ELSEVIER

Physica D 116 (1998) 121–142

PHYSICA D

Internal modes of envelope solitons

Dmitry E. Pelinovsky [a,1], Yuri S. Kivshar [b], Vsevolod V. Afanasjev [c]

[a] *Department of Mathematics, Monash University, Clayton, Vic. 3168, Australia*

[b] *Australian Photonics Cooperative Research Centre, Optical Sciences Centre, Research School of Physical Sciences and Engineering, The Australian National University, Canberra, ACT 0200, Australia*

[c] *Optical Communications Group, School of Electrical Engineering, University of New South Wales, Sydney, NSW 2052, Australia*

Received 5 March 1997; received in revised form 16 August 1997; accepted 30 September 1997
Communicated by A.C. Newell

Abstract

We extend the concept of internal mode to envelope solitons and show that this mode is responsible for long-lived, weakly damped periodic oscillations of the soliton amplitude observed in numerical simulations. We present analytical and numerical results for solitons of the generalized nonlinear Schrödinger equation and analyze the example of the cubic–quintic nonlinearity in more detail. We obtain also analytical criteria for the existence and bifurcations of the soliton internal mode and calculate the rate of the radiation-induced damping of the soliton oscillations induced by excitation of the internal mode. Copyright © 1998 Elsevier Science B.V.

PACS: 42.65.Tg; 63.20.Pw; 52.35.Mw; 03.40.Kf
Keywords: Internal mode; Eigenvalue problem; Envelope soliton

1. Introduction

Integrable models of nonlinear physics are known to possess stable soliton solutions–spatially localized structures that interact elastically retaining their identity [1]. However, the integrable systems appear as a limit of more general physical models; they describe the physical systems only with a certain approximation, and very often one needs to account for the effects produced by nonintegrability to nonlinear equations and their solutions for solitary waves. It is generally believed that inelastic interaction between solitary waves due to emission of radiation is major property which differs from nonlinear waves in integrable and nonintegrable models with respect to physical applications (see, e.g., the review [2] and references therein). However, as has been already observed for *kinks*, which are topological solitary waves of the Klein–Gordon type models, soliton interactions may differ drastically when the colliding kinks possess the so-called *internal modes* [3]. In respect to applications in solid state physics, the kink's internal mode can be treated as 'phonons' coupled to the localized state, e.g., phonons localized at a dislocation in a generalized version of the Frenkel–Kontorova model [4].

[1] Present address: Department of Mathematics, University of Toronto, Toronto, Ontario, Canada, M5S 3G3.

PII S0167-2789(97)00247-9

The internal modes of kinks, usually called '*shape modes*', have been analyzed for different models and their role is understood (see, e.g., [3,5]). Indeed, if one considers the models described by the equations which deviate from the exactly integrable sine-Gordon equation, the small disturbances around the kink soliton include usually a zero-frequency localized mode, which is the Goldstone mode associated with the translational invariance, and the linear (continuous) modes associated with the waves scattered by the kink. Additionally, the spectrum may be modified essentially by the appearance of novel localized modes, or *kink's shape modes*, which have the frequencies below the lowest phonon frequency. The existence of these localized modes has important consequences on the kink dynamics because they can temporarily store energy taken away from the kink kinetic energy which can later be restored giving rise to resonant structures in the kink–antikink collisions [3,5]. The similar effect have been predicted for the resonant kink–impurity interactions when the impurity supports localized oscillation at the impurity site [6].

In the physical models we find solitary waves of different types and structure [1,2]. In particular, solitary waves discussed in nonlinear optics are *envelope solitons* which are characterized by spatially localized solutions for the slowly varying envelope of the electric field (see, e.g., [7,8]). In systems with weak nonlinearity, such solitary waves are described by the integrable cubic nonlinear Schrödinger (NLS) equation, and its solitons *do not possess any internal modes*. This certainly is a proper model for temporal solitons propagating along optical fibers [8], but the model is inappropriate for spatial solitons (sometimes called *self-guided beams*) propagating in waveguides or bulk materials where much higher powers are required. In particular, as has been recently demonstrated theoretically and experimentally, self-guided beams can be observed in materials with a strong photorefractive effect [9], in vapors with a strong saturation of the effective index [10], and also they can exist due to the phase-matched two- and three-wave parametric interactions in diffractive $\chi^{(2)}$ nonlinear crystals [11,12]. In all these cases, propagation of envelope solitary waves is observed in *non-Kerr materials* which are described by more general models than the cubic NLS equation.

Then the natural question arises: *Do envelope solitons possess internal modes for these general nonintegrable NLS-type models, and if it is so, what are their properties?* The importance of this question follows from the results of numerical simulations of the dynamics of optical solitons in the systems with $\chi^{(2)}$ nonlinearities [13,14] where solitary waves are known to display long-lived oscillations of their amplitude, similar to those observed earlier in numerical simulations of the self-focusing effects in a medium with nonlinearity saturation [15]. The problem of the existence of the soliton internal modes can also be approached from the other side, considering the cubic nonlinearity but modified dispersive properties of the system, e.g., due to discreteness. For example, it is well established that discrete models can support highly localized nonlinear modes, the so-called *discrete breathers* [16,17]. Such localized modes resemble the envelope solitons but excited on a few lattice cites, and sometimes they can be approximated by the discrete NLS equation [18]. Numerical simulations of such nonlinear modes in one-dimensional and two-dimensional discrete lattice [19] reveal many features resembling the dynamics of the solitary waves of continuous NLS-type models of the generalized nonlinearities, including the excitation of long-lived periodic oscillations.

Therefore, in many models describing envelope solitary waves, *continuous and discrete*, numerical simulations of the nonstationary dynamics of spatially localized structures display the existence of long-lived oscillations of the soliton amplitude that are *practically undamped and persist for a long time*. We show in this paper on the example of the generalized NLS equation that this effect can be naturally associated with the existence of localized modes of envelope solitons (often called *internal modes*), similar to the case of the topological solitons of the Klein–Gordon models. From the physical point of view, similar to the case of kinks, the internal mode of an envelope solitary wave can be treated as a localized linear excitation associated with the existence of a bound state, e.g., *bound or trapped photons* of a self-guided optical beam. From the mathematical point of view, the internal mode is described by a nontrivial discrete eigenvalue of the associated scattering problem which can appear due to nonintegrability of the nonlinear model. The frequency of this mode is in the gap of linear spectrum band and therefore the oscillations with this mode are localized near the soliton. Therefore, having the frequency different from the soliton frequency, the mode manifests itself as a periodic beating of the soliton amplitude, similar to the 'wobbling' oscillation of the kink shape in the Klein–Gordon models [3,5].

The paper is organized as follows. In Section 2 we introduce the model described by the generalized NLS equation and consider, in more details, the particular example of the cubic–quintic nonlinearity where solitary waves can be found in an explicit analytical form. Section 3 presents the analysis of an eigenvalue problem which describes linear excitations upon the envelope solitary wave. Existence of the soliton internal modes is discussed in Section 4 where we employ the so-called Evan's function and study its analytical properties to analyze bifurcations associated with the soliton internal mode which can emerge from either the edge of the continuous-spectrum band or the instability domain. In Section 5 we calculate the radiation-induced damping of the initially excited oscillations of the soliton amplitude associated with the existence of the soliton internal mode. This radiative damping is a direct consequence of nonintegrability of the physical system (i.e. no exact solutions periodic in time exist) and it is accounted by a generation of higher-order harmonics. At last, Section 6 concludes the paper.

2. Model and stationary localized solutions

We consider the generalized NLS equation which describes, in particular, propagation of a self-guided beam of the fundamental frequency in a dielectric optical waveguide. In the dimensionless case this equation reads

$$\mathrm{i}\frac{\partial \Psi}{\partial t} + \frac{\partial^2 \Psi}{\partial x^2} + F(|\Psi|^2)\Psi = 0, \tag{1}$$

where, in the case of spatial optical solitons propagating in a slab waveguide, t and x are the longitudinal and transverse coordinates, respectively. The function Ψ describes a complex (normalized) envelope amplitude of the fundamental wave and $F(I)$ is proportional to the nonlinearity-induced change in the material refractive index which depends on the wave intensity $I = |\Psi|^2$. It is natural to assume the condition $F(0) = 0$, i.e., this nonlinear correction vanishes for small intensities.

Self-guided beams (bright spatial solitons) are described by localized stationary solutions of Eq. (1) of the form $\Psi(x,t) = \Phi(x;\omega)\mathrm{e}^{\mathrm{i}\omega t}$, where ω is the nonlinearity-induced correction to the carrier frequency of the fundamental wave. In application to the theory of spatial solitions, this parameter is referred to as the soliton propagation constant. Function $\Phi(x;\omega)$ satisfies the following differential equation.

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} - \omega\Phi + F(\Phi^2)\Phi = 0. \tag{2}$$

A simple analysis reveals that a localized solution of Eq. (2) exists for $\omega > 0$ provided there is at least one root, $I = I_m(\omega)$ of the equation $\int_0^{I_m} F(I)\mathrm{d}I - \omega I_m = 0$. Under this condition, a bright soliton solution is described by an *even, positive, single-humped* function $\Phi(x;\omega)$ with the amplitude proportional to $\sqrt{I_m(\omega)}$. In the limit $x \to \pm\infty$ the function $\Phi(x;\omega)$ vanishes exponentially:

$$\Phi(x;\omega) \to \chi(\omega)\mathrm{e}^{-\sqrt{\omega}|x|} \quad \text{as } x \to \pm\infty, \tag{3}$$

where $\chi(\omega)$ is a constant coefficient.

