P.L., Alfano, R.R., 1989. Phys. Rev. A 39, 3406.
Akhmediev, N.N., Korneev, V.I., Nabiev, R.F., 1992. Opt. Lett. 17, 393.
Alexander, J.C., Pego, R.L., Sachs, R.L., 1997. Phys. Lett. A 226, 187.
Alfano, R.R., Shapiro, S.L., 1970. Phys. Rev. Lett. 24, 584, 592.
Allen, M.A., Rowlands, G., 1993. J. Plasma Phys. 50, 413.
Allen, M.A., Rowlands, G., 1995. J. Plasma Phys. 53, 63.
Allen, M.A., Rowlands, G., 1997. Phys. Lett. A 235, 145.
Andersen, D., Bondeson, A., Lisak, M., 1979a. J. Plasma Phys. 21, 259.
Andersen, D., Bondeson, A., Lisak, M., 1979b. Phys. Scripta 20, 343.
Anders, I., 1995. Physica D 87, 160.
Anile, A.M., Hunter, J.K., Pantano, P., Russo, G., 1993. Ray Methods for Nonlinear Waves in Fluids and Plasmas. Longman Scientific & Technical, Essex, UK.
Ashkin, A., Dziedzic, J.M., Smith, P.W., 1982. Opt. Lett. 7, 276.
Askar'yan, G.A., 1962. Zh. Eksp. Teor. Fiz. 42, 1567 [Sov. Phys. JETP 15 (1962) 1088].
Baboiu, D.-M., Stegeman, G.I., 1998. Opt. Lett. 23, 31.

Bang, O., Rasmussen, J.J., Christiansen, P.L., 1994. Nonlinearity 7, 205.
Benjamin, T.B., Feir, J.E., 1967. J. Fluid Mech. 27, 417.
Bergé, L., 1998a. Phys. Rep. 303, 259.
Bergé, L., 1998b. Phys. Rev. E 58, 6606.
Bergé, L., Mezentsev, V.K., Rasmussen, J.J., Wyller, 1995. Phys. Rev. A. 52, R28.
Bergé, L., Rasmussen, J.J., 1996a. Phys. Rev. A 53, 4476.
Bergé, L., Rasmussen, J.J., 1996b. Phys. Plasmas 3, 824.
Bergé, L., Kuznetsov, E.A., Rasmussen, J.J., 1996. Phys. Rev. E 53, R1340.
Bergé, L., Rasmussen, J.J., Schmidt, M.R., 1998. Phys. Scripta T75, 18.
Berkhoer, A.L., Zakharov, V.E., 1970. Zh. Eksp. Teor. Fiz. 58, 903 [Sov. Phys. JETP 31 (1970) 486].
Bertolotti, M., Li Voti, R., Marchetti, S., Sibilia, C., 1997. Opt. Commun. 133, 578.
Bespalov, V.I., Talanov, V.I., 1966. Pisma Zh. Eksp. Teor. Fiz. 3, 471 [JETP Lett. 3, 307].
Bettinson, D., Rowlands, G., 1996a. Phys. Rev. E 54, 6102.
Bettinson, D., Rowlands, G., 1996b. Phys. Rev. E 55, 5427.
Bettinson, D., Rowlands, G., 1998a. Phys. Rev. E 57, 169.
Bettinson, D., Rowlands, G., 1998b. J. Plasma Phys. 59, 543.
Bialynicki-Birula, I., Mycielski, J., 1979. Phys. Scripta 20, 539.
Bjorkholm, J.E., Ashkin, A., 1974. Phys. Rev. Lett. 32, 129.
Boardman, A.D., Xie, K., 1993. Radio Sci. 28, 891.
Bondeson, A., 1979. Phys. Rev. Lett. 43, 1117.
Boyd, R.W., 1992. Nonlinear Optics. Academic Press, San Diego (Chapters 6 and 7).
Buryak, A.V., Kivshar, Yu.S., 1994. Opt. Lett. 19, 1612.
Buryak, A.V., Kivshar, Yu.S., 1995. Phys. Lett. A 197, 407.
