Instability analysis of internal solitary waves in a nearly uniformly stratified fluid

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(Received 27 January 1997; accepted 11 June 1997)

Long finite-amplitude internal solitary waves propagating in a stratified fluid with nearly uniform stratification are considered within an asymptotic approximation leading to a nonlocal evolution equation of the Korteweg–de Vries (KdV) type. Analytical properties of this equation and its solitary wave solutions are studied and a criterion for solitary wave instability is derived. This criterion coincides with that for solitary waves in a local generalized KdV equation. Applications of these results reveal that strengthening of the stratification might lead to destabilization of smooth solitary waves and their blow-up into vortex-type wave structures. [S1070-6631(97)02210-1]

I. INTRODUCTION

Internal waves in the ocean and atmosphere have been under intense experimental and theoretical studies for the last 30 years or even more (see, for instance, the reviews by Ostrovsky and Stepanyants[1] and Grimshaw[2]). It has been well understood that the density stratification can support not only the dynamics of linear dispersive wave packets but also the propagation of relatively long, but localized disturbances, called solitary waves. These waves are typically single-humped isolated waves of elevation or depression and they can propagate for long distances without distortion of their shapes. Internal solitary waves commonly occur in coastal seas, fjords and lakes,[3] as well as in the atmospheric boundary layer.[4]

A simple but very effective method for the analytical description of these solitary waves is based on an asymptotic multi-scale expansion technique which uses the assumptions of a long wavelength and a small wave amplitude (see, e.g., Ref. 2). This method allows one to reduce a full set of hydrodynamics equations describing the motion of a stratified fluid to one of a set of basic evolution equations such as the Korteweg–de Vries (KdV) equation, the Benjamin–Ono (BO) equation, or the intermediate long-wave equation, depending on the type of dispersion occurring in the wave environment.[2] These evolution equations have been found to predict reasonably well the basic properties of the waves observed experimentally such as the dependence of the solitary wave speed and width versus the amplitude, stability properties, etc. Moreover, extensions of these evolution equations to the second order have also been done[5,6] to analyze higher-order nonlinear corrections to the wave characteristics due to an increase in the solitary wave amplitudes.

These evolution equations take into account a certain balance of weakly nonlinear and dispersive effects and are generally valid in a wide interval of the parameters of the given environment, but can fail in a number of so-called critical cases. These critical cases occur when the coefficients of the governing evolution equations, which are to be evaluated through the given stratification profiles, vanish or diverge. For instance, the coefficient of the usual quadratic nonlinearity in the KdV equation describing long internal solitary waves vanishes for a symmetric stratification and, therefore, the KdV equation is not applicable in this critical case. However, in order to find a nontrivial evolution equation in this case it is usually sufficient to extend the nonlinearity expansion to a higher order and to add a cubic nonlinear term to the small quadratic approximation. This procedure thus leads to a modified KdV equation which displays basically the same properties of the wave motion but modifies some quantative characteristics of solitary waves.[6]

In this paper we study a more drastic modification of the underlying evolution equation which occurs in a nearly uniformly stratified fluid within the well-known Boussinesq limit.[7] In this very special but commonly occurring situation, the leading-order term in the asymptotic multi-scale expansion is an exact solution of the Euler equations for an arbitrary amplitude which is not scaled.[8,9] Therefore, the usual nonlinearity in the evolution equation vanishes identically within this limit and one needs to keep the wave amplitude of arbitrary magnitude in order to study the dynamics of long internal waves in nearly uniformly stratified fluid. Thus, a regularized asymptotic multi-scale expansion should be based within this critical limit using only a small parameter related to the long wavelength. This regularization was done by Grimshaw and Yi[10] where they derived a nonlocal generalization of the KdV equation for resonant generation of finite-amplitude internal waves by the flow over topography. A similar nonlocal equation was also derived a little earlier by Warn[11] for propagation of solitary Rossby waves on a weak shear. Recently, extensive numerical simulations of the Euler equations were carried out by Rottman *et al.*[12] for the uniformly stratified fluid flow over topography and their results reveal excellent agreement with predictions of the nonlocal evolution equation derived by Grimshaw and Yi.[10] Furthermore, a generalization of the asymptotic approach was developed by Derzho and Velarde[13] to cover the wave dispersion of the BO type in an infinitely deep fluid with weak shear and also by Derzho and Grimshaw[14] to construct an asymptotic solution describing steady-state vortex-type wave structures of supercritical amplitudes. The problem of matching the asymptotic expansions of Grimshaw and Yi[10] with the small-amplitude radiation shelves was recently considered by Prasad and Akylas.[15]

