

LINEAR AND NONLINEAR INSTABILITY OF THE PEAKED PERIODIC WAVE IN THE REDUCED OSTROVSKY EQUATION

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ABSTRACT. Stability of the peaked periodic waves in the reduced Ostrovsky equation has remained an open problem for a long time. In order to solve this problem we obtain sharp bounds on the exponential growth of the L^2 norm of co-periodic perturbations to the peaked periodic wave, from which it follows that the peaked periodic wave is orbitally unstable. We also prove that the peaked periodic wave with parabolic profile is the unique peaked wave in the space of periodic L^2 functions with zero mean and a single minimum per period.

1. INTRODUCTION

We address solutions of the Cauchy problem for the reduced Ostrovsky equation [29] written in the form

$$(1.1) \quad \begin{cases} u_t + uu_x = \partial_x^{-1}u, & t > 0, \\ u|_{t=0} = u_0, \end{cases}$$

where u_0 is a 2π -periodic function with zero mean defined in the Sobolev space $H_{\text{per}}^s(-\pi, \pi)$ for some $s \geq 0$, which we simply write as H_{per}^s . We denote the subspace of 2π -periodic functions with zero mean in H_{per}^s by \dot{H}_{per}^s . The operator $\partial_x^{-1} : \dot{H}_{\text{per}}^s \rightarrow \dot{H}_{\text{per}}^{s+1}$ denotes the anti-derivative with zero mean, which can be defined by using Fourier series.

The reduced Ostrovsky equation is also known under the names of Ostrovsky–Hunter and Ostrovsky–Vakhnenko equation due to contributions of Hunter [24] and Vakhnenko [36].

Local solutions to the Cauchy problem (1.1) with $u_0 \in \dot{H}_{\text{per}}^s$ exist for $s > \frac{3}{2}$ [33], and we refer to [28] for a discussion on how the well-posedness in $H^s(\mathbb{R})$ is extended to \dot{H}_{per}^s . For sufficiently large initial data, the local solutions break in finite time, similar to the inviscid Burgers equation [28]. However, if the initial data u_0 is suitably small, then the local solutions for $s = 3$ are continued for all times [18, 19]. Weak bounded solutions with shock discontinuities were constructed in [6, 7]. Weak solutions of the Cauchy problem (1.1) as the limiting solution of the Cauchy problem for the regularized Ostrovsky equation were considered in [5].

The reduced Ostrovsky equation with smooth solutions is completely integrable as it can be reduced to the integrable Tzitzeica equation by a coordinate transformation [27]. This property enables a construction of a bi-infinite set of conserved quantities in the time evolution [4] and the inverse scattering transform with the Riemann–Hilbert approach [1]. Two integrable semi-discretizations of the reduced Ostrovsky equation have been obtained by using bilinear forms [15].

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Stability of smooth and peaked periodic waves in the reduced Ostrovsky equation has been recently addressed in a number of publications [12, 17, 20, 21, 34]. By using higher-order conserved quantities the smooth small-amplitude periodic waves were shown in [12] to be unconstrained minimizers of a higher-order energy function. This result holds for *subharmonic* perturbations, that is, perturbations whose period is a multiple of the period of the smooth periodic waves. Since the higher-order conserved quantities are well-defined in the space \dot{H}_{per}^3 , where global well-posedness has been proven [19], it follows from the minimization properties that smooth small-amplitude periodic waves are both spectrally and orbitally stable. The minimization properties were confirmed numerically for smooth periodic waves of large amplitude all the way up to the limiting peaked wave of parabolic profile with maximal amplitude, for which the numerical results were inconclusive [12].

Spectral stability of smooth periodic waves with respect to *co-periodic* perturbations, that is, perturbations with the same period as the period of the periodic wave, was shown in [17] by using the standard variational formulation of the periodic waves as critical points of energy subject to fixed momentum. This result holds also for the generalized reduced Ostrovsky equation with power nonlinearity. Independently, spectral stability of smooth periodic waves in the reduced Ostrovsky equation was shown in [21] by using a coordinate transformation of the spectral stability problem to an eigenvalue problem studied earlier in [34].

Regarding the peaked periodic waves, some conflicting results were recently obtained. In [21], the peaked wave with the parabolic profile was addressed and claimed to be “unstable in the absence of periodic boundary conditions”. The proof was obtained by a construction of explicit solutions of the spectral stability problem for a positive (unstable) eigenvalue¹, which produces “wild boundary conditions” on perturbations of the peaked wave [21]. In contrast, families of peaked periodic waves of small amplitude, which were previously unknown in the context of the reduced Ostrovsky equation, were constructed in [20] and these families were claimed to be spectrally stable with respect to co-periodic perturbations.

In this paper we give a simple and definition conclusion about existence and stability of peaked periodic waves in the reduced Ostrovsky equation. This is the first time, to the best of our knowledge, that linear as well as nonlinear instability of peaked periodic waves is proven by means of semigroup theory and energy estimates.

The following theorem presents the main result of this paper.

Theorem 1. *The peaked periodic wave with parabolic profile is the unique (up to spatial translations) peaked travelling wave solution of the reduced Ostrovsky equation in \dot{L}_{per}^2 having a single minimum per period. The solution is Lipschitz continuous and exists in \dot{H}_{per}^s with $s < 3/2$. The orbit generated by spatial translations of the peaked periodic wave is linearly and nonlinearly unstable with respect to perturbations in H_{per}^s with $s > 3/2$.*

The first part of Theorem 1 allows us to disprove existence of the families of peaked periodic small-amplitude waves constructed in [20]. Apparently, these families of solutions are artefacts of the construction method which relies on change of coordinates that transforms the reduced Ostrovsky equation into the semi-linear Klein–Gordon equation, see Remark 4.

¹The explicit solution constructed in Section 5 of [21] is only valid for the periodic wave with a period being equal to two due to a simple algebraic error, but a more general solution can be constructed for the periodic wave of any period. Nevertheless, this explicit solution is not relevant for co-periodic perturbations in the space \dot{L}_{per}^2 .

The last part of Theorem 1 gives a definite conclusion on linear and nonlinear instability of the peaked periodic wave with parabolic profile with respect to co-periodic perturbations. Compared to the formal construction of special solutions of the spectral stability problem without periodic boundary conditions in [21], we make no claims on the spectral stability problem related to the peaked periodic wave. The standard approach in [32] allows to conclude on the nonlinear instability of a travelling wave from its linear instability if the spectral assumption is satisfied, that is, if a part of the spectrum of the linearized operator at the travelling wave lies in the right half of the complex plane. We cannot use this standard approach here because we do not know if the spectral assumption is satisfied, see Remark 7.

In order to prove instability of the peaked periodic waves, we obtain sharp bounds on the exponential growth of the L^2 norm of the co-periodic perturbations in the linearized time-evolution problem, see Lemma 7. These bounds are used to prove nonlinear orbital instability of the peaked periodic wave in the Cauchy problem (1.1), see Lemma 10.

Peaked periodic waves in a similar Whitham type equation were recently studied in [9, 10], where various estimates on Hölder regularity of the peaked periodic wave were obtained. Compared to these works our analysis of Hölder regularity of the peaked periodic wave relies on Fourier theory and the existence of a first integral. Indeed, the reduced Ostrovsky equation for smooth periodic waves is equivalent to a second-order differential equation with a conserved quantity. Although this equivalence can not be used when dealing with peaked periodic waves, we can still use a first-order invariant of the second-order differential equation to analyze the behavior of the smooth parts of the peaked periodic waves together with sharp estimates of the solution at the singularity, see Remark 3 and Lemma 2.

The paper is organized as follows. Section 2 contains the proof that the peaked wave with the parabolic profile is unique up to spatial translations in the space of functions in \dot{L}_{per}^2 with a single minimum per period. Section 3 gives the proof of linear instability of the peaked periodic wave with respect to co-periodic perturbations. Section 4 describes relevant details for the proof of nonlinear orbital instability of the peaked periodic wave. Theorem 1 is proven by collecting results of Sections 2, 3, and 4.

2. PEAKED PERIODIC WAVE

The periodic travelling waves in the reduced Ostrovsky equation are given by $u(x, t) = U(x - ct)$, where $c \in \mathbb{R}$ is the wave speed and U is a bounded 2π -periodic wave profile with zero mean. The wave profile U is to be found from the boundary-value problem

$$(2.1) \quad \begin{cases} [c - U(z)]U'(z) + (\partial_z^{-1}U)(z) = 0, & \text{for every } z \in (-\pi, \pi) \text{ such that } U(z) \neq c, \\ U(-\pi) = U(\pi), \quad \int_{-\pi}^{\pi} U(z)dz = 0, \end{cases}$$

where $z = x - ct$ is the travelling wave coordinate with wave speed $c > 0$. If $U \in \dot{L}_{\text{per}}^2$, then $\partial_z^{-1}U \in \dot{H}_{\text{per}}^1$. By Sobolev's embedding, it follows that $\partial_z^{-1}U \in C_{\text{per}}$ so that the anti-derivative $\partial_z^{-1}U$ with zero mean can be expressed by the pointwise formula

$$(2.2) \quad (\partial_z^{-1}U)(z) = \int_0^z U(z')dz' - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^z U(z')dz'dz, \quad z \in [-\pi, \pi].$$

In what follows, we assume that U is at least continuous on $[-\pi, \pi]$, that is, we assume that $U \in C_{\text{per}}$. For $\alpha \in (0, 1)$, let C_{per}^α be the space of α -Hölder 2π -periodic continuous functions

such that

$$(2.3) \quad |U(x) - U(y)| \leq K|x - y|^\alpha, \quad \text{for all } x, y \in [-\pi, \pi],$$

for some $K \in \mathbb{R}$. We will adopt the following definition of single-lobe periodic waves.

Definition 1. We say that $U \in C_{\text{per}}$ is a single-lobe periodic wave if there exists $z_0 \in (-\pi, \pi)$ such that U is non-increasing on $[-\pi, z_0]$ and non-decreasing on $[z_0, \pi]$.