Cahn, J.W., Hilliard, J.E., 1958. J. Chem. Phys. 28, 258.
Calogero, F., Eckhaus, W., 1987. Inverse Problems 3, 229.
Campillo, A.J., Shapiro, S.L., Suydam, B.R., 1973. Appl. Phys. Lett. 23, 628.
Campillo, A.J., Shapiro, S.L., Suydam, B.R., 1974. Appl. Phys. Lett. 24, 178.
Chernev, P., Petrov, V., 1992a. Opt. Lett. 17, 172.
Chernev, P., Petrov, V., 1992b. Opt. Commun. 87, 28.
Chi, S., Guo, Q., 1995. Opt. Lett. 20, 1598.
Chiao, R.I., Garmire, E., Townes, C., 1964. Phys. Rev. Lett. 13, 478.
Christiansen, P.L., Gaididei, Yu., Johansson, M., Rasmussen, K.Ø., Yakimenko, I., 1996a. Phys. Rev. E 54, 924.
Christiansen, P.L., Gaididei, Yu.B., Mezentsev, V.K., Musher, S.L., Rasmussen, K.Ø., Rasmussen, J.J., Ryzhenkova, I.V., Turitsyn, S.K., 1996b. Phys. Scripta T67, 160.
Christodoulides, D.N., Joseph, R.I., 1988. Opt. Lett. 13, 794.
Craig, W., Sulem, C., Schanz, U., 1997. Ann. Inst. H. Poincare, Analyse Non lineaire 14, 615.
Darmanyan, S., Relke, I., Lederer, F., 1997. Phys. Rev. E 55, 7662.
De Rossi, A., Trillo, S., Buryak, A.V., Kivshar, Yu.S., 1997a. Opt. Lett. 22, 2116.
De Rossi, A., Trillo, S., Buryak, A.V., Kivshar, Yu.S., 1997b. Phys. Rev. E 56, 4959.
Degtyarev, L.M., Zakharov, V.E., L.I. Rudakov, 1975. Zh. Eksp. Teor. Fiz. 68, 115 [Sov. Phys. JETP 41 (1975) 57].
Diddams, S.A., Eaton, H.K., Zozulya, A.A., Clement, T.S., 1998. Opt. Lett. 23, 379.
D'yachenko, A.I., Kuznetsov, E.A., 1994. ZhETF Pisma 59, 103 [JETP Lett. 39 (1994) 108].
D'yachenko, A.I., Kuznetsov, E.A., 1995. Physica D 87, 301.
Edmundson, D.E., 1997. Phys. Rev. E 55, 7636.
Eleonskii, V.M., Oganesyants, L.G., Silin, V.P., 1973. Zh. Eksp. Teor. Fiz. 63, 532.
Feit, M.D., Fleck, J.A., 1988. J. Opt. Soc. Am. B 5, 623.
Fibich, G., 1996. Phys. Rev. Lett. 76, 4356.
Fibich, G., Malkin, V.M., Papanicolaou, G.C., 1995. Phys. Rev. A 52, 4218.
Fibich, G., Papanicolaou, G.C., 1998. Phys. Lett. A 239, 167.
Fibich, G., Papanicolaou, G.C., 1999. SIAM J. Appl. Math. 60, in press.
Firth, W.J., Paré, C., 1988. Opt. Lett. 13, 1096.

Firth, W.J., Fitzgerald, A., Paré, C., 1990. J. Opt. Soc. Am. B 7, 1087.
Firth, W.J., Skryabin, D.V., 1997. Phys. Rev. Lett. 79, 2450.
Fraiman, G.M., 1985. Sov. Phys. JETP 61, 228.
Frycz, P., Infeld, E., 1989a. J. Plasma Phys. 41, 441.
Frycz, P., Infeld, E., 1989b. Phys. Rev. Lett. 63, 384.
Frycz, P., Infeld, E., Samson, J.C., 1992. Phys. Rev. Lett. 69, 1057.