Although the derivation of nonlocal evolution equations for the description of solitary waves in a nearly uniformly stratified fluid is now well understood, analytical properties of the nonlocal equations have not been studied in great detail as yet. In particular, the dynamical properties of solitary waves (stability, evolution and interaction) have not been considered. In spite of this, the authors of Ref. 13 have started a discussion of the effects of destabilization of long solitary waves in the presence of a strong or symmetrical shear flow applying as a guide the well-known results on stability analysis valid for local models such as the generalized KdV and BO equations.[16]

The aim of this paper is to find the instability criterion for steady-state solitary waves supported by a deviation of the stratification profile from the uniform case as well as by the traditional non-Boussinesq terms. We extend the analysis of solitary wave instability developed by Pego and Weinstein[17] and Pelinovsky and Grimshaw[18] for the local generalized KdV equation to the present nonlocal case. However, our results reveal that the same conventional instability criterion given by the dependence of the solitary wave momentum on its velocity (for a review, see Ref. 19) is valid also for the nonlocal evolution equation.

The results of our paper are described as follows. First, in Section II we re-derive the nonlocal evolution equation for long internal waves following basically the analysis of Grimshaw and Yi.[10] However, we use some modifications of this approach which enable us to represent the nonlinear term of the evolution equation in a simplified form convenient for further analysis. Then, in Section III we discuss the analytical properties of the underlying model including the Lagrangian representation, conserved quantities and steady-state solitary wave solutions. The main results concerning solitary wave instability are described in Section IV where the conventional instability criterion is rigorously proved. The nonlinear dynamics of unstable internal solitary waves with subcritical amplitudes is analyzed in Section V within an adiabatic perturbation theory for solitary waves.[19] We show that the instability development could lead to the formation of vortex structures with supercritical amplitudes. Finally, we discuss in Section VI applications of the general instability theory to some particular stratifications and reveal that strengthening of the stratification distribution might lead to destabilization of the steady-state internal solitary waves and to the generation of vortices in the stratified fluid.

II. DERIVATION OF A NONLOCAL EVOLUTION EQUATION

We consider the two-dimensional motion of an inviscid, incompressible fluid in the presence of density stratification. It is convenient to describe this motion in terms of the standard variables, the vertical particle displacement $\eta(x,z,t)$ and the stream function $\psi(x,z,t)$, where x and z are horizontal and vertical coordinates, respectively, and t is the time. The density of the fluid is expressed through $R(z-\eta)$, where $R(z)$ is a given profile of the basic stratification. The horizontal and vertical velocities, u and w, are expressed through ψ by means of the relations, $u=\psi_z$ and $w=-\psi_x$. The primary equations for the stratified fluid's motion are then given by[10]

$$\Delta\psi_t + J(\Delta\psi,\psi) - N^2(z-\eta)\,\eta_x + \frac{1}{g}N^2(z-\eta) \times[\eta_x(\psi_{xt}+J(\psi_x,\psi)) + (\eta_z-1)(\psi_{zt}+J(\psi_z,\psi))]=0, \tag{1}$$

$$\eta_t + J(\eta,\psi) + \psi_x = 0. \tag{2}$$

Here $J(a,b)=a_x b_z - a_z b_x$ is the Jacobian operator, $\Delta\psi = \psi_{xx}+\psi_{zz}$ is the Laplacian operating on ψ, g is the gravity constant, and $N^2(z)$ is the buoyancy frequency defined by

$$N^2 = -\frac{gR_z}{R}.$$

In order to simplify consideration of the internal wave motion we impose ''rigid lid'' boundary conditions at the plane surfaces $z=0$ and $z=h$,

$$\psi|_{z=0} = \psi|_{z=h} = 0. \tag{3}$$

We analyze the system of governing equations (1) and (2) under the assumption that the internal waves are long enough so that the following scaling holds:

