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On a structure of the explicit solutions to the Davey–Stewartson equations

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Abstract

A general representation of explicit solutions to the Davey–Stewartson (DS) equations is obtained within the framework of the dressing method. The structure of the solutions with nonzero boundary values at infinity is established to coincide with the functional structure of the solutions to equations of the generalized KP hierarchy. The exponential and rational-type solutions are found both for the DS equations and for their one-dimensional reduction, the nonlinear Schrödinger equation.

1. Introduction

New interesting wave phenomena were recently discovered in two-dimensional (2D) nonlinear dispersive media described by the well-known Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) models [1]. For instance, nonconventional processes of soliton resonance [2,3] and related instability of quasi-plane waves [4], dynamics of anomalous scattering of 2D structures such as lump solutions falling off rationally [5] and dromion solutions falling off exponentially [6–10] were investigated in different aspects. These discoveries stimulate a new interest in soliton equations of mathematical physics.

Investigation of the KP and DS models is based on analysis of various structures of their explicit solutions. For the KP equation

$$\partial_1(-4\partial_3 u + 6u\partial_1 u + \partial_1^3 u) + 3\partial_2^2 u = 0, \quad \partial_k = \frac{\partial}{\partial t_k} \tag{1}$$

and related equations of the KP hierarchy [11] such a structure is manifested in the existence of a determinant representation [12,13]

$$\tau = \det\left(c_n \delta_{nm} + \int_{t_1}^{\infty} f_n^+(t_1) f_m^-(t_1)\, dt_1 \right)_{1 \le n,m \le N}, \tag{2}$$

where matrix elements depend on arbitrary solutions of the linear equations

$$\partial_k f_n^+ = \partial_1^k f_n^+, \quad \partial_k f_n^- = (-1)^{k-1} \partial_1^k f_n^-, \qquad k \ge 1 \tag{3}$$

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and arbitrary constants c_n. The determinant (2) is a solution of Eq. (1) for the variable τ: $u = 2\partial_1^2 \ln \tau$. To simplify the notation we keep in Eq. (2) a sign of dependence of the functions $f_n^\pm$ on the variable t_1 and omit their dependences on the other variables. It is important to note that the determinant representation is rather general because other representations, including the well-known Wronskian one [14], can be obtained from (2) by trivializing the functions f_n^- or f_n^+ [15].

Unlike the equations of the KP hierarchy, the structure of the explicit solutions to the DS equations has not been investigated in detail, although some representations in the form of two-directional and double Wronskians [14], the grammian-type determinant [8,9] as well as the τ-function expressed by vacuum expectation values of Clifford algebra operators [10] were found and used to analyze some classes of their exact solutions. It seems important to find a general representation of the solutions to the DS equations. Such a representation can be found in the framework of the dressing method introduced by Zakharov & Shabat [16] and applied to the DS equations first by Anker & Freeman [17] and then, in an implicit form, by Nakamura [18,19].

In this paper the explicit structure of the solutions is derived and analyzed for the DS2 equations

$$2i\partial_t \Psi + \partial_x^2 \Psi - \partial_y^2 \Psi + 2(n + |\Psi|^2 - \rho^2)\Psi = 0, \tag{4}$$

$$\partial_x^2 n + \partial_y^2 n + 2\partial_x^2 |\Psi|^2 = 0, \tag{5}$$

where n is a real function, Ψ and Ψ^* are complex conjugated functions, $|\Psi|^2 = \Psi \cdot \Psi^*$, and ρ is an arbitrary real parameter. The determinant representation obtained by the dressing method generalizes (2) and transforms to it for the solutions with nonzero boundary values at infinity ($|\Psi| \to \rho, n \to 0$ at $x, y \to \infty$, except, perhaps, for a finite number of directions on the x, y plane). The exponential and rational-type solutions with these boundary conditions are found in an explicit form. These solutions generalize those obtained earlier [17–22].