Remark 1. Due to the condition $U(-\pi) = U(\pi)$ and the symmetry of equation

$$(c - U(z))U'(z) + \int_0^z U(z')dz' - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^z U(z')dz'dz = 0$$

with respect to the reflection $z \mapsto -z$, the single-lobe periodic waves in Definition 1 have even profile U with $z_0 = 0$. In this case, $(\partial_z^{-1}U)(z) = \int_0^z U(z')dz'$ is odd.

A family of smooth 2π -periodic waves to the boundary-value problem (2.1) satisfying $U(z) < c$ for every $z \in [-\pi, \pi]$ was constructed in our previous work [17] in an open interval of the speed parameter c . By Theorem 1(a) and Lemma 3 in [17], we have the following result.

Lemma 1. *There exists $c_* > 1$ such that for every $c \in (1, c_*)$, the boundary-value problem (2.1) admits a unique smooth periodic wave in the sense of Definition 1 with the profile $U \in \dot{H}_{\text{per}}^\infty$ satisfying $U(z) < c$ for every $z \in [-\pi, \pi]$.*

Remark 2. For the smooth periodic wave $U \in \dot{H}_{\text{per}}^\infty$ to the boundary-value problem (2.1), the periodic boundary conditions are satisfied for all derivatives of U .

At $c = c_*$, the periodic wave with parabolic profile has been known since the original work of Ostrovsky [29]. It is easy to check that the boundary-value problem (2.1) is satisfied by $U(z) = (z^2 - 3c)/6$, whereas the zero mean condition is satisfied if $c = c_* := \pi^2/9$. This yields the exact expression for the peaked periodic wave with zero mean

$$(2.4) \quad U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],$$

periodically continued beyond $[-\pi, \pi]$. Note that $U_*(\pm\pi) = \pi^2/9 = c_*$ and $U'_*(\pi) = -U'_*(-\pi) = \frac{\pi}{3}$. The peaked periodic wave (2.4) can be represented by the Fourier cosine series

$$U_*(z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{3n^2} \cos(nz),$$

which is well defined in \dot{H}_{per}^s for $s < 3/2$.

Remark 3. The peaked periodic wave (2.4) belongs to solutions of the boundary-value problem (2.1) with profile $U_* \in \dot{H}_{\text{per}}^1$ satisfying $U_*(z) < c$ for every $z \in (-\pi, \pi)$ and $U_*(\pm\pi) = c$. The first derivative of $U_* \in \dot{H}_{\text{per}}^1$ has a finite jump singularity across the end points $z = \pm\pi$. More precisely, the profile U_* is Lipschitz continuous at $\pm\pi$, that is, there exist constants $0 < c_1 < c_2$ such that

$$c_1|z - \pi| \leq |U_*(z) - c_*| \leq c_2|z - \pi| \quad \text{for } |z - \pi| \ll 1,$$

which can be readily checked in view of the explicit expression (2.4).

The next result states that the only single-lobe periodic wave with profile $U \in C_{\text{per}}$ satisfying the boundary-value problem (2.1) and having a singularity in the derivative at $z = \pm\pi$ is the peaked periodic wave U_* given in (2.4). This disproves existence of all other peaked or cusped periodic waves, in particular all small-amplitude peaked periodic waves constructed in [20], see Remark 4.

Lemma 2. *The boundary-value problem (2.1) does not admit single-lobe periodic waves in the sense of Definition 1 which are C_{per}^α with $\alpha \in [0, 1)$. The only periodic wave with a singularity in the derivative at $z = \pm\pi$ is the peaked wave with parabolic profile (2.4), which is Lipschitz at the peak.*

Proof. Let $U \in \dot{L}_{\text{per}}^2 \cap C_{\text{per}}$ be a single-lobe periodic wave solution of the boundary-value problem (2.1). By Remark 1, $U \in \dot{L}_{\text{per}}^2$ is even, $\partial_z^{-1}U \in \dot{H}_{\text{per}}^1$ is odd, and $\partial_z^{-1}U$ is represented by the Fourier sine series which converges absolutely and uniformly, so that $(\partial_z^{-1}U)(\pm\pi) = 0$.

• Let us first consider the case where $U(z) < c$ for every $z \in (-\pi, \pi)$ and $U(\pm\pi) = c$. Let $\alpha \in (0, 1)$. We assume to the contrary that there exists a solution U of the boundary-value problem (2.1) with $U \in C_{\text{per}}^\alpha$. If $U \in C_{\text{per}}^\alpha$ with $\alpha \in (0, 1)$, then $\partial_z^{-1}U \in C_{\text{per}}^1$. Since $U(\pm\pi) = c$ and $(\partial_z^{-1}U)(\pm\pi) = 0$ we find that $c - U(z) \sim (\pi - z)^\alpha$ and $(\partial_z^{-1}U)(z) \sim (\pi - z)$ at $z = \pm\pi$. Since U satisfies the boundary value problem (2.1) we have that

$$(2.5) \quad U'(z) = -\frac{(\partial_z^{-1}U)(z)}{c - U(z)}, \quad z \in (-\pi, \pi)$$

which yields $U'(z) \sim (\pi - z)^{1-\alpha}$ at $z = \pm\pi$. Equation (2.5) also implies that $U' \in C^1(-\pi, \pi)$ so we find that $U' \in C_{\text{per}}^{1-\alpha}$. Since $1 - \alpha \in (0, 1)$ we conclude that $U \in C_{\text{per}}^1$ in contradiction to the assumption that $U \in C_{\text{per}}^\alpha$ with $\alpha \in (0, 1)$. The case $\alpha = 0$ can be proven in exactly the same way². Therefore, if there exists a single-lobe periodic wave with a singularity in the derivative at $z = \pm\pi$ and profile $U(z) < c$ for every $z \in (-\pi, \pi)$, then U must be at least Lipschitz continuous at $z = \pm\pi$.

We now show that the only peaked periodic solution with peak at $U(\pm\pi) = c$ is the solution with the parabolic profile (2.4). Since $U(z) < c$ for every $z \in (-\pi, \pi)$ and $U \in C^1(-\pi, \pi)$, the first-order invariant with the solution U holds for $z \in (-\pi, \pi)$,

$$(2.6) \quad \begin{aligned} E &= \frac{1}{2} [c - U(z)]^2 [U'(z)]^2 + \frac{c}{2} U(z)^2 - \frac{1}{3} U(z)^3 \\ &= \frac{1}{2} [(\partial_z^{-1}U)(z)]^2 + \frac{c}{2} U(z)^2 - \frac{1}{3} U(z)^3. \end{aligned}$$

Since $(\partial_z^{-1}U)(z)$ is continuous in $z = \pm\pi$ with $(\partial_z^{-1}U)(\pm\pi) = 0$, E is continuous and constant up to the boundary at $z = \pm\pi$ and we have $E|_{z=\pm\pi} = c^3/6 =: E_c$. The level with $E = E_c$ corresponds to a single trajectory in the phase plane (U, U') (see the bold curve in Figure 1), which for $U(z) < c$ gives rise to the peaked periodic wave with parabolic profile (2.4).

• Let us now analyze the situation where there exists $z_1 \in (0, \pi)$ such that $U(\pm z_1) = c$ and equation (2.5) holds separately for $z \in (0, z_1)$ and for $z \in (z_1, \pi)$. There are two possibilities, either $(\partial_z^{-1}U)(\pm z_1) = 0$ or $(\partial_z^{-1}U)(\pm z_1) \neq 0$.

² The case $\alpha = 0$ refers to solutions $U \in C_{\text{per}}$ in view of Definition 1.

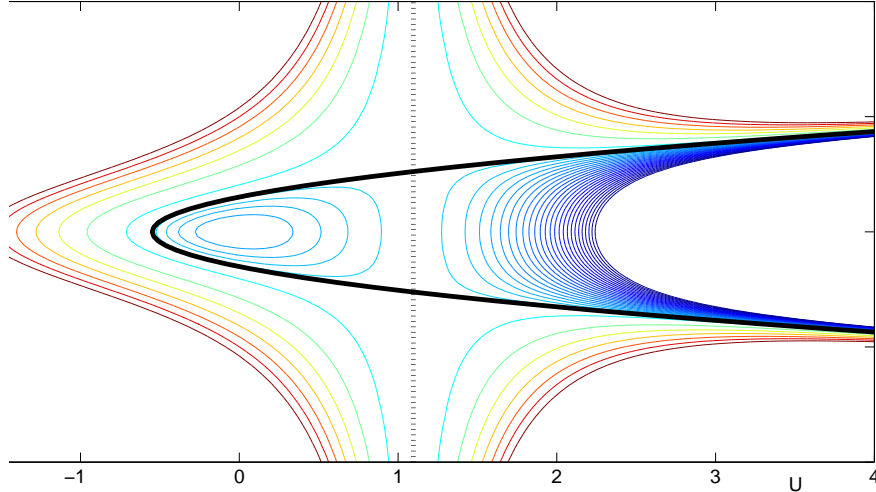


FIGURE 1. Phase plane portrait obtained from level curves of the first-order invariant (2.6) for some $c > 0$. The dashed black line indicates the singularity line $U = c$. The solid black curve to the left of the singular line corresponds to the parabolic profile (2.4).

If $(\partial_z^{-1}U)(\pm z_1) = 0$, then by the same argument as above the first-order invariant E is continuous and constant on $[-z_1, z_1]$ with $E|_{z=\pm z_1} = E_{z_1}$. The level with $E = E_{z_1}$ corresponds to a single trajectory in the phase plane (U, U') . For $z \in (z_1, \pi]$ the corresponding solution can either be continued uniquely into the region with $U(z) > c > 0$ or $z = z_1$ represents a turning point (a local maximum for U) and the solution can be continued uniquely into the region with $U(z) < c$ for $z \gtrsim z_1$ ³. However, the first variant implies that $(\partial_z^{-1}U)(z) = \int_0^z U(z')dz' > 0$ for every $z \in (z_1, \pi]$ which contradicts $(\partial_z^{-1}U)(\pi) = 0$. The second continuation is possible but does not belong to the class of single-lobe periodic waves, see Remark 6.