$$X=\epsilon(x-ct), \quad \tau=\epsilon^3 t, \tag{4}$$

where ϵ is a small parameter and c is the limiting speed of infinitely long waves (see (12) below). In addition, we suppose that the stratification is nearly uniform and the deviation of the buoyancy frequency from the constant value N_0^2 is measured as

$$N^2(z) = N_0^2 + \epsilon^2 M(z). \tag{5}$$

Finally, the Boussinesq approximation is applied and it allows us to consider the last term in (1) to be small of the order of $g^{-1}=\epsilon^2\sigma$, where the parameter σ determines the non-Boussinesq effects.

Within these approximations, the governing equations (1) and (2) are now rewritten in the form

$$J\left(\psi_{zz} + \frac{1}{c^2}N_0^2\psi + \epsilon^2 F_2, \psi - cz\right) - N_0^2\left[\eta - \frac{1}{c}\psi\right]_X + \epsilon^2\psi_{zz\tau} + O(\epsilon^4) = 0, \tag{6}$$

$$J\left(\eta - \frac{1}{c}\psi, \psi - cz\right) + \frac{1}{c}\epsilon^2\psi_\tau + O(\epsilon^4) = 0, \tag{7}$$

where

$$F_2 = \psi_{XX} + \frac{1}{c^2}M\left(z - \frac{1}{c}\psi\right)\psi + \frac{\sigma}{c}N_0^2\left[\frac{1}{2}\psi_z^2 - c\psi_z\right].$$

To proceed with the reduction of these equations, we introduce the Lagrangian coordinate ζ according to the substitution

$$\zeta = z - \frac{1}{c}\psi(X,z,\tau), \tag{8}$$

and consider ψ, η, and z to be dependent on ζ, X, and τ. Then, Equations (6) and (7) can be integrated (see Ref. 10) with the boundary conditions

$$\psi,\eta\to 0 \quad \text{as} \quad X\to +\infty. \tag{9}$$

As a result, we find the following closed boundary-value problem for the stream function ψ subject to the boundary conditions (3) and (9),

$$\psi_{zz}+\frac{1}{c^2}N_0^2\psi+\epsilon^2 G_2[\psi]+O(\epsilon^4)=0, \tag{10}$$

where

$$G_2=F_2+\frac{N_0^2}{c^2}\int_{+\infty}^{X}\left[\left(1+\frac{\partial z'}{\partial\zeta}\right)\frac{\partial z'}{\partial\tau}+(z'-z)\frac{\partial}{\partial\zeta}\left(\frac{\partial z'}{\partial\tau}\right)\right]dX'$$

and $z'=z(X',\tau,\zeta)$.

The boundary condition (9) means that there is no radiation in the direction $X\to+\infty$. On the other hand, such a restriction is not imposed as $X\to-\infty$. Indeed, it can be easily shown that the small-amplitude radiative waves always propagate to the left in the reference frame moving with the limiting speed c. Therefore, if the evolution process generates radiation, it propagates only to the left and this leads to the appearance of secular shelf-type divergences of the stream function ψ in the limit $X\to-\infty$ (see formula (21) below and also the discussion in Ref. 11).

Now we expand the solution of (10) in an asymptotic series,

$$\psi=\psi_0(X,z,\tau)+\epsilon^2\psi_2(X,z,\tau)+O(\epsilon^4). \tag{11}$$

The leading-order term can be found by separation of variables,

$$\psi_0=cA(X,\tau)W(z),\quad W(z)=\sin\left[\frac{\pi n z}{h}\right],\quad c=\frac{N_0 h}{\pi n}, \tag{12}$$

where $n=\pm1,\pm2,\dots$. This is just a set of standard internal modes supported by the uniform stratification. The variable A stands for an amplitude of the given (nth) internal mode. We suppose that the mode is localized at the leading order, i.e., the following boundary conditions are met,

$$A\to 0 \quad \text{as} \quad X\to\pm\infty. \tag{13}$$

Next, the correction ψ_2 of the asymptotic series (11) satisfies an inhomogeneous boundary-value problem following directly from (10) subject to the following compatability condition:

$$\int_0^h W(z)G_2[\psi_0]dz=0. \tag{14}$$

This condition leads to an evolution equation for $A(X,\tau)$. In the leading order, the Lagrangian coordinate ζ is given by

$$\zeta=z-A(X,\tau)W(z), \tag{15}$$

which means that the variable z can be regarded as an effective function of only two variables, $z=z(A,\zeta)$. This transformation is valid only for *subcritical* amplitudes A limited by

$$|A|<\frac{h}{\pi n}. \tag{16}$$

Therefore, the wave disturbances considered in the framework of this long-scale asymptotic approach should satisfy the amplitude restriction given by (16).