It should be mentioned that the inverse scattering problem for the DS equations with nonzero boundary conditions was considered on the basis of the nonlocal Riemann–Hilbert problem and the DBAR technique by Bogdanov [23]. The construction of explicit solutions can be realized in the framework of this approach as well. However, in this paper we apply the Zakharov–Shabat dressing method in its original formulation [16] because it enables us to get a convenient form of the explicit solutions which could be useful for possible applications.

2. The determinant representation

Eqs. (4),(5) are known as the conditions of commuting a pair of linear matrix operators [12]:

$$\mathbf{M}_1 = i\partial_y + J\partial_x + Q(\Psi), \quad \mathbf{M}_2 = i\partial_t + J\partial_x^2 + Q(\Psi)\partial_x + R,$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Q(\Psi) = \begin{pmatrix} 0 & \Psi \\ -\Psi^* & 0 \end{pmatrix}$ and the elements of matrix R are related to the variables Ψ, n by the formulas

$$R_{11} - R_{22} = n + |\Psi|^2 - \rho^2, \quad R_{12} = (\partial_x - i\partial_y)\Psi/2, \quad R_{21} = -(\partial_x + i\partial_y)\Psi^*/2.$$

The dressing method [16,17] enables us to construct new complicated solutions to Eqs. (4),(5) from a simple solution which is supposed to have the form of an unperturbed wave $\Psi_0 = \rho, n_0 = 0$. This goal is achieved by solving a linear integral Gelfand–Levitan–Marchenko equation for matrix functions K and F of dimensions 2×2:

$$F(x,z) + K(x,z) + \int_x^{+\infty} K(x,s) \otimes F(s,z) ds = 0. \tag{6}$$

The function F entering the kernel of the integral equation is given by an arbitrary solution of the linear equations

$$i\partial_y F + J \otimes \partial_x F + \partial_z F \otimes J + Q(\rho) \otimes F - F \otimes Q(\rho) = 0, \tag{7}$$

$$i\partial_t F + J \otimes \partial_x^2 F - \partial_z^2 F \otimes J + Q(\rho) \otimes \partial_x F + \partial_z F \otimes Q(\rho) = 0. \tag{8}$$

Then, the function K that is determined from Eq. (6) generates a new solution to the nonlinear equations (4),(5) in accordance with the relationships

$$\Psi = \rho + 2K_{12}(x,x), \tag{9}$$

$$\Psi^* = \rho + 2K_{21}(x,x), \tag{10}$$

$$n = 2\partial_x (K_{11}(x,x) - K_{22}(x,x)) \tag{11}$$

The determinant representation of the functions Ψ, n can be found by separating the variables in Eqs. (6) and (7),(8) [13]. For a matrix case of (6), an adequate variable separation is achieved by factorizing matrix F by vector functions $\mathbf{f}_n^+ = (f_{n1}^+, f_{n2}^+)^\top$ and $\mathbf{f}_n^- = (f_{n1}^-, f_{n2}^-)$:

$$F(x,z) = \sum_{n=1}^{N} C_n^{-1} \mathbf{f}_n^+(x) \otimes \mathbf{f}_n^-(z), \tag{12}$$

where C_n are arbitrary real constants. The substitution of (12) into (7),(8) allows us to determine linear differential equations for the functions $f_{nj}^{\pm}, j = 1,2$:

$$(\partial_x + i\partial_y) f_{n1}^{\pm} + \rho f_{n2}^{\pm} = 0, \quad (\partial_x - i\partial_y) f_{n2}^{\pm} + \rho f_{n1}^{\pm} = 0, \quad \partial_t f_{nj}^{\pm} = \pm \partial_x \partial_y f_{nj}^{\pm}. \tag{13}$$