If $(\partial_z^{-1}U)(\pm z_1) \neq 0$, then the contradiction arises from the fact that, since $(\partial_z^{-1}U)(z)$ is continuous at z_1 and equation (2.5) holds separately for $z \in (0, z_1)$ and for $z \in (z_1, \pi)$, the change of the sign of $U'(z)$ across z_1 is determined by the change of the sign of $c - U(z)$ across z_1 . Indeed, if $U(z) < c$ both for $z \lesssim z_1$ and $z \gtrsim z_1$, then the sign of $U'(z)$ remains the same for $z \in (0, z_1)$ and $z \in (z_1, \pi)$. But this is impossible since $U'(z)$ must change sign for $z \lesssim z_1$ and $z \gtrsim z_1$ if $U(z) < c$ on both sides of z_1 . If on the other hand $U(z) < c$ for $z \lesssim z_1$ and $U(z) > c$ for $z \gtrsim z_1$, then the sign of $U'(z)$ flips again, in contradiction with the monotone increase of $U(z)$ for all $z \in (0, \pi)$. Hence, both possibilities with $(\partial_z^{-1}U)(\pm z_1) \neq 0$ yield a contradiction.

Combing all these arguments we find that the only single-lobe peaked periodic wave has parabolic profile (2.4) and is Lipschitz at the peak $U(\pm\pi) = c$. \square

³The notation $z \gtrsim z_1$ means that $0 < z - z_1 < \varepsilon$ for some small $\varepsilon > 0$, and equivalently for the reverse inequality.

Remark 4. Small-amplitude peaked periodic solutions for the reduced Ostrovsky equation with the square root singularity

$$(2.7) \quad U(z) = c + \mathcal{O}(\sqrt{\pi^2 - z^2}) \quad \text{as } z \rightarrow \pm\pi,$$

were formally constructed in [20]. However, our analysis in the proof of Lemma 2 shows⁴ that such solutions can not exist, since the expansion (2.7) implies that $U \in C_{\text{per}}^{1/2}$. We conclude that the small-amplitude peaked periodic waves in [20] are artefacts of the construction method, which relies on a transformation of the semi-linear Klein–Gordon equation to the reduced Ostrovsky equation.

Remark 5. Three peaked solitary waves were formally constructed in [35], one of which is a loop soliton studied in many publications [14, 36, 37] and the other two with the peak at $z = 0$ are based on the two possibilities analyzed in the proof of Lemma 2 for the case $\partial_z^{-1}U(0) = 0$. The latter peaked solitary waves were also constructed in [34] by using the transformation of the semi-linear Klein–Gordon equation to the reduced Ostrovsky equation. However, the same argument as in the proof of Lemma 2 eliminates both of these possibilities and only leaves the possibility of the loop soliton, which is given by a multi-valued function.

Remark 6. There is a simple way to obtain other peaked periodic waves in the boundary-value problem (2.1). One can flip the periodic wave with the parabolic profile at a point $z_0 \in (0, \pi)$ and pack two such waves over one period. This possibility is allowed in the proof of Lemma 2, but not in the class of single-lobe periodic waves. Similarly, one can pack three and more periods of the peaked wave with parabolic profiles. Definition 1 eliminates this type of non-uniqueness of the peaked periodic wave in the boundary-value problem (2.1).

3. LINEAR INSTABILITY OF THE PEAKED PERIODIC WAVE

We add a *co-periodic* perturbation v to the travelling wave U , that is, a perturbation with the same period 2π . Truncating the quadratic terms and moving with the reference frame of the travelling wave yields the linearized evolution problem in the form

$$(3.1) \quad \begin{cases} v_t + \partial_z [(U(z) - c)v] = \partial_z^{-1}v, & t > 0, \\ v|_{t=0} = v_0. \end{cases}$$

The linearized evolution equation can be formulated in the form $v_t = \partial_z L v$ defined by the self-adjoint operator

$$(3.2) \quad L = P_0 (\partial_z^{-2} + c - U(z)) P_0 : \dot{L}_{\text{per}}^2 \rightarrow \dot{L}_{\text{per}}^2,$$

where $P_0 : L_{\text{per}}^2 \rightarrow \dot{L}_{\text{per}}^2$ is the projection operator that removes the mean value of 2π -periodic functions. The form $v_t = \partial_z L v$ is related to the formulation of the reduced Ostrovsky equation in the travelling wave coordinate $z = x - ct$ as a Hamiltonian system defined by the symplectic operator ∂_z and the conserved energy function $H_c(u) = H(u) + cQ(u)$, where

$$(3.3) \quad H(u) = \int_{-\pi}^{\pi} \left[-(\partial_z^{-1}u)^2 - \frac{1}{3}u^3 \right] dz, \quad Q(u) = \int_{-\pi}^{\pi} u^2 dz$$

⁴Note that the solutions of [20] have nonzero mean value, so Lemma 2 does not apply directly. However, the arguments in the proof lead to the same conclusion also for solutions with nonzero mean. Indeed, if U has a non-zero mean, $\partial_z^{-1}U(z)$ may not be zero at $z = \pm\pi$. However, if we translate the solution by half a period so that the singularity is placed at $z = 0$, then $\partial_z^{-1}U(0) = 0$ by oddness of $\partial_z^{-1}U$ and we can use the same contradiction as the one obtained from (2.5).

are the conserved energy and momentum functionals for the reduced Ostrovsky equation (1.1). The periodic wave $u = U$ is a critical point of $H_c(u)$ and the self-adjoint operator L is the Hessian operator of the energy function $H_c(u)$ at the periodic wave $u = U$.

Thanks to the translational invariance of the boundary-value problem (2.1), we have that $L\partial_z U = 0$, where $\partial_z U \in \dot{L}_{\text{per}}^2$, holds for both the smooth periodic waves of Lemma 1 and the peaked periodic wave (2.4) in Lemma 2. Associated to the translational eigenvector is the symplectic orthogonality constraint $\langle U, v \rangle = 0$, which is used to study both the spectrum of $\partial_z L$ and the evolution of the Cauchy problem (3.1) see [3, 23, 30]. In what follows, $\langle \cdot, \cdot \rangle$ and $\|v\|_{L_{\text{per}}^2}$ denote the inner product and the L^2 norm with integration over $[-\pi, \pi]$, respectively.

We distinguish two concepts of stability of the 2π -periodic wave with respect to linearization.

Definition 2. The travelling wave U is said to be spectrally stable if the spectral problem $\lambda v = \partial_z L v$ with $v \in \dot{H}_{\text{per}}^1$ satisfying $\langle U, v \rangle = 0$ has no eigenvalues $\lambda \notin i\mathbb{R}$. Otherwise, it is said to be spectrally unstable.

Definition 3. The travelling wave U is said to be linearly stable if for every $v_0 \in \dot{H}_{\text{per}}^1$ satisfying $\langle U, v_0 \rangle = 0$, there exists $C > 0$ and a unique global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$ to the Cauchy problem (3.1) such that

$$(3.4) \quad \|v(t)\|_{H_{\text{per}}^1} \leq C \|v_0\|_{H_{\text{per}}^1}, \quad t > 0.$$

Otherwise, it is said to be linearly unstable.

In [17], we have proved that the smooth periodic waves of Lemma 1 are spectrally stable in the sense of Definition 2. Here we intend to show that the peaked periodic wave U_* of Lemma 2 given in (2.4) is linearly unstable in the sense of Definition 3. The linear instability is due to the sharp exponential growth of the unique global solution to the Cauchy problem (3.1) with $U = U_*$:

$$(3.5) \quad C \|v_0\|_{L_{\text{per}}^2} e^{\pi t/6} \leq \|v(t)\|_{L_{\text{per}}^2} \leq \|v_0\|_{L_{\text{per}}^2} e^{\pi t/6}, \quad t > 0,$$

for some $C \in (0, 1)$. We will obtain these bounds in two steps. In the first step, Section 3.1, we apply the method of characteristics to the truncated linearized equation (3.1) without the dispersive term $\partial_z^{-1} v$ and obtain the sharp bounds (3.5) for all initial conditions $v_0 \in \dot{H}_{\text{per}}^1$ satisfying the constraint

$$(3.6) \quad \int_{-\pi}^{\pi} z v_0(z)^2 dz = 0.$$

In the second step, Section 3.2, we will show that the bounds (3.5) remain true in the full linearized equation (3.1) for a subset of possible initial conditions $v_0 \in \dot{H}_{\text{per}}^1$ satisfying the constraint (3.6) and the additional constraint

$$(3.7) \quad \int_{-\pi}^{\pi} z^2 v_0(z) dz = 0,$$

which arises due to the orthogonality condition $\langle U, v \rangle = 0$ in Definition 3 and the zero-mean condition on v_0 . Regarding spectral stability or instability of the peaked periodic wave (2.4), we will show in Section 3.3 that the spectrum of the linear self-adjoint operator L in (3.2) is given by a continuous spectrum on $[0, \pi^2/6]$, which includes the embedded eigenvalue $\lambda_0 = 0$ with the eigenvector $\partial_z U$, and a simple negative eigenvalue $\lambda_1 < 0$. As a result, no spectral

gap appears between $\lambda_0 = 0$ and the continuous spectrum, hence it is impossible to solve the spectral stability problem by applying the standard methods from [3, 23, 30].

Remark 7. We give no claims of spectral stability or instability for the peaked periodic wave (2.4). The formal instability result in Section 5 in [21] violates the periodic boundary conditions on the perturbation v and hence does not provide an answer to the spectral instability question.