Using the function $z=z(A,\zeta)$ given by (15) we present (14) in the form of a nonlocal evolution equation,

$$\frac{2N_0^2}{c^3}\int_{+\infty}^{X}K(A,A')\frac{\partial A'}{\partial\tau}dX'+\frac{\partial^2 A}{\partial X^2}+f(A)=0, \tag{17}$$

where $A'=A(X',\tau)$, while the integral kernel $K(A,A')$ and the nonlinear function $f(A)$ are given by

$$K=\frac{1}{h}\int_0^h d\zeta\left[\frac{\partial z}{\partial A}\frac{\partial z'}{\partial A'}-z\frac{\partial z}{\partial A}\frac{\partial}{\partial\zeta}\frac{\partial z'}{\partial A'}-z'\frac{\partial z'}{\partial A'}\frac{\partial}{\partial\zeta}\frac{\partial z}{\partial A}\right], \tag{18}$$

$$f=\frac{2}{c^2h}A\int_0^h dz W^2(z)M(z-AW(z))+\frac{\sigma N_0^3}{3ch}\times A^2[1-(-1)^n], \tag{19}$$

where $z'=z(A',\zeta)$.

The nonlocal evolution equation (17) with $K(A,A')$ given by (18) was derived by Grimshaw and Yi[10] but the nonlinear function $f(A)$ in their paper (see formula (3.23b) in Ref. 10) looks different from our form (19). As a matter of fact, this function arises from nonlinear effects of two different types. The first type is induced by a deviation of the stratification profile from the uniform case, while the second one is due to the non-Boussinesq terms in the primitive equations. Using our approach we have found it possible to reduce the nonlinearity of the second type to a solely quadratic form (19) [cf. (3.23b), (3.28) in Ref. 10].

Finally, using a simple transformation,

$$\zeta\to h\zeta,\quad z\to hz,\quad A\to hA,\quad X\to x,\quad \tau\to\frac{2N_0^2}{c^3}t,$$

we reduce (17) to the dimensionless form,

$$\int_{+\infty}^{x}K(A,A')\frac{\partial A'}{\partial t}dx'+\frac{\partial^2 A}{\partial x^2}+f(A)=0, \tag{20}$$

where $K(A,A')$ and $f(A)$ are given by (18) and (19) with the depth h renormalized by unity. We call this evolution equation the nonlocal generalized KdV equation because it generalizes the conventional KdV equation by an arbitrary nonlinear function and a nonlocal nonlinear evolution operator.

It follows from (10) and (11) that even if the amplitude of the internal mode A is localized at both infinities according to (13), the correction ψ_2 still has a shelf $\psi^-(z,\tau)=\lim_{X\to-\infty}\psi_2(X,z,\tau)$, which obeys,

$$\psi_{zz}^-+\frac{1}{c^2}N_0^2\psi^-=\frac{N_0^2}{c^2}\int_{-\infty}^{+\infty}\left[\left(1+\frac{\partial z}{\partial\zeta}\right)\frac{\partial z}{\partial A}+(z-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\partial z}{\partial A}\right)\right]\frac{\partial A}{\partial\tau}dX. \tag{21}$$

This equation describes the generation of small-amplitude radiating waves behind the localized nonlinear disturbance. Indeed, in the small-amplitude limit the nonlocal KdV equation (20) describes harmonic waves propagating to the left, $A \sim \exp[i(kx+k^3t)]$, while the nonlinear solitary wave disturbance propagates to the right (see the discussion in Section III). The generation of these radiating waves limits the applicability of the asymptotic scheme used in the analysis described above to a time scale for which τ is $O(1)$, that is dimensional time of $O(\epsilon^{-3})$. In order to extend this analysis to a longer time scale, one needs to calculate the radiation according to (21) and then, in the next-order approximation, evaluate the dissipation-type corrections of the radiative losses for the internal wave dynamics. However, this analysis is beyond the scope of our paper, where we consider only the wave evolution in the framework of the nonlocal generalized KdV equation (20). The problems related to the shelf generation were recently investigated by Prasad and Akylas.[15] It should be noted that the right-hand side of (21) is zero, and hence ψ^- is zero, for a steadily propagating wave.