On the other hand, the substitution of (12) into (6) reduces the integral equation to a system of linear algebraic equations which has a unique solution

$$K(x,z) = -\sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\Delta_{nm}(x)}{\Delta_N(x)} \mathbf{f}_n^+(x) \otimes \mathbf{f}_m^-(z), \tag{14}$$

where Δ_{nm} is the co-factor of element h_{nm} of the form

$$h_{nm} = C_n \delta_{nm} + \langle \mathbf{f}_m^-, \mathbf{f}_n^+ \rangle, \quad \langle \mathbf{f}_m^-, \mathbf{f}_n^+ \rangle = \int_x^{+\infty} \left(f_{n1}^+(s) f_{m1}^-(s) + f_{n2}^+(s) f_{m2}^-(s) \right) ds, \tag{15}$$

and $\Delta_N \equiv H_{N,N}^N = \det(h_{nm})_{1 \le n,m \le N}$. The determinant Δ_N is highly important for investigation of the solutions to the DS equations because it determines the functions $n, |\Psi|^2$ as

$$n = 2\partial_x^2 \ln \Delta_N, \quad |\Psi|^2 = \rho^2 - (\partial_x^2 + \partial_y^2) \ln \Delta_N. \tag{16}$$

These functions are meaningful for Eqs. (4),(5) only if they are real. Therefore, the determinant Δ_N must also be real. For other representations of the solutions to the DS equations, the real conditions make analysis difficult and give rise to laborious calculations (see, e.g., papers [7,22]). For our determinant representation, this problem has a simple solution. Namely, at real x, y the determinant Δ_N is real for arbitrary N if

$$C_n = C_n^*, \quad f_{n2}^{\pm} = \sigma_n f_{n1}^{\pm *}, \qquad 1 \le n \le N, \tag{17}$$

and for even $N = 2M$ if

$$C_{n+M} = C_n^*, \quad f_{n+M1}^{\pm} = \sigma_n f_{n2}^{\pm *}, \quad f_{n+M2}^{\pm} = \sigma_{n+M} f_{n1}^{\pm *}, \qquad 1 \le n \le M, \tag{18}$$

where $\sigma_n = \pm 1, 1 \le n \le N$. Obviously, the conditions (17),(18) are completely consistent with the system (13).

The determinant Δ_N can be regarded as a direct generalization of (2). Its form seems to be more convenient than the block matrix form which was introduced earlier [17–19].

3. Other forms of explicit solutions

There exists a close mathematical correspondence between the solutions to the DS and KP equations. Under the condition $\rho \neq 0$, the determinant Δ_N with the matrix elements h_{nm} can be transformed to (2). For this purpose it is necessary to consider complex values of the variables $t_k, k = 1, 2$ and to introduce new variables $t_k, k = -1, -2$ so that the functions $f_n^{\pm}, 1 \le n \le N$ satisfy an additional set of integro-differential equations

$$\partial_k f_n^+ = \rho^{-2k} \partial_1^k f_n^+, \quad \partial_k f_n^- = (-1)^{k-1} \rho^{-2k} \partial_1^k f_n^-, \qquad k \le -1. \tag{19}$$

Here we suppose that the functions $f_n^{\pm}$ decrease exponentially as $t_1 \to +\infty$ so that the inverse operator $\partial_1^{-1} = -\int_{t_1}^{+\infty} dt_1$ is well-defined.

If we transform the independent variables $x = -(t_1 + t_{-1}), y = i(t_1 - t_{-1}), t = -2i(t_2 - t_{-2})$ and use the relation

$$(\partial_x + i\partial_y) f_{n1}^+ f_{n1}^- = (\partial_x - i\partial_y) f_{n2}^+ f_{n2}^-,$$

then the matrix elements h_{nm} containing the vector - functions $\mathbf{f}_n^+ = (f_{n1}^+, f_{n2}^+)^{\mathrm{T}}$ and $\mathbf{f}_n^- = (f_{n1}^-, f_{n2}^-)$ are reduced to the form used in (2) with the scalar functions $f_{n1}^{\pm} \equiv f_n^{\pm}$ and the constants $C_n = -2c_n$. In the new variables these functions depend on $t_{\pm 1}, t_{\pm 2}$ in agreement with the linear equations (3) and (19). Therefore, the functional structure of the explicit solutions to the DS equations with nonzero boundary values at infinity completely coincides with the structure of the solutions to the generalized KP hierarchy with an additional set of independent variables. It accounts for similarity of the soliton and rational solutions of these equations which was revealed by a bilinear method [20].

Besides (2), there exist other functional representations of the solutions to the KP hierarchy. Now we shall show that the relationship between the solutions to the DS and KP equations discussed above can be extended to the other representations. Indeed, it is well known for the Wronskian form of the solutions [14].

The procedure of transforming the determinant Δ_N uses the formal series obtained from the matrix element h_{nm} by integrating by parts

$$h_{nm} = c_n \delta_{nm} + \sum_{k=0}^{\infty} (-1)^{k+1} \partial_1^k f_n^+ \partial_1^{-k-1} f_m^- \tag{20}$$

$$= c_n \delta_{nm} + \sum_{k=0}^{\infty} (-1)^{k+1} \partial_1^{-k-1} f_n^+ \partial_1^k f_m^-. \tag{21}$$

Let some functions f_m^- for $M+1 \le m \le N$ be simple exponential solutions to the systems (3) and (19)

$$f_m^- = c_m^- \exp\left(\phi^-(q_m)\right), \tag{22}$$

$$\phi^-(q_m) = \sum_{k=1}^{\infty}(-1)^{k+1} q_m^k t_k + \sum_{k=1}^{\infty}(-1)^{k+1}(\rho^2/q_m)^k t_{-k}. \tag{23}$$

Then, the expressions (20),(21) become power series in terms of $1/q_m$ and q_m, respectively, and the higher terms of these series are decreasing for (20) if $q_m \to \infty$ and for (21) if $q_m \to 0$. Since the functions (16) do not change when the determinant Δ_N is multiplied by an arbitrary constant and/or exponential factors like (22), under the condition $c_m = 0$ for $M+1 \le m \le N$ a solution to Eqs. (4),(5) is independent of the trivialized functions f_m^-. Then, the limit $q_m \to 0$ for $M+1 \le m \le M+\mu$ and $q_m \to \infty$ for $M+1+\mu \le m \le N$ transforms the functional structure of the determinant Δ_N which becomes as follows

$$\Delta_N \equiv H^N_{N;M,\mu} = \det \begin{pmatrix} \left(c_n\delta_{nm} + \int_{t_1}^{+\infty} f_n^+(s) f_m^-(s)\,ds\right)_{1\le n\le N, 1\le m\le M} \\ \left(\partial_1^{m-(M+1+\mu)} f_n^+\right)_{1\le n\le N, M+1\le m\le N} \end{pmatrix}. \tag{24}$$

The Wronskian form [14] appears as a partial case of the determinant $H^N_{N;M,\mu}$ at $M = 0$. We are able to construct a still more general representation of Δ_N by choosing the functions f_n^+ at $K+1 \le n \le N$ in the exponential form which is adjoint to (22),(23)

$$f_n^+ = c_n^- \exp\left(\phi^+(p_n)\right), \tag{25}$$

$$\phi^+(p_n) = \sum_{k=1}^{\infty} p_n^k t_k + \sum_{k=1}^{\infty}(\rho^2/p_n)^k t_{-k}. \tag{26}$$

For $\mu \ne 0$ and $\mu \ne N-M-1$, after the limit transitions, when p_n tends to zero or to infinity, the order of Δ_N decreases. However, at $\mu = 0$ and $p_n \to 0, c_n = 0$ for $K+1 \le n \le N$ we obtain a new functional form of an explicit solution

$$\Delta_N \equiv H^N_{K;M,0} = \det \begin{pmatrix} \left(c_n\delta_{nm} + \int_{t_1}^{+\infty} f_n^+(s) f_m^-(s)\,ds\right)_{1\le n\le K, 1\le m\le M} \\ \left(\partial_1^{m-(M+1)} f_n^-\right)_{1\le n\le K, M+1\le m\le N} \\ \left((-\partial_1)^{-n+K} f_m^-\right)_{K+1\le n\le N, 1\le m\le M} \\ \left(\delta_{(n-K)(m-M)}\right)_{K+1\le n\le N, M+1\le m\le N} \end{pmatrix}. \tag{27}$$

Note that besides the determinants $H^N_{N;M,\mu}$, $H^N_{K;M,0}$ there exist determinants $H^N_{K,\kappa;N}$, $H^N_{K,0;M}$ which are antisymmetrical with respect to the functions $f_n^{\pm}$. The other limit transition at $\mu = N-M-1$ and $p_n \to \infty$ for $K+1 \le n \le N$ transforms the determinant $H^N_{N;M,\mu}$ to $H^N_{K,0;M}$ which is antisymmetrical to (27).

Thus, the DS and KP equations have an identical, extremely various functional structure of explicit solutions. It is important that the form of the complex variable Ψ obtained in the framework of the dressing method (formulas (9),(10)) can be rewritten using the determinant (27):

$$\Psi = \rho\frac{H^{N+1}_{N;N,0}}{\Delta_N}, \quad \Psi^* = \rho\frac{H^{N+1}_{N,0;N}}{\Delta_N}. \tag{28}$$

It is obvious that the determinants $H^{N+1}_{N;N,0}$, $H^{N+1}_{N,0;N}$ depend on the same functions $f_n^{\pm}$ for $1 \le n \le N$ as the determinant Δ_N. Moreover, they do not differ functionally from Δ_N and, as is well known, are the result of the action of the transformation groups changing the phases of the independent variables of the function Δ_N [11].

4. Exponential and rational-type solutions

Employing the found representations of the determinant Δ_N, we analyze here the exponential and rational-type solutions with nonzero boundary values at infinity both for Eqs. (4),(5) and for their 1D analog, the nonlinear Schrödinger (NLS) equation.

The exponential-type solutions are generated by choosing the functions $f_n^{\pm}$ in the form of an arbitrary superposition of the exponents (16a,b). The choice of the unique exponent with the constants $c_n^{\pm} = i(p_n+q_n)^{1/2}$ and $c_n = 1$ corresponds to the pure "N-soliton solution" of Eqs. (4),(5). According to (16), it is expressed by the determinant Δ_N, which can be expanded in the well-known polynomial of exponentials [20]:

$$\Delta_N = \det\left(\delta_{nm} + d_{nm}\exp((\eta_n + \eta_m)/2)\right)_{1\le n,m\le N} \tag{29}$$

$$= \sum_{\mu=(0,1)} \exp\left(\sum_{1\le n\le N} \mu_n \eta_n + \sum_{1\le n<m\le N} \mu_n \mu_m A_{nm} \right), \tag{30}$$

where $\eta_n = \phi^{+}(p_n) + \phi^{-}(q_n)$, $d_{nm} = ((p_n+q_n)(p_m+q_m))^{1/2}/(p_n+q_m)$, and

$$A_{nm} = \ln\left(\frac{(p_n - p_m)(q_n - q_m)}{(p_n + q_m)(p_m + q_n)}\right).$$

The reality conditions for the determinant Δ_N which are equivalent to (17) in the original variables yield that the complex parameters p_n, q_n for any n belong to the circumference of radius ρ.

The expressions (29),(30) reduce to the "N-soliton solution" of the NLS equation when the condition $\partial_y \Psi = 0$ is met. In the new variables this condition has the form

$$\partial_1 \Psi = \partial_{-1}\Psi. \tag{31}$$

A sufficient condition is that all the phases η_n of (29),(30) should satisfy (31). This leads to the following equations

$$p_n q_n = \rho^2 \quad \text{for any } n, \tag{32}$$

which coincide with the known Hirota reduction [21]. In our analysis this reduction emerges naturally from the investigated relationship between the functional structure of the solutions to the DS and KP equations.

The rational-type solutions for which Δ_N is an ordinary polynomial (probably multiplied by an arbitrary exponential factor) appear by choosing $c_n = 0$ for any n and

$$f_n^{+} = \partial_{p_n}^{m_n} \exp(\phi^{+}(p_n)), \quad f_n^{-} = \partial_{q_n}^{l_n} \exp(\phi^{-}(q_n)). \tag{33}$$

If there are no equal values among the parameters p_n, q_n the polynomial solutions can be represented in an operator form [24]

$$\Delta_N = \partial^{m_1+\dots+m_N+l_1+\dots+l_N}_{p_1,p_2,\dots,p_N,q_1,q_2,\dots,q_N} \exp\left(\sum_{1\le n\le N} (\eta_n + \ln(p_n+q_n)) + \sum_{1\le n<m\le N} A_{nm} \right). \tag{34}$$

It is not difficult to show that the expression (34) transforms for $m_n = 1, l_n = 0$ for any n to the known polynomials of N degree which are associated with the dynamics of 2D lump solitons in the KP1 and DS1 equations [20]. However, there are broader classes of the polynomials Δ_N. Such polynomials are generated by the functions (33) with equal parameters p_n, q_n for some or for all n. The rational solutions to the NLS equation can be found precisely among such degenerate polynomials of the DS equations because the Hirota conditions (32) are rewritten for rational solutions as $p_n^2 = q_n^2 = -\rho^2$. Using the technique of the recent paper [15] we shall show that the determinants $H^N_{N;M,0}$, $H^N_{K,0;N}$ with the functions $f_n^\pm$

$$f_n^\pm = S^{n-1}(\lambda) P^\pm_{2n-1} \cdot \exp(\phi^\pm(p)) \qquad (35)$$

are rational solutions to the NLS equation if

$$p^2 = -\rho^2, \quad \lambda = -1/p. \qquad (36)$$

Here we introduced the Schur polynomials

$$P_n^\pm(\theta_1^\pm, \theta_2^\pm, \ldots, \theta_n^\pm) = \exp(-\phi^\pm(p))\partial_p^n \exp(\phi^\pm(p)),$$

the phases $\theta_n^\pm = \partial_p^n \phi^\pm / n!$ and the vertex operator

$$S(\lambda) = \exp\left(-\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \partial_{\theta_n^\pm}\right).$$

It follows from the factorized form of the determinants $H^N_{N;M,0}$, $H^N_{K,0;N}$ with the functions (35) [15] and from the Bineu–Cauchy formula that a general "$N \times M$" or "$K \times N$" solution satisfies Eq. (31) if the partial Wronskians $H^N_{N;0,0}$, $H^N_{0,0;N}$ satisfy the same equation. Since these Wronskians are equivalent to the Wronskians $W_N^\pm[P_1^\pm, S(\lambda)P_3^\pm, \ldots, S^{N-1}(\lambda)P^\pm_{2N-1}]$ and therefore depend only on the variables $\theta_n^\pm$, it is convenient to rewrite Eq. (31) in these variables

$$\left(1 + \frac{\rho^2}{p^2}\right)\partial_{\theta_1^\pm} W_N^\pm + \frac{\rho^2}{p^3}\left(\sum_{m=0}^{\infty}(-p)^{-m}\partial_{\theta^\pm_{2+m}} W_N^\pm\right) = 0.$$

Direct calculations prove that both the parentheses in this equation are identically equal to zero under the conditions (36). Therefore, both the partial Wronskians and the general determinants $H^N_{N;M,0}$, $H^N_{K,0;N}$ with the functions (35) are rational solutions to the NLS equation.

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