Although our analysis of the solitary waves of model (1) is general, throughout the paper we demonstrate our results for the particular case of the nonlinear function $F(I)$ corresponding to the cubic–quintic nonlinearity. Without loss of generality, this function can be rescaled to the following form.

$$F(I) = 4I + 3\sigma I^2, \tag{4}$$

where $\sigma = \pm 1$. This function takes into account a quintic-nonlinearity correction to the cubic (or Kerr) nonlinearity which should be taken into account for larger light intensities. This correction may be either focusing ($\sigma = +1$) or defocusing ($\sigma = -1$). The generalized NLS equation with the nonlinearity (4) allows to find an explicit solution for a solitary wave:

$$\Phi(x;\omega) = \left\{ \frac{\omega}{1 + \sqrt{1+\sigma\omega}\cosh(2\sqrt{\omega}x)} \right\}^{1/2}. \tag{5}$$

In the limit of small amplitudes, when $\omega \ll 1$, the solitary wave (5) transforms into the conventional soliton of the NLS equation

$$\Phi(x;\omega) \to \frac{\sqrt{\omega}}{\sqrt{2}\cosh(\sqrt{\omega}x)} \quad \text{as } \omega \to 0. \tag{6}$$

For larger intensities the quintic nonlinearity distorts the shape of the NLS soliton. For the defocusing case when $\sigma = -1$, the existence of a solitary wave is limited by the critical value $\omega = 1$ which defines the critical soliton amplitude, $I_m(\omega) \to 1$. Therefore, for $\sigma = -1$ the solution (5) exists only for any positive ω, and in the limit of large amplitudes when $\omega \gg 1$, its shape becomes closer to that described by the critical (quintic) NLS equation:

$$\Phi(x;\omega) \to \left\{ \frac{\sqrt{\omega}}{\cosh(2\sqrt{\omega}x)} \right\}^{1/2} \quad \text{as } \omega \to \infty. \tag{7}$$

3. Linear excitations of an envelope soliton

3.1. Linear eigenvalue problem

Here we analyze small (linear) perturbations excited upon the soliton solution. To do this, we linearize the generalized NLS equation (1) around the solitary wave $\Phi(x;\omega)$ by applying the following substitution for the linear perturbation [20]:

$$\Psi(x,t) = [\Phi(x;\omega) + (U-W)e^{i\Omega t} + (U^* + W^*)e^{i\Omega t}]e^{-i\omega t}, \tag{8}$$

where $U = U(x;\omega,\Omega)$ and $W = W(x;\omega,\Omega)$ are generally complex functions, Ω is a complex eigenvalue and the asterisk stands for the complex conjugation. The substitution (8), after neglecting nonlinear terms, reduces Eq. (1) to the linear eigenvalue problem which can be written in the following matrix form:

$$\mathcal{L}\mathcal{Y} = \Omega\mathcal{Y}, \tag{9}$$

where

$$\mathcal{Y} = \begin{pmatrix} U \\ W \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} 0 & \mathcal{L}_0 \\ \mathcal{L}_1 & 0 \end{pmatrix},$$

and the linear operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are associated with the soliton solution $\Phi(x;\omega)$:

$$\mathcal{L}_0 = -\frac{\partial^2}{\partial x^2} + \omega - F(\Phi^2), \tag{10}$$

$$\mathcal{L}_1 = -\frac{\partial^2}{\partial x^2} + \omega - F(\Phi^2) - 2\Phi^2 F'(\Phi^2). \tag{11}$$

Besides the eigenfunction $\mathcal{Y}$ we consider also an adjoint eigenfunction $\mathcal{Z} = (U^a, W^a)$, where U^a and W^a satisfy the same linear eigenvalue problem (9). Then, the following Wronskian-type quantity,

$$\mathcal{W}(\mathcal{Z}, \mathcal{Y}) = \mathcal{Z}\frac{\partial \mathcal{Y}}{\partial x} - \frac{\partial \mathcal{Z}}{\partial x}, \mathcal{Y} = U^a \frac{\partial U}{\partial x} + W^a \frac{\partial W}{\partial x} - \frac{\partial U^a}{\partial x} U - \frac{\partial W^a}{\partial x} W, \tag{12}$$

does not depend on x, and hence, it plays an important role in the analysis of the linear eigenvalue problem (9).

We define four linearly independent fundamental solutions to the problem (9) by imposing the following boundary conditions:

$$\mathcal{Y}_j \to \mathbf{y}_j \mathrm{e}^{\mu_j x} \quad \text{as } x \to +\infty, \tag{13}$$

$$\mathcal{Z}_j \to \mathbf{z}_j \mathrm{e}^{-\mu_j x} \quad \text{as } x \to -\infty, \tag{14}$$

where $j = \overline{1, 4}$, $\mu_1 = -\mu_3 = -\sqrt{\omega - \Omega}$, $\mu_2 = -\mu_4 = -\sqrt{\omega + \Omega}$,

$$\mathbf{y}_1 = \mathbf{y}_3 = \mathbf{e}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{y}_2 = \mathbf{y}_4 = \mathbf{e}_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and $\mathbf{z}_j = (1/4\mu_j)\mathbf{y}_j^{\mathrm{t}}$ with "t" standing for transposed matrices. These boundary conditions follow from the asymptotic solutions to Eq. (9) in the limits $x \to \pm\infty$. In the opposite limits, the fundamental eigenfunctions can be superposed through the set of asymptotic solutions as follows:

$$\mathcal{Y}_j \to \sum_{k=1}^{4} \mathcal{D}_{jk}(\Omega)\mathbf{y}_k \mathrm{e}^{\mu_k x} \quad \text{as } x \to -\infty, \tag{15}$$

$$\mathcal{Z}_j \to \sum_{k=1}^{4} \mathcal{D}_{kj}(\Omega)\mathbf{z}_k \mathrm{e}^{-\mu_k x} \quad \text{as } x \to +\infty. \tag{16}$$

The coefficients $\mathcal{D}_{jk}(\Omega)$, referred to as *scattering coefficients*, can be expressed through the Wronskians (12),

$$\mathcal{D}_{jk}(\Omega) = \mathcal{W}(\mathcal{Z}_k, \mathcal{Y}_j). \tag{17}$$

The detailed analysis of the scattering coefficients should be carried out independently in different regions of the spectral parameter Ω. However, there are general relations between these coefficients following from the symmetries of the linear problem (9). Indeed, because the function $\Phi(x;\omega)$ is even in x, the eigenfunctions $\mathcal{Y}_j(-x;\omega,\Omega)$ also satisfy Eq. (9), and therefore, we can construct $\mathcal{Z}_j$ as follows:

$$\mathcal{Z}_j(x;\omega,\Omega) = \frac{1}{4\mu_j}\mathcal{Y}_j^{\mathrm{t}}(-x;\omega,\Omega). \tag{18}$$

This property leads to the symmetry relations

$$\frac{\mathcal{D}_{jk}}{\mathcal{D}_{kj}} = \frac{\mu_j}{\mu_k}. \tag{19}$$

3.2. Scattering problem and continuous-wave spectrum

Continuous-wave (nonvanishing) solutions of Eq. (9) exist for real Ω such that $|\Omega| \geq \omega$. The continuous-wave spectrum is shown schematically in Fig. 1 by shaded regions. In the region $\Omega > \omega$ we construct a uniform continuous-wave eigenfunction $\mathcal{Y}^+(x, k)$, where k is a real parameter specifying the spectral eigenvalue, $\Omega = \omega + k^2$, from

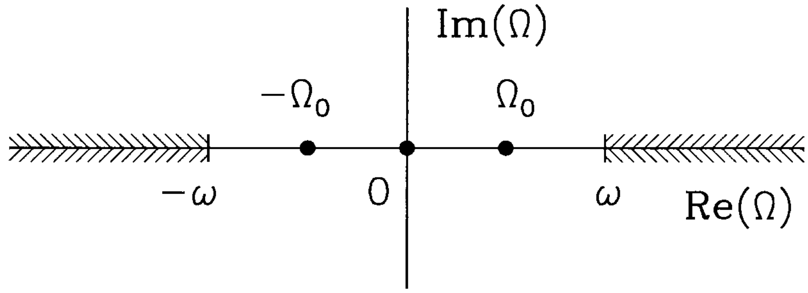

Fig. 1. Schematic structure of the spectrum of linear eigenvalue problem (9)–(11) in the case of a single internal mode of a solitary wave, e.g., for the cubic–quintic NLS equation (1) and (4) at $\sigma = +1$. Shaded regions correspond to the continuous-wave spectrum $|\Omega| > \omega$. The symmetric eigenvalues $\Omega = \pm\Omega_0$ define the frequency of the soliton internal mode.

the fundamental solutions $\mathcal{Y}_1$, or alternatively $\mathcal{Y}_3 = \mathcal{Y}_1^*$, and $\mathcal{Y}_2$ by removing the exponentially growing term in Eq. (15) proportional to $e^{\mu_2 x}$. For example, for negative values of the parameter k, the continuous-wave eigenfunction $\mathcal{Y}^+(x,k)$ is defined by

$$\mathcal{Y}^+(x,k) = \mathcal{Y}_1(x;\Omega,\omega) - \frac{\mathcal{D}_{12}(\Omega)}{\mathcal{D}_{22}(\Omega)}\mathcal{Y}_2(x;\Omega,\omega) \quad \text{for } \Omega > \omega.$$

Then, neglecting exponentially small terms in the limits $x \to \pm\infty$ we find the following boundary conditions for the continuous-wave eigenfunction:

$$\mathcal{Y}^+(x,k) \to \begin{cases} \mathbf{e}_+ e^{ikx}, & x \to +\infty, \\ A(k)\mathbf{e}_+ e^{ikx} + B(k)\mathbf{e}_+ e^{-ikx}, & x \to -\infty, \end{cases} \tag{20}$$

where the coefficients $A(k)$ and $B(k)$ are given for negative k by the formulas

$$A(k) = \frac{\mathcal{D}_{11}(\Omega)\mathcal{D}_{22}(\Omega) - \mathcal{D}_{12}(\Omega)\mathcal{D}_{21}(\Omega)}{\mathcal{D}_{22}(\Omega)}, \tag{21}$$

$$B(k) = \frac{\mathcal{D}_{13}(\Omega)\mathcal{D}_{22}(\Omega) - \mathcal{D}_{12}(\Omega)\mathcal{D}_{23}(\Omega)}{\mathcal{D}_{22}(\Omega)}. \tag{22}$$

To find the continuous-wave eigenfunctions and the coefficients $A(k)$ and $B(k)$ for positive k we use the symmetry relations

$$\mathcal{Y}^+(x,-k) = \mathcal{Y}^{+*}(x,k), \qquad A(-k) = A^*(k), \qquad B(-k) = B^*(k). \tag{23}$$

In addition, it follows from the Wronskian relations (see, e.g., [21]) that the coefficients $A(k)$ and $B(k)$ satisfy the following scattering relation:

$$|A(k)|^2 = 1 + |B(k)|^2. \tag{24}$$

It is clear from Eqs. (21) and (22) that zeros of the function $\mathcal{D}_{22}(\Omega)$ for $|\Omega| \geq \omega$ are poles of the coefficients $A(k)$ $B(k)$. The *main assumption* of this paper is the condition $\mathcal{D}_{22}(\Omega) > 0$ for $\Omega \geq \omega$. Under this restriction, the coefficients $A(k)$ and $B(k)$ are not singular for real k and the localized eigenmodes embedded into the continuous-wave spectrum are absent (for an example of such an eigenmode see [14]).

The other branch of the continuous spectrum located for negative Ω could be constructed by using an obvious symmetry of Eq. (9) (see e.g., [20]), $\Omega \to -\Omega$, $U \to U$, and $W \to -W$. For convenience, we define the other continuous-wave eigenfunction $\mathcal{Y}^-$ by the following boundary conditions:

$$\mathcal{Y}^{-}(x,k) \to \begin{cases} \mathbf{e}_{-}\mathrm{e}^{-\mathrm{i}kx}, & x \to +\infty, \\ A^*(k)\mathbf{e}_{-}\mathrm{e}^{-\mathrm{i}kx} + B^*(k)\mathbf{e}_{-}\mathrm{e}^{\mathrm{i}kx}, & x \to -\infty. \end{cases} \tag{25}$$

The real parameter k specifies now the eigenvalue as $\Omega = -(\omega + k^2)$.

The continuous-wave eigenfunctions could be explicitly found for the integrable cubic NLS equation which follows from Eq. (1) for the normalized function $F(I) = 4I$. In this case, the soliton solutions are given by the asymptotic expression (6) while the continuous-wave eigenfunctions $\mathcal{Y}^{\pm}(x,k)$ have the form

$$\mathcal{Y}^{\pm}(x;k) = \frac{\mathrm{e}^{\pm \mathrm{i}kx}}{(k \pm \mathrm{i}\sqrt{\omega})^2}\{[k^2 - \omega \pm 2\mathrm{i}k\sqrt{\omega}\tanh(\sqrt{\omega}x)]\mathbf{e}_{\pm} + \omega\,\mathrm{sech}^2(\sqrt{\omega}x)(\mathbf{e}_{+} + \mathbf{e}_{-})\}. \tag{26}$$

For this particular case, we find from Eq. (26) that $B(k) = 0$ and

$$A(k) = \left(\frac{k - \mathrm{i}\sqrt{\omega}}{k + \mathrm{i}\sqrt{\omega}}\right)^2. \tag{27}$$

3.3. Discrete spectrum and neutral modes

The neutral modes of the discrete eigenvalue spectrum are localized solutions of Eq. (9) at $\Omega = 0$, and they are related to the infinitesimal spatial translation and gauge transformation of the soliton $\Phi(x;\omega)$. They are explicitly expressed through the function $\Phi(x;\omega)$ as

$$\mathcal{Y}_{d1}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\partial \Phi}{\partial x}(x;\omega), \qquad \mathcal{Y}_{d2}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Phi(x;\omega). \tag{28}$$

Besides these neutral modes, we need to introduce the associated discrete-spectrum modes $\mathcal{Y}_{aj}$, where $j = 1, 2$, which satisfy the inhomogeneous equations, $\mathcal{L}\mathcal{Y}_{aj} = \mathcal{Y}_{dj}$. These associated modes correspond to an infinitesimal change of the soliton velocity and propagation constant, and they can be found in the explicit form

$$\mathcal{Y}_{a1} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \frac{x}{2}\Phi(x;\omega), \qquad \mathcal{Y}_{a2} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \frac{\partial \Phi}{\partial \omega}(x;\omega). \tag{29}$$

3.4. Internal modes and soliton stability

For the cubic NLS equation, i.e., Eq. (1) with the nonlinearity $F(I) = 4I$, the system of linear eigenfunctions defined by Eqs. (26)–(29) is complete (see, e.g., [22]), so that other localized solutions to Eq. (9), different from the neutral modes defined in Section 3.3, are absent. However, for a more general (non-Kerr) nonlinearity, additional discrete-spectrum modes may appear for some nonzero values of the eigenvalue Ω, and these localized modes are associated with the internal dynamics of a solitary wave. In particular, the soliton dynamics may display the linear (exponential-type) instability if the eigenvalue Ω has a negative imaginary part; or the periodic long-term oscillations if Ω is real. We call the localized eigenfunctions $\mathcal{Y}_{\mathrm{in}}(x)$ corresponding to real values of Ω the *soliton internal modes*. It is clear that the internal modes may exist only inside the gap of the continuum spectrum, i.e., for $|\Omega| < \omega$.

According to the general analysis of the linear problem (9) and its generalizations [23,24], there exists at most one eigenvalue Ω with negative (or, alternatively, positive) imaginary part. This result follows from Theorem 5.8 of Grillakis et al. [23] and Theorem 3.1 of Pego and Weinstein [24] provided by the condition that the linear operator $\mathcal{L}_0$ (see Eq. (10)) has no negative eigenvalues while the operator $\mathcal{L}_1$ (see Eq. (11)) has strictly one such mode. The unique eigenvalue to the linear problem (9) with a negative imaginary part exists only inside the instability domain which is defined by the inequality

$$\frac{\mathrm{d}N_{\mathrm{s}}(\omega)}{\mathrm{d}\omega} < 0, \tag{30}$$

where

$$N_{\mathrm{s}}(\omega) = \frac{1}{2}\int_{-\infty}^{+\infty} \Phi^2(x;\omega)\mathrm{d}x \tag{31}$$

is the power invariant calculated for the solitary wave solution. In our previous paper [25] we have shown different regimes of the instability development of envelope solitons occurring within the instability domain (30). In the present paper we study general features of dynamics of stable bright solitons when system (9) does not admit any complex eigenvalues Ω. It has been shown (see e.g., [26]) that all possible eigenvalues of Eq. (9) are real inside the stability domain defined by the condition opposite to that in Eq. (30),

$$\frac{\mathrm{d}N_{\mathrm{s}}(\omega)}{\mathrm{d}\omega} > 0, \tag{32}$$

although neither a number of real eigenvalues nor the general features of the dynamics of solitary waves which possess internal modes can be predicted or studied by the methods of the previous papers. Below, we present the general approach for constructing the internal modes and also derive the analytical criterion for their existence.

4. Existence and properties of soliton internal modes

4.1. The Evans' function and internal modes

We consider the linear eigenvalue problem (9) for real Ω located in the gap of the continuous spectrum, $|\Omega| < \omega$. Because of the obvious symmetry we confine ourselves by the case $\Omega > 0$. The fundamental solutions $\mathcal{Y}_1$ and $\mathcal{Y}_2$ vanish exponentially as $x \to +\infty$ (see Eq. (13)) while two other solutions, $\mathcal{Y}_3$ and $\mathcal{Y}_4$, are exponentially diverging. Therefore, the only possible localized solution $\mathcal{Y}_{\mathrm{in}}$ to Eq. (9) can be superposed through the fundamental solutions as follows:

$$\mathcal{Y}_{\mathrm{in}}(x) = \alpha_1\mathcal{Y}_1(x;\omega,\Omega) + \alpha_2\mathcal{Y}_2(x;\omega,\Omega) \tag{33}$$

for certain constants α_1 and α_2. However, in the limit $x \to -\infty$ both the fundamental functions are generally diverging because the exponential terms are proportional to $\mathrm{e}^{\mu_1 x}$ and $\mathrm{e}^{\mu_2 x}$. Only if the following determinant of the scattering coefficients (15) and (16),

$$\mathcal{D}(\Omega) = \mathcal{D}_{11}(\Omega)\mathcal{D}_{22}(\Omega) - \mathcal{D}_{12}(\Omega)\mathcal{D}_{21}(\Omega), \tag{34}$$

vanishes for a certain value of Ω, the linear superposition (33) can be chosen *exponentially decaying at both the infinities.* Thus, the eigenvalues for the internal modes are determined by zeros of the function $\mathcal{D}(\Omega)$ which is called *Evans' function* (see [27] and references therein). As a matter of fact, similar arguments leading to the idea of using the function (34) have been recently used by Malkin and Shapiro [20] for analyzing the spectrum of linear soliton excitations in the two-dimensional NLS equation.

As can be easily shown, the Evan's function (34) is a continuous and real function for real values of Ω such that $|\Omega| < \omega$. As a result, the existence of internal modes inside the spectrum gap $|\Omega| < \omega$ could be studied by analyzing the asymptotic behavior of this function in the limits $\Omega \to 0$ and $\Omega \to \omega^-$.

In order to deal with the limit $\Omega \to 0$, it is useful to find an integral formula for the derivatives of $\mathcal{D}_{jk}(\Omega)$. We follow the general technique described by Pego and Weinstein [24,27] and obtain the following final result:

$$\mathcal{D}'_{jk}(\Omega) = \int_{-\infty}^{+\infty} \left\{ U_k^a W_j + W_k^a U_j + \frac{1}{2}\mathcal{D}_{jk}(\Omega)[\mu'_j(\Omega) + \mu'_k(\Omega)] \right\} \mathrm{d}x + \frac{1}{2}\mathcal{D}_{jk}(\Omega) \left\{ \frac{1}{\mu_j}\mu'_j(\Omega) - \frac{1}{\mu_k}\mu'_k(\Omega) \right\}, \quad j,k = 1,2, \tag{35}$$

where the primes stand for derivatives with respect to Ω.

Now we construct the eigenfunctions $\mathcal{Y}_1$ and $\mathcal{Y}_2$ at $\Omega = 0$ with the boundary conditions (13) by superposing the neutral discrete-spectrum modes (28):

$$\mathcal{Y}_1\Big|_{\Omega=0} = -\frac{1}{\chi\sqrt{\omega}}\mathcal{Y}_{d1} + \frac{1}{\chi}\mathcal{Y}_{d2}, \tag{36}$$

$$\mathcal{Y}_2\Big|_{\Omega=0} = -\frac{1}{\chi\sqrt{\omega}}\mathcal{Y}_{d1} - \frac{1}{\chi}\mathcal{Y}_{d2} \tag{37}$$

where the boundary condition (3) has been used. The adjoint functions $\mathcal{Z}_j$ are given by Eq. (18). Then from Eqs. (15) and (35) we find that $\mathcal{D}_{jk}(0) = \mathcal{D}'_{jk}(0) = 0$ for $j,k = 1,2$. Further, we define the derivatives $\partial\mathcal{Y}_j/\partial\Omega = 0$ by the same formulas (36) and (37) but with $\mathcal{Y}_{dj}$ replaced by $\mathcal{Y}_{aj}$ (see Eq. (29)). By differentiating formula (35) we finally obtain

$$\mathcal{D}''_{11}(0) = \mathcal{D}''_{22}(0) = \frac{1}{4\chi^2\omega^{3/2}}N_{\mathrm{s}}(\omega) + \frac{1}{2\chi^2\sqrt{\omega}}\frac{\mathrm{d}N_{\mathrm{s}}(\omega)}{\mathrm{d}\omega},$$

$$\mathcal{D}''_{12}(0) = \mathcal{D}''_{21}(0) = \frac{1}{4\chi^2\omega^{3/2}}N_{\mathrm{s}}(\omega) - \frac{1}{2\chi^2\sqrt{\omega}}\frac{\mathrm{d}N_{\mathrm{s}}(\omega)}{\mathrm{d}\omega},$$

where $N_{\mathrm{s}}(\omega)$ is defined by Eq. (31). As a result, the asymptotic behavior of the Evans' function (34) for $\Omega \to 0$ is defined by the following expresion:

$$\mathcal{D}(\Omega) = \frac{1}{24}\mathcal{D}^{IV}(0)\Omega^4 + \mathrm{O}(\Omega^6), \qquad \mathcal{D}^{IV}(0) = \frac{3}{\chi^4\omega^2}N_{\mathrm{s}}(\omega)\frac{\mathrm{d}N_{\mathrm{s}}(\omega)}{\mathrm{d}\omega}. \tag{38}$$

Thus, we come to the conclusion that inside the stability domain (32) the Evans' function $\mathcal{D}(\Omega)$ is always positive for small Ω.

Now we consider the asymptotic behavior of $\mathcal{D}(\Omega)$ as $\Omega \to \omega^-$. To do this, we first analyze the behavior of the coefficients $A(k)$ and $B(k)$ for the continuous-wave eigenfunction $\mathcal{Y}^+(x,k)$ (see Eq. (20)) as $k \to 0$. In this limit, the boundary conditions (20) transform as follows:

$$\mathcal{Y}^+(x,k)\Big|_{k\to 0} \to \begin{cases} \mathbf{e}_+, & x \to +\infty, \\ [A(k)+B(k)]\mathbf{e}_+ + \mathrm{i}k[A(k)-B(k)]x\mathbf{e}_+, & x \to -\infty. \end{cases} \tag{39}$$

In a generic case, the secular term (i.e., that growing linearly in x as $x \to -\infty$) is generated by the linear eigenvalue problem (9) at $\Omega = \omega$ according to the formula

$$\left.\frac{\partial U^+(x,0)}{\partial x}\right|_{x\to-\infty} + \left.\frac{\partial W^+(x,0)}{\partial x}\right|_{x\to-\infty} = 4b_{-1}, \tag{40}$$

where

$$b_{-1}=\frac{1}{4}\int_{-\infty}^{+\infty}\mathrm{d}x\{F(\Phi^2)[U^+(x,0)+W^+(x,0)]+2\Phi^2F'(\Phi^2)U^+(x,0)\}. \tag{41}$$

It follows from Eqs. (39) and (40) that if $b_{-1}\neq 0$ the coefficients $A(k)$ and $B(k)$ are diverging as $k\to 0$ and they can be found from the asymptotic formula

$$A(k)=-B(k)=-\frac{\mathrm{i}}{k}b_{-1}\quad \text{as } k\to 0. \tag{42}$$

Now, employing the same analysis to the function $\mathcal{D}(\Omega)$ in the limit $\Omega\to\omega^-$, i.e., inside the gap of the continuous-wave spectrum, we find the asymptotic formula (cf. Eqs. (21) and (42)),

$$\mathcal{D}(\Omega)\to-\frac{b_{-1}\mathcal{D}_{22}(\Omega)}{\sqrt{\omega-\Omega}}\quad \text{as } \Omega\to\omega^-. \tag{43}$$

We apply here the assumption that the coefficient $\mathcal{D}_{22}(\Omega)$ is positive for $\Omega\geq\omega$. In this case, the coefficient b_{-1} defines the sign of the function $D(\Omega)$ in the limit $\Omega\to\omega^-$, and hence determines the existence of an internal mode inside the gap of the continuous-wave spectrum. If $b_{-1}>0$, the Evans' function $\mathcal{D}(\Omega)$ changes its sign from positive for small Ω to negative for $\Omega\to\omega^-$. Therefore, there exists *at least one root* of the equation $\mathcal{D}(\Omega)=0$ in the internal $0<\Omega<\omega$ which is associated with the soliton internal mode. If $b_{-1}<0$, then the function $\mathcal{D}(\Omega)$ either has no roots (and associated soliton internal modes) being positive everywhere for $0<\Omega<\omega$ or even number of roots are expected. Thus, the singular behavior of the coefficients $A(k)$ and $B(k)$ at the edge of the continuous spectrum given by Eq. (42) determines *the sufficient condition* $b_{-1}>0$ for the existence of the internal mode of an envelope soliton.

In the special case, when $b_{-1}=0$, the coefficients $A(k)$ and $B(k)$ are not singular as $k\to 0$. Moreover, it can be shown from Eqs. (23), (24) and (39) that $B(0)=0$ and $A(0)=1$. In this case, the Evans' function is not singular as well and it has the asymptotic representation

$$\mathcal{D}(\Omega)\to\mathcal{D}_{22}(\Omega)\quad \text{as } \Omega\to\omega^-.$$

In particular, this special situation *appears always* for integrable models because soliton solutions are associated with the reflectionless potentials of the corresponding linear eigenvalue problems for which $B(k)$ vanishes identically for all k. For example, for the cubic NLS equation we can find from Eq. (27) that the Evans' function $\mathcal{D}(\Omega)$ tends to the following limit:

$$\mathcal{D}(\Omega)\to\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2\quad \text{as } \Omega\to\omega^-.$$

Therefore, the internal mode is absent for the soliton of the integrable cubic NLS equation, but it can be induced by a perturbation which would lead to a positive value of b_{-1}. We note that this positive b_{-1} might be generated by both a correction to the cubic NLS equation and by a perturbation of the solitary wave profile within the cubic NLS equation.

As a particular example, we study numerically the Evans' function (34) for the generalized NLS equation with the cubic–quintic nonlinearity (4). The results are presented in Figs. 2(a) and (b) for two possible signs of σ. For $\sigma=+1$ the Evans' function $\mathcal{D}(\Omega)$ tends to $-\infty$ as $\Omega\to\omega^-$ and there always exists only one zero of this function at $\Omega=\Omega_0(\omega)$ (see Fig. 2(a)) located inside the gap of the continuous-wave spectrum $|\Omega|<\omega$. The complete spectrum of the linear problem (9) corresponding to this case is shown schematically in Fig. 1. In the case $\sigma=-1$

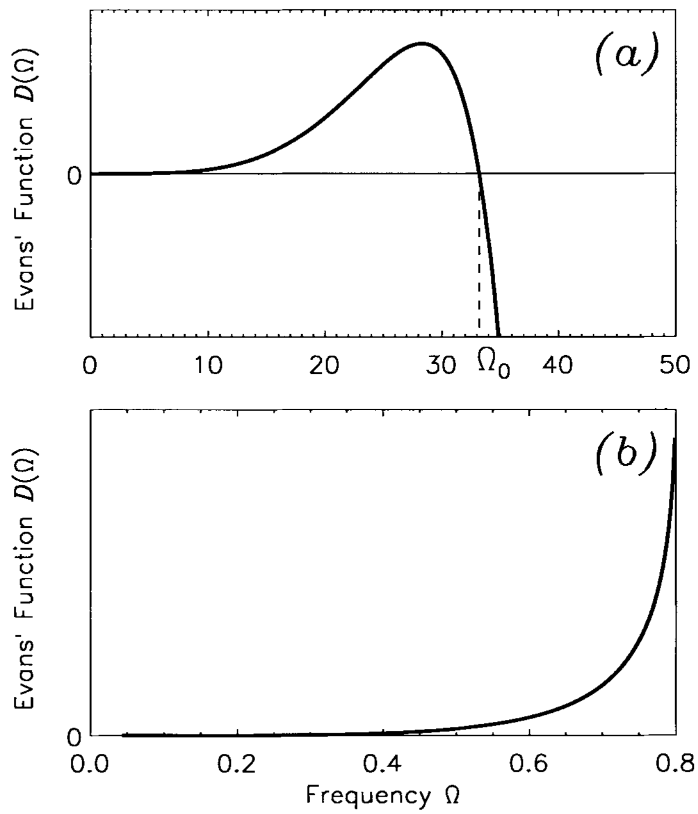

Fig. 2. The Evans' function $D(\Omega)$ calculated numerically for the soliton of the cubic–quintic NLS equation (1) and (4) in the case: (a) $\sigma = +1$ and (b) $\sigma = -1$. A single zero $D(\Omega_0) = 0$ in the case of (a) corresponds to the soliton internal mode.

the Evans' function tends to $+\infty$ as $\Omega \to \omega^-$ and, therefore, there are no zeros of this function within the gap of the continuous-wave spectrum (see Fig. 2(b)). The profiles $U_{\rm in}(x)$ and $W_{\rm in}(x)$ of the internal mode corresponding to the eigenvalue Ω_0 at $\sigma = +1$ have been calculated from Eq. (9) with the help of the numerical shooting method. We present these eigenfunctions, together with the soliton profile $\Phi(x;\omega)$, in Fig. 3 for $\omega = 5$. Thus, for both focusing cubic and quintic nonlinearity (4) a stable bright soliton of a finite amplitude *always has a unique internal mode* which determines the long-term oscillatory dynamics of the solitary wave. Dependence of the frequency Ω_0 of this mode vs. the soliton frequency (or propagation constant) ω has been obtained numerically and it is shown in Fig. 4(a). It is clear that the frequency of the internal mode is always inside the gap between the continuous- and discrete-spectrum modes, i.e., in the internal $0 < \Omega_0 < \omega$.

We mention that the existence of several internal modes of different spatial symmetries within the interval $0 < \Omega < \omega$ is not generally prohibited by the properties of the linear system (9) unlike that happen for complex Ω [23,24]. However, we confine our analysis only to the *fundamental internal* mode, i.e., that having the smallest value of Ω and the profile of the component $U_{\rm in}(x)$ with two symmetric nodes (see Fig. 3). For the considered example of the cubic–quintic nonlinearity, this fundamental internal mode turns out to be unique and it appears only due to rather universal bifurcations at the edges of the interval $0 < \Omega < \omega$. The bifurcation in the limit $\Omega \to 0$ can occur at the edge of the stability domain (32) when a pair of real eigenvalues merges and moves to the imaginary axis. On the other hand, a bifurcation in the limit $\Omega \to \omega^-$ can occur when the generalized NLS equation (1) reduces to the integrable cubic NLS equation with $B(k) = 0$. Both these bifurcations are analyzed below.

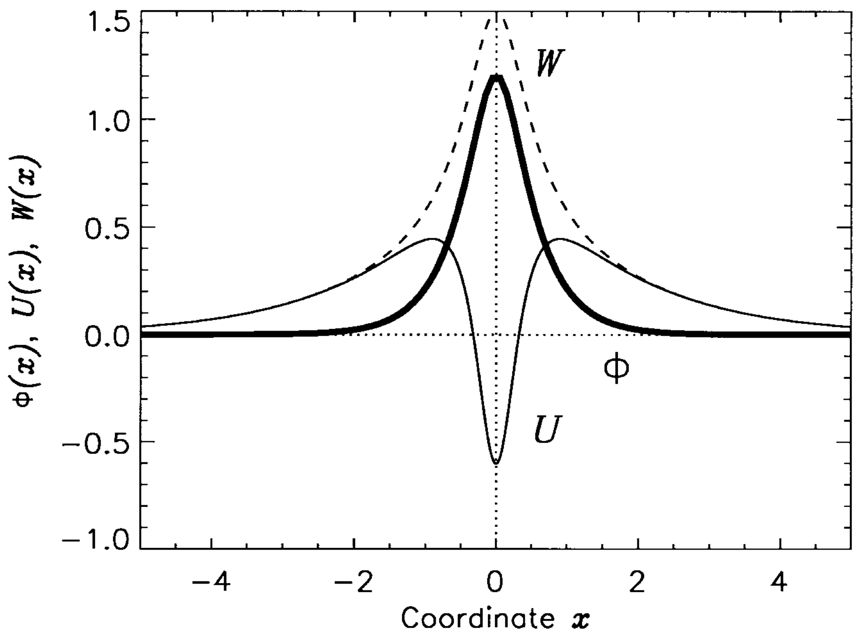

Fig. 3. Examples of the soliton profile $\Phi(x;\omega)$ (thick solid) and the functions $U_{\rm in}(x)$ (thin solid) and $W_{\rm in}(x)$ (dashed) describing a localized eigenmode of the discrete spectrum for the cubic–quintic NLS equation (1) and (4) at $\sigma = +1$ and $\omega = 5$.

4.2. *An eigenvalue emerging from the edge of the continuous spectrum*

In Section 4.1 we have shown that the bifurcation of the internal mode may take place in the vicinity of the integrable case, where $B(k) = 0$ (see the definition in Eq. (20)), and the Evans' function (34) is not diverging in the limit $\Omega \to \omega^-$ for this case. Here we analyze this bifurcation in a general form and consider the nonlinear function $F(I)$ in Eq. (1) of the form

$$F(I) = 4I + \epsilon f(I), \tag{44}$$

where ϵ is a small positive parameter which scales the amplitude of the perturbation of the cubic (Kerr) nonlinearity described by the function $f(I)$. The soliton profile $\Phi(x;\omega)$ can be found from Eqs. (2) and (44) as the perturbation expansion,

$$\Phi(x;\omega) = \Phi_0(x;\omega) + \epsilon\Phi_1(x;\omega) + \mathrm{O}(\epsilon^2), \tag{45}$$

where the NLS soliton $\Phi_0(x;\omega)$ is given by Eq. (6) while $\Phi_1(x;\omega)$ satisfies the inhomogeneous linear equation, $\mathcal{L}_1^0\Phi_1 = f(\Phi_0^2)\Phi_0$. We denote the linear operators (10) and (11) in the zero-order (NLS) approximation as $\mathcal{L}_0^0$ and $\mathcal{L}_1^0$ which are

$$\mathcal{L}_0^0 = -\frac{\partial^2}{\partial x^2} + \omega - 2\,\omega\,\mathrm{sech}^2(\sqrt{\omega}x), \qquad \mathcal{L}_1^0 = -\frac{\partial^2}{\partial x^2} + \omega - 6\,\omega\,\mathrm{sech}^2(\sqrt{\omega}x).$$

Substituting expansions (44) and (45) into Eq. (9) we find a linear perturbed eigenvalue problem of the first-order approximation,

$$(\mathcal{L}^0 + \epsilon\mathcal{L}^1)\mathcal{Y} = \Omega\mathcal{Y}, \tag{46}$$

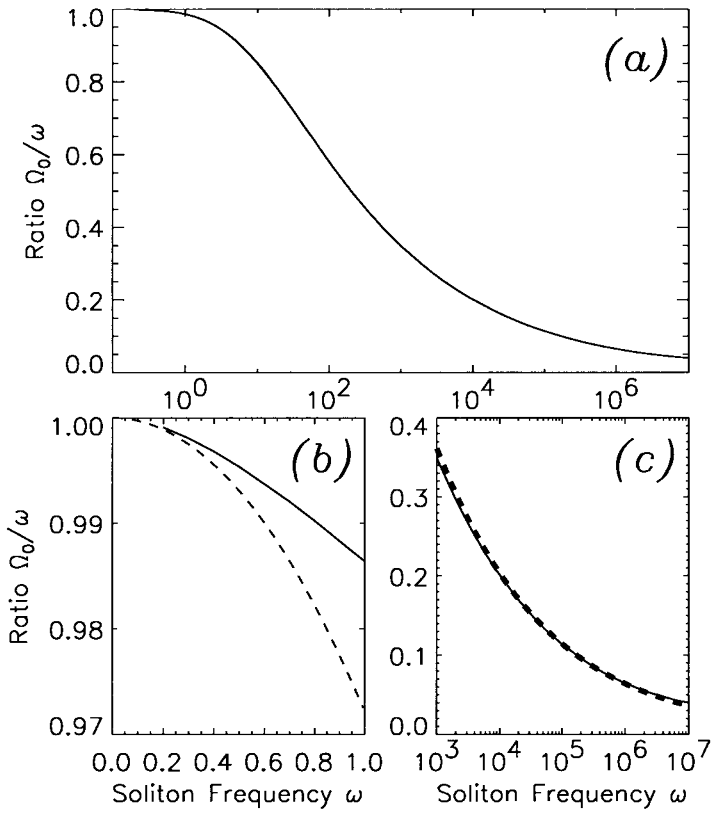

Fig. 4. (a)–(c) Dependence of the internal mode frequency Ω_0 on the soliton frequency ω for the solitary wave of the cubic–quintic NLS equation (1) and (4) at $\sigma = +1$. Solid curve shows the numerical result, dashed curves in (b) and (c) show the asymptotic results (58) and (68), respectively. Note different scales in (a)–(c).

where

$$\mathcal{L}^0 = \begin{pmatrix} 0 & \mathcal{L}_0^0 \\ \mathcal{L}_1^0 & 0 \end{pmatrix}, \qquad \mathcal{L}^1 = \begin{pmatrix} 0 & \mathcal{L}_0^1 \\ \mathcal{L}_1^1 & 0 \end{pmatrix},$$

and

$$\mathcal{L}_0^1 = -f(\Phi_0^2) - 8\Phi_0\Phi_1, \qquad \mathcal{L}_1^1 = -f(\Phi_0^2) - 2\Phi_0^2 f'(\Phi_0^2) - 24\Phi_0\Phi_1.$$

In the leading-order approximation ($\epsilon = 0$), a complete set of eigenfunctions consists of two branches of the continuous spectrum (26) and also the neutral and associated discrete-spectrum eigenmodes (28) and (29). Therefore, we seek a solution to the perturbed problem (46) by expanding the function $\mathcal{Y}$ through this complete set as follows:

$$\mathcal{Y}(x) = \int_{-\infty}^{+\infty} \mathrm{d}k[\alpha^+(k)\mathcal{Y}^+(x,k) + \alpha^-(k)\mathcal{Y}^-(x,k)] + \sum_{n=1,2} (\alpha_n \mathcal{Y}_{dn} + \beta_n \mathcal{Y}_{an}), \tag{47}$$

where $\alpha^{\pm}(k)$, α_n and β_n are coefficients of this expansion. It is clear that the discrete spectrum located at $\Omega = 0$ is not relevant to the problem of bifurcation at $\Omega = \omega$. Henceforth, we neglect all components of the discrete spectrum in the subsequent calculations. A dangerous role of the discrete-spectrum modes in a variational analysis of oscillations of bright solitons supported by the dynamics of the continuous-wave packets has been recently discussed by Kaup and Lakoba [21].

Following the standard analysis (see, e.g., [22]) we define an inner product for the continuous-wave eigenfunctions $\mathcal{Y}^{\pm}(x,k)$ according to the formula

$$\langle \mathcal{Y}(k'), \mathcal{Y}(k) \rangle = \frac{1}{2} \int_{-\infty}^{+\infty} \{U(-x,k')W(x,k) + W(-x,k')U(x,k)\} \mathrm{d}x. \tag{48}$$

Then we use the Wronskian relations (12) and the boundary conditions (20) and (25) for $B(k) = 0$ to find that the cross-product of $\mathcal{Y}^{\pm}(x,k)$ are zero, while the self-products are given by

$$\langle \mathcal{Y}^{+}(k'), \mathcal{Y}^{+}(k) \rangle = 2\pi A(k) \delta(k-k'), \tag{49}$$

$$\langle \mathcal{Y}^{-}(k'), \mathcal{Y}^{-}(k) \rangle = -2\pi A^{*}(k) \delta(k-k'), \tag{50}$$

where $A(k)$ is defined by Eq. (27). Using these formulas for the inner products, we substitute expansion (47) into Eq. (46) and reduce the linear problem to the system of linear integral equations

$$a^{\pm}(k) = \pm \frac{\epsilon}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}k' \left\{ \frac{K_{+}^{\pm}(k,k') a^{+}(k')}{A(k')(\Omega - \Omega_{k'})} + \frac{K_{-}^{\pm}(k,k') a^{-}(k')}{A^{*}(k')(\Omega + \Omega_{k'})} \right\}, \tag{51}$$

where $a^{+}(k) = A(k)(\Omega - \Omega_k)\alpha^{+}(k)$, $a^{-}(k) = A^{*}(k)(\Omega + \Omega_k)\alpha^{-}(k)$, $\Omega_k = (\omega + k^2)$, and

$$K_{\mp}^{\pm}(k,k') = \frac{1}{2} \int_{-\infty}^{+\infty} \{U^{\pm}(-x,k)\mathcal{L}_1^1 U^{\mp}(x,k') + W^{\pm}(-x,k)\mathcal{L}_0^1 W^{\mp}(x,k')\} \mathrm{d}x.$$

Now we assume that the perturbation leads to a bifurcation of the internal mode into the gap of the continuous spectrum. This assumption implies that the system of integral equations (51) exhibits bounded solutions for $a^{\pm}(k)$ at a certain eigenvalue $\Omega = \Omega_0 < \omega$. According to this, we introduce the parametrization, $\Omega = \omega - \epsilon^2\kappa^2$, and notice that the integrands in the first integrals of Eq. (51) have poles at $k = \pm \mathrm{i}\epsilon\kappa$ provided $\kappa > 0$ (ϵ is supposed to be positive). These poles lead to a singular behavior of the first integrals as $\epsilon \to 0$. In order to evaluate this singular behavior of the integrals we notice from Eq. (26) that the function $\mathcal{Y}^{+}(x,k)$ vanishes exponentially fast as $|k| \to \infty$ for $x > 0$ and $\mathrm{Im}(k) > 0$, or for $x < 0$ and $\mathrm{Im}(k) < 0$. Therefore, we close the integration contours in Eq. (51) through infinity $\mathrm{Im}(k) > 0$ for $x > 0$, or $\mathrm{Im}(k) < 0$ for $x < 0$. We suppose that the coefficient $a^{+}(k)$ is normalized as follows:

$$|a^{+}(k)| \to 1 \quad \text{as} \quad \mathrm{Im}(k) \neq 0, \quad |k| \to \infty. \tag{52}$$

Besides, we suppose that the coefficients $a^{\pm}(k)$ are not singular as $k \to 0$. Of course, these coefficients, as well as the kernel function $A^{-1}(k)\mathcal{Y}^{+}(x,k)$, might have some poles in the complex plane of k. but these poles do not give any singular contribution into the integrals of Eq. (51) as $\epsilon \to 0$. Furthermore, the second integral in Eq. (51) does not lead to a singularity for $\epsilon \to 0$ as well. Therefore, we neglect all these terms and obtain finally an explicit asymptotic solution of system (51) in the limit $\epsilon \to 0$:

$$a^{\pm}(k) = \mp \frac{1}{2\kappa} a^{+}(0) K_{+}^{\pm}(k,0). \tag{53}$$

This explicit solution is self-consistent provided the parameter κ is determined by the equation

$$\kappa = -\frac{1}{4} \int_{-\infty}^{+\infty} \mathrm{d}x \{U^+(-x, 0)\mathcal{L}_1^1 U^+(x, 0) + W^+(-x, 0)\mathcal{L}_0^1 W^+(x, 0)\}, \quad \kappa > 0. \tag{54}$$

Thus, the system of integral equations (51) does have a solution for the internal mode with the eigenvalue $\Omega = \Omega_0 = \omega - \epsilon^2\kappa^2$ if the parameter κ defined by Eq. (54) turns out to be positive. It can be shown from Eqs. (46), (41), and (54) that the following relation is asymptotically valid for $\epsilon \to 0$, $b_{-1} = \epsilon\kappa$. Therefore, the results of the bifurcation analysis agree with the criterion of the existence of the fundamental internal mode found in Section 4.1.

If the internal mode exists, i.e., $\kappa > 0$, then it follows from Eq. (47) for $\epsilon \to 0$ that the profile of the internal mode approaches at infinity to the following shape:

$$\mathcal{Y}_{\text{in}}(x) \to -\left(\frac{a^+(0)}{2\mathrm{i}\epsilon\kappa}\right)\mathcal{Y}^+(x, \mp\mathrm{i}\epsilon\kappa) \quad \text{as } x \to \pm\infty. \tag{55}$$

We see that the internal mode is exponentially localized in the limit $|x| \to \infty$, and it coincides with the profiles of the continuous-spectrum eigenfunctions continued analytically into the complex plane of k. The limiting behavior for $a^+(k)$ imposed by Eq. (52) provides this exponential localization for the internal mode, and therefore is self-consistent with the approach developed here.

The results described above can be easily applied to evaluate the asymptotic limit of small soliton amplitudes for the frequency of the internal mode supported by the cubic–quintic nonlinearity (4). For this case, the first-order correction $\Phi_1(x;\omega)$ has the form

$$\Phi_1(x;\omega) = -\sigma\frac{\sqrt{\omega^3}\cosh(2\sqrt{\omega}x)}{8\sqrt{2}\cosh^3(\sqrt{\omega}x)}. \tag{56}$$

As follows from Eq. (26), the limiting eigenfunction $\mathcal{Y}^+(x, 0)$ is given by

$$\mathcal{Y}^+(x, 0) = \mathbf{e}_+ - \operatorname{sech}^2(\sqrt{\omega}x)(\mathbf{e}_+ + \mathbf{e}_-). \tag{57}$$

Calculating Eq. (54) with Eqs. (56) and (57), we find that $\kappa = (\sigma/6)\omega^{3/2}$. Therefore, the discrete eigenvalue emerges from the continuous spectrum only for $\sigma = +1$, and the approximate analytical expression for the frequency of the internal mode $\Omega = \Omega_0(\omega)$ can be found in an explicit analytic form

$$\Omega_0 = \omega\left[1 - \frac{\omega^2}{36} + \mathrm{O}(\omega^4)\right] \quad \text{as } \omega \to 0. \tag{58}$$

This result is presented in Fig. 4(b) by a dashed line. The solid line shows the results of numerical simulations from Fig. 4(a). Thus, the asymptotic expression (58) is approximately valid for relatively small ω such that $\omega \leq 0.3$

4.3. *An eigenvalue emerging from the instability domain*

It follows directly from the linear system (9) that any set of localized solutions $\mathcal{Y}_n(x)$ for real $\Omega = \Omega_n$ satisfy the orthogonality conditions

$$(\Omega_n - \Omega_m)\int_{-\infty}^{+\infty}[U_nW_m + W_nU_m]\mathrm{d}x = 0 \quad \text{for } n \neq m. \tag{59}$$

Applying this general condition to the internal mode $\mathcal{Y}_n = \mathcal{Y}_{\rm in}$ for $\Omega_m \Omega_0$, and the neutral discrete-spectrum modes (28), $\mathcal{Y}_m = \mathcal{Y}_{dj}$ for $\Omega_m = 0$ and $j = 1, 2$, we transform the orthogonality conditions (59) to the form

$$D_1(\omega, \Omega_0) = \Omega_0 \int_{-\infty}^{+\infty} \frac{\partial \Phi(x;\omega)}{\partial x} W_{\rm in}(x;\omega,\Omega_0){\rm d}x = 0, \tag{60}$$

$$D_2(\omega, \Omega_0) = \Omega_0 \int_{-\infty}^{+\infty} \Phi(x;\omega) U_{\rm in}(x;\omega,\Omega_0){\rm d}x = 0. \tag{61}$$

The first condition, Eq. (60), is relevant if the function $W_{\rm in}$ is odd in x, whereas the second condition, Eq. (61), is applicable for the even function $U_{\rm in}$. Here, we are interested in the bifurcation of the internal mode from the neutral discrete-spectrum modes (28). It can be easily shown that this bifurcation is only possible for the neutral mode $\mathcal{Y}_{d2}$, and as a result, the function $\mathcal{Y}_{\rm in}$ is turned out to be even, see Eq. (28). To find an asymptotic representation for this function in the limit $\Omega_0 \to 0$, we follow the analysis of our previous paper [25] and introduce the asymptotic expression for solutions of Eq. (9),

$$\mathcal{Y}_{\rm in} = \mathcal{Y}_{d2}(x;\omega) + \Omega_0 \mathcal{Y}_{a2}(x;\omega) + \sum_{n=2}^{\infty} \Omega_0^n \mathcal{Y}_n(x;\omega). \tag{62}$$

Here $\mathcal{Y}_{d2}$ and $\mathcal{Y}_{a2}$ are given by Eqs. (28) and (29) while the higher-order corrections $\mathcal{Y}_n$ have to be found by inverting the linear inhomogeneous equations,

$$\mathcal{L}\mathcal{Y}_n = \mathcal{Y}_{n-1}, \quad n \geq 2, \tag{63}$$

and taking into account the orthogonality (solvability) condition (61). For instance, the second-order correction is

$$\mathcal{Y}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} W_2(x;\omega),$$

where $W_2(x;\omega)$ satisfies the equation $\mathcal{L}_0 W_2 = -\partial\Phi/\partial\omega$, and the solvability condition (61) leads to the relation ${\rm d}N_{\rm s}(\omega)/{\rm d}\omega = 0$, i.e., to the threshold between the instability (30) and stability (32) domains. Let us define the marginal value for this threshold as $\omega = \omega_{\rm c}$, i.e., ${\rm d}N_{\rm s}(\omega)/{\rm d}\omega|_{\omega=\omega_{\rm c}} = 0$, and extend the solvability condition (61) to the fourth-order approximation valid for $(\omega - \omega_{\rm c}) \approx {\rm O}(\epsilon^2)$. Then the detailed analysis (see [25]) reveals that the roots of $\mathcal{D}_2(\omega, \Omega_0)$ can be approximated as follows:

$$\mathcal{D}_2(\omega, \Omega_0) = -\Omega_0^2 \frac{{\rm d}N_{\rm s}(\omega)}{{\rm d}\omega} + \Omega_0^4 M_{\rm s}(\omega_{\rm c}) + {\rm O}(\Omega_0^6) = 0, \tag{64}$$

where the positive coefficient $M_{\rm s}(\omega_{\rm c})$ is defined by the expression

$$M_{\rm s}(\omega_{\rm c}) = \int_{-\infty}^{+\infty} \left(\frac{1}{\Phi(x;\omega)} \int_0^x \Phi(x';\omega) \frac{\partial \Phi(x';\omega)}{\partial \omega} {\rm d}x' \right)^2 \Bigg|_{\omega=\omega_{\rm c}} {\rm d}x. \tag{65}$$

It follows from this equation that within the stability domain (32), there always exists an internal mode near the marginal stability threshold, and its frequency $\Omega_0(\omega)$ is given by

$$\Omega_0(\omega) = \left\{ \frac{1}{M_{\rm s}(\omega_{\rm c})} \frac{{\rm d}N_{\rm s}(\omega)}{{\rm d}\omega} \right\}^{1/2}. \tag{66}$$

Thus, a transition from instability for solitary waves always generates an internal discrete-spectrum mode to the linear problem (9) which is described by the even eigenfunction $\mathcal{Y}_{\rm in}$ (see Eqs. (28), (29) and (62)). The relation between the stability bifurcation and existence of oscillatory solutions to the linear problem has been recently pointed out by Malkin and Shapiro [20] who have carried out the similar bifurcation analysis for the two-dimensional NLS equation.

Let us apply this analysis to the particular case of the cubic–quintic nonlinearity (4) for $\sigma = +1$. In this case, the soliton power $N_{\rm s}(\omega)$ can be found with the help of Eqs. (5) and (31)

$$N_{\rm s}(\omega) = \tfrac{1}{2}\tan^{-1}(\sqrt{\omega}). \tag{67}$$

The derivative $N_{\rm s}'(\omega)$ vanishes in the limit $\omega \to \infty$ when the bright soliton approaches the profile (7) of the soliton solutions of the critical (quintic) NLS equation. The coefficient $M_{\rm s}(\omega_{\rm c})$ (see Eq. (65)) in this limit can be calculated as $M_{\rm s}(\omega_{\rm c}) = \pi^3/(512\omega^3)$ so that the eigenvalue $\Omega_0(\omega)$ has the following asymptotic representation:

$$\frac{\Omega_0}{\omega} = \frac{8\sqrt{2}}{\pi^{3/2}\omega^{1/4}} + {\rm O}(\omega^{-3/4}) \quad \text{as } \omega \to \infty, \tag{68}$$

which is shown in Fig. 4(c) as a dashed curve compared with the numerical result (solid curve) from Fig. 4(a).

5. Radiative damping of the soliton oscillations

Being excited, the internal mode oscillation of a finite amplitude generates higher-order harmonics with the frequencies multiple integer to the frequency Ω_0. Thus, even if the frequency Ω_0 lies in the gap of the continuous spectrum, as shown in Fig. 1, the multiple frequencies might fall within the continious-spectrum band inducing radiation propagating away from the soliton. This escaping radiation should induce a damping mechanism by which the amplitude oscillation decays. In this section we evaluate the rate of this radiation-induced damping for different cases.

5.1. Radiation due to generation of a double frequency

First, we consider a simple case when the oscillating internal mode generates linear wave with the double frequency $2\Omega_0$. This situation can be realized provided $\omega/2 < \Omega_0 < \omega$. To describe nonlinear effects leading to the radiative damping, we consider a standard multi-scale asymptotic expansion which assumes that the oscillation amplitude is a small parameter. Then we introduce a nonlinear generalization of expansion (8) given by the asymptotic series

$$\Psi = \{\Phi + \epsilon[a(U_{\rm in} - W_{\rm in}){\rm e}^{{\rm i}\Omega_0 t} + a^*(U_{\rm in}^* + W_{\rm in}^*){\rm e}^{-{\rm i}\Omega_0 t}] + \epsilon^2\Phi_2 + \epsilon^3\Phi_3 + {\rm O}(\epsilon^4)\}{\rm e}^{{\rm i}\omega t}, \tag{69}$$

where $U_{\rm in}$ and $W_{\rm in}$ are components of the internal mode, $a = a(T)$ is the oscillation amplitude, $T = \epsilon^2 t$ is slow time of the amplitude evolution induced by the nonlinear effects and ϵ is an effective small parameter. Substituting the asymptotic expansion (69) into Eq. (1), we derive a system of equations for Φ_2, Φ_3 and higher-order corrections. Thus, within the second-order approximation we present the formal solution as follows:

$$\Phi_2 = |a|^2 U_0 + [a^2(U_2 - W_2){\rm e}^{2{\rm i}\Omega_0 t} + a^{*2}(U_2^* + W_2^*){\rm e}^{-2{\rm i}\Omega_0 t}], \tag{70}$$

where the functions U_0, U_2, and W_2 are to be found from the linear inhomogeneous equations

$$\mathcal{L}_1 U_0 = 2[3\Phi F'(\Phi^2) + 2\Phi^3 F''(\Phi^2)]U_{\rm in}^2 + 2\Phi F'(\Phi^2) W_{\rm in}^2, \tag{71}$$

$$\mathcal{L}_1 U_2 - 2\Omega_0 W_2 = [3\Phi F'(\Phi^2) + 2\Phi^3 F''(\Phi^2)]U_{\rm in}^2 - \Phi F'(\Phi^2) W_{\rm in}^2, \tag{72}$$

$$\mathcal{L}_0 W_2 - 2\Omega_0 U_2 = 2\Phi F'(\Phi^2) U_{\rm in} W_{\rm in}. \tag{73}$$

Since the eigenfunctions of the homogeneous linear system following from (72) and (73) are nonlocalized, the right-hand side of this system generates a solution which is not spatially localized as well. From the physical motivation it is clear that the radiation escaping the perturbed soliton has the form of a wave propagating to the right ($x \to +\infty$) and a wave propagating to the left ($x \to -\infty$). Thus, we can introduce the amplitudes $a_2^{\pm}$ of the generated waves according to the following boundary conditions:

$$(U_2, W_2) \to a_2^{\pm} \exp[\mp {\rm i}\sqrt{2\Omega_0 - \omega}\, x] \quad \text{as } x \to \pm\infty. \tag{74}$$

The amplitudes of the radiation field can be found from system (72) and (73) as follows:

$$a_2^+ = -B^*(k) a_2^- - \frac{1}{4{\rm i}k} \int_{-\infty}^{+\infty} {\rm d}x \{\Phi F'(\Phi^2)[3U_{\rm in}^2 U^{+*}(x,k) + 2U_{\rm in} W_{\rm in} W^{+*}(x,k) - W_{\rm in}^2 U^{+*}(x,k)] + 2\Phi^3 F''(\Phi^2) U_{\rm in}^2 U^{+*}(x,k)\}, \tag{75}$$

$$a_2^- = -\frac{1}{4{\rm i}k A(k)} \int_{-\infty}^{+\infty} {\rm d}x \{\Phi F'(\Phi^2)[3U_{\rm in}^2 U^{+}(x,k) + 2U_{\rm in} W_{\rm in} W^{+}(x,k) - W_{\rm in}^2 U^{+}(x,k)] + 2\Phi^3 F''(\Phi^2) U_{\rm in}^2 U^{+}(x,k)\}, \tag{76}$$

where $k = -\sqrt{2\Omega_0 - \omega}$ while $A(k)$, $U^+(x,k)$ and $W^+(x,k)$ are the spectral data and the continuous-spectrum functions defined by Eq. (20).

Next, we analyze the third-order approximation Φ_3 and remove an exponentially divergent term at the fundamental internal mode frequency Ω_0. This procedure leads to the following equation for the slowly varying amplitude $a = a(T)$:

$$2{\rm i}\alpha \frac{{\rm d}a}{{\rm d}T} + (\beta + {\rm i}\gamma)|a|^2 a = 0, \tag{77}$$

where the coefficients α, β, and γ are defined by the following expressions:

$$\alpha = \int_{-\infty}^{+\infty} U_{\rm in} W_{\rm in}\, {\rm d}x = \frac{1}{\Omega_0} \int_{-\infty}^{+\infty} W_{\rm in} \mathcal{L}_0 W_{\rm in}\, {\rm d}x > 0, \tag{78}$$

$$\gamma = 4\sqrt{2\Omega_0 - \omega}(|a_2^+|^2 + |a_2^-|^2) > 0, \tag{79}$$

and

$$\beta = -\int_{-\infty}^{+\infty} \mathrm{d}x\{2\Phi F'(\Phi^2)[(3U_{\mathrm{in}}^2 + W_{\mathrm{in}}^2)U_0 + (3U_{\mathrm{in}}^2 - W_{\mathrm{in}}^2)\mathrm{Re}\, U_2 + 2U_{\mathrm{in}}W_{\mathrm{in}}\mathrm{Re}\, W_2]$$

$$+ F'(\Phi^2)[3U_{\mathrm{in}}^4 + 2U_{\mathrm{in}}^2 W_{in}^2 + 3W_{\mathrm{in}}^4] + 4\Phi^3 F''(\Phi^2)U_{\mathrm{in}}^2(U_0 + \mathrm{Re}U_2)$$

$$+ 4\Phi^2 F''(\Phi^2)(3U_{\mathrm{in}}^4 + U_{\mathrm{in}}^2 W_{\mathrm{in}}^2) + 4\Phi^4 F'''(\Phi^2)U_{\mathrm{in}}^4\}. \tag{80}$$

The coefficient α given by (78) is the norm for the internal mode $\mathcal{Y}_{\mathrm{in}}(x)$ of the linear problem (9), as it follows from the orthogonality conditions (59) at $\Omega_n = \Omega_m$ and $\mathcal{Y}_n(x) = \mathcal{Y}_m(x)$. This norm is positive because the operator $\mathcal{L}_0$ is positive definite for all W different from Φ. For example, in the limit $\Omega_0 \to 0$, the norm can be calculated from Eqs. (62) and (64) to be $\alpha = \Omega_0^3 M_{\mathrm{s}}(\omega_{\mathrm{cr}}) + \mathrm{O}(\Omega_0^5)$.

The coefficient β in Eq. (77) defined by Eq. (80) determines a nonlinearity-induced correction to the frequency of the internal mode and this correction is typically *negative*. The *positive* coefficient γ, defined by Eq. (79), describes the dissipative effects induced by the generation of the wave packets with the double frequency. As a result of the dissipative effects, the amplitude of the internal mode decays according to the analytical solution of Eq. (77) written for $Q = |a|^2$ as follows:

$$Q = \frac{\alpha Q_0}{\alpha + \gamma Q_0 T}, \tag{81}$$

where $Q_0 = Q(0)$. Thus, the generation of the linear waves at the double frequency $2\Omega_0$ by the oscillating internal mode leads to the inverse linear decay (81) of the mode energy Q. This result has been confirmed by numerical simulations. Fig. 5 gives an example of the long-term evolution (a) and its view of much shorter timescales (b) of the solitary wave of the cubic–quintic NLS equation (1) and (4) with an initially excited internal mode. The law of the oscillation decay observed is given, with a good accuracy, by the analytical result (81).

As was observed in numerical simulations (see, e.g., [28–30]), the evolution of a perturbed soliton is accompanied by intermediate oscillations even in the case of the cubic NLS equation where the internal mode does not exist. This means that these intermediate oscillations are induced by wave packets of the continuous spectrum which are decaying because of the linear (dispersion) properties. Indeed, it was found (see Eqs. (26) and (55) in [29]) that the dominating frequency of the oscillations varies during the time evolution, approaching the frequency $\Omega = \omega$ of the edge of the continuous spectrum being effectively in the spectrum gap, $\Omega = \omega - 2|\alpha(0)|^2/T$, where $\alpha(k)$ is expressed through the scattering date. In that sense, the existence of the intermediate relaxation oscillation of a scaled soliton in the integrable cubic NLS equation can be explained by using the concept of a virtual quasi-mode existing for a finite time interval inside the spectrum gap due to the nonlinearity-induced frequency shift with the amplitude decaying as $T^{-1/2}$. We notice that precisely the same decay rate for this virtual quasi-mode follows from our approach as given by Eq. (81). Thus, the inverse linear decay law is valid for both the radiative loses induced by the generation of the waves at the double frequency and by linear dispersive effects.

5.2. Radiation due to generation of higher harmonics

Here we generalize the analysis presented above to evaluate the decay rate induced by the generation of the n-multiple-frequency harmonics of the continuous spectrum. Because the direct asymptotic technique becomes very cumbersome in higher orders of the multiscale expansions, we apply here an equivalent method based on the balance

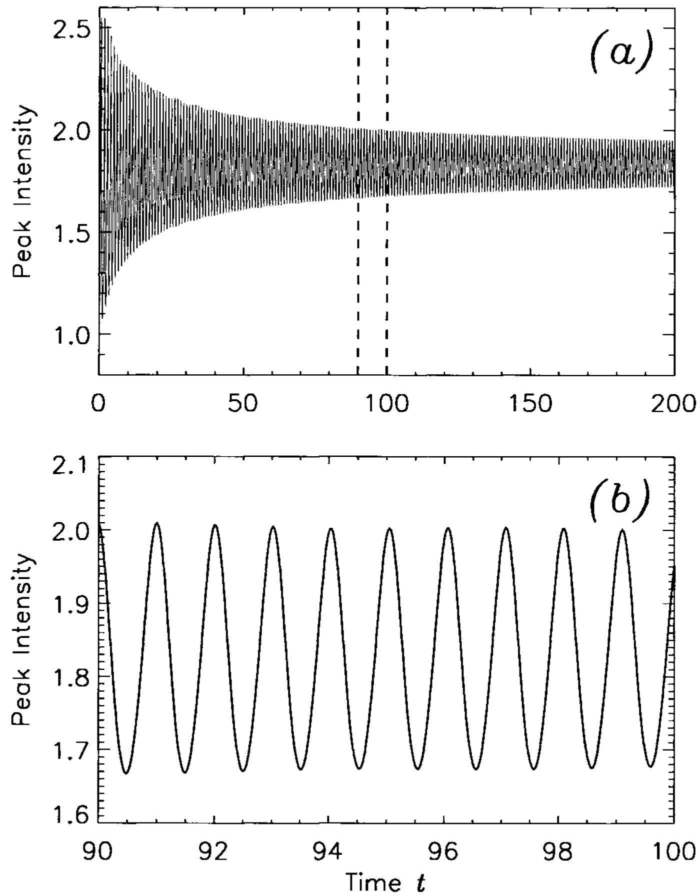

Fig. 5. (a) Example of the power law decay of the soliton amplitude oscillations shown for the soliton of the cubic–quintic NLS equation (1) and (4) at $\sigma = +1$ and $\omega = 5$. (b) Part of (a) shown in a different scale.

equations (see, e.g., [31] [2]). This method allows us to take into account only the nonlinear dissipative effects which seem to be more important for describing the long-term evolution of the soliton internal mode while neglecting the effects caused by the nonlinearity-induced shift of the mode frequency.

The balance equation for the power N can be written as follows:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{i}}{2}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)\Bigg|_{x=-\infty}^{x=+\infty}, \tag{82}$$

where

$$N = \frac{1}{2}\int_{-\infty}^{+\infty} |\Psi|^2\,\mathrm{d}x.$$

Using the asymptotic expansion (69) and also the properties of the linear system (see Eq. (61)), we find the leading-order of the expansion of Nin the form

[2] The same approach was employed in the paper by Buryak and Akhmediev [32], where more details and comparison with numerical simulations can be found.

$$N = N_s(\omega) + \epsilon^2 \alpha |a|^2 + O(\epsilon^3), \tag{83}$$

where we have used the relation

$$\int_{-\infty}^{+\infty} (U_{\text{in}}^2 + W_{\text{in}}^2 + \Phi U_0)\text{d}x = \alpha,$$

which can be proven by comparing the results that follow from the power balance equaion (82) and from the Hamiltonian balance equation, see [25].

Next, we suppose that the lowest frequency in the admissible interval for the wave generation is the nth multiple frequency of Ω_0. It implies that the frequency Ω_0 belongs to the nth zone in the gap of the continuous spectrum,

$$\frac{\omega}{n} < \Omega_0 \le \frac{\omega}{n-1}. \tag{84}$$

In this case, the radiative nonlocalized component, that appears in the order of $O(\epsilon^n a^n)$ of the asymptotic expansion, has the asymptotics given by Eq. (74) with $2\Omega_0$ replaced by $n\Omega_0$ and $a_2^{\pm}$ by $a_n^{\pm}$. The balance between the nonlinear dissipative and evolution terms in Eq. (82) leads to the evolution of the energy $Q = |a|^2$ of the internal mode oscillations with the slow time $T_n = \epsilon^{2n-2} t$ according to the equation

$$\frac{\text{d}Q}{\text{d}T_n} = -\frac{\gamma_n}{\alpha} Q^n, \tag{85}$$

where $\gamma_n = 4\sqrt{n\Omega_0 - \omega}(|a_n^+|^2 + |a_n^-|^2)$ (cf. Eq. (79)). Finally, we evaluate the decay rate of the internal oscillation induced by the generation of the nth multiple frequency wave packets

$$Q = \frac{Q_0}{[1 + (n-1)\alpha^{-1}\gamma_n Q_0^{n-1} T_n]^{1/(n-1)}}. \tag{86}$$

It should be noted that, in the asymptotic limit $\Omega_0 \ll \omega$, the analytical theory leading to Eq. (86) becomes invalid, because the generated radiation is beyond the asymptotic expansions in powers of the small parameter ϵ. In this case, the radiation is exponentially small in ϵ and this should modify the right-hand side of Eq. (86). A special asymptotic technique should be applied to find a correct analytical law of the oscillation decay, and the corresponding details will be published elsewhere.

6. Conclusions

Taking a rather universal model for envelope solitons described by the generalized NLS equation, we have analyzed the existence and properties of internal modes of solitary waves which may appear when the nonlinear equation for the wave envelope deviates from the exactly integrable cubic NLS equation. The internal mode of a solitary wave manifests itself through long-lived periodic oscillation of the soliton amplitude which persists for many periods. We have shown that there exists *no threshold* for the internal mode to emerge from the edge of the continuous spectrum, so that any small pertubation of the cubic NLS equation (with an appropriate sign) can generate a soliton internal mode, and as a result, will modify qualitatively the soliton dynamics. We have calculated the rate of a weak, radiation-induced relaxation damping of the initially excited amplitude oscillation of a solitary wave associated with the existence of the internal mode. We have also pointed out that the existence of the soliton internal modes in nonintegrable models can be naturally linked to the problem of stability of solitary waves.

The approach developed in this paper and the results obtained for the NLS equation with a general nonlinearity are rather universal to find their applications in other nonlinear problems of different physical context. We believe that our study will stimulate the further analysis of the soliton internal modes restricted up to now by a number of kink-bearing models.

Acknowledgements

Yuri Kivshar acknowledges useful discussions of the oscillating envelope solitons with D.K. Campbell, M. Peyrard, A.W. Snyder and V. Zakharov.

References

[1] A.C. Newell, Solitons in Mathematics and Physics, SIAM, Philadelphia, PA, 1985.
[2] Yu.S. Kivshar, B.A. Malomed, Rev. Modern Phys. 63 (1989) 761.
[3] D.K. Campbell, J.F. Schonfeld, C.A. Wingate, Physica D 9 (1983) 1.
[4] O.M. Braun, Yu.S. Kivshar, Nonlinear dynamics of the Frenkel–Kontorova model, Phys. Rep., in press.
[5] M. Peyrard, D.K. Campbell, Physica D 9 (1983) 33; D.K. Campbell, M. Peyrard, P. Sodano, Physica D 19 (1986) 165; M. Peyrard, M. Remoissenet, Phys. Rev. B 26 (1992) 2886.
[6] Yu.S. Kivshar, F. Zhang, L. Vázquez, Phys. Rev. Lett. 67 (1991) 1177; F. Zhang, Yu. S. Kivshar, L. Vázquez, Phys. Rev. A 45 (1992) 6019; 46 (1992) 5214.
[7] A.C. Newell, J.V. Moloney, Nonlinear Optics, Chapter 2, Addison-Wesley, Redwood City, 1992.
[8] A. Hasegawa, Y. Kodama, Solitons in Optical Communications Chapter 3, Oxford University Press, Oxford, 1995.
[9] G.C. Duree, J.L. Shultz, G.J. Salamo, M. Segev, A. Yariv, B. Crosignani, P. DiPorto, E.J. Sharp, R.R. Neurgaonkar, Phys. Rev. Lett. 71 (1993) 533; M. Segev, G.C. Valley, B. Crosignani, P. DiPorto, A. Yariv, Phys. Rev. Lett. 73 (1994) 3211; M. Shih, P. Leach, M. Segev, M.H. Garrett, G. Salamo, G.C. Valley, Opt. Lett. 21 (1996) 324.
[10] V. Tikhonenko, J. Christou, B. Luther-Davies, J. Opt. Soc. Am. B 12 (1995) 2046; Phys. Rev. Lett. 76 (1996) 2698.
[11] Yu. N. Karamzin, A.P. Sukhorukov, Zh. Eksp. Teor. Fiz. 68 (1975) 834 [Sov. Phys. JETP 41 (1976) 414]; A.V. Buryak, Yu.S. Kivshar, Opt. Lett. 19 (1994) 1612.
[12] W.E. Torruellas, Z. Wang, D.J. Hagan, E.W. VanStryland, G.I. Stegeman, L. Torner, C.R. Menyuk, Phys. Rev. Lett. 74 (1995) 5036.
[13] D.E. Pelinovsky, A.V. Buryak, Yu.S. Kivshar, Phys. Rev. Lett. 75 (1995) 591.
[14] C. Etrich, U. Peschel, F. Lederer, B.A. Malomed, Yu.S. Kivshar, Phys. Rev. E 54 (1996) 4321.
[15] V.E. Zakharov, V.V. Sobolev, V.C. Synakh, Sov. Phys. JETP 33 (1971) 77; A.W. Snyder, S. Hewlett, D.J. Mitchell, Phys. Rev. E 51 (1995) 6297.
[16] A.J. Sievers, J.B. Page, in: G.K. Horton, A.A. Maradudin (Eds.), Dynamical Properties of Solids, vol. 7, North-Holland, Amsterdam, 1994 p. 137.
[17] R.S. MacKay, S. Aubry, Nonlinearity 7 (1994) 1623.
[18] Yu.S. Kivshar, Phys. Lett. A 173 (1993) 172; Phys. Rev. E 48 (1993) 4132.
[19] V.K. Mezentsev, S.L. Musher, I.V. Ryzhenkova, S.K. Turitsyn, JETP Lett. 60 (1994) 829; E.W. Laedke, K.H. Spatschek, S.K. Turitsyn, V.K. Mezentsev, Phys. Rev. E 52 (1995) 5549; R. Dusi, G. Viliani, N. Wagner, Phys. Rev. B 54 (1996) 9809.
[20] V.M. Malkin, E.G. Shapiro, Physica D 53 (1991) 25.
[21] D.J. Kaup, T.I. Lakoba, J. Math. Phys. 37 (1996) 3442.
[22] D.J. Kaup, J. Math. Anal. Appl. 54 (1976) 849; D.J. Kaup, Phys. Rev. A 42 (1990) 5689.
[23] M. Grillakis, J. Shatah, W. Strauss, J. Funct. Anal. 94 (1990) 308.
[24] R.L. Pego, M.I. Weinstein, Phil. Trans. Roy. Soc. London. A 340 (1992) 47.
[25] D.E. Pelinovsky, V.V. Afanasjev, Yu.S. Kivshar, Phys. Rev. E 53 (1996) 1940.
[26] J. Shatah, W. Strauss, Commun. Math. Phys. 100 (1985) 173.
[27] R.L. Pego, M.I. Weinstein, in: W.F. Ames, E.M. Harrell II, J.V. Herod (Eds.), Differential Equations with Applications to Mathematical Physics, Academic Press, San Diego, 1993.
[28] J. Satsuma, N. Yajima, Progr. Theoret. Phys. Suppl. 55 (1974) 284.
[29] E.A. Kuznetsov, A.V. Mikhailov, I.A. Shimokhin, Physica D 87 (1995) 201.
[30] J.P. Gordon, J. Opt. Soc. Am. B 9 (1992) 91; W.L. Kath, N.F. Smyth, Phys. Rev. E 51 (1995) 1484.
[31] B.A. Malomed, Phys. Lett. A 154 (1991) 441.
[32] A.V. Buryak, N.N. Akhmediev, Phys. Rev. E 50 (1994) 3126.