III. SOLITARY WAVE SOLUTIONS AND CONSERVED QUANTITIES

The analysis of the nonlocal evolution equation (20) is based on two main properties of the integral kernel $K(A,A')$ which follows from the explicit representation (18) at $h=1$,

$$K(A,A')=K(A',A), \tag{22}$$

$$\int_0^A K(A,A')dA'=A. \tag{23}$$

The first property is an obvious symmetry of K in A and A' while the other enables us to reduce (20) to a local equation for the steady-state solutions propagating with a constant speed v, $A=u_s(x-vt)$. Supposing zero boundary conditions at infinity, $x\to\pm\infty$ [see (13)], we find from (20) and (23) a simple differential equation for $u_s(x)$,

$$\frac{\partial^2 u_s}{\partial x^2}-vu_s+f(u_s)=0. \tag{24}$$

This equation exactly occurs for steady-state solutions of a local generalized KdV equation (see Refs. 17 and 18) and, therefore, the solitary wave solutions in both equations are identical. Supposing the function $f(u)$ to satisfy the conditions $f(0)=f'(0)=0$, we conclude from (24) that solitary waves propagate with *positive* velocities v and exist provided there is a value $u=u^*$ so that

$$\int_0^{u^*} f(u)du-\frac{1}{2}vu^{*2}=0. \tag{25}$$

Under this condition the solitary waves are described by an even nodeless function $u=u_s(x)$ which has a maximum value $u=u^*$ at $x=0$ and approaches zero as $|x|\to\infty$ at an exponential rate,

$$u_s(x)\to\beta(v)e^{-\sqrt{v}|x|} \quad \text{as} \quad x\to\pm\infty, \tag{26}$$

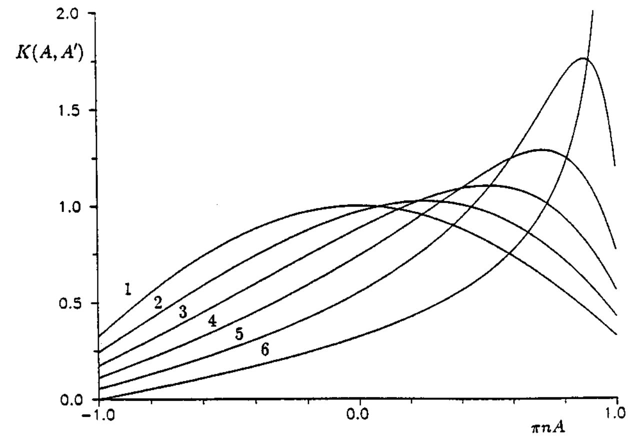

FIG. 1. The function $K(A,A')$ given by (18) within the domain $-1 \leqslant \pi nA \leqslant 1$ for different values of $\pi nA'$, 1–0, 2–0.2, 3–0.4, 4–0.6, 5–0.8, 6–1.0.

where $\beta(v)$ is a constant amplitude. Some typical nonlinearities $f(A)$ and the corresponding solitary wave solutions are discussed in Section VI while here we carry out our analysis for a general form of the nonlinear function.

In spite of the disappearance of the integral kernel $K(A,A')$ in this steady-state problem, the time evolution of solitary waves depends essentially on the properties of this nonlinear integral function. In order to point out that this function is not trivial we find the first terms in the Taylor expansion of K in a small-amplitude limit A, $A'\to 0$,

$$K=1-\frac{(\pi n)^2}{4}[3A^2-8AA'+3A'^2]+\frac{(\pi n)^4}{64}\times[5A^4-128A^3A'+270A^2A'^2-128AA'^3+5A'^4]+O(A^6). \tag{27}$$

The full dependence of $K(A,A')$ evaluated numerically within the domain $-(\pi n)^{-1}\leqslant A\leqslant(\pi n)^{-1}$ [see Eq. (16)] is shown in Fig. 1 for different values of A' (cf. Fig. 4 in Ref. 12). We conclude from this figure that this function is always positive within the domain and it diverges to infinity for A, $A'\to(\pi n)^{-1}$.

In order to study the evolution (stability) properties of the solitary wave solutions, we find the following conserved quantities of (20),

$$M[A]=\int_{-\infty}^{+\infty}dx\int_0^A K(0,A')dA' = \int_{-\infty}^{+\infty}dx\int_0^1 d\zeta(z-\zeta)W(\zeta)\left[1+\frac{\partial z}{\partial\zeta}\right], \tag{28}$$

$$P[A]=\frac{1}{2}\int_{-\infty}^{+\infty}A^2dx. \tag{29}$$

The first conserved quantity is a mass constant which is conserved only if A satisfies the boundary conditions (13). The other conserved quantity (29) is the momentum of the nonlinear wave field. This quantity together with the Hamiltonian of the evolution equation plays an important role in a traditional energetic theory of solitary wave stability (see, e.g., Ref. 16). In particular, the traditional proofs of stability and instability rely on the facts that (i) the solitary wave solutions are a local minimizer of the energy functional H subject to the constraint of fixed momentum P and (ii) an operator of the second variation of the modified Hamiltonian functional (e.g., $H_v = H + vP$) has at most one negative eigenvalue.

Because the steady-state solutions $A = u_s(x - vt)$ for the nonlocal KdV equation (20) are the same as in a local case, the energy functional H is given by

$$H[A] = \int_{-\infty}^{+\infty} \left[\frac{1}{2} \left(\frac{\partial A}{\partial x} \right)^2 - \int_0^A f(u) du \right] dx. \tag{30}$$

However, it is not easy to prove that this functional is a constant of motion for (20). In Appendix A we analyze a Lagrangian representation for a related nonlocal KdV equation with a symmetric nonlocal operator where the form of the Lagrangian implies, indeed, that the function $H[A]$ is the third conserved quantity of this symmetric evolution equation. However, the nonlocal equation (20) arising in the physical problem has an asymmetric nonlocal operator due to a special radiation condition (9) and, hence, is unlikely to have a Lagrangian representation and a third conserved quantity. Indeed, our results from an asymptotic multi-scale analysis described in Section V reveal that the integral quantity (30) is no longer a constant of motion for (20).

For the aforementioned reasons, a formal proof of the solitary wave stability and instability based on the traditional energetic methods[16] seems to be difficult and we develop instead a direct analytical theory involving the so-called Evans's function (see Ref. 17 and references therein). However, this direct method described in the next section predicts the same criterion for solitary wave instability as the traditional theory.

IV. CRITERION FOR SOLITARY WAVE INSTABILITY

We consider the linearized stability of steady-state solitary wave solutions satisfying (24) and reduce (20) to a linear eigenvalue problem. To do this, we substitute $A(x,t) = u_s(x - vt) + Y(x - vt;\lambda) e^{\lambda t}$, where λ is an eigenvalue and $Y(x;\lambda)$ satisfies

$$\left[\frac{\partial^2}{\partial x^2} - v + f'(u_s) \right] Y(x;\lambda) = -\lambda \int_{+\infty}^{x} K(u_s, u_s') Y(x';\lambda) dx', \tag{31}$$

where $f'(u_s) = df(u_s)/du_s$. In the asymptotic limits $x \to \pm\infty$ the nonlocal eigenvalue problem (31) reduces to the local form

$$\left[\frac{\partial^3}{\partial x^2} - v \frac{\partial}{\partial x} + \lambda \right] Y(x;\lambda) = 0 \quad \text{as } x \to \pm\infty, \tag{32}$$

which has the solution

$$Y(x;\lambda) = \sum_{j=1}^{3} y_j^{\pm} e^{\mu_j x}, \tag{33}$$

where $y_j^{\pm}$ are arbitrary constants and μ_j, $j = 1,2,3$ are roots of the characteristic polynomial,