3.1. Linear instability of truncated evolution. For the peaked periodic wave (2.4), we obtain the simple expression

$$(3.8) \quad U_*(z) - c_* = \frac{1}{6}(z^2 - \pi^2), \quad z \in [-\pi, \pi].$$

Truncating the linearized evolution problem (3.1) by removing the term $\partial_z^{-1}v$ and using the explicit expression (3.8), we can write the truncated evolution problem in the form

$$(3.9) \quad \begin{cases} v_t + \frac{1}{6}\partial_z [(z^2 - \pi^2)v] = 0, & t > 0, \\ v|_{t=0} = v_0, \end{cases}$$

where the initial data v_0 is taken in \dot{H}_{per}^1 . The evolution problem can be solved by the method of characteristics along the family of characteristic curves $z = Z(s, t)$, where $s \in [-\pi, \pi]$ is a parameter for the initial data and $t \geq 0$ is the evolution time. Defining

$$(3.10) \quad \begin{cases} \frac{d}{dt}Z(s, t) = \frac{1}{6}[Z(s, t)^2 - \pi^2], & t > 0, \\ Z(s, 0) = s, \end{cases}$$

and setting $V(s, t) := v(Z(s, t), t)$ yields the evolution problem in the form

$$(3.11) \quad \begin{cases} \frac{d}{dt}V(s, t) = -\frac{1}{3}Z(s, t)V(s, t), & t > 0, \\ V(s, 0) = v_0(s). \end{cases}$$

The family of characteristic curves is obtained by integrating the differential equation (3.10) with the parameter $s \in [-\pi, \pi]$. Because $Z = \pm\pi$ are critical points of the differential equation (3.10), the family of characteristic curves remain inside the invariant region $[-\pi, \pi]$ for every $t \geq 0$. The family of characteristic curves can be obtained in the explicit form

$$(3.12) \quad Z(s, t) = \pi \frac{s \cosh(\pi t/6) - \pi \sinh(\pi t/6)}{\pi \cosh(\pi t/6) - s \sinh(\pi t/6)}, \quad s \in [-\pi, \pi], \quad t \in \mathbb{R}.$$

Note that $Z(\pm\pi, t) = \pm\pi$ for every $t \in \mathbb{R}$. For later use of the chain rule we compute

$$(3.13) \quad e^{\frac{1}{3}\int_0^t Z(s, t')dt'} = \frac{\partial}{\partial s}Z(s, t) = \frac{\pi^2}{[\pi \cosh(\pi t/6) - s \sinh(\pi t/6)]^2}, \quad s \in [-\pi, \pi], \quad t \in \mathbb{R}.$$

The explicit solution for V in characteristic variables is obtained by integrating the differential equation (3.11) with the parameter $s \in [-\pi, \pi]$:

$$V(s, t) = v_0(s)e^{-\frac{1}{3}\int_0^t Z(s, t')dt'}.$$

In view of (3.13), the explicit solution is given by

$$(3.14) \quad V(s, t) = \frac{1}{\pi^2}[\pi \cosh(\pi t/6) - s \sinh(\pi t/6)]^2 v_0(s), \quad s \in [-\pi, \pi], \quad t \in \mathbb{R}.$$

By using the explicit solutions (3.12) and (3.14), we are able to state and prove the following linear instability result for the truncated evolution problem (3.9).

Lemma 3. For every $v_0 \in \dot{H}_{\text{per}}^1$, there exists a unique global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$ to the Cauchy problem (3.9) satisfying the upper bound

$$(3.15) \quad \|v(t)\|_{L_{\text{per}}^2} \leq \|v_0\|_{L_{\text{per}}^2} e^{\pi t/6}, \quad t > 0.$$

If $\int_{-\pi}^{\pi} s v_0(s)^2 ds = 0$, then the global solution satisfies the lower bound

$$(3.16) \quad \frac{1}{2} \|v_0\|_{L_{\text{per}}^2} e^{\pi t/6} \leq \|v(t)\|_{L_{\text{per}}^2}, \quad t > 0.$$

Proof. Existence of a global solution in the explicit form (3.12) and (3.14) is obtained from the method of characteristics. By using the chain rule and (3.13), we verify that the mean-zero constraint is preserved in the time evolution:

$$\int_{-\pi}^{\pi} v(z, t) dz = \int_{-\pi}^{\pi} V(s, t) \frac{\partial Z}{\partial s} ds = \int_{-\pi}^{\pi} v_0(s) ds = 0.$$

The explicit expression (3.14) implies that $V(\cdot, t) \in H_{\text{per}}^1$ if $v_0 \in \dot{H}_{\text{per}}^1$ and $t \in \mathbb{R}$. On the other hand, the explicit expression (3.12) implies that for every $\tau > 0$, there exists $C_{\tau} > 0$ such that

$$\frac{\partial}{\partial s} Z(s, t) \geq C_{\tau}, \quad s \in [-\pi, \pi], \quad t \in [-\tau, \tau].$$

Hence, the chain rule implies that $v(\cdot, t) \in \dot{H}_{\text{per}}^1$ if $v_0 \in \dot{H}_{\text{per}}^1$ and $t \in \mathbb{R}$. Uniqueness of such global solution follows by the standard theory (see Theorem 3.1 in [2]).

It remains to prove the sharp exponential growth in the bounds (3.15) and (3.16). By the chain rule, we obtain

$$\int_{-\pi}^{\pi} v(z, t)^2 dz = \int_{-\pi}^{\pi} V(s, t)^2 \frac{\partial Z}{\partial s} ds = \frac{1}{\pi^2} \int_{-\pi}^{\pi} [\pi \cosh(\pi t/6) - s \sinh(\pi t/6)]^2 v_0(s)^2 ds.$$

From here, we have the upper bound

$$\|v(t)\|_{L_{\text{per}}^2}^2 \leq e^{\pi t/3} \|v_0\|_{L_{\text{per}}^2}^2$$

and the lower bound under the additional condition $\int_{-\pi}^{\pi} s v_0(s)^2 ds = 0$:

$$\|v(t)\|_{L_{\text{per}}^2}^2 = \cosh(\pi t/6)^2 \|v_0\|_{L_{\text{per}}^2}^2 + \frac{1}{\pi^2} \sinh(\pi t/6)^2 \|s v_0\|_{L_{\text{per}}^2}^2 \geq \frac{1}{4} e^{\pi t/3} \|v_0\|_{L_{\text{per}}^2}^2.$$

Taking the square root from these bounds yields (3.15) and (3.16). \square

Remark 8. The global solution in Lemma 3 remains bounded in L^1 . This follows from the chain rule:

$$\int_{-\pi}^{\pi} |v(z, t)| dz = \int_{-\pi}^{\pi} |V(s, t)| \frac{\partial Z}{\partial s} ds = \int_{-\pi}^{\pi} |v_0(s)| ds.$$

Since

$$(3.17) \quad \|v_0\|_{L_{\text{per}}^1} \leq (2\pi)^{1/2} \|v_0\|_{L_{\text{per}}^2},$$

hence $v_0 \in \dot{H}_{\text{per}}^1$ implies $v_0 \in L^1$. Extending this bound to the time-dependent solution,

$$(3.18) \quad \|v(t)\|_{L_{\text{per}}^1} \leq (2\pi)^{1/2} \|v(t)\|_{L_{\text{per}}^2}, \quad t > 0,$$

shows that the L^1 norm of the global solution $v(t)$ may remain bounded even if the L^2 norm of this solution grows exponentially.

Remark 9. Truncating a quadratic form associated with the self-adjoint operator L in (3.2) and using the chain rule yield the energy conservation for the truncated evolution (3.9):

$$\int_{-\pi}^{\pi} (\pi^2 - z^2) v(z, t)^2 dz = \int_{-\pi}^{\pi} [\pi^2 - Z(s, t)^2] V(s, t)^2 \frac{\partial Z}{\partial s} ds = \int_{-\pi}^{\pi} (\pi^2 - s^2) v_0(s)^2 ds.$$

The energy conservation shows that the truncated evolution leads to the exponential growth of $\|v(t)\|_{L_{\text{per}}^2}^2$ and $\|zv(t)\|_{L_{\text{per}}^2}^2$ but the difference between the two squared norms remains bounded.

Remark 10. For the smooth periodic waves of Lemma 1 satisfying $U(z) < c$ for every $z \in [-\pi, \pi]$, the truncated energy $\int_{-\pi}^{\pi} (c - U)v^2 dz$ is coercive in the L^2 norm, hence the energy conservation

$$\int_{-\pi}^{\pi} [c - U(z)] v(z, t)^2 dz = \int_{-\pi}^{\pi} [c - U(z)] v_0(s)^2 ds$$

implies a global time-independent bound on $\|v(t)\|_{L_{\text{per}}^2}^2$, where $v(t)$ is a solution of the truncation of the linear evolution equation (3.1) without the $\partial_z^{-1}v$ term.

Remark 11. For the smooth periodic waves of Lemma 1, the characteristic curves reach boundaries $z = \pm\pi$ in finite time because $z = \pm\pi$ are not critical points of the differential equations for the characteristic curves. On the other hand, for the peaked periodic wave (2.4), the characteristic curves reach boundaries $z = \pm\pi$ in infinite time. The latter property induces exponential growth of the global solutions to the Cauchy problem (3.9), as is shown in Lemma 3.

3.2. Linear instability of full evolution. Here we consider the full linearized evolution problem (3.1) with (3.8) and rewrite the evolution problem in the form

$$(3.19) \quad \begin{cases} v_t + \frac{1}{6} \partial_z [(z^2 - \pi^2)v] = \partial_z^{-1} v, & t > 0, \\ v|_{t=0} = v_0, \end{cases}$$

where the initial data v_0 is taken in \dot{H}_{per}^1 .

Lemma 4. *For every $v_0 \in \dot{H}_{\text{per}}^1$ there exists a unique global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$ of the Cauchy problem (3.19).*

Proof. By Lemma 3, the Cauchy problem (3.9) with $v_0 \in \dot{H}_{\text{per}}^1$ has a unique global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$. In the framework of semigroup theory, the evolution equation (3.9) can be written in the form $v_t = A_0 v$, where

$$A_0 := \frac{1}{6} \partial_z (\pi^2 - z^2) v.$$

Existence of a unique global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$ implies that the operator A_0 with domain $D(A_0) = \dot{H}_{\text{per}}^1$ is the infinitesimal generator of a strongly continuous semigroup $(S_0(t))_{t \geq 0}$ on \dot{L}_{per}^2 given by (3.14). Since $\partial_z^{-1} : \dot{H}_{\text{per}}^1 \rightarrow \dot{H}_{\text{per}}^1$ is a bounded operator, the Bounded Perturbation Theorem (see Theorem III,1.3 on p. 158 in [11]) implies that the operator

$$A := A_0 + \partial_z^{-1}$$

also generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ on \dot{L}_{per}^2 . Therefore, the evolution equation in the Cauchy problem (3.19) can be viewed as a bounded perturbation of the evolution equation in the Cauchy problem (3.9). The assertion of the lemma follows by Proposition 6.2 on p. 145 in [11]. \square

In what follows, we obtain bounds on the global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$. First, we note the following upper bound on the growth of the global solution to the Cauchy problem (3.19).

Lemma 5. *A global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$ to the Cauchy problem (3.19) in Lemma 4 satisfies the upper bound*

$$(3.20) \quad \|v(t)\|_{L_{\text{per}}^2} \leq \|v_0\|_{L_{\text{per}}^2} e^{\pi t/6}, \quad t > 0.$$

Proof. Note the following integration

$$\int_{-\pi}^{\pi} v(\partial_z^{-1}v) dz = \frac{1}{2}(\partial_z^{-1}v)^2 \Big|_{z=-\pi}^{z=\pi} = 0,$$

since $\partial_z^{-1}v \in H_{\text{per}}^2$ and hence $\partial_z^{-1}v \in C_{\text{per}}$ by Sobolev's embedding. Integrating by parts yields the following balance equation

$$\frac{d}{dt} \frac{1}{2} \|v(t)\|_{L_{\text{per}}^2}^2 = \frac{1}{6} \int_{-\pi}^{\pi} v \partial_z [(\pi^2 - z^2)v] dz = -\frac{1}{6} \int_{-\pi}^{\pi} (\pi^2 - z^2)v \partial_z v dz = -\frac{1}{6} \int_{-\pi}^{\pi} z v^2 dz.$$

Hence

$$\frac{d}{dt} \|v(t)\|_{L_{\text{per}}^2}^2 \leq \frac{\pi}{3} \|v(t)\|_{L_{\text{per}}^2}^2$$

and Gronwall's inequality yields the desired bound (3.20). \square

In order to obtain the lower bound on the L^2 norm of the global solution to the Cauchy problem (3.19), we use the generalized method of characteristics and treat $\partial_z^{-1}v(z, t)$ as a source term in (3.9). This term satisfies the following useful bound (also proven in [28]).

Lemma 6. *If $g := \partial_z^{-1}v \in \dot{H}_{\text{per}}^1$, then*

$$(3.21) \quad \|g\|_{L_{\text{per}}^\infty} \leq \|v\|_{L_{\text{per}}^1}.$$

Proof. By Sobolev embedding of H_{per}^1 into C_{per} , g is a continuous 2π -periodic function with zero mean. Therefore, there exists $\zeta \in [-\pi, \pi]$ such that $g(\zeta) = 0$. For every $z \in [-\pi, \pi]$, we can write

$$g(z) = \int_{\zeta}^z v(z') dz',$$

from which bound (3.21) follows. Note that L^2 is continuously embedded into L^1 because of the bound (3.17). \square

By using the family of characteristic curves $z = Z(s, t)$ with $s \in [-\pi, \pi]$ and $t \geq 0$, where Z is defined by the same initial-value problem (3.10), and setting $V(s, t) := v(Z(s, t), t)$ and $G(s, t) := g(Z(s, t), t)$, we obtain the evolution problem in the form

$$(3.22) \quad \begin{cases} \frac{d}{dt} V(s, t) = -\frac{1}{3} Z(s, t) V(s, t) + G(s, t), & t > 0, \\ V(s, 0) = v_0(s). \end{cases}$$

The family of characteristic curves Z is still given by the same explicit form (3.12). Integrating the differential equation (3.22) with an integrating factor yields the explicit solution for V in the form

$$(3.23) \quad V(s, t) = \left[v_0(s) + \int_0^t G(s, t') e^{\frac{1}{3} \int_0^{t'} Z(s, t'') dt''} dt' \right] e^{-\frac{1}{3} \int_0^t Z(s, t') dt'}$$

By using the explicit solution (3.23), we are able to prove the linear instability result for the Cauchy problem (3.19).

Lemma 7. *There exists $v_0 \in \dot{H}_{\text{per}}^1$ and $C > 0$ such that the unique global solution $v \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$ to the Cauchy problem (3.19) in Lemma 4 satisfies the lower bound*

$$(3.24) \quad \|v(t)\|_{L_{\text{per}}^2} \geq C \|v_0\|_{L_{\text{per}}^2} e^{\pi t/6}, \quad t > 0.$$

Proof. By the chain rule, the explicit expression (3.23) with the help of (3.13) yields the following equation:

$$\begin{aligned} \int_{-\pi}^{\pi} v(z, t)^2 dz &= \int_{-\pi}^{\pi} V(s, t)^2 \frac{\partial Z}{\partial s} ds \\ &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} [\pi \cosh(\pi t/6) - s \sinh(\pi t/6)]^2 \\ &\quad \times \left[v_0(s) + \int_0^t \frac{\pi^2 G(s, t')}{[\pi \cosh(\pi t'/6) - s \sinh(\pi t'/6)]^2} dt' \right]^2 ds. \end{aligned}$$

Let us assume the same constraint $\int_{-\pi}^{\pi} s v_0(s)^2 ds = 0$ as in Lemma 3. Neglecting positive terms in the lower bound, we obtain

$$(3.25) \quad \|v(t)\|_{L_{\text{per}}^2}^2 \geq \frac{1}{4} e^{\pi t/3} \|v_0\|_{L_{\text{per}}^2}^2 - 2 \int_{-\pi}^{\pi} \int_0^t |v_0(s)| |G(s, t')| \frac{[\pi \cosh(\pi t/6) - s \sinh(\pi t/6)]^2}{[\pi \cosh(\pi t'/6) - s \sinh(\pi t'/6)]^2} dt' ds.$$

Let us define for any $t > 0$,

$$K(t, t', s) := \frac{\pi \cosh(\pi t/6) - s \sinh(\pi t/6)}{\pi \cosh(\pi t'/6) - s \sinh(\pi t'/6)}, \quad t' \in [0, t], \quad s \in [-\pi, \pi].$$

We prove that for every $0 \leq t' \leq t$,

$$(3.26) \quad \sup_{s \in [-\pi, \pi]} K(t, t', s) = e^{\pi(t-t')/6}.$$

Indeed, $K(t, t', s) = e^{\pi(t-t')/6} M(t, t', s)$, where

$$M(t, t', s) := \frac{(\pi - s) + (\pi + s)e^{-\pi t/3}}{(\pi - s) + (\pi + s)e^{-\pi t'/3}},$$

and M is monotonically decreasing since $\partial_s M(t, t', s) \leq 0$ for every $t' \in [0, t]$ and $s \in [-\pi, \pi]$. Therefore, M has a maximum at $s = -\pi$, where $M(t, t', -\pi) = 1$.

By using (3.18), (3.20), (3.21), (3.25), and (3.26), we obtain

$$\begin{aligned} \|v(t)\|_{L_{\text{per}}^2}^2 &\geq \frac{1}{4} e^{\pi t/3} \|v_0\|_{L_{\text{per}}^2}^2 - 2 \|v_0\|_{L_{\text{per}}^1} \int_0^t \|g(t')\|_{L^\infty(-\pi, \pi)} e^{\pi(t-t')/3} dt' \\ &\geq \frac{1}{4} e^{\pi t/3} \|v_0\|_{L_{\text{per}}^2}^2 - 2 \|v_0\|_{L_{\text{per}}^1} \int_0^t \|v(t')\|_{L_{\text{per}}^1} e^{\pi(t-t')/3} dt' \\ &\geq \frac{1}{4} e^{\pi t/3} \|v_0\|_{L_{\text{per}}^2}^2 - 2\sqrt{2\pi} \|v_0\|_{L_{\text{per}}^1} \|v_0\|_{L_{\text{per}}^2} e^{\pi t/3} \int_0^t e^{-\pi t'/6} dt' \end{aligned}$$

Hence,

$$\|v(t)\|_{L_{\text{per}}^2}^2 e^{-\pi t/3} \geq \|v_0\|_{L_{\text{per}}^2} \left(\frac{1}{4} \|v_0\|_{L_{\text{per}}^2} - \frac{12\sqrt{2}}{\sqrt{\pi}} \|v_0\|_{L_{\text{per}}^1} \right)$$

and since $\|v_0\|_{L_{\text{per}}^2}$ can be much larger than $\|v_0\|_{L_{\text{per}}^1}$ by the bound (3.17), there exist $v_0 \in \dot{H}_{\text{per}}^1$ and $C^2 \in (0, 1/4)$ such that

$$(3.27) \quad \|v_0\|_{L_{\text{per}}^1} \leq \frac{\sqrt{\pi}(1 - 4C^2)}{48\sqrt{2}} \|v_0\|_{L_{\text{per}}^2},$$

and hence

$$(3.28) \quad \|v(t)\|_{L_{\text{per}}^2}^2 e^{-\pi t/3} \geq C^2 \|v_0\|_{L_{\text{per}}^2}^2.$$

This yields the desired bound (3.24). \square

Remark 12. Let us show that there exist functions $v_0 \in \dot{H}_{\text{per}}^1$ satisfying the constraints (3.6), (3.7), and (3.27). Indeed, if v_0 is odd, then v_0^2 is even, hence the two constraints (3.6) and (3.7) are satisfied simultaneously. From the class of odd initial data, we need to pick functions that satisfy the inequality (3.27) for a fixed $C^2 \in (0, 1/4)$. For example, we can consider the following odd function in \dot{H}_{per}^1

$$v_0(x) = \frac{x(\pi^2 - x^2)}{1 + a^2 x^2}, \quad x \in [-\pi, \pi],$$

where $a > 0$ is a parameter. Since $v_0(\pm\pi) = 0$ and $v_0(-x) = v_0(x)$, we have $v_0 \in \dot{H}_{\text{per}}^1$. We obtain by direct computation,

$$\|v_0\|_{L_{\text{per}}^1} = \left(\frac{\pi^2}{a^2} + \frac{1}{a^4} \right) \log(1 + \pi^2 a^2) - \frac{\pi^2}{a^2}$$

and

$$\|v_0\|_{L_{\text{per}}^2}^2 = \frac{1}{a^3} \left[\left(\pi^4 + \frac{6\pi^2}{a^2} + \frac{5}{a^4} \right) \arctan(\pi a) - \frac{\pi(15 + 13\pi^2 a^2)}{3a^3} \right].$$

Since $\|v_0\|_{L_{\text{per}}^1} = \mathcal{O}(\log(a)a^{-2})$ decays to zero as $a \rightarrow \infty$ faster than $\|v_0\|_{L_{\text{per}}^2} = \mathcal{O}(a^{-3/2})$, inequality (3.27) can be satisfied for sufficiently large a .

Remark 13. In the presence of the source term G , we are not able to show that $\|v(t)\|_{L_{\text{per}}^1}$ remains bounded as $t \rightarrow \infty$, see Remark 8. By using the integral

$$\int_{-\pi}^{\pi} \frac{\pi^2}{[\pi \cosh(\pi t'/6) - s \sinh(\pi t'/6)]^2} ds = 2\pi, \quad t' \in [0, t],$$

we obtain the bound

$$\|v(t)\|_{L_{\text{per}}^1} \leq \|v_0\|_{L_{\text{per}}^1} + 2\pi \int_0^t \|g(t')\|_{L_{\text{per}}^\infty} dt',$$

in view of (3.13) and (3.23). Thanks to the bound (3.21), the inequality is closed as follows:

$$\|v(t)\|_{L_{\text{per}}^1} \leq \|v_0\|_{L_{\text{per}}^1} + 2\pi \int_0^t \|v(t')\|_{L_{\text{per}}^1} dt'.$$

By Gronwall's inequality, this bound gives the fast exponential growth

$$\|v(t)\|_{L_{\text{per}}^1} \leq \|v_0\|_{L_{\text{per}}^1} e^{2\pi t},$$

which cannot be sharp because $\|v(t)\|_{L^1_{\text{per}}}$ is bounded by a slowly growing exponential function that follows from the bounds (3.18) and (3.20).

Remark 14. There exists a conserved energy for the Cauchy problem (3.19), see Remark 9, which is given by

$$(3.29) \quad \langle Lv(t), v(t) \rangle = \langle Lv_0, v_0 \rangle,$$

where the self-adjoint operator L is defined by (3.2). However, the conserved quantity (3.29) does not prevent $\|v(t)\|_{L^2_{\text{per}}}$ to grow exponentially fast as $t \rightarrow \infty$ because the bounded operator L is not coercive under the constraint (3.7), under which it is non-negative, see Lemma 8.

3.3. Spectrum of the linear self-adjoint operator L . Here we consider the spectrum $\sigma(L)$ of the linear self-adjoint operator L defined by (3.2). We will prove that $\sigma(L)$ consists of the continuous spectrum on $[0, \pi^2/6]$, which includes the embedded eigenvalue $\lambda_0 = 0$ with the eigenvector $\partial_z U$, and a simple negative eigenvalue $\lambda_1 < 0$. No spectral gap appears between $\lambda_0 = 0$ and the continuous spectrum. The following lemma gives the corresponding result.

Lemma 8. *The spectrum of the self-adjoint operator L given by (3.2) is*

$$(3.30) \quad \sigma(L) = \{\lambda_1\} \cup \left[0, \frac{\pi^2}{6}\right],$$

where $\lambda_1 < 0$ is the unique zero of the transcendental equation

$$(3.31) \quad (\pi^2 + 3\lambda) \log \frac{\sqrt{\pi^2 - 6\lambda} + \pi}{\sqrt{\pi^2 - 6\lambda} - \pi} - 3\pi \sqrt{\pi^2 - 6\lambda} = 0, \quad \lambda < 0.$$

Proof. By the spectral theorem (see, e.g., Definition 8.39, Theorem 8.70, and Theorem 8.71 in [31]), the spectrum of the self-adjoint operator L in \dot{L}^2_{per} denoted by $\sigma(L)$ may consist of only two disjoint sets on the real line: the point spectrum of eigenvalues with eigenvectors in \dot{L}^2_{per} denoted by $\sigma_p(L)$ and the continuous spectrum denoted by $\sigma_c(L)$, where the resolvent operator exists but is unbounded.

The self-adjoint operator L in (3.2) is given by the sum of a bounded operator L_0 and a compact operator K given by

$$(3.32) \quad L_0 := \frac{1}{6} P_0 (\pi^2 - z^2) P_0 : \dot{L}^2_{\text{per}} \rightarrow \dot{L}^2_{\text{per}}$$

and

$$(3.33) \quad K := P_0 \partial_z^{-2} P_0 : \dot{L}^2_{\text{per}} \rightarrow \dot{L}^2_{\text{per}}.$$

Moreover, the compact operator is in the trace class since $\sum_{n=1}^{\infty} n^{-2} < \infty$. By Kato's Theorem [25] (see Theorem 4.4 on p. 542 in [26]), $\sigma_c(L) = \sigma_c(L_0)$. We show that $[0, \pi^2/6] \subseteq \sigma_c(L_0)$ by considering the odd functions in \dot{L}^2_{per} , which can be represented by the Fourier sine series. Let us denote the space of odd functions in \dot{L}^2_{per} by $L^2_{\text{per,odd}}$. Then,

$$L_0 f = \frac{1}{6} (\pi^2 - z^2) f, \quad \forall f \in L^2_{\text{per,odd}}.$$

Then, $\sigma_c(L_0)$ in $L^2_{\text{per,odd}}$ coincides with the range of the multiplicative function $h(z) = \frac{1}{6} (\pi^2 - z^2)$ for $z \in [-\pi, \pi]$, which is $[0, \pi^2/6]$. Hence, $[0, \pi^2/6] \subseteq \sigma_c(L_0)$ in \dot{L}^2_{per} .

Let us show that $[0, \pi^2/6] \equiv \sigma_c(L_0)$ by working with the resolvent equation $(L_0 - \lambda I)f = g$ for given $g \in \dot{L}_{\text{per}}^2$ and $\lambda \notin [0, \pi^2/6]$. The resolvent equation can be written in the component form for $z \in [-\pi, \pi]$:

$$\frac{1}{6}(\pi^2 - 6\lambda - z^2)f(z) - k(f) = g(z), \quad k(f) := \frac{1}{12\pi} \int_{-\pi}^{\pi} (\pi^2 - z^2)f(z)dz,$$

where $f \in \dot{L}_{\text{per}}^2$ is supposed to satisfy the zero-mean constraint $\int_{-\pi}^{\pi} f(z)dz = 0$. Computing the solution explicitly,

$$f(z) = \frac{6}{\pi^2 - 6\lambda - z^2} [g(z) + k(f)],$$

and using the zero mean constraint, we can define $k(f)$ in terms of g :

$$k(f) = \frac{\int_{-\pi}^{\pi} \frac{g(z)}{\pi^2 - 6\lambda - z^2} dz}{\int_{-\pi}^{\pi} \frac{1}{\pi^2 - 6\lambda - z^2} dz}.$$

For every $\lambda \notin [0, \pi^2/6]$, there exist positive constants $C_\lambda, C'_\lambda > 0$ such that

$$\sup_{z \in [-\pi, \pi]} \frac{6}{|\pi^2 - 6\lambda - z^2|} \leq C_\lambda, \quad \left| \int_{-\pi}^{\pi} \frac{6}{\pi^2 - 6\lambda - z^2} dz \right| \geq C'_\lambda.$$

As a result, we obtain the bound

$$\|f\|_{L_{\text{per}}^2} \leq C_\lambda \left[\|g\|_{L_{\text{per}}^2} + |k(f)|\sqrt{2\pi} \right] \leq C_\lambda [1 + 2\pi(C'_\lambda)^{-1}C_\lambda] \|g\|_{L_{\text{per}}^2}.$$

Therefore, the resolvent operator $(L_0 - \lambda I)^{-1} : \dot{L}_{\text{per}}^2 \rightarrow \dot{L}_{\text{per}}^2$ is bounded for every $\lambda \notin [0, \pi^2/6]$ so that $\sigma_c(L_0) = [0, \pi^2/6]$.

In order to study $\sigma_p(L) \in \mathbb{R} \setminus [0, \pi^2/6]$, we consider the spectral problem for operator L with the spectral parameter $\lambda \notin [0, \pi^2/6]$:

$$(3.34) \quad \frac{1}{6}P_0(\pi^2 - z^2)w + P_0\partial_z^{-2}w = \lambda w, \quad w \in \dot{L}_{\text{per}}^2.$$

Since $\partial_z^{-2}w \in H_{\text{per}}^2$, bootstrapping arguments show that $w \in H_{\text{loc}}^2$, where functions in H_{loc}^2 are defined on any compact subset in $(-\pi, \pi)$. Therefore, on a compact subset in $(-\pi, \pi)$ the spectral problem (3.34) can be differentiated twice, after which it is rewritten as the second-order differential equation

$$(3.35) \quad (\pi^2 - z^2 - 6\lambda) \frac{d^2w}{dz^2} - 4z \frac{dw}{dz} + 4w(z) = 0, \quad w \in H_{\text{loc}}^2,$$

with the two linearly independent solutions for $\lambda \in \mathbb{R} \setminus [0, \pi^2/6]$,

$$w_1(z) = z$$

and

$$w_2(z) = \begin{cases} -1 + \frac{z^2}{2(\pi^2 - z^2 - 6\lambda)} + \frac{3z}{4\sqrt{\pi^2 - 6\lambda}} \log \frac{\sqrt{\pi^2 - 6\lambda} + z}{\sqrt{\pi^2 - 6\lambda} - z}, & \lambda < 0, \\ -1 + \frac{z^2}{2(\pi^2 - z^2 - 6\lambda)} - \frac{3z}{2\sqrt{6\lambda - \pi^2}} \arctan \frac{z}{\sqrt{6\lambda - \pi^2}}, & \lambda > \frac{\pi^2}{6}. \end{cases}$$

The first solution corresponds to the eigenvector $\partial_z U$ of the spectral problem (3.34) for the eigenvalue $\lambda_0 = 0$, which is embedded into $\sigma_c(L) = [0, \pi^2/6]$. Since eigenvectors of the self-adjoint operator for distinct eigenvalues are orthogonal, we are looking for solutions w of the

spectral problem (3.34) such that $\langle w, w_1 \rangle = 0$. Therefore, we take⁵ $w = w_2$ and extend it from H_{loc}^2 to \dot{L}_{per}^2 . This extension is achieved if and only if w has zero mean, that is,

$$(3.36) \quad 0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} w_2(z) dz = \begin{cases} -\frac{3}{4} + \frac{\pi^2+3\lambda}{4\pi\sqrt{\pi^2-6\lambda}} \log \frac{\sqrt{\pi^2-6\lambda}+\pi}{\sqrt{\pi^2-6\lambda}-\pi} & \lambda < 0, \\ -\frac{3}{4} - \frac{\pi^2+3\lambda}{2\pi\sqrt{6\lambda-\pi^2}} \arctan \frac{\pi}{\sqrt{6\lambda-\pi^2}} & \lambda > \frac{\pi^2}{6}. \end{cases}$$

The piecewise graph of the right-hand side of the zero-mean constraint (3.36) on $(-\infty, 0)$ and $(\pi^2/6, \infty)$ is shown on Figure 2. The first line of the zero-mean constraint (3.36) is equivalent to the transcendental equation (3.31) and it has only one simple zero at $\lambda_1 \approx -0.2262$. The second line of (3.36) does not have any zeros. Hence, $\lambda_1 < 0$ is the only eigenvalue in $\sigma_p(L)$. \square

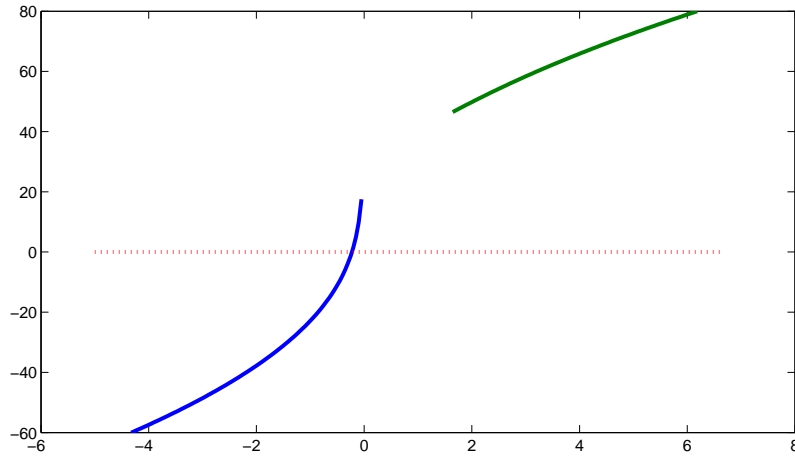


FIGURE 2. The graph of the right-hand side of the zero-mean constraint (3.36) on $(-\infty, 0)$ and $(\pi^2/6, \infty)$ as a function of the spectral parameter λ . Only one simple zero $\lambda_1 < 0$ exists.

Remark 15. For the smooth periodic waves of Lemma 1, we proved in [17] that the spectrum of L includes a simple negative eigenvalue, a simple zero eigenvalue with the eigenvector $\partial_z U$, and the rest of the spectrum is bounded away from zero. Hence, the spectral gap is present in the case of smooth periodic waves, which enabled us to prove in [17] spectral stability of the smooth periodic waves according to Definition 2. By the standard analysis involving the conserved quantity (3.29), see [22], this spectral stability result transfers to the linear stability of the smooth periodic waves according to Definition 3.

4. NONLINEAR INSTABILITY

Here we transfer the linear instability result of Lemma 7 to the proof of nonlinear instability of the peaked periodic wave (2.4) in the Cauchy problem (1.1). Our proof cannot be deduced from the standard approach in [32] because we do not know if the spectral assumption on the spectral instability is satisfied for the peaked periodic wave.

⁵Note that $\langle w_2, w_1 \rangle = 0$ because w_1 is odd and w_2 is even.

Because of the translational symmetry of the Cauchy problem (1.1), we need to consider an orbit $\{U_*(z - a), a \in [-\pi, \pi]\}$ of the peaked periodic wave (2.4). The following definition of orbital stability is widely used in the literature.

Definition 4. The travelling wave U_* is said to be orbitally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $u_0 \in \dot{H}_{\text{per}}^1$ satisfying $\|u_0 - U_*\|_{H_{\text{per}}^1} < \delta$, there exists a unique global solution $u \in C(\mathbb{R}, \dot{H}_{\text{per}}^1)$ to the Cauchy problem (1.1) with $u(0) = u_0$ such that for every $t > 0$,

$$(4.1) \quad \inf_{a \in [-\pi, \pi]} \|u(t, \cdot) - U_*(\cdot - a)\|_{H_{\text{per}}^1} < \epsilon.$$

Otherwise, the periodic wave U_* is said to be orbitally unstable.

Remark 16. The local well-posedness theory for the Cauchy problem (1.1) in [33] requires $u_0 \in \dot{H}_{\text{per}}^s$ with $s > 3/2$, which is not suitable for the peaked periodic wave $U_* \in \dot{H}_{\text{per}}^s$ with $s < 3/2$. We avoid this obstacle by introducing the decomposition $u = U_* + v$, where $v \in \dot{H}_{\text{per}}^s$ with $s > 3/2$ is defined in a smoother space than U_* .

Remark 17. The large-norm smooth solutions to the Cauchy problem (1.1) are not global as their H_{per}^1 norm may blow up in a finite time [28]. This would have been a difficult obstacle in the proof of orbital stability of the peaked periodic wave⁶. However, for the proof of orbital instability of the peaked periodic wave, continuation of local solutions for all $t > 0$ is not required.

The translational parameter a in (4.1) does not have to be defined by minimizing the H_{per}^1 norm. One can define a by minimizing the L_{per}^2 norm as is done e.g. in [16]. To this end, we introduce the following decomposition of the local solution $u(t, \cdot) \in \dot{H}_{\text{per}}^1$ to the Cauchy problem (1.1):

$$(4.2) \quad u(t, x) = U_*(x - c_*t - a(t)) + v(t, x - c_*t - a(t)), \quad \langle \partial_x U_*, v \rangle = 0,$$

where $v \in H^s$ with $s > 3/2$, see Remark 16. The co-periodic perturbation v to the travelling wave U_* satisfies the evolution problem in the form

$$(4.3) \quad \begin{cases} v_t + \frac{1}{6} \partial_z [(z^2 - \pi^2)v] + v \partial_z v = \partial_z^{-1} v + a'(t)(\partial_z U_* + \partial_z v), & t > 0, \\ v|_{t=0} = v_0, \end{cases}$$

where $z = x - c_*t - a(t)$. By projecting the evolution problem to $\partial_z U_* \in \dot{L}_{\text{per}}^2$ and using the orthogonality condition $\langle \partial_z U_*, v \rangle = 0$, we obtain the modulation equation determining the evolution of the modulation parameter a :

$$(4.4) \quad \begin{cases} a'(t) = -\frac{\langle \partial_z U_*, \partial_z L v \rangle - \langle \partial_z U_*, v \partial_z v \rangle}{\|\partial_z U_*\|_{L_{\text{per}}^2}^2 + \langle \partial_z U_*, \partial_z v \rangle}, & t > 0, \\ a(0) = 0, \end{cases}$$

where L is the same self-adjoint operator as in (3.2). We note the following useful simplification of $\langle \partial_z U_*, \partial_z L v \rangle$.

Lemma 9. *If $v \in \dot{H}_{\text{per}}^1$, then*

$$(4.5) \quad \langle \partial_z U_*, \partial_z L v \rangle = -\frac{2}{3} \langle U_*, v \rangle.$$

⁶In a similar context, the work of [8] on the Camassa–Holm equation avoided this obstacle by introducing a weaker definition of orbital stability of peaked waves over short intervals of existence of local solutions.

Proof. By using the definitions from (2.4) and (3.2), we obtain

$$\langle \partial_z U_*, \partial_z Lv \rangle = \frac{1}{3} \int_{-\pi}^{\pi} z \partial_z^{-1} v dz + \frac{1}{3} \int_{-\pi}^{\pi} z \partial_z [(c_* - U_*)v] dz.$$

Since $\partial_z^{-1} v$ and U_* are C_{per} and $c_* - U_*(\pm\pi) = 0$, integration by parts yields

$$\langle \partial_z U_*, \partial_z Lv \rangle = - \int_{-\pi}^{\pi} U_* v dz - \frac{1}{3} \int_{-\pi}^{\pi} (c_* - U_*) v dz.$$

Since $\int_{-\pi}^{\pi} v dz = 0$, we obtain (4.5). \square

We now prove the orbital instability result (in the sense of Definition 4) for the Cauchy problem (4.3)–(4.4) near the peaked periodic wave U_* .

Lemma 10. *There exists $\varepsilon > 0$ such that for every small $\delta > 0$, there exists $v_0 \in \dot{H}_{\text{per}}^s$ with $s > 3/2$ satisfying $\|v_0\|_{H_{\text{per}}^s} \leq \delta$, for which the unique local solution $v \in C([0, T], \dot{H}_{\text{per}}^s)$ with $a \in C^1([0, T], \mathbb{R})$ to the Cauchy problem (4.3)–(4.4) with $T = \mathcal{O}(\delta^{-1})$ satisfies $\|v(t_1)\|_{L_{\text{per}}^2} \geq \varepsilon$ for some $t_1 \in (0, T)$. Therefore, the peaked periodic wave U_* is orbitally unstable in the sense of Definition 4.*

Proof. By the local well-posedness result⁷ in [33], for every $v_0 \in \dot{H}_{\text{per}}^s$ with $s > 3/2$ there exists a unique solution $v \in C([0, T], \dot{H}_{\text{per}}^s)$ of the Cauchy problem (4.3) such that T is inverse proportional to $\|v_0\|_{H_{\text{per}}^s}$. The presence of the translational parameter a is treated similarly as in Lemmas 6.1 and 6.3 in [16], from which existence of the unique solution $a \in C^1([0, T], \mathbb{R})$ of the Cauchy problem (4.4) follows.

The quasi-linear evolution equation (4.3) can be viewed as the inhomogeneous Cauchy problem

$$v_t = Av + F,$$

where A is the infinitesimal generator of the strongly continuous semigroup $(S(t))_{t \geq 0}$ studied in Lemma 4 and

$$(4.6) \quad F := a'(t)(\partial_z U_* + \partial_z v) - v \partial_z v.$$

Since $v(t) \in \dot{H}_{\text{per}}^s \subset D(A)$ for $t \in [0, T]$, we have $\partial_z v(t), v(t) \partial_z v(t) \in \dot{L}_{\text{per}}^2$ and hence $F \in L^1((0, T), \dot{L}_{\text{per}}^2)$. Therefore, every solution $v(t)$ to the Cauchy problem (4.3) also satisfies the integral formulation

$$(4.7) \quad v(t) = S(t)v_0 + \int_0^t S(t-t')F(t')dt'$$

for $t \in [0, T]$, see Definition 7.2, p. 437 in [11]. We use the integral formulation (4.7) to estimate the \dot{L}_{per}^2 norm of the local solution $v(t) \in \dot{H}_{\text{per}}^s$ for $t \in [0, T]$ by using the triangle inequality.

Let $\delta > 0$ be arbitrary and $v_0 \in \dot{H}_{\text{per}}^s$ with $s > 3/2$ satisfy $\|v_0\|_{H_{\text{per}}^s} \leq \delta$. By Lemma 7⁸, there exists $v_0 \in \dot{H}_{\text{per}}^s$ with $s > 3/2$ such that

$$(4.8) \quad \|S(t)v_0\|_{L_{\text{per}}^2} \geq C\|v_0\|_{L_{\text{per}}^2} e^{\pi t/6}, \quad t > 0,$$

⁷Even though this result is obtained in $H^s(\mathbb{R})$ [33], the same iterative method works also in \dot{H}_{per}^s , see Section 2 in [28].

⁸Lemma 7 and Remark 12 can be generalized for every $v_0 \in \dot{H}_{\text{per}}^s$ with $s \geq 1$, in particular for $s > 3/2$.

for some $C > 0$. The initial condition which provides (4.8) must satisfy the constraints (3.6), (3.7), and (3.27). In addition, it must satisfy the constraint $\langle \partial_z U_*, v_0 \rangle = 0$ to enable the decomposition (4.2). This additional constraint is not an obstacle to obtain the lower bound (4.8) in Lemma 7 but just results in a more complicated expression in the example of such v_0 in Remark 12.

By Lemma 5, for every $F(t) \in \dot{L}_{\text{per}}^2$ with $t \in [0, T]$, we have

$$(4.9) \quad \left\| \int_0^t S(t-t')F(t')dt' \right\|_{L_{\text{per}}^2} \leq \int_0^t e^{\pi(t-t')/6} \|F(t')\|_{L_{\text{per}}^2} dt'.$$

In view of the integral formulation (4.7) we obtain from (4.8) and (4.9) that

$$(4.10) \quad \|v(t)\|_{L_{\text{per}}^2} \geq C\|v_0\|_{L_{\text{per}}^2} e^{\pi t/6} - \int_0^t e^{\pi(t-t')/6} \|F(t')\|_{L_{\text{per}}^2} dt'.$$

Our goal is to prove that after a certain point in time this bound is greater than a fixed $\varepsilon > 0$, i.e. that $\|v(t_1)\|_{L_{\text{per}}^2} \geq \varepsilon$ for some $t_1 \in (0, T)$. We pursue a proof by contradiction. Assume⁹ that there exists $B > 0$ such that the solution $v \in C([0, T], \dot{H}_{\text{per}}^s)$ satisfies $\|v(t)\|_{H_{\text{per}}^1} \leq B\delta$ for every $t \in [0, T]$. Since $\delta > 0$ is small, the denominator in the modulation equation (4.4) is strictly positive and bounded from below. The numerator in the modulation equation (4.4) consists of two terms

$$(4.11) \quad \langle \partial_z U_*, \partial_z Lv \rangle \quad \text{and} \quad \langle \partial_z U_*, v \partial_z v \rangle.$$

The first term in (4.11) is estimated from Lemma 9 and the momentum conservation

$$Q(u(t)) = Q(u_0),$$

where $Q(u) = \|u\|_{L_{\text{per}}^2}^2$. Indeed, by using the decomposition (4.2), we obtain

$$Q(u_0) = Q(U_*) + 2\langle U_*, v \rangle + Q(v),$$

where $Q(v) \leq B^2\delta^2$. On the other hand, $Q(u_0) = Q(U_*) + Q(v_0)$ since $\langle U_*, v_0 \rangle = 0$ is satisfied from the constraint (3.7). Hence $|Q(u_0) - Q(U_*)| \leq \delta^2$, so that the expression (4.5) yields the following estimate for the first term in (4.11):

$$|\langle \partial_z U_*, \partial_z Lv \rangle| \leq \frac{1}{3}(1 + B^2)\delta^2.$$

The second term in (4.11) is quadratic in v , hence it follows from (4.4) that there exists $B' > 0$ such that

$$(4.12) \quad |a'(t)| \leq B'\delta^2$$

for every $t \in [0, T]$. By using the definition of F in (4.6), it follows from (4.8), (4.9), (4.10) and (4.12) that

$$(4.13) \quad \|v(t)\|_{L_{\text{per}}^2} \geq e^{\pi t/6} \left[C\|v_0\|_{L_{\text{per}}^2} - \frac{6}{\pi} \left(B'\delta^2 \|\partial_z U_*\|_{L_{\text{per}}^2} + B'B\delta^3 + B^2\delta^2 \right) \right],$$

for every $t \in [0, T]$. Since $\delta > 0$ is small, one can always find $v_0 \in \dot{H}_{\text{per}}^s$ with $s > \frac{3}{2}$ such that

$$C(\delta) := C\|v_0\|_{L_{\text{per}}^2} - \frac{6}{\pi} \left(B'\delta^2 \|\partial_z U_*\|_{L_{\text{per}}^2} + B'B\delta^3 + B^2\delta^2 \right) > 0,$$

⁹We are using Definition 4 with $\epsilon = B\delta$, a more general argument with a small $\epsilon(\delta)$ is readily available.

since $\|v_0\|_{L^2_{\text{per}}} = \mathcal{O}(\delta)$ as $\delta \rightarrow 0$. Hence, for a fixed $\varepsilon > 0$ there exists $t_1 \in [0, T]$ such that

$$(4.14) \quad \|v(t)\|_{L^2_{\text{per}}} \geq e^{\pi t/6} C(\delta) \geq \varepsilon, \quad t \in [t_1, T],$$

which yields a contradiction with the assumption that the solution $v \in C([0, T], \dot{H}^s_{\text{per}})$ remains small in the \dot{H}^1_{per} norm. To see why this last bound holds fix $\varepsilon = 1$ and observe that in view of the exponential growth of the lower bound in (4.14) there exists $t_1 > 0$ such that $e^{\pi t/6} C(\delta) \geq 1$ for $t > t_1$. Finally, since $\delta |\log \delta| \rightarrow 0$ as $\delta \rightarrow 0$ and $e^{t_1} \delta = \mathcal{O}(1)$ as $\delta \rightarrow 0$, we obtain that $t_1 = \mathcal{O}(|\log \delta|) \ll \mathcal{O}(\delta^{-1}) = T$, since T is inverse proportional to $\|v_0\|_{H^s_{\text{per}}} = \mathcal{O}(\delta)$ as $\delta \rightarrow 0$. This implies that $t_1 < T$ which completes the proof. \square

Remark 18. Local solutions to the Cauchy problem (4.3) may blow up in a finite time. It was proven¹⁰ in [19] that the local solution to the original Cauchy problem (1.1) in H^3_{per} remains bounded in the H^3_{per} norm if the initial condition $u_0 \in H^3_{\text{per}}$ satisfies the constraint $1 - 3u''_0(x) > 0$ for every $x \in [-\pi, \pi]$. Since $u_0(x) = U_*(x) + v_0(x)$ and the peaked periodic wave (2.4) satisfies

$$1 - 3U''_*(x) = 0, \quad x \in (-\pi, \pi),$$

any small perturbation in v_0 may violate the global well-posedness constraint if $v''_0(x) > 0$ for some $x \in (-\pi, \pi)$. This blow-up in finite time does not contradict the nonlinear instability result of Lemma 10: since $T = \mathcal{O}(\delta^{-1})$ is large, there is enough time to observe instability of the peaked wave even if the solution v , for which we obtain the instability estimate (4.14), breaks in finite time.

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