Fuerst, R.A., Baboiu, D.M., Lawrence, B., Torruellas, W.E., Stegeman, G.I., Trillo, S., Wabnitz, S., 1997a. Phys. Rev. Lett. 78, 2756.
Fuerst, R.A., Lawrence, B., Torruellas, W.E., Stegeman, G.I., 1997b. Opt. Lett. 22, 19.
Gaididei, Yu.B., Mingaleev, S.F., Yakimenko, I.I., Christiansen, P.L., Johansson, M., Rasmussen, K.Ø., 1996. Phys. Scripta T67, 151.
Gaididei, Yu.B., Christiansen, P.L., 1998. Opt. Lett. 23, 1090.
Garmire, E., Chiao, R.Y., Townes, C.H., 1966. Phys. Rev. Lett. 16, 347.
Gauthier, D.J., Malcuit, M.S., Gaeta, A.L., Boyd, R.W., 1990. Phys. Rev. Lett. 64, 1721.
Goldman, M.V., 1984. Rev. Mod. Phys. 56, 709.
Goldman, M.V., Rypdal, K., Hafizi, B., 1980. Phys. Fluids 23, 945.
Gorshkov, K.A., Pelinovsky, D.E., 1995a. Physica D 85, 468.
Gorshkov, K.A., Pelinovsky, D.E., 1995b. Inverse Problems 11, 603.
Grantham, J.W., Gibbs, H.M., Khitrova, G., Valley, J.F., Xu Jiajin, 1991. Phys. Rev. Lett. 66, 1422.
Grischkowsky, D., 1970. Phys. Rev. Lett. 24, 866.
Grynberg, G., Le Biham, E., Verkerk, P., Simoneau, P., Leite, J.R.R., Bloch, D., Le Boiteux, S., Ducloy, M., 1988. Opt. Commun. 67, 363.
Gustafson, T.K., Kelley, P.L., Chiao, R.Y., Brewer, R.G., 1968. Appl. Phys. Lett. 12, 165.
Hadžievski, Lj.R., Škorić, M.M., 1991. Phys. Fluids B 3, 2452.
Haus, H.A., 1966. Appl. Phys. Lett. 8, 128.
He, H., Werner, W.J., Drummond, P.D., 1996. Phys. Rev. E 54, 896.
Iizuka, T., Kivshar, Yu.S., 1999. Phys. Rev. E 59, 7148.
Infeld, E., 1985. J. Plasma Phys. 33, 171.
Infeld, E., Frycz, P., 1987. J. Plasma Phys. 37, 97.
Infeld, E., Rowlands, R., 1990. Nonlinear Waves, Solitons and Chaos. Cambridge University Press, Cambridge.
Infeld, E., Lenkowska-Czerwińska, T., 1997. Phys. Rev. E 55, 6101.
Infeld, E., Senatorski, A., Skorupski, A.A., 1994. Phys. Rev. Lett. 72, 1345.
Infeld, E., Senatorski, A., Skorupski, A.A., 1995. Phys. Rev. E 51, 3183.
Iturbe-Castillo, M.D., Torres-Cisneros, M., Sánchez-Mondragón, J.J., Chávez-Cerda, S., Stepanov, S.I., Vysloukh, V.A., Torres-Cisneros, G.E., 1995. Opt. Lett. 20, 1853.
Janssen, P.A.E.M., Rasmussen, J.J., 1983. Phys. Fluids 26, 1279.
Jones, C.A., Putterman, S.J., Roberts, P.H., 1986. J. Phys. A: Math. Gen. 19, 2991.
Josserand, C., Rica, S., 1997. Phys. Rev. Lett. 78, 1215.
Kachanov, Yu.S., Ryzhov, O.S., Smith, F.T., 1993. J. Fluid Mech. 251, 273.
Kadomtsev, B.B., 1976. Collective Phenomena in Plasmas. Nauka, Moscow (in Russian).
Kadomtsev, B.B., Petviashvili, V.I., 1970. Dokl. Acad. Nauk SSSR 192, 753 [Sov. Phys. Doklady 15 (1970) 539].
Kamchatnov, A.M., 1997. Phys. Rep. 286, 199.
Kanashov, A.A., Rubenchik, A.M., 1981. Physica D 4, 122.
Karamzin, Yu.N., Sukhorukov, A.P., 1974. Zh. Eksp. Teor. Fiz. 68, 834 [Sov. Phys. JETP 41 (1976) 414].
Karpman, V.I., 1996. Phys. Rev. E 53, R1336.
Katyshev, Yu.V., Makhankov, V.G., 1976. Phys. Lett. A 57, 10.
Kaup, D.J., 1990. Phys. Rev. A 42, 5689.
Kaw, P.K., Nishikawa, K., Yoshida, Y., Hasegawa, A., 1975. Phys. Rev. Lett. 35, 88.
Kelley, P., 1965. Phys. Rev. Lett. 15, 1005.
Kivshar, Yu.S., 1993. Opt. Lett. 18, 1147.
Kivshar, Yu.S., 1998a. Opt. Quantum Electron. 30, 571.

Kivshar, Yu.S., 1998b. In: Boardman, A.D., Pavlov, L., Tanev, S. (Eds.), Advanced Photonics with Second-Order Optically Nonlinear Processes. Kluwer, Amsterdam, p. 451.
Kivshar, Yu.S., Malomed, B.A., 1989. Rev. Mod. Phys. 61, 763.
Kivshar, Yu.S., Peyrard, M., 1992. Phys. Rev. A 46, 3198.
Kivshar, Yu.S., Flytzanis, N., 1992. Phys. Rev. A 46, 7972.
Kivshar, Yu.S., Turitsyn, S.K., 1994. Phys. Rev. E 49, R2536.
Kivshar, Yu.S., Yang, X., 1994. Phys. Rev. E 50, R40.
Kivshar, Yu.S., Luther-Davies, B., 1998. Phys. Rep. 298, 81.
Kivshar, Yu.S., Christou, J., Tikhonenko, V., Luther-Davies, B., Pismen, L., 1998. Opt. Commun. 152, 198.
Klimontovich, Yu.L., Luzgin, S.N., 1979. Pisma Zh. Eksp. Teor. Fiz. 30, 645 [JETP Lett. 30 (1979) 610].
Ko, K., Kuehl, H.H., 1979. Phys. Fluids 22, 1343.
Komashko, A., Musher, S., Turitsyn, S.K., Rubenchik, A., Feit, M., 1995. Pis'ma Zh. Eksp. Teor. Fiz. 62, 849 [JETP Lett. 62 (1995) 860].
Kruglov, V.I., Vlasov, R.A., 1985. Phys. Lett. A 111, 401.
Kruglov, V.I., Logvin, Yu.A., Volkov, V.M., 1992. J. Mod. Opt. 39, 2277.
Kuznetsov, E.A., Musher, S.L., Shafarenko, A.V., 1983. Pis'ma Zh. Eksp. Teor. Fiz. 37, 204 [JETP Lett. 37 (1983) 241].
Kuznetsov, E.A., Spector, M.D., Falkovich, G., 1984. Physica D 10, 379.
Kuznetsov, E.A., Rasmussen, J.J., 1995. Phys. Rev. E 51, 4479.
Kuznetsov, E.A., Rubenchik, A.M., Zakharov, V.E., 1986. Phys. Rep. 142, 103.
Kuznetsov, E.A., Musher, S.L., 1986. Zh. Eksp. Teor. Fiz. 91, 1605 [Sov. Phys. JETP 64 (1986) 947].
Kuznetsov, E.A., Turitsyn, S.K., 1988. Zh. Eksp. Teor. Fiz. 94, 119 [Sov. Phys. JETP 67 (1988) 1583].
Law, C.T., Swartzlander Jr., G.A., 1993. Opt. Lett. 18, 586.
Laedke, E.W., Spatschek, K.H., 1978. Phys. Rev. Lett. 41, 1798.
Laedke, E.W., Spatschek, K.H., 1979. J. Math. Phys. 20, 1838.
Laedke, E.W., Spatschek, K.H., 1982. J. Plasma Phys. 28, 469.
Laedke, E.W., Spatschek, K.H., Zocha, K.B., 1986. Phys. Fluids 29, 1127.
Laedke, E.W., Spatschek, K.H., Turitsyn, S.K., 1994. Phys. Rev. Lett. 73, 1055.
Litvak, A.G., Mironov, V.A., Fraiman, G.M., Yunakovskii, A.D., 1975. Fiz. Plazmy 1, 60 [Sov. J. Plasma Phys. 1 (1975) 31].
Litvak, A.G., Sergeev, A.M., Zharova, N.A., 1979. Pis'ma Zh. Tekh. Fiz. 5, 86 [Sov. Tech. Phys. Lett. 5 (1979) 33].
Litvak, A.G., Petrova, T.A., Fraiman, G.M., Sher, E.M., Yunakovsky, A.D., 1991. Physica D 52, 36.
Lomdahl, P.S., Olsen, O.H., Christiansen, P.L., 1980. Phys. Lett. A 78, 125.
Luther, G.G., McKinstrie, C.J., 1990. J. Opt. Soc. Am. B 7, 1125.
Luther, G.G., Moloney, J.V., Newell, A.C., Wright, E.M., 1994a. Opt. Lett. 19, 789.
Luther, G.G., Moloney, J.V., Newell, A.C., Wright, E.M., 1994b. Opt. Lett. 19, 863.
Luther, G.G., Newell, A.C., Moloney, J.V., 1994c. Physica D 74, 59.
Ma, Y.C., 1984. Stud. Appl. Math. 70, 201.
Makhankov, V.G., 1978. Phys. Rep. 35, 1.
Malkin, V.M., 1990. Phys. Lett. A 151, 285.
Mamaev, A.V., Saffman, M., Zozulya, A.A., 1996a. Phys. Rev. Lett. 76, 2262.
Mamaev, A.V., Saffman, M., Anderson, D.Z., Zozulya, A.A., 1996b. Phys. Rev. A 54, 870.
Mamaev, A.V., Saffman, M., Zozulya, A.A., 1996c. Europhys. Lett. 35, 25.
Maradudin, A.A., 1983. In: Borissov, M. (Ed.), Nonlinear Surface Electromagnetic Waves in Optical and Acoustic Waves in Solids. World Scientific, Singapore, p. 72.
Marburger, J.H., Dawes, E., 1968. Phys. Rev. Lett. 21, 556.
Martin, D.U., Yuen, H.C., Saffman, P.G., 1980. Wave Motion 2, 215.
Marquié, P., Bilbault, J.M., Remoissenet, M., 1994. Phys. Rev. E 49, 828.
Marquié, P., Bilbault, J.M., Remoissenet, M., 1995. Physica D 87, 371.
Maxon, S., Viecelli, J., 1974. Phys. Rev. Lett. 32, 4.
McDonald, G.S., Syed, K.S., Firth, W.J., 1993. Opt. Commun. 95, 281.
McKinstrie, C.J., Russell, D.A., 1988. Phys. Rev. Lett. 61, 2929.
Melville, W.K., 1982. J. Fluid Mech. 115, 165.

Milsted, C.S., Contrell, C.D., 1996. Phys. Rev. A 53, 3536.
Mizumura, M., Noguchi, A., 1975. Elect. Commun. Japan A 58, 9.
Mlejnek, M., Wright, E.M., Moloney, J.V., 1998. Phys. Rev. E 58, 4903.
Moloney, J.V., 1987. Phys. Rev. A 36, 4563.
Murakami, Y., Tajiri, M., 1992. J. Phys. Soc. Jpn. 61, 791.
Musslimani, Z.H., Segev, M., Nepomnyashchy, A., Kivshar, Yu.S., 1999. Phys. Rev. E 60, R1170.
Myra, J.R., Lin, C.H., 1980. Phys. Fluids 23, 225.
Newell, A.C., Moloney, J.V., 1992. Nonlinear Optics. Addison-Wesley, Redwood City.
Novick-Cohen, A., Segel, L.A., 1984. Physica D 10, 277.
Ostrovsky, L.A., 1966. Zh. Exp. Teor. Fiz. 51, 1189 [Sov. Phys. JETP 24 (1967) 797].
Ostrovsky, L.A., Shrira, V.I., 1976. Zh. Eksp. Teor. Fiz. 71, 1412 [Sov. Phys. JETP 44 (1976) 738].
Pelinovsky, D.E., 1994. Phys. Lett. A 196, 181.
Pelinovsky, D.E., 1998. Physica D 119, 301.
Pelinovsky, D.E., 2000. J. Math. Comput., in press.
Pelinovsky, D.E., Shrira, V.I., 1995. Phys. Lett. A 206, 195.
Pelinovsky, D.E., Grimshaw, R.H.J., 1996. Physica D 98, 139.
Pelinovsky, D.E., Grimshaw, R.H.J., 1997. In: Debnath, L., Choudhury, S.R. (Eds.), Nonlinear Instability Analysis. Computational Mechanics Publications, Southampton, p. 245.
Pelinovsky, D.E., Stepanyants, Yu.A., 1993. Zh. Eksp. Teor. Fiz. 104, 3387 [Sov. Phys. JETP 77 (1993) 602].
Pelinovsky, D.E., Stepanyants, Yu.A., 1994. Zh. Eksp. Teor. Fiz. 105, 1635 [Sov. Phys. JETP 78 (1994) 883].
Pelinovsky, D.E., Stepanyantz, Yu.A., Kivshar, Yu.S., 1995. Phys. Rev. E 51, 5016.
Pelinovsky, D.E., Afanasjev, V.V., Kivshar, Yu.S., 1996a. Phys. Rev. E 53, 1940.
Pelinovsky, D.E., Kivshar, Yu.S., Afanasjev, V.V., 1996b. Phys. Rev. E 54, 2015.
Pereira, N.R., Sudan, R.N., Denavit, J., 1977. Phys. Fluids 20, 271, 936.
Pereira, N.R., Sen, A., Bers, A., 1978. Phys. Fluids 21, 117.
Pérez-García, V.M., Porras, M.A., Vázquez, L., 1995. Phys. Lett. A 202, 176.
Pesenson, M.Z., 1991. Phys. Fluids A 3, 3001.
Petrov, D.V., Torner, L., Martorell, J., Vilaseca, R., Torres, J.P., Cojocaru, C., 1998. Opt. Lett. 23, 1444.
Pilipetskii, N.F., Rustamov, A.R., 1965. Pisma Zh. Eksp. Teor. Fiz. 2, 88.
Pouget, J., Remoissenet, M., Tamga, J.M., 1993. Phys. Rev. B 47, 14866.
Rabinovich, M.I., Trubetskov, D., 1984. Theory of Oscillations and Waves. Nauka, Moscow.
Ranka, J.K., Schirmer, R.W., Gaeta, A.L., 1996. Phys. Rev. Lett. 77, 3783.
Rasmussen, J.J., Rypdal, K., 1986. Phys. Scripta 33, 481.
Rasmussen, K.Ø., Bang, O., Christiansen, P.L., 1994. Phys. Lett. A 184, 241.
Rasmussen, K.Ø., Gaididei, Yu.B., Bang, O., Christiansen, P.L., 1995. Phys. Lett. A 204, 121.
Relke, I., 1998. Phys. Rev. E 57, 6105.
Rothenberg, J.E., 1992a. Opt. Lett. 17, 583.
Rothenberg, J.E., 1992b. Opt. Lett. 17, 1340.
Rubenchik, A.M., Shapiro, E.G., Turitsyn, S.K., 1995. Phys. Lett. A 198, 201.
Ryan, A.T., Agrawal, G.P., 1995. Opt. Lett. 20, 306.
Rypdal, K., Rasmussen, J.J., 1989. Phys. Scripta 40, 192.
Saffman, M., 1998. Phys. Rev. Lett. 81, 65.
Schmidt, G., 1975. Phys. Rev. Lett. 34, 724.
Senatorski, A., Infeld, E., 1998. Phys. Rev. E 57, 6050.
Shen, Y.R., 1984. Principles of Nonlinear Optics. Wiley, New York (Chapter 17).
Sheppard, A.P., Haelterman, M., 1998. Opt. Lett. 23, 1820.
Shih, M.F., Segev, M., Valley, G.C., Salamo, G., Crosignani, B., Di Porto, P., 1995. Electron. Lett. 31, 826.
Shih, M.F., Leach, P., Segev, M., Garrett, M.H., Salamo, G., Valley, G.C., 1996. Opt. Lett. 21, 324.
Shinozaki, A., Oono, Y., 1993. Phys. Rev. E 47, 804.
Shivamoggi, B.K., Rollins, D.K., Fanjul, R., 1993. Phys. Scripta 47, 15.
Shrira, V.I., 1980. Zh. Eksp. Teor. Fiz. 79, 87 [Sov. Phys. JETP 52, 44].
Shrira, V.I., 1989. Dokl. Akad. Nauk USSR 308, 732 (in Russian).

Shrira, V.I., Pesenson, M.Z., 1983. In: Sagdeev, R.Z. (Ed.), Nonlinear and Turbulent Processes in Physics, Vol. 2. Harwood Academic, New York, p. 1061.
Skryabin, D.V., Firth, W.J., 1998a. Phys. Rev. E 58, R1252.
Skryabin, D.V., Firth, W.J., 1998b. Opt. Commun. 148, 79.
Skryabin, D.V., Firth, W.J., 1998c. Phys. Rev. Lett. 81, 3379.
Skryabin, D.V., Firth, W.J., 1999. Phys. Rev. E 60, 1019.
Snyder, A.W., Mitchell, D.J., Poladian, L., Ladouceuer, F., 1991. Opt. Lett. 16, 21.
Snyder, A.W., Mitchell, D.J., 1997a. Opt. Lett. 23, 16.
Snyder, A.W., Mitchell, D.J., 1997b. Science 276, 1538.
Soljačić, M., Sears, S., Segev, M., 1998. Phys. Rev. Lett. 81, 4851.
Soto-Crespo, J.M., Heatley, D.R., Wright, E.M., Akhmediev, N.N., 1991. Phys. Rev. A 44, 636.
Soto-Crespo, J.M., Wright, E.M., Akhmediev, N.N., 1992. Phys. Rev. A 45, 3168.
Spatschek, K.H., Shukla, P.K., Yu, M.Y., 1975. Phys. Lett. A 54, 419.
Spiegel, E.A., 1980. Physica D 1, 236.
Stegeman, G., Hagan, D.J., Torner, L., 1996. Opt. Quantum Electron. 28, 1691.
Stepanyants, Yu.A., 1981. Wave Motion 3, 335.
Strickland, D., Corkum, P.B., 1991. Proc. Soc. Photo-Opt. Instrum. Eng. 1413, 54.
Strickland, D., Corkum, P.B., 1994. J. Opt. Soc. Am. B 11, 492.
Su, M.Y., 1982. Phys. Fluids 25, 2167.
Sulem, C., Sulem, P.-L., 1999. Nonlinear Schrödinger Equation: Self-focusing Instability and Wave Collapse. Springer, Berlin.
Suter, D., Blasberg, T., 1993. Phys. Rev. A 48, 4583.
Tai, K., Hasegawa, A., Tomita, A., 1986. Phys. Rev. Lett. 56, 135.
Talanov, V.I., 1964. Izv. VUZov-Radiofizika 7, 564 (in Russian).
Tikhonenko, V., Christou, J., Luther-Davies, B., 1995. J. Opt. Soc. Am. B 12, 2046.
Tikhonenko, V., Christou, J., Luther-Davies, B., Kivshar, Yu.S., 1996a. Opt. Lett. 21, 1129.
Tikhonenko, V., Christou, J., Luther-Davies, B., 1996b. Phys. Rev. Lett. 76, 2698.
Torner, L., Mazilu, D., Michalache, D., 1996. Phys. Rev. Lett. 77, 2455.
Torres, J.P., Soto-Crespo, J.M., Torner, L., Petrov, D.V., 1998. J. Opt. Soc. Am. B 15, 625.
Torruellas, W.E., Wang, Z., Hagan, D.J., VanStryland, E.M., Stegeman, G.I., Torner, L., Menyuk, C.R., 1995. Phys. Rev. Lett. 74, 5036.
Trubnikov, B.A., Zhdanov, S.K., 1987. Phys. Rep. 155, 137.
Turitsyn, S.K., 1985. Teor. Mat. Fiz. 64, 226.
Turitsyn, S.K., 1993a. Opt. Lett. 18, 110.
Turitsyn, S.K., 1993b. Phys. Rev. E 47, R13.
Turitsyn, S.K., 1995. Pis'ma Zh. Eksp. Teor. Fiz. 61, 469 [JETP Lett. 61, 469 (1995)].
Vlasov, S.N., Sheinina, E.V., 1983. Izv. Vyssh. Ucheb. Zaved. Radiofiz. 26, 20 [Sov. Radiophys. 27 (1983) 19].
Vidal, F., Johnston, T.W., 1996. Phys. Rev. Lett. 77, 1282.
Vysotina, N., Rosanov, N., Smirnov, V., 1990. J. Opt. Soc. Am. B 7, 998.
Wadati, M., Iizuka, T., Yajima, T., 1991. Physica D 51, 388.
Washimi, H., 1974. Progr. Theor. Phys., Suppl. No. 55, 138.
Whitham, G.B., 1974. Linear and Nonlinear Waves. Wiley, New York.
Wilcox, J.Z., Wilcox, T.J., 1975. Phys. Rev. Lett. 34, 1160.
Wright, E.M., Lawrence, B.L., Torruellas, W., Stegeman, G., 1995. Opt. Lett. 20, 2481.
Yankauskas, Z.K., 1966. Izv. VUZov Radiofiz. 9, 412 [Sov. Radiophys. 9 (1966) 261].
Yajima, N., 1974. Prog. Theor. Phys. 52, 1066.
Yajima, N., 1983. Prog. Theor. Phys. 69, 678.
Yuen, H.C., Ferguson, W.E., 1978. Phys. Fluids 21, 2116.
Yuen, H.C., Lake, B.M., 1975. Phys. Fluids 18, 958.
Yuen, H.C., Lake, B.M., 1986. In: Solitons in Action. Academic Press, New York (Chapter 5).
Zakharov, V.E., 1967. Zh. Eksp. Teor. Fiz. 53, 1735 [Sov. Phys. JETP 26 (1968) 994].
Zakharov, V.E., 1968. J. Appl. Mech. Tech. Phys. 2, 190 (translation from Russian).

Zakharov, V.E., 1975. Pis'ma Zh. Eksp. Teor. Fiz. 22, 364 [JETP Lett. 22 (1975) 172].
Zakharov, V.E., 1986. Izvestia VUZov Radiofizika 29, 1073 (in Russian).
Zakharov, V.E. (Ed.), 1991. Special Issue "Wave Collapse". Physica D 52, 1–312.
Zakharov, V.E., Kuznetsov, E.A., 1974. Sov. Phys. JETP 39, 285.
Zakharov, V.E., Rubenchik, A.M., 1973. Zh. Eksp. Teor. Fiz. 65, 997 [Sov. Phys. JETP 38 (1974) 494].
Zakharov, V.E., Sobolev, V.V., Synakh, V.C., 1971. Sov. Phys. JETP 33, 77.
Zharova, N.A., Litvak, A.G., Petrova, T.A., Sergeev, A.M., Yunakovskii, A.D., 1986. Pis'ma Zh. Eksp. Teor. Fiz. 44, 12 [JETP Lett. 44 (1986) 13].
Zozulya, A.A., Diddams, S.A., Clement, T.S., 1998. Phys. Rev. A 58, 3303.