$$P(\mu) = \mu^3 - v\mu + \lambda = 0. \tag{34}$$

We are interested in the construction of a bounded solution to (31) for $\text{Re}(\lambda) > 0$ which indicates the linearized instability of a solitary wave with respect to a small localized perturbation. Therefore, we consider only the case $\text{Re}(\lambda) > 0$, when the cubic equation (34) admits only one root μ_1 with $\text{Re}(\mu_1) < 0$ and two other roots $\mu_{2,3}$ with $\text{Re}(\mu_{2,3}) > 0$. Solutions to (31) are then decaying in the limit $x \to +\infty$ only if they have the asymptotic form

$$Y(x;\lambda) \to y_1 e^{\mu_1 x} \quad \text{as } x \to +\infty, \tag{35}$$

where y_1 is constant. This boundary condition specifies the whole function $Y(x;\lambda)$. However, this function is generally nonlocalized in the limit $x \to -\infty$ because of the first diverging exponential term in the asymptotic representation (33). Only if the coefficient in front of the first term vanishes, which occurs for certain values of λ, then a convergent solution $Y(x;\lambda)$ does exist and the steady-state solitary wave is unstable.

To characterize the divergent term in $Y(x;\lambda)$ we introduce an associated transposed system,

$$\left[\frac{\partial^2}{\partial x^2} - v + f'(u_s) \right] \frac{\partial Z(x;\lambda)}{\partial x} = \lambda \int_{-\infty}^{x} K(u_s, u_s') \frac{\partial Z(x';\lambda)}{\partial x'} dx' \tag{36}$$

with the boundary condition

$$Z(x;\lambda) \to z_1 e^{-\mu_1 x} \quad \text{as } x \to -\infty, \tag{37}$$

where z_1 is constant. Then, we define an analytical function $D(\lambda)$ which is referred to as the Evans's function (see Ref. 17 and references therein) in the following form:

$$D(\lambda) = Z \frac{\partial^2 Y}{\partial x^2} - \frac{\partial Z}{\partial x} \frac{\partial Y}{\partial x} + \frac{\partial^2 Z}{\partial x^2} Y + [f'(u_s) - v] ZY - \lambda \int_{-\infty}^{x} dx' \int_{x}^{+\infty} dx'' \frac{\partial K(u_s', u_s'')}{\partial u_s'} \frac{\partial u_s'}{\partial x'} \times Z(x';\lambda) Y(x'',\lambda). \tag{38}$$

It can be directly shown from (31), (36), and (38) that the function $D(\lambda)$ does not depend on x. As a consequence, this function defines the diverging exponential term in the asymptotic representation of $Y(x;\lambda)$ and $Z(x;\lambda)$ following from (35), (37), and (38),

$$Y(x;\lambda) \to D(\lambda) y_1 e^{\mu_1 x} \quad \text{as } x \to -\infty, \tag{39}$$

$$Z(x;\lambda) \to D(\lambda) z_1 e^{-\mu_1 x} \quad \text{as } x \to +\infty, \tag{40}$$

where the constants y_1 and z_1 are supposed to satisfy the normalization condition

$$P'(\mu_1) y_1 z_1 = 1, \tag{41}$$

with $P'(\mu) = 3\mu^2 - v$. Thus, the eigenvalue λ for $\text{Re}(\lambda) > 0$ is defined by the zeros of the Evans's function.[17] Furthermore, the function $D(\lambda)$ is obviously real for real λ. Using an asymptotic transformation of the linear eigenvalue problems (31) and (36) in the limit $|\lambda|\to\infty$, $\mathrm{Re}(\lambda)>0$ so that $\mu\to-\lambda^{1/3}$ we can show (see Ref. 17 for details) that the Evans's function satisfies the boundary condition

$$D(\lambda)\to 1 \quad \text{as } |\lambda|\to\infty, \tag{42}$$

provided the normalization (41) is met. Thus, if $D(\lambda)<0$ for small real λ, then the function $D(\lambda)$ changes its sign for larger λ according to (42) and, hence, a zero of $D(\lambda)$ and the associated instability eigenvalue λ always exