LINEAR AND NONLINEAR INSTABILITY OF THE PEAKED PERIODIC WAVE IN THE REDUCED OSTROVSKY EQUATION

ANNA GEYER AND DMITRY E. PELINOVSKY

Abstract. Stability of the peaked periodic wave in the reduced Ostrovsky equation has remained open for long time. We obtain sharp bounds on the exponential growth of the L^2 norm of co-periodic perturbations to the peaked periodic wave, from which it follows that the peaked periodic wave is orbitally unstable. We also prove that the peaked periodic wave with the parabolic profile is a unique peaked wave in the space of single-lobe periodic L^2 functions with zero mean.

1. Introduction

We address solutions of the Cauchy problem for the reduced Ostrovsky equation [28] written in the form

$$\left\{ \begin{array}{l} u_t + uu_x = \partial_x^{-1} u, \quad t > 0, \\ u|_{t=0} = u_0, \end{array} \right. \tag{1.1}$$

where u_0 is a 2π-periodic function with zero mean defined in a Sobolev space $H^s_{\rm per}(-\pi,\pi)$ with some $s \geq 0$ and ∂_x^{-1} denotes the anti-derivative in $L^2_{\rm per}(-\pi,\pi)$ with zero mean. We denote the subspace of $L^2_{\rm per}(-\pi,\pi)$ for the 2π-periodic functions with zero mean by $\dot{L}^2_{\rm per}(-\pi,\pi)$ and use similar notations $\dot{H}^s_{\rm per}(-\pi,\pi)$ for functions in $H^s_{\rm per}(-\pi,\pi)$ with zero mean. The reduced Ostrovsky equation is also known under the names of Ostrovsky–Hunter and Ostrovsky–Vakhnenko equations due to contributions of Hunter [23] and Vakhnenko [35, 36].

Local solutions to the Cauchy problem (1.1) exist for $s > \frac{3}{2}$ [32]. For sufficiently large initial data, the local solutions break in finite time, similar to the inviscid Burgers equation [27]. However, if the initial data u_0 is suitably small, then the local solutions for $s = 3$ are continued for all times [17, 18]. Weak bounded solutions with shock discontinuities were constructed in [6, 7]. Weak solutions of the Cauchy problem (1.1) as the limiting solution of the Cauchy problem for the regularized Ostrovsky equation were considered in [5].

The reduced Ostrovsky equation with smooth solutions is completely integrable as it can be reduced to the integrable Tzizeica equation by a coordinate transformation [26]. This property enables a construction of a bi-infinite set of conserved quantities in the time evolution [4] and the inverse scattering transform with the Riemann–Hilbert approach [1]. Two integrable semi-discretizations of the reduced Ostrovsky equation have been obtained by using bilinear forms [14].

Stability of smooth and peaked periodic waves in the reduced Ostrovsky equation has been recently addressed in a number of publications [11, 16, 19, 20, 33]. By using higher-order conserved quantities, the smooth small-amplitude periodic waves were shown in [11] to be unconstrained minimizers of a higher-order energy function. This result holds for *subharmonic* perturbations, that is, perturbations with a multiple period to the period of the smooth periodic waves. Since

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the higher-order conserved quantities are well-defined in the space $\dot{H}^3$, where global well-posedness has been proven [18], it follows from the minimization properties that smooth small-amplitude periodic waves are both spectrally and orbitally stable. The minimization properties were confirmed numerically for smooth large-amplitude periodic waves all way to the limiting peaked wave of a parabolic profile, where the numerical results were inconclusive [11].

Spectral stability of smooth periodic waves with respect to *co-periodic* perturbations, that is, perturbations with the same period as the period of the periodic wave, was shown in [16] by using the standard variational formulation of the periodic waves as critical points of energy subject to fixed momentum. This result holds also for generalized reduced Ostrovsky equation with power nonlinearity. Independently, spectral stability of smooth periodic waves in the reduced Ostrovsky equation was shown in [20] by using a coordinate transformation of the spectral stability problem to a new eigenvalue problem studied earlier in [33].

Regarding the peaked periodic waves, some confusing results were recently obtained. In [20], the peaked wave with the parabolic profile was addressed and claimed to be *unstable in the absence of periodic boundary conditions*. The proof was obtained by a construction of explicit solutions of the spectral stability problem for a positive (unstable) eigenvalue[1]. However, the explicit construction produces *wild boundary conditions* on perturbations to the peaked wave [20].

On the contrast, families of peaked small-amplitude periodic waves were constructed in [19] and these families were claimed to be spectrally stable with respect to co-periodic perturbations. These peaked small-amplitude periodic waves were previously unknown in the context of the reduced Ostrovsky equation.

Finally, peakons (peaked solitary waves) were constructed with a similar coordinate transformation and these peakons were claimed to be spectrally unstable in [33]. In a very similar context of the Camassa–Holm equation, peakons were claimed to be orbitally stable on finite spans of the time evolution [8].

In this paper, we address existence and stability of peaked periodic waves in the reduced Ostrovsky equation and give a simple and definition conclusion. Informally speaking, the main result of this paper is the following theorem.

Theorem 1. *The peaked periodic wave with the parabolic profile is a unique peaked wave of the reduced Ostrovsky equation in the space of single-lobe functions in $\dot{L}^2_{\rm per}(-\pi,\pi)$ up to spatial translations. An orbit of the peaked periodic wave generated by spatial translations is linearly and nonlinearly unstable with respect to perturbations in $H^s_{\rm per}(-\pi,\pi)$ with $s > 3/2$.*

In comparison with the previous literature, the first part of Theorem 1 allows us to eliminate the families of peaked small-amplitude periodic waves constructed in [19]. Apparently, these families are artifacts of the coordinate transformations. The second part of Theorem 1 gives definite conclusion on linear instability of the peaked periodic wave with the parabolic profile with respect to co-periodic perturbations. Compared to the formal construction of some solutions to the spectral stability problem without periodic boundary conditions in [20], our study avoids analysis of the spectral stability problem. In the present time, we have no claims on the spectral stability problem related to the peaked periodic wave. Instead, we obtain sharp bounds on the exponential growth of the L^2 norm of the co-periodic perturbations in the linearized time-evolution problem. Then, we

[1] The explicit solution constructed in Section 5 of [20] is only valid for the 2-periodic wave due to a simple algebraic error, however, a more general solution can be constructed for the periodic wave of any period. Nevertheless, this explicit solution is not relevant for co-periodic perturbations in the space $\dot{L}^2_{\rm per}(-\pi,\pi)$.

transfer the linear instability to the Cauchy problem (1.1) and prove nonlinear orbital instability of the peaked periodic wave[2].

Construction of peaked periodic waves in a similar Whitham equation was recently studied in [9, 10], where different estimates on the Hölder regularity of the peaked periodic wave were obtained. Compared to these works, our analysis of Hölder regularity of the peaked periodic wave relies on the Fourier theory and the strong formulation of the boundary-value problem. The boundary-value problem for the smooth periodic waves in the reduced Ostrovsky equation is equivalent to the second-order differential equation, however, this can not be used when dealing with the peaked periodic waves. We nevertheless can use the first-order invariant of the second-order differential equation to analyze behavior of the smooth parts of the peaked periodic waves.

The paper is organized as follows. Section 2 reports construction of the peaked periodic wave and the proof that the peaked wave with the parabolic profile is unique in the space of single-lobe functions in $\dot{L}^2_{\rm per}(-\pi,\pi)$ up to the spatial translation. Section 3 gives the proof of linear instability of the peaked periodic wave with respect to co-periodic perturbations. Section 4 describes relevant details for the proof of nonlinear orbital instability of the peaked periodic wave. Theorem 1 is proven by collecting results of Sections 2, 3, and 4 together.

2. Peaked periodic wave

The travelling periodic waves in the reduced Ostrovsky equation are given by $u(x,t) = U(x-ct)$, where $c \in \mathbb{R}$ is the wave speed and U is a 2π-periodic wave profile with zero mean. The wave profile U is to be found from the following boundary-value problem:

$$\left\{ \begin{array}{l} [c - U(z)]\, U'(z) + (\partial_z^{-1} U)(z) = 0, \quad \text{for every } \; z \in (-\pi,\pi) \; \text{ such that } \; U(z) \neq c, \\ U(-\pi) = U(\pi), \\ \int_{-\pi}^{\pi} U(z) dz = 0, \end{array} \right. \tag{2.1}$$

where $z = x - ct$ is the travelling wave coordinate. If $U \in \dot{L}^2_{\rm per}(-\pi,\pi)$, then $\partial_z^{-1} U \in H^1_{\rm per}(-\pi,\pi)$. By Sobolev's embedding, it follows that $\partial_z^{-1} U \in C^0_{\rm per}([-\pi,\pi])$ so that the anti-derivative $\partial_z^{-1} U$ with the zero mean can be expressed by the pointwise formula:

$$(\partial_z^{-1} U)(z) := \int_0^z U(z') dz' - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^z U(z') dz' dz, \quad z \in [-\pi,\pi]. \tag{2.2}$$

In what follows, we assume that U is at least continuous on $[-\pi,\pi]$, that is, we assume that $U \in C^0_{\rm per}([-\pi,\pi])$. For $\alpha \in (0,1)$, let $C^{\alpha}([-\pi,\pi])$ be the space of α-Hölder 2π-periodic continuous functions such that

$$|U(x) - U(y)| \lesssim |x-y|^{\alpha}, \quad \text{for all } x,y \in [-\pi,\pi]. \tag{2.3}$$

We will adopt the following definition of the single-lobe periodic waves.

Definition 1. We say that $U \in C^0_{\rm per}([-\pi,\pi])$ is a single-lobe periodic wave if there exists $z_0 \in (-\pi,\pi)$ such that U is non-increasing on $[-\pi,z_0]$ and non-decreasing on $[z_0,\pi]$.

Remark 1. Due to the condition $U(-\pi) = U(\pi)$ and the symmetry of equation

$$(c - U(z))U'(z) + \int_0^z U(z') dz' - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^z U(z') dz' dz = 0$$

[2]We cannot use the standard approach in [31] to conclude on the nonlinear instability from the linear instability because we do not know if the spectral assumption of [31] is true, namely, we do not know if the linearized operator at the peaked periodic wave has a part of the spectrum in the right half of the complex plane.

with respect to reflection $z \mapsto -z$, the single-lobe periodic waves in the Definition 1 have even profile U with $z_0 = 0$. In this case, $(\partial_z^{-1} U)(z) = \int_0^z U(z')dz'$ is odd.

A family of smooth 2π-periodic waves to the boundary-value problem (2.1) satisfying $U(z) < c$ for every $z \in [-\pi, \pi]$ was constructed in our previous work [16] in an open interval of the speed parameter c. By Theorem 1(a) and Lemma 3 in [16], we have the following result.

Lemma 1. *There exists $c_* > 1$ such that for every $c \in (1, c_*)$, the boundary-value problem (2.1) admits a unique smooth periodic wave in the sense of Definition 1 with the profile $U \in \dot{H}^\infty_{\rm per}(-\pi, \pi)$ satisfying $U(z) < c$ for every $z \in [-\pi, \pi]$.*

Remark 2. For the smooth periodic wave $U \in \dot{H}^\infty_{\rm per}(-\pi, \pi)$ to the boundary-value problem (2.1), the periodic boundary conditions are satisfied for all derivatives of U.

At $c = c_*$, the periodic wave with a parabolic profile has been known since the original work of Ostrovsky [28]. It is easy to check that the first two lines of the boundary-value problem (2.1) are satisfied by $U(z) = (z^2 - 3c)/6$, whereas the zero mean condition is satisfied if $c = c_* = \pi^2/9$. This yields the exact expression for the peaked periodic wave:

$$c = \frac{\pi^2}{9}: \quad U(z) = \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi], \tag{2.4}$$

periodically continued beyond $[-\pi, \pi]$. The peaked periodic wave (2.4) can be represented by the Fourier cosine series

$$U(z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{3n^2} \cos(nz),$$

which is well defined in $\dot{H}^s_{\rm per}(-\pi, \pi)$ for $s < 3/2$.

Remark 3. The peaked periodic wave (2.4) belongs to solutions of the boundary-value problem (2.1) with profile $U \in \dot{H}^1_{\rm per}(-\pi, \pi)$ satisfying $U(z) < c$ for every $z \in (-\pi, \pi)$ and $U(\pm\pi) = c$. The first derivative of $U \in \dot{H}^1_{\rm per}(-\pi, \pi)$ has a finite jump singularity across the end points $z = \pm\pi$.

The next result states that the only single-lobe periodic wave with profile $U \in C^0_{\rm per}(-\pi, \pi)$ satisfying the boundary-value problem (2.1) and having a peaked singularity at $z = \pm\pi$ is the peaked wave (2.4). This eliminates all other small-amplitude peaked periodic waves constructed in [19] by means of a transformation of the small-amplitude solutions to the semi-linear Klein–Gordon equation into peaked solutions to the reduced Ostrovsky equation (1.1).

Lemma 2. *The boundary-value problem (2.1) admits no single-lobe peaked periodic waves in the sense of Definition 1 which are $C^\alpha([-\pi, \pi])$ with $\alpha \in (0, 1)$. The only peaked periodic wave with the peak at $U(\pm\pi) = c$ is the peaked wave with the parabolic profile (2.4).*

Proof. By Remark 1, $U \in \dot{L}^2_{\rm per}(-\pi, \pi)$ is even and $\partial_z^{-1} U \in \dot{H}^1_{\rm per}(-\pi, \pi)$ is odd. Hence, $\partial_z^{-1} U$ is represented by the Fourier sine series which converges absolutely and uniformly, so that $(\partial_z^{-1} U)(\pm\pi) = 0$. Let us first consider the case that $U(\pm\pi) = c$, so that $U(z) < c$ for every $z \in (-\pi, \pi)$. If $U \in C^\alpha([-\pi, \pi])$ with $\alpha \in (0, 1)$, then $\partial_z^{-1} U \in C^1([-\pi, \pi])$. By using the Hölder inequality (2.3) and $(\partial_z^{-1} U)(\pm\pi) = 0$, we obtain from the equation

$$U'(z) = -\frac{(\partial_z^{-1} U)(z)}{c - U(z)}, \quad z \in (-\pi, \pi) \tag{2.5}$$

that $U' \in C^{1-\alpha}([-\pi,\pi])$. Since $1-\alpha \in (0,1)$, this implies that $U \in C^1([-\pi,\pi])$ in the contradiction to the assumption that $U \in C^\alpha([-\pi,\pi])$ with $\alpha \in (0,1)$. Thus, it follows from the equation (2.5) that the peaked periodic wave with profile satisfying $U(z) < c$ for every $z \in (-\pi,\pi)$ belongs to the class $U \in C^1([-\pi,\pi])$.

Let us show that the only peaked periodic wave $U \in C^1([-\pi,\pi])$ with the peak at $U(\pm\pi) = c$ is the peaked wave with the parabolic profile (2.4). Since $U(z) < c$ for every $z \in (-\pi,\pi)$ and $U \in C^1([-\pi,\pi])$, the following first-order invariant

$$E = \frac{1}{2}\left[c - U(z)\right]^2 \left[U'(z)\right]^2 + \frac{c}{2}U(z)^2 - \frac{1}{3}U(z)^3 = \frac{1}{2}\left[(\partial_z^{-1}U)(z)\right]^2 + \frac{c}{2}U(z)^2 - \frac{1}{3}U(z)^3 \tag{2.6}$$

is constant for every $z \in (-\pi,\pi)$ with finite limits as $z \to \pm\pi$. Since $(\partial_z^{-1}U)(\pm\pi) = 0$ and $U(\pm\pi) = c$, there is only one value $E = E_c := c^3/6$ for the conserved quantity (2.6). However, $E = E_c$ selects the only trajectory on the phase plane (see bolded curve on Figure 1), which corresponds to the peaked periodic wave with the parabolic profile (2.4).

Let us now analyze a possibility of $U(\pm z_0) = c$ for $z_0 \in (0,\pi)$. There are two possibilities: $(\partial_z^{-1}U)(\pm z_0) = 0$ and $(\partial_z^{-1}U)(\pm z_0) \neq 0$.

If $(\partial_z^{-1}U)(\pm z_0) = 0$, then the same argument as above selects the bolded curve on Figure 1 which is either continued uniquely to the region with $U(z) > c$ for every $z \in (z_0,\pi]$ or flipped and continued uniquely to the region with $U(z) < c$ for some $z \gtrsim z_0$. However, the first possibility is impossible since it implies that $(\partial_z^{-1}U)(z) > 0$ for every $z \in (z_0,\pi]$ which contradicts $(\partial_z^{-1}U)(\pi) = 0$. The second possibility is possible but does not belong to the class of single-lobe periodic waves (see Remark 6).

If $(\partial_z^{-1}U)(\pm z_0) \neq 0$, then the contradiction arises from equation

$$U'(z) = -\frac{(\partial_z^{-1}U)(z)}{c - U(z)}, \quad z \in (0,z_0) \quad \text{and} \quad U'(z) = -\frac{(\partial_z^{-1}U)(z)}{c - U(z)}, \quad z \in (z_0,\pi). \tag{2.7}$$

Since $(\partial_z^{-1}U)(z)$ is continuous at z_0, the change of the sign of $U'(z)$ across z_0 is determined by the change of the sign of $c - U(z)$ across z_0. If $U(z) < c$ both for $z \lesssim z_0$ and $z \gtrsim z_0$, then the sign of $U'(z)$ remains the same, in contradiction with the monotone increase of $U(z)$ before $z \lesssim z_0$ and decrease after $z \gtrsim z_0$. If $U(z) < c$ for $z \lesssim z_0$ and $U(z) > c$ for $z \gtrsim z_0$, then the sign of $U'(z)$ flips, in contradiction with the monotone increase of $U(z)$ both for $z \lesssim z_0$ and $z \gtrsim z_0$. Hence, both cases with $(\partial_z^{-1}U)(\pm z_0) \neq 0$ are impossible due to contradictions.

Combing together, the only single-lobe peaked periodic wave has parabolic profile (2.4) and satisfies $U(\pm\pi) = c$ at the peak height. $\square$

Remark 4. Solutions with the square root singularity,

$$U(z) = c + \mathcal{O}(\sqrt{\pi^2 - z^2}) \quad \text{as} \quad z \to \pm\pi, \tag{2.8}$$

were formally constructed in [19]. However, existence of such solutions contradicts the arguments in the proof of Lemma 2, since expansion (2.8) implies that $U \in C^{1/2}([-\pi,\pi])$. Moreover, the peaked periodic waves considered on Figure 3 in [19] have nonzero mean value[3]. Hence, the peaked periodic waves in [19] are artifacts of the construction method that relies on the transformation of the semi-linear Klein–Gordon equation to the reduced Ostrovsky equation.

[3]The nonzero mean value plays no role in the arguments arising from the contradictions (2.7) with $z_0 = 0$ if the peaked point is moved from $z = \pm\pi$ to $z = 0$ by a translation of the peaked periodic wave by a half of its period.

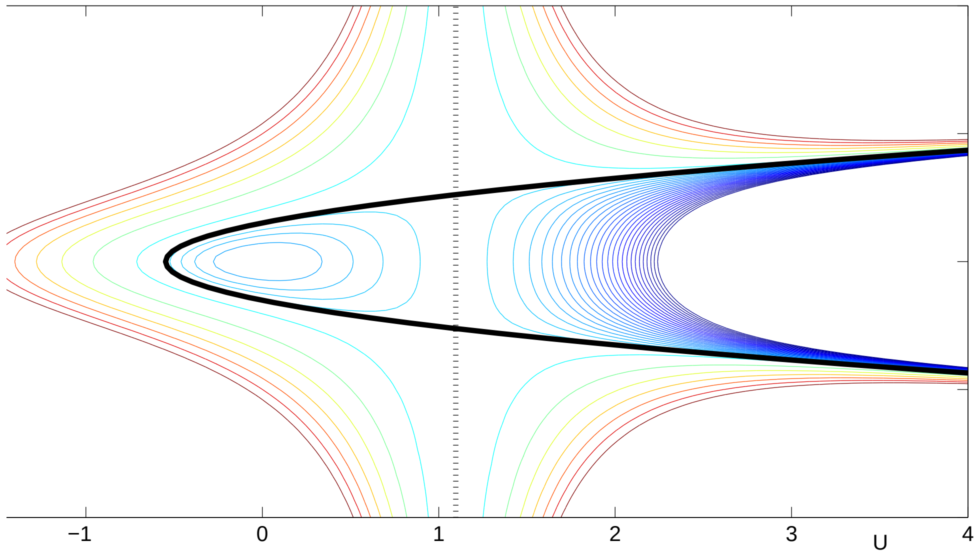

FIGURE 1. Phase plane portrait obtained from level curves of the first-order invariant (2.6) for some $c > 0$. The solid black curve corresponds to the parabolic profile (2.4). The dashed black line indicates the singularity line $U = c$.

Remark 5. Three peaked solitary waves were formally constructed in [34], one of which is a loop soliton studied in many publications [13, 35, 36] and the other two are based on the two possibilities analyzed below equation (2.7). The latter solutions were constructed in [33] by using the transformation of the semi-linear Klein–Gordon equation to the reduced Ostrovsky equation. However, the same argument as in the proof of Lemma 2 eliminates both the possibilities and only leaves a possibility of the loop soliton, which is given by a multi-valued function.

Remark 6. There is a cheap way to obtain other peaked periodic waves beyond the class of single-lobe periodic waves of Definition 1. One can flip the periodic wave with the parabolic profile at a point $z_0 \in (0, \pi)$ and pack two such waves over one period. This possibility is mentioned in the proof of Lemma 2. Similarly, one can pack three and more periods of the peaked wave with the parabolic profiles. These possibility is a well-known argument for non-uniqueness of 2π-periodic waves beyond the class of single-lobe periodic waves in Definition 1.

3. LINEAR INSTABILITY OF THE PEAKED PERIODIC WAVE

We add a *co-periodic* perturbation v to the travelling wave U, that is, a perturbation with the same period 2π. Truncating the quadratic terms and moving with the reference frame of the travelling wave yield the linearized evolution problem in the form

$$\begin{cases} v_t + \partial_z \left[(U(z) - c)v\right] = \partial_z^{-1} v, \quad t > 0, \\ v|_{t=0} = v_0. \end{cases} \tag{3.1}$$

The linearized evolution equation can be formulated in the form $v_t = \partial_z L v$ defined by the self-adjoint operator

$$L = P_0 \left(\partial_z^{-2} + c - U(z)\right) P_0 : \dot{L}^2_{\rm per}(-\pi,\pi) \to \dot{L}^2_{\rm per}(-\pi,\pi), \tag{3.2}$$

where $P_0 : L^2_{\rm per}(-\pi,\pi) \to \dot{L}^2_{\rm per}(-\pi,\pi)$ is the projection operator that removes the mean value of 2π-periodic functions. The form $v_t = \partial_z L v$ is related to the formulation of the reduced Ostrovsky equation in the travelling wave coordinate $z = x - ct$ as a Hamiltonian system defined by the symplectic operator ∂_z and the conserved energy function $H_c(u) = H(u) + cQ(u)$, where

$$H(u) = \int_{-\pi}^{\pi} \left[-(\partial_z^{-1} u)^2 - \frac{1}{3} u^3 \right] dz, \quad Q(u) = \int_{-\pi}^{\pi} u^2 dz \tag{3.3}$$

are the conserved energy and momentum functionals to the reduced Ostrovsky equation (1.1). The periodic wave $u = U$ is a critical point of $H_c(u)$ and the self-adjoint operator L is the Hessian operator of the energy function $H_c(u)$ at the periodic wave $u = U$.

Thanks to the translational invariance of the boundary-value problem (2.1), we have $L\partial_z U = 0$, where $\partial_z U \in \dot{L}^2_{\rm per}(-\pi,\pi)$ holds for both the smooth periodic waves of Lemma 1 and the periodic wave (2.4). Associated to the translational eigenvector is the symplectic orthogonality constraint $\langle U, v\rangle_{L^2(-\pi,\pi)} = 0$, which is used to study both the spectrum of $\partial_z L$ and the evolution of the Cauchy problem (3.1) [3, 22, 29].

Let us distinguish two concepts of stability of the 2π-periodic wave with respect to linearization.

Definition 2. The travelling wave U is said to be spectrally stable if the spectral problem $\lambda v = \partial_z L v$ with $v \in \dot{H}^1_{\rm per}(-\pi,\pi)$ satisfying $\langle U, v\rangle_{L^2(-\pi,\pi)} = 0$ has no eigenvalues $\lambda \notin i\mathbb{R}$. Otherwise, it is said to be spectrally unstable.

Definition 3. The travelling wave U is said to be linearly stable if for every $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ satisfying $\langle U, v\rangle_{L^2(-\pi,\pi)} = 0$, there exists $C > 0$ and a unique global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$ to the Cauchy problem (3.1) such that

$$\|v(t)\|_{H^1_{\rm per}} \leq C \|v_0\|_{H^1_{\rm per}}, \quad t > 0. \tag{3.4}$$

Otherwise, it is said to be linearly unstable.

In [16], we have proved that the smooth periodic waves of Lemma 1 are spectrally stable in the sense of Definition 2. Here we intend to show that the peaked periodic wave (2.4) is linearly unstable in the sense of Definition 3. The linear instability is due to the sharp exponential growth of the unique global solution to the Cauchy problem (3.1) with the peaked periodic wave (2.4):

$$C\|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/6} \leq \|v(t)\|_{L^2(-\pi,\pi)} \leq \|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/6}, \quad t > 0, \tag{3.5}$$

for some $C \in (0,1)$. We will obtain these bounds in two steps. In the first step, Section 3.1, we apply the method of characteristics to the truncated linearized equation (3.1) without the dispersive term $\partial_z^{-1} v$ and obtain the sharp bounds (3.5) for all initial conditions $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ satisfying the constraint

$$\int_{-\pi}^{\pi} z v_0(z)^2 dz = 0. \tag{3.6}$$

In the second step, Section 3.2, we will show that the bounds (3.5) remain true in the full linearized equation (3.1) for a subset of possible initial conditions $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ satisfying the constraint

(3.6) and the additional constraint

$$\int_{-\pi}^{\pi} z^2 v_0(z) dz = 0, \tag{3.7}$$

which arises due to the orthogonality condition $\langle U, v\rangle_{L^2(-\pi,\pi)} = 0$ in Definition 3 and the zero-mean condition on v_0.

Regarding spectral stability or instability of the peaked periodic wave (2.4), we will show in Section 3.3 that the spectrum of the linear self-adjoint operator L in (3.2) is given by a continuous spectrum on $[0, \pi^2/6]$, which includes the embedded eigenvalue $\lambda_0 = 0$ with the eigenvector $\partial_z U$, and a simple negative eigenvalue $\lambda_1 < 0$. As a result, no spectral gap appears between $\lambda_0 = 0$ and the continuous spectrum, hence it is impossible to solve the spectral stability problem by applying the standard methods from [3, 22, 29]. We give no claims of spectral stability or instability for the peaked periodic wave (2.4)[4].

3.1. **Linear instability of truncated evolution.** For the peaked periodic wave (2.4), we obtain the following simple expression:

$$U(z) - c = \frac{1}{6}(z^2 - \pi^2), \quad z \in [-\pi, \pi]. \tag{3.8}$$

Truncating the linearized evolution problem (3.1) by removing the term $\partial_z^{-1} v$ and using the explicit expression (3.8), we can write the truncated evolution problem in the form:

$$\left\{ \begin{array}{l} v_t + \frac{1}{6}\partial_z \left[(z^2 - \pi^2)v\right] = 0, \quad t > 0, \\ v|_{t=0} = v_0, \end{array} \right. \tag{3.9}$$

where the initial data v_0 is taken in $\dot{H}^1_{\rm per}(-\pi, \pi)$. The evolution problem can be solved by the method of characteristics along the family of characteristic curves $z = Z(s,t)$, where $s \in [-\pi, \pi]$ is a parameter for the initial data and $t \geq 0$ is the evolution time. Defining

$$\left\{ \begin{array}{l} \frac{d}{dt} Z(s,t) = \frac{1}{6}\left[Z(s,t)^2 - \pi^2\right], \quad t > 0, \\ Z(s,0) = s, \end{array} \right. \tag{3.10}$$

and setting $V(s,t) := v(Z(s,t),t)$ yields the evolution problem in the form

$$\left\{ \begin{array}{l} \frac{d}{dt} V(s,t) = -\frac{1}{3} Z(s,t) V(s,t), \quad t > 0, \\ V(s,0) = v_0(s). \end{array} \right. \tag{3.11}$$

The family of characteristic curves is obtained by integrating the differential equation (3.10) with the parameter $s \in [-\pi, \pi]$. Because $Z = \pm\pi$ are critical points of the differential equation (3.10), the family of characteristic curves remain inside the invariant region $[-\pi, \pi]$ for every $t \in \mathbb{R}$. The family of characteristic curves can be obtained in the explicit form:

$$Z(s,t) = \pi \frac{s\cosh(\pi t/6) - \pi \sinh(\pi t/6)}{\pi \cosh(\pi t/6) - s \sinh(\pi t/6)}, \quad s \in [-\pi, \pi], \ \ t \in \mathbb{R}. \tag{3.12}$$

Note that $Z(\pm\pi, t) = \pm\pi$ for every $t \in \mathbb{R}$. In order to use the chain rule, we compute:

$$e^{\frac{1}{3}\int_0^t Z(s,t')dt'} = \frac{\partial}{\partial s} Z(s,t) = \frac{\pi^2}{[\pi\cosh(\pi t/6) - s\sinh(\pi t/6)]^2}, \quad s \in [-\pi, \pi], \ \ t \in \mathbb{R}. \tag{3.13}$$

[4]The formal result in Section 5 in [20] violates the periodic boundary conditions on the perturbation v and is deemed to be irrelevant for the question of spectral stability of the peaked periodic wave (2.4).

The explicit solution for V in characteristic variables is obtained by integrating the differential equation (3.11) with the parameter $s \in [-\pi, \pi]$. In view of (3.13), the explicit solution is given by

$$V(s,t) = v_0(s) e^{-\frac{1}{3}\int_0^t Z(s,t')dt'},$$

or equivalently, by

$$V(s,t) = \frac{1}{\pi^2}[\pi \cosh(\pi t/6) - s\sinh(\pi t/6)]^2 v_0(s), \quad s \in [-\pi,\pi], \quad t \in \mathbb{R}. \tag{3.14}$$

By using the explicit solutions (3.12) and (3.14), we are able to state and prove the following linear instability result for the truncated evolution problem (3.9).

Lemma 3. *For every $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$, there exists a unique global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$ to the Cauchy problem (3.9). If $\int_{-\pi}^{\pi} s v_0(s)^2 ds = 0$, then the global solution satisfies the following bound*

$$\frac{1}{2}\|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/6} \leq \|v(t)\|_{L^2(-\pi,\pi)} \leq \|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/6}, \quad t > 0. \tag{3.15}$$

Proof. Existence of a global solution in the explicit form (3.12) and (3.14) is obtained from the method of characteristics. By using the chain rule (3.13), we verify that the mean-zero constraint ins preserved in the time evolution:

$$\int_{-\pi}^{\pi} v(z,t)dz = \int_{-\pi}^{\pi} V(s,t)\frac{\partial Z}{\partial s} ds = \int_{-\pi}^{\pi} v_0(s) ds = 0.$$

The explicit expression (3.14) implies that $V(\cdot,t) \in H^1_{\rm per}(-\pi,\pi)$ if $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ and $t \in \mathbb{R}$. On the other hand, the explicit expression (3.12) implies that for every $\tau > 0$, there exists $C_\tau > 0$ such that

$$\frac{\partial}{\partial s} Z(s,t) \geq C_\tau, \quad s \in [-\pi,\pi], \quad t \in [-\tau,\tau].$$

Hence, the chain rule implies that $v(\cdot,t) \in \dot{H}^1_{\rm per}(-\pi,\pi)$ if $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ and $t \in \mathbb{R}$. Uniqueness of such global solution follows by the standard theory (see Theorem 3.1 in [2]).

It remains to prove the sharp exponential growth (3.15). By the chain rule, we obtain

$$\int_{-\pi}^{\pi} v(z,t)^2 dz \;=\; \int_{-\pi}^{\pi} V(s,t)^2 \frac{\partial Z}{\partial s} ds = \frac{1}{\pi^2}\int_{-\pi}^{\pi} [\pi\cosh(\pi t/6) - s\sinh(\pi t/6)]^2 v_0(s)^2 ds.$$

From here, we have the upper bound

$$\|v(t)\|^2_{L^2(-\pi,\pi)} \leq e^{\pi t/3}\|v_0\|^2_{L^2(-\pi,\pi)}$$

and the lower bound under the additional condition $\int_{-\pi}^{\pi} s v_0(s)^2 ds = 0$:

$$\|v(t)\|^2_{L^2(-\pi,\pi)} = \cosh(\pi t/6)^2 \|v_0\|^2_{L^2(-\pi,\pi)} + \frac{1}{\pi^2}\sinh(\pi t/6)^2 \|s v_0\|^2_{L^2(-\pi,\pi)} \geq \frac{1}{4} e^{\pi t/3}\|v_0\|^2_{L^2(-\pi,\pi)}.$$

Taking square root from these bounds yields (3.15). □

Remark 7. The global solution in Lemma 3 remains bounded in $L^1(-\pi,\pi)$. This follows from the chain rule:

$$\int_{-\pi}^{\pi} |v(z,t)| dz = \int_{-\pi}^{\pi} |V(s,t)| \frac{\partial Z}{\partial s} ds = \int_{-\pi}^{\pi} |v_0(s)| ds.$$

Since

$$\|v_0\|_{L^1(-\pi,\pi)} \leq (2\pi)^{1/2} \|v_0\|_{L^2(-\pi,\pi)}, \tag{3.16}$$

hence $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ implies $v_0 \in L^1(-\pi,\pi)$. Extending this bound to the time-dependent solution,

$$\|v(t)\|_{L^1(-\pi,\pi)} \leq (2\pi)^{1/2} \|v(t)\|_{L^2(-\pi,\pi)}, \quad t > 0, \tag{3.17}$$

shows that the L^1 norm of the global solution $v(t)$ may remain bounded even if the L^2 norm of this solution grows exponentially.

Remark 8. Truncating a quadratic form associated with the self-adjoint operator L in (3.2) and using the chain rule yield the energy conservation for the truncated evolution (3.9):

$$\int_{-\pi}^{\pi} (\pi^2 - z^2) v(z,t)^2 dz = \int_{-\pi}^{\pi} \left[\pi^2 - Z(s,t)^2\right] V(s,t)^2 \frac{\partial Z}{\partial s} ds = \int_{-\pi}^{\pi} (\pi^2 - s^2) v_0(s)^2 ds.$$

The energy conservation shows that the truncated evolution leads to the exponential growth of $\|v(t)\|^2_{L^2(-\pi,\pi)}$ and $\|zv(t)\|^2_{L^2(-\pi,\pi)}$ but the difference between the two squared norms remains bounded.

Remark 9. For the smooth periodic waves of Lemma 1 satisfying $U(z) < c$ for every $z \in [-\pi,\pi]$, the truncated energy $\int_{-\pi}^{\pi}(c-U)v^2 dz$ is coercive in the L^2 norm, hence the energy conservation

$$\int_{-\pi}^{\pi} \left[c - U(z)\right] v(z,t)^2 dz = \int_{-\pi}^{\pi} \left[c - U(z)\right] v_0(s)^2 ds$$

implies a global bound on $\|v(t)\|^2_{L^2(-\pi,\pi)}$ given by the initial data, where $v(t)$ is a solution of the truncation of the linear evolution equation (3.1) without the $\partial_z^{-1} v$ term.

Remark 10. For the smooth periodic waves of Lemma 1, the characteristic curves reach boundaries $z = \pm\pi$ in a finite time because $z = \pm\pi$ are not critical points of the differential equations for the characteristic curves. On the other hand, for the peaked periodic wave (2.4), the characteristic curves reach boundaries $z = \pm\pi$ in the infinite time. The latter property induces an exponential growth of the global solutions to the Cauchy problem (3.9), as is shown in Lemma 3.

3.2. **Linear instability of full evolution.** Here we consider the full linearized evolution problem (3.1) with (3.8). Let us rewrite the evolution problem in the form:

$$\left\{ \begin{array}{l} v_t + \frac{1}{6}\partial_z \left[(z^2 - \pi^2) v\right] = \partial_z^{-1} v, \quad t > 0, \\ v|_{t=0} = v_0, \end{array} \right. \tag{3.18}$$

where the initial data v_0 is taken in $\dot{H}^1_{\rm per}(-\pi,\pi)$. Since ∂_z^{-1} is a bounded perturbation to the truncated evolution problem (3.9) with a global solution in class $C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$, there exists a unique global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$ to the Cauchy problem (3.18). In what follows, we obtain bounds on the global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$.

First, we note the following upper bound on the growth of the global solution to the Cauchy problem (3.18).

Lemma 4. *A global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$ to the Cauchy problem (3.18) satisfies the following bound*

$$\|v(t)\|_{L^2(-\pi,\pi)} \leq \|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/6}, \quad t > 0. \tag{3.19}$$

Proof. Note the following integration

$$\int_{-\pi}^{\pi} v(\partial_z^{-1} v) dz = \frac{1}{2} (\partial_z^{-1} v)^2 |_{z=-\pi}^{z=\pi} = 0,$$

since $\partial_z^{-1} v \in H^2_{\rm per}(-\pi,\pi)$ and hence $\partial_z^{-1} v \in C_{\rm per}(-\pi,\pi)$ by Sobolev's embedding. Integrating by parts yields the following balance equation

$$\frac{d}{dt}\frac{1}{2}\|v(t)\|^2_{L^2(-\pi,\pi)} = \frac{1}{6}\int_{-\pi}^{\pi} v\partial_z \left[(\pi^2-z^2)v\right] dz = -\frac{1}{6}\int_{-\pi}^{\pi} (\pi^2-z^2) v \partial_z v dz = -\frac{1}{6}\int_{-\pi}^{\pi} z v^2 dz.$$

Hence

$$\frac{d}{dt}\|v(t)\|^2_{L^2(-\pi,\pi)} \leq \frac{\pi}{3}\|v(t)\|^2_{L^2(-\pi,\pi)}$$

and Gronwall's inequality yields the desired bound (3.19). □

In order to obtain the lower bound on the L^2 norm of the global solution to the Cauchy problem (3.18), we would like to use the generalized method of characteristics, where we treat $g(z,t) := \partial_z^{-1} v(z,t)$ as a source term. This term is estimated from the following useful bound (also proven in [27]).

Lemma 5. *If $g := \partial_z^{-1} v \in \dot{H}^1_{\rm per}(-\pi,\pi)$, then*

$$\|g\|_{L^\infty(-\pi,\pi)} \leq \|v\|_{L^1(-\pi,\pi)}. \tag{3.20}$$

Proof. By Sobolev embedding of $H^1_{\rm per}(-\pi,\pi)$ to $C^0_{\rm per}([-\pi,\pi])$, g is a continuous 2π-periodic function with zero mean. Therefore, there exists $\zeta \in [-\pi,\pi]$ such that $g(\zeta) = 0$. For every $z \in [-\pi,\pi]$, we can write

$$g(z) = \int_{\zeta}^{z} v(z') dz',$$

from which bound (3.20) follows. Note that $L^2(-\pi,\pi)$ is continuously embedded into $L^1(-\pi,\pi)$ because of the bound (3.16). □

By using the family of characteristic curves $z = Z(s,t)$ with $s \in [-\pi,\pi]$ and $t \geq 0$, where Z is defined by the same initial-value problem (3.10), and setting $V(s,t) := v(Z(s,t),t)$ and $G(s,t) := g(Z(s,t),t)$, we obtain the evolution problem in the form

$$\left\{ \begin{array}{l} \frac{d}{dt} V(s,t) = -\frac{1}{3} Z(s,t) V(s,t) + G(s,t), \quad t > 0, \\ V(s,0) = v_0(s). \end{array} \right. \tag{3.21}$$

The family of characteristic curves Z is still obtained in the same explicit form (3.12). Integrating the differential equation (3.21) with an integrating factor yields the explicit solution for V in the form

$$V(s,t) = \left[v_0(s) + \int_0^t G(s,t') e^{\frac{1}{3}\int_0^{t'} Z(s,t'')dt''} dt' \right] e^{-\frac{1}{3}\int_0^t Z(s,t')dt'} \tag{3.22}$$

By using the explicit solution (3.22), we are able to prove the linear instability result for the Cauchy problem (3.18).

Lemma 6. *There exists $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ and $C > 0$ such that the unique global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$ to the Cauchy problem (3.18) satisfies the following bound*

$$\|v(t)\|_{L^2(-\pi,\pi)} \geq C\|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/6}, \quad t > 0. \tag{3.23}$$

Proof. By the chain rule, the explicit expression (3.22) with the help of (3.13) yields the following equation:

$$\begin{aligned}\int_{-\pi}^{\pi} v(z,t)^2 dz &= \int_{-\pi}^{\pi} V(s,t)^2 \frac{\partial Z}{\partial s} ds \\ &= \frac{1}{\pi^2}\int_{-\pi}^{\pi} [\pi\cosh(\pi t/6) - s\sinh(\pi t/6)]^2 \left[v_0(s) + \int_0^t \frac{\pi^2 G(s,t')}{[\pi\cosh(\pi t'/6) - s\sinh(\pi t'/6)]^2} dt' \right]^2 ds.\end{aligned}$$

Let us assume the same constraint $\int_{-\pi}^{\pi} s v_0(s)^2 ds = 0$ as in Lemma 3. Neglecting positive terms in the lower bound, we obtain

$$\begin{aligned}\|v(t)\|^2_{L^2(-\pi,\pi)} &\geq \frac{1}{4} e^{\pi t/3} \|v_0\|^2_{L^2(-\pi,\pi)} \\ &\quad -2\int_{-\pi}^{\pi}\int_0^t |v_0(s)||G(s,t')| \frac{[\pi\cosh(\pi t/6) - s\sinh(\pi t/6)]^2}{[\pi\cosh(\pi t'/6) - s\sinh(\pi t'/6)]^2} dt' ds.\end{aligned} \tag{3.24}$$

Let us define for any $t > 0$,

$$K(t,t',s) := \frac{\pi\cosh(\pi t/6) - s\sinh(\pi t/6)}{\pi\cosh(\pi t'/6) - s\sinh(\pi t'/6)}, \quad t' \in [0,t], \quad s \in [-\pi,\pi].$$

We prove that for every $0 \leq t' \leq t$,

$$\sup_{s\in[-\pi,\pi]} K(t,t',s) = e^{\pi(t-t')/6}. \tag{3.25}$$

Indeed, $K(t,t',s) = e^{\pi(t-t')/6} M(t,t',s)$, where

$$M(t,t',s) := \frac{(\pi - s) + (\pi + s)e^{-\pi t/3}}{(\pi - s) + (\pi + s)e^{-\pi t'/3}},$$

and M is monotonically decreasing since $\partial_s M(t,t',s) \leq 0$ for every $t' \in [0,t]$ and $s \in [-\pi,\pi]$. Therefore, M has a maximum at $s = -\pi$, where $M(t,t',-\pi) = 1$.

By using (3.17), (3.19), (3.20), (3.24), and (3.25), we obtain

$$\begin{aligned}\|v(t)\|^2_{L^2(-\pi,\pi)} &\geq \frac{1}{4} e^{\pi t/3} \|v_0\|^2_{L^2(-\pi,\pi)} - 2\|v_0\|_{L^1(-\pi,\pi)} \int_0^t \|g(t')\|_{L^\infty(-\pi,\pi)} e^{\pi(t-t')/3} dt' \\ &\geq \frac{1}{4} e^{\pi t/3} \|v_0\|^2_{L^2(-\pi,\pi)} - 2\|v_0\|_{L^1(-\pi,\pi)} \int_0^t \|v(t')\|_{L^1(-\pi,\pi)} e^{\pi(t-t')/3} dt' \\ &\geq \frac{1}{4} e^{\pi t/3} \|v_0\|^2_{L^2(-\pi,\pi)} - 2\sqrt{2\pi}\|v_0\|_{L^1(-\pi,\pi)} \|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/3} \int_0^t e^{-\pi t'/6} dt'\end{aligned}$$

Hence,

$$\|v(t)\|^2_{L^2(-\pi,\pi)} e^{-\pi t/3} \geq \frac{1}{4}\|v_0\|^2_{L^2(-\pi,\pi)} - \frac{12\sqrt{2}}{\sqrt{\pi}} \|v_0\|_{L^1(-\pi,\pi)} \|v_0\|_{L^2(-\pi,\pi)}$$

and since $\|v_0\|_{L^2(-\pi,\pi)}$ can be much larger than $\|v_0\|_{L^1(-\pi,\pi)}$ by the bound (3.16), there exist $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ and $C^2 \in (0,1/4)$ such that

$$\|v(t)\|^2_{L^2(-\pi,\pi)} e^{-\pi t/3} \geq C^2 \|v_0\|^2_{L^2(-\pi,\pi)}. \tag{3.26}$$

This yields the desired bound (3.23). $\square$

Remark 11. Let us show that there exists functions $v_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ satisfying the constraints (3.6) and (3.7) as well as the constraint used in the derivation of (3.26). Indeed, if v_0 is odd, then v_0^2 is even, hence the two constraints (3.6) and (3.7) are satisfied simultaneously. From the class of odd initial data, we need to pick functions that satisfy the inequality

$$\|v_0\|_{L^1(-\pi,\pi)} \leq \frac{\sqrt{\pi}(1-4C^2)}{48\sqrt{2}} \|v_0\|_{L^2(-\pi,\pi)}, \tag{3.27}$$

for a fixed $C^2 \in (0,1/4)$. For example, we can consider the following odd function in $\dot{H}^1_{\rm per}(-\pi,\pi)$

$$v_0(x) = \frac{x(\pi^2 - x^2)}{1 + a^2 x^2}, \quad x \in [-\pi,\pi],$$

where $a > 0$ is a parameter. We obtain by direct computation,

$$\|v_0\|_{L^1(-\pi,\pi)} = \left(\frac{\pi^2}{a^2} + \frac{1}{a^4} \right) \log(1 + \pi^2 a^2) - \frac{\pi^2}{a^2}$$

and

$$\|v_0\|^2_{L^2(-\pi,\pi)} = \frac{1}{a^3} \left[\left(\pi^4 + \frac{6\pi^2}{a^2} + \frac{5}{a^4} \right) \arctan(\pi a) - \frac{\pi(15 + 13\pi^2 a^2)}{3a^3} \right].$$

Since $\|v_0\|_{L^1(-\pi,\pi)} = \mathcal{O}(\log(a) a^{-2})$ decays to zero as $a \to \infty$ faster than $\|v_0\|_{L^2(-\pi,\pi)} = \mathcal{O}(a^{-3/2})$, inequality (3.27) can be satisfied for sufficiently large a.

Remark 12. In the presence of the source term G, we are not able to show that $\|v(t)\|_{L^1(-\pi,\pi)}$ remains bounded as $t \to \infty$, see Remark 7. By using the integral

$$\int_{-\pi}^{\pi} \frac{\pi^2}{[\pi \cosh(\pi t'/6) - s \sinh(\pi t'/6)]^2} ds = 2\pi, \quad t' \in [0,t],$$

we obtain the bound

$$\|v(t)\|_{L^1(-\pi,\pi)} \quad \leq \quad \|v_0\|_{L^1(-\pi,\pi)} + 2\pi \int_0^t \|g(t')\|_{L^\infty(-\pi,\pi)} dt',$$

in view of (3.13) and (3.22). Thanks to the bound (3.20), the inequality is closed as follows:

$$\|v(t)\|_{L^1(-\pi,\pi)} \quad \leq \quad \|v_0\|_{L^1(-\pi,\pi)} + 2\pi \int_0^t \|v(t')\|_{L^1(-\pi,\pi)} dt'.$$

By Gronwall's inequality, this bound gives the fast exponential growth

$$\|v(t)\|_{L^1(-\pi,\pi)} \leq \|v_0\|_{L^1(-\pi,\pi)} e^{2\pi t},$$

which cannot be sharp because of the slow exponential growth that follows from the bounds (3.17) and (3.19).

Remark 13. There exists a conserved energy for the Cauchy problem (3.18), see Remark 8, which is given by

$$\langle L v(t), v(t) \rangle_{L^2(-\pi,\pi)} = \langle L v_0, v_0 \rangle_{L^2(-\pi,\pi)}, \tag{3.28}$$

where the self-adjoint operator L is defined by (3.2). However, the conserved quantity (3.28) does not prevent $\|v(t)\|_{L^2(-\pi,\pi)}$ to grow exponentially fast as $t \to \infty$ because the bounded operator L has no spectral gap between the zero eigenvalue and the positive continuous spectrum, see Lemma 7 below.

3.3. **Spectrum of the linear self-adjoint operator L.** Here we consider the spectrum of the linear self-adjoint operator L defined by (3.2). We will prove that it consists of the continuous spectrum on $[0,\pi^2/6]$, which includes the embedded eigenvalue $\lambda_0 = 0$ with the eigenvector $\partial_z U$, and a simple negative eigenvalue $\lambda_1 < 0$. No spectral gap appears between $\lambda_0 = 0$ and the continuous spectrum.

The following lemma gives the corresponding result.

Lemma 7. *The spectrum of the self-adjoint operator L given by (3.2) is*

$$\sigma(L) = \{\lambda_1\} \cup \left[0, \frac{\pi^2}{6}\right], \tag{3.29}$$

where $\lambda_1 < 0$ is the unique zero of the transcendental equation

$$(\pi^2 + 3\lambda) \log \frac{\sqrt{\pi^2 - 6\lambda} + \pi}{\sqrt{\pi^2 - 6\lambda} - \pi} - 3\pi\sqrt{\pi^2 - 6\lambda} = 0, \quad \lambda < 0. \tag{3.30}$$

Proof. By the spectral theorem (see, e.g., Definition 8.39, Theorem 8.70, and Theorem 8.71 in [30]), the spectrum of the self-adjoint operator L in $\dot{L}^2_{\rm per}(-\pi,\pi)$ denoted by $\sigma(L)$ may consist of only two disjoint sets on the real line: the point spectrum of eigenvalues with eigenvectors in $\dot{L}^2_{\rm per}(-\pi,\pi)$ denoted by $\sigma_p(L)$ and the continuous spectrum denoted by $\sigma_c(L)$, where the resolvent operator exists but is unbounded.

The self-adjoint operator L in (3.2) is given by the sum of a bounded operator L_0 and a compact operator K given by

$$L_0 := \frac{1}{6} P_0 \left(\pi^2 - z^2\right) P_0 : \; \dot{L}^2_{\rm per}(-\pi,\pi) \to \dot{L}^2_{\rm per}(-\pi,\pi) \tag{3.31}$$

and

$$K := P_0 \partial_z^{-2} P_0 : \; \dot{L}^2_{\rm per}(-\pi,\pi) \to \dot{L}^2_{\rm per}(-\pi,\pi). \tag{3.32}$$

Moreover, the compact operator is in the trace class since $\sum_{n=1}^{\infty} n^{-2} < \infty$. By Kato's Theorem [24] (see Theorem 4.4 on p. 542 in [25]), $\sigma_c(L) = \sigma_c(L_0)$. We show that $[0,\pi^2/6] \subseteq \sigma_c(L_0)$ by considering the odd functions in $\dot{L}^2_{\rm per}(-\pi,\pi)$, which can be represented by the Fourier sine series. Let us denote the space of odd functions in $\dot{L}^2_{\rm per}(-\pi,\pi)$ by $L^2_{\rm per,odd}(-\pi,\pi)$. Then,

$$L_0 f = \frac{1}{6}(\pi^2 - z^2) f, \quad \forall f \in L^2_{\rm per,odd}(-\pi,\pi).$$

Then, $\sigma_c(L_0)$ in $L^2_{\rm per,odd}(-\pi,\pi)$ coincides with the range of the multiplicative function $h(z) = \frac{1}{6}(\pi^2 - z^2)$ for $z \in [-\pi,\pi]$, which is $[0,\pi^2/6]$. Hence, $[0,\pi^2/6] \subseteq \sigma_c(L_0) = \sigma_c(L)$ in $\dot{L}^2_{\rm per}(-\pi,\pi)$.

Let us show that $[0,\pi^2/6] \equiv \sigma_c(L_0)$ by working with the resolvent equation $(L_0 - \lambda I) f = g$ for given $g \in \dot{L}^2_{\rm per}(-\pi,\pi)$ and $\lambda \notin [0,\pi^2/6]$. The resolvent equation can be written in the component form for $z \in [-\pi,\pi]$:

$$\frac{1}{6}(\pi^2 - 6\lambda - z^2) f(z) - k(f) = g(z), \quad k(f) := \frac{1}{12\pi} \int_{-\pi}^{\pi} (\pi^2 - z^2) f(z) dz,$$

where $f \in \dot{L}^2_{\rm per}(-\pi,\pi)$ is supposed to satisfy the zero-mean constraint $\int_{-\pi}^{\pi} f(z) dz = 0$. Computing the solution explicitly,

$$f(z) = \frac{6}{\pi^2 - 6\lambda - z^2} \left[g(z) + k(f) \right],$$

and using the zero mean constraint, we can define $k(f)$ in terms of g:

$$k(f) = \frac{\int_{-\pi}^{\pi} \frac{g(z)}{\pi^2-6\lambda-z^2} dz}{\int_{-\pi}^{\pi} \frac{1}{\pi^2-6\lambda-z^2} dz}.$$

For every $\lambda \notin [0, \pi^2/6]$, there exists positive constants $C_\lambda, C'_\lambda > 0$ such that

$$\sup_{z\in[-\pi,\pi]} \frac{6}{|\pi^2 - 6\lambda - z^2|} \leq C_\lambda, \quad \left| \int_{-\pi}^{\pi} \frac{6}{\pi^2 - 6\lambda - z^2} dz \right| \geq C'_\lambda.$$

As a result, we obtain the bound

$$\|f\|_{L^2(-\pi,\pi)} \leq C_\lambda \left[\|g\|_{L^2(-\pi,\pi)} + |k(f)|\sqrt{2\pi} \right] \leq C_\lambda \left[1 + 2\pi (C'_\lambda)^{-1} C_\lambda\right] \|g\|_{L^2(-\pi,\pi)}.$$

Therefore, the resolvent operator $(L_0 - \lambda I)^{-1} : \dot{L}^2_{\rm per}(-\pi, \pi) \to \dot{L}^2_{\rm per}(-\pi, \pi)$ is bounded for every $\lambda \notin [0, \pi^2/6]$ so that $\sigma_c(L_0) = [0, \pi^2/6]$.

In order to study $\sigma_p(L) \in \mathbb{R}\backslash[0, \pi^2/6]$, we consider the spectral problem for operator L with the spectral parameter $\lambda \notin [0, \pi^2/6]$:

$$\frac{1}{6} P_0 \left(\pi^2 - z^2\right) w + P_0 \partial_z^{-2} w = \lambda w, \quad w \in \dot{L}^2_{\rm per}(-\pi, \pi). \tag{3.33}$$

Since $\partial_z^{-2} w \in H^2_{\rm per}(-\pi, \pi)$, bootstrapping arguments show that $w \in H^2_{\rm loc}(-\pi, \pi)$, where $H^2_{\rm loc}(-\pi, \pi)$ are defined on any compact subset in $(-\pi, \pi)$. Therefore, on a compact subset in $(-\pi, \pi)$, the spectral problem (3.33) can be differentiated twice, after which it is rewritten as the following second-order differential equation

$$\left(\pi^2 - z^2 - 6\lambda\right) \frac{d^2 w}{dz^2} - 4z \frac{dw}{dz} + 4w(z) = 0, \quad w \in H^2_{\rm loc}(-\pi, \pi), \tag{3.34}$$

with the following two linearly independent solutions for $\lambda \in \mathbb{R}\backslash[0, \pi^2/6]$,

$$w_1(z) = z$$

and

$$w_2(z) = \begin{cases} -1 + \frac{z^2}{2(\pi^2-z^2-6\lambda)} + \frac{3z}{4\sqrt{\pi^2-6\lambda}} \log \frac{\sqrt{\pi^2-6\lambda}+z}{\sqrt{\pi^2-6\lambda}-z}, & \lambda < 0, \\ -1 + \frac{z^2}{2(\pi^2-z^2-6\lambda)} - \frac{3z}{2\sqrt{6\lambda-\pi^2}} \arctan \frac{z}{\sqrt{6\lambda-\pi^2}}, & \lambda > \frac{\pi^2}{6}. \end{cases}$$

The first solution corresponds to the eigenvector $\partial_z U$ of the spectral problem (3.33) for the eigenvalue $\lambda_0 = 0$, which is embedded into $\sigma_c(L) = [0, \pi^2/6]$. Since eigenvectors of the self-adjoint operator for distinct eigenvalues are orthogonal, we are looking for solutions w of the spectral problem (3.33) such that $\langle w, w_1 \rangle_{L^2(-\pi,\pi)} = 0$. Therefore, we take[5] $w = w_2$ and extend it from $H^2_{\rm loc}(-\pi, \pi)$ to $\dot{L}^2_{\rm per}(-\pi, \pi)$. This extension is achieved if and only if w has zero mean, that is,

$$0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} w_2(z) dz = \begin{cases} -\frac{3}{4} + \frac{\pi^2+3\lambda}{4\pi\sqrt{\pi^2-6\lambda}} \log \frac{\sqrt{\pi^2-6\lambda}+\pi}{\sqrt{\pi^2-6\lambda}-\pi} & \lambda < 0, \\ -\frac{3}{4} - \frac{\pi^2+3\lambda}{2\pi\sqrt{6\lambda-\pi^2}} \arctan \frac{\pi}{\sqrt{6\lambda-\pi^2}} & \lambda > \frac{\pi^2}{6}. \end{cases} \tag{3.35}$$

The piecewise graph of the right-hand side of the zero-mean constraint (3.35) on $(-\infty, 0)$ and $(\pi^2/6, \infty)$ is shown on Figure 2. The first line of the zero-mean constraint (3.35) is equivalent to the transcendental equation (3.30) and it has only one simple zero at $\lambda_1 \approx -0.2262$. The second line of (3.35) does not have any zeros. Hence, $\lambda_1 < 0$ is the only eigenvalue in $\sigma_p(L)$. $\square$

[5]Note that $\langle w_2, w_1 \rangle_{L^2(-\pi,\pi)} = 0$ because w_1 is odd and w_2 is even.

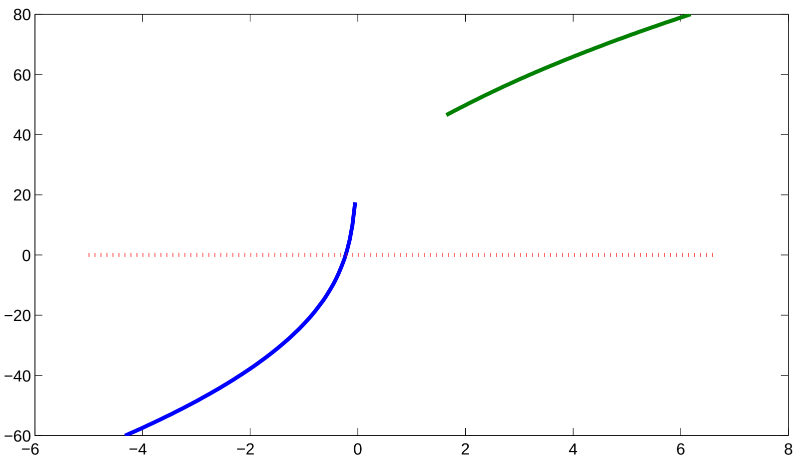

Figure 2. A graph of the right-hand side of the zero-mean constraint (3.35) on $(-\infty, 0)$ and $(\pi^2/6, \infty)$.

Remark 14. For the smooth periodic waves of Lemma 1, we proved in [16] that the spectrum of L includes a simple negative eigenvalue, a simple zero eigenvalue with the eigenvector $\partial_z U$, and the rest of the spectrum is bounded away from zero. Hence, the spectral gap is present in the case of smooth periodic waves, which enabled us to prove in [16] spectral stability of the smooth periodic waves according to Definition 2. By the standard analysis involving the conserved quantity (3.28), see [21], this spectral stability result transfers to the linear stability of the smooth periodic waves according to Definition 3

4. Nonlinear instability

Here we transfer the linear instability result of Lemma 6 to the proof of nonlinear instability of the peaked periodic wave (2.4) in the Cauchy problem (1.1). Our proof cannot be deduced from the standard approach in [31] because we do not know if the spectral assumption on the spectral instability is satisfied for the peaked periodic wave.

Because of the translational symmetry of the Cauchy problem (1.1), we need to consider an orbit $\{U(x-a), \;\; a \in [-\pi,\pi]\}$ of the peaked periodic wave (2.4). The following definition of the orbital stability of the periodic wave is widely used in the literature, see, e.g., [15].

Definition 4. The travelling wave U is said to be orbitally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $u_0 \in \dot{H}^1_{\rm per}(-\pi,\pi)$ satisfying $\|u_0 - U\|_{H^1_{\rm per}} < \delta$, there exists a unique global solution $u \in C(\mathbb{R}, \dot{H}^1_{\rm per}(-\pi,\pi))$ to the Cauchy problem (1.1) with $u(0) = u_0$ such that for every $t > 0$,

$$\inf_{a \in [-\pi,\pi]} \|u(t,\cdot + a) - U(\cdot)\|_{H^1_{\rm per}} < \epsilon. \tag{4.1}$$

Otherwise, the periodic wave U is said to be orbitally unstable.

Remark 15. The local well-posedness theory for the Cauchy problem (1.1) in [32] requires $u_0 \in \dot{H}^s_{\rm per}(-\pi,\pi)$ with $s > 3/2$, which contradicts to the low regularity of the peaked periodic wave

$U \in \dot{H}^s_{\rm per}(-\pi,\pi)$ with $s < 3/2$. We avoid this obstacle by introducing the decomposition $u = U + v$, where v is defined in a smoother space than U.

Remark 16. The large-norm smooth solutions to the Cauchy problem (1.1) are not global as their $H^1_{\rm per}$ norm may blow up in a finite time [27]. This would have been a difficult obstacle in the proof of orbital stability of the peaked periodic wave[6]. However, since we are proving orbital instability of the peaked periodic wave, we do not worry if the local solution can not be continued for all $t > 0$.

The translational parameter a in (4.1) does not have to be defined by minimizing the $H^1_{\rm per}$ norm. One can define a by minimizing the L^2 norm, as is done, e.g., in [15]. Therefore, we introduce the following decomposition of the local solution $u(t,\cdot) \in \dot{H}^1_{\rm per}(-\pi,\pi)$ to the Cauchy problem (1.1):

$$u(t,x) = U(x-ct-a(t)) + v(t,x-ct-a(t)), \quad \langle \partial_z U, v\rangle_{L^2(-\pi,\pi)} = 0, \tag{4.2}$$

where $z = x - ct - a(t)$. The co-periodic perturbation v to the travelling wave U satisfies the evolution problem in the form

$$\left\{ \begin{array}{l} v_t + \frac{1}{6}\partial_z\left[(z^2-\pi^2)v\right] + v\partial_z v = \partial_z^{-1} v + a'(t)(\partial_z U + \partial_z v), \quad t > 0, \\ v|_{t=0} = v_0. \end{array} \right. \tag{4.3}$$

By projecting the evolution problem to $\partial_z U \in \dot{L}^2_{\rm per}(-\pi,\pi)$ and using the orthogonality condition in (4.2), we obtain the modulation equation determining evolution of the modulation parameter a:

$$\left\{ \begin{array}{l} a'(t) = -\frac{\langle \partial_z U, \partial_z L v\rangle_{L^2(-\pi,\pi)} - \langle \partial_z U, v\partial_z v\rangle_{L^2(-\pi,\pi)}}{\|\partial_z U\|^2_{L^2(-\pi,\pi)} + \langle \partial_z U, \partial_z v\rangle_{L^2(-\pi,\pi)}}, \quad t > 0, \\ a(0) = 0, \end{array} \right. \tag{4.4}$$

where L is the same self-adjoint operator as in (3.2). We note the following useful simplification of $\langle \partial_z U, \partial_z L v\rangle_{L^2(-\pi,\pi)}$.

Lemma 8. *If $v \in \dot{H}^1_{\rm per}(-\pi,\pi)$, then*

$$\langle \partial_z U, \partial_z L v\rangle_{L^2(-\pi,\pi)} = -\frac{2}{3}\langle U, v\rangle_{L^2(-\pi,\pi)}. \tag{4.5}$$

Proof. By using the definitions from (2.4) and (3.2), we obtain

$$\langle \partial_z U, \partial_z L v\rangle_{L^2(-\pi,\pi)} = \frac{1}{3}\int_{-\pi}^{\pi} z\partial_z^{-1} v dz + \frac{1}{3}\int_{-\pi}^{\pi} z\partial_z\left[(c-U)v\right] dz.$$

Since $\partial_z^{-1} v$ and U are $C_{\rm per}(-\pi,\pi)$ and $c - U(\pm\pi) = 0$, integration by parts yield

$$\langle \partial_z U, \partial_z L v\rangle_{L^2(-\pi,\pi)} = -\int_{-\pi}^{\pi} U v dz - \frac{1}{3}\int_{-\pi}^{\pi}(c-U) v dz.$$

Since $\int_{-\pi}^{\pi} v dz = 0$, we obtain (4.5). □

Let $S(t) : \dot{L}^2_{\rm per}(-\pi,\pi) \to \dot{L}^2_{\rm per}(-\pi,\pi)$ denote the semi-group of the linearized system (3.18) which exists thanks to Lemma 4. By Duhamel's principle, the Cauchy problem (4.3) can be rewritten in the equivalent integral form

$$v(t) = S(t)v_0 + \int_0^t S(t-t')F(t')dt', \quad F := a'(t)(\partial_z U + \partial_z v) - v\partial_z v, \tag{4.6}$$

[6]In a similar context of the Camassa–Holm equation, the work of [8] avoided this obstacle by introducing a weaker (nonconventional) definition of orbital stability of peaked waves over short intervals of existence of local solutions.

for all $t > 0$, for which $\partial_z v(t), v(t)\partial_z v(t) \in \dot{L}^2_{\rm per}(-\pi,\pi)$. By the same local well-posedness result in [32], there exists a local solution of the Cauchy problem (4.3) for every $v_0 \in \dot{H}^s_{\rm per}(-\pi,\pi)$ with $s > 3/2$. For this solution, we have $\partial_z v(t), v(t)\partial_z v(t) \in \dot{L}^2_{\rm per}(-\pi,\pi)$ so that the same local solution $v(t)$ to the Cauchy problem (4.3) satisfies the integral formulation (4.6).

By using Definition 4, we prove the orbital instability result for the Cauchy problem (4.3) near the peaked periodic wave U.

Lemma 9. *There exists $\epsilon > 0$ such that for every small $\delta > 0$, there exists $v_0 \in \dot{H}^s_{\rm per}(-\pi,\pi)$ with $s > 3/2$ satisfying $\|v_0\|_{H^s_{\rm per}} \leq \delta$, for which the unique local solution $v \in C([0,t_0],\dot{H}^s_{\rm per}(-\pi,\pi))$ with $a \in C([0,t_0],\mathbb{R})$ to the Cauchy problem (4.3)–(4.4) with $t_0 = \mathcal{O}(\delta^{-1})$ satisfies $\|v(t_1)\|_{L^2(-\pi,\pi)} \geq \epsilon$ for some $t_1 \in (0,t_0)$. Therefore, the peaked periodic wave U is orbitally unstable in the sense of Definition 4.*

Proof. By the local well-posedness theory [32], the unique local solution $v \in C([0,t_0],\dot{H}^s_{\rm per}(-\pi,\pi))$ exists for every $v_0 \in \dot{H}^s_{\rm per}(-\pi,\pi)$ with $s > 3/2$ and t_0 is inverse proportional to $\|v_0\|_{H^s_{\rm per}}$, hence, $t_0 = \mathcal{O}(\delta^{-1})$ is large. We estimate the L^2 norm of the local solution $v(t)$ by using the integral formulation (4.6) and the triangle inequality

$$\|v(t)\|_{L^2(-\pi,\pi)} \geq \|S(t)v_0\|_{L^2(-\pi,\pi)} - \left\| \int_0^t S(t-t')F(t')dt' \right\|_{L^2(-\pi,\pi)}, \tag{4.7}$$

By Lemma 6, there exists $v_0 \in \dot{H}^s_{\rm per}(-\pi,\pi)$ with $s > \frac{3}{2}$ and $C > 0$ such that

$$\|S(t)v_0\|_{L^2(-\pi,\pi)} \geq C\|v_0\|_{L^2(-\pi,\pi)} e^{\pi t/6}, \quad t > 0. \tag{4.8}$$

We recall that v_0 in Lemma 6 satisfies the constraints (3.6), (3.7), and (3.27).

By Lemma 4, for every $F(t) \in L^2_{\rm per}(-\pi,\pi)$ with $t \in [0,t_0]$, we have

$$\left\| \int_0^t S(t-t')F(t')dt' \right\|_{L^2(-\pi,\pi)} \leq \int_0^t e^{\pi(t-t')/6} \left\|F(t')\right\|_{L^2(-\pi,\pi)} dt'. \tag{4.9}$$

In order to obtain a contradiction, let us assume by using Definition 4 that there exists $B > 0$ such that the solution $v \in C([0,t_0],\dot{H}^s_{\rm per}(-\pi,\pi))$ satisfies $\|v(t)\|_{H^1_{\rm per}} \leq B\delta$ for every $t \in [0,t_0]$. Since $\delta > 0$ is small[7], the denominator in the modulation equation (4.4) is strictly positive and bounded from below. The numerator in the modulation equation (4.4) consists of two terms

$$\langle \partial_z U, \partial_z L v\rangle_{L^2(-\pi,\pi)} \quad \text{and} \quad \langle \partial_z U, v\partial_z v\rangle_{L^2(-\pi,\pi)}. \tag{4.10}$$

The first term in (4.10) is estimated from Lemma 8 and the momentum conservation

$$Q(u(t)) = Q(u_0),$$

where $Q(u) = \|u\|^2_{L^2(-\pi,\pi)}$. By using the decomposition (4.2), we obtain

$$Q(u_0) = Q(U) + 2\langle U, v\rangle_{L^2(-\pi,\pi)} + Q(v),$$

where $Q(v) \leq B^2\delta^2$. On the other hand, $Q(u_0) = Q(U) + Q(v_0)$ since $\langle U, v_0\rangle_{L^2(-\pi,\pi)} = 0$ is satisfied from the constraint (3.7). Hence $|Q(u_0) - Q(U)| \leq \delta^2$, so that the expression (4.5) yields the

[7] We are using Definition 4 with $\epsilon = B\delta$, a more general argument with a small $\epsilon(\delta)$ is readily available.

following estimate for the first term in (4.10):

$$\left|\langle \partial_z U, \partial_z L v\rangle_{L^2(-\pi,\pi)}\right| \leq \frac{1}{3}(1+B^2)\delta^2.$$

The second term in (4.10) is quadratic in v, hence it follows from (4.4) that there exists $B' > 0$ such that

$$|a'(t)| \leq B'\delta^2 \tag{4.11}$$

for every $t \in [0, t_0]$. By using the definition of F in (4.6), it follows from (4.7), (4.8), (4.9), and (4.11) that

$$\|v(t)\|_{L^2(-\pi,\pi)} \geq e^{\pi t/6}\left[C\|v_0\|_{L^2(-\pi,\pi)} - \frac{6}{\pi}\left(B'\delta^2\|\partial_z U\|_{L^2(-\pi,\pi)} + B'B\delta^3 + B^2\delta^2\right)\right], \tag{4.12}$$

for every $t \in [0, t_0]$. Since $\delta > 0$ is small, one can always find $v_0 \in \dot{H}^s_{\rm per}(-\pi,\pi)$ with $s > \frac{3}{2}$ such that $\|v_0\|_{L^2(-\pi,\pi)} = \mathcal{O}(\delta)$ as $\delta \to 0$ and

$$C\|v_0\|_{L^2(-\pi,\pi)} - \frac{6}{\pi}\left(B'\delta^2\|\partial_z U\|_{L^2(-\pi,\pi)} + B'B\delta^3 + B^2\delta^2\right) > 0.$$

The lower bound in (4.12) grows exponentially fast, so that there exists a time instant t_1 such that $t_1 = \mathcal{O}(1)$ as $\delta \to 0$ and

$$e^{\pi t/6}\left[C\|v_0\|_{L^2(-\pi,\pi)} - \frac{6}{\pi}\left(B'\delta^2\|\partial_z U\|_{L^2(-\pi,\pi)} + B'B\delta^3 + B^2\delta^2\right)\right] \geq \epsilon, \quad t \in [t_1, t_0],$$

for any fixed ϵ (say, $\epsilon = 1$). This yields a contradiction with the assumption that the solution $v \in C([0,t_0], \dot{H}^s_{\rm per}(-\pi,\pi))$ remains small in the $\dot{H}^1_{\rm per}(-\pi,\pi)$ norm. $\square$

Remark 17. Local solutions to the Cauchy problem (4.3) may blow up in a finite time. It was proven[8] in [18] that the local solution to the original Cauchy problem (1.1) in $H^3_{\rm per}(-\pi,\pi)$ remain bounded in $H^3_{\rm per}(-\pi,\pi)$ norm if the initial condition $u_0 \in H^3_{\rm per}(-\pi,\pi)$ satisfies the constraint $1 - 3u_0''(x) > 0$ for every $x \in [-\pi,\pi]$. Since $u_0(x) = U(x) + v_0(x)$ and the peaked periodic wave (2.4) satisfies

$$1 - 3U''(x) = 0, \quad x \in (-\pi,\pi),$$

then small perturbation in v_0 may violate the global well-posedness constraint if $v_0''(x) > 0$ for some $x \in (-\pi,\pi)$. This blow-up in a finite time does not contradict the nonlinear instability result of Lemma 9.

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[8]The result of [18] was proven on the real line $\mathbb{R}$ but it can be extended verbatim to the circle $(-\pi,\pi)$.

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(A. Geyer) Delft Institute of Applied Mathematics, Faculty Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

E-mail address: `A.Geyer@tudelft.nl`

(D. Pelinovsky) Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

E-mail address: `dmpeli@math.mcmaster.ca`

(D. Pelinovsky) Department of Applied Mathematics, Nizhny Novgorod State Technical University, 24 Minin street, 603950 Nizhny Novgorod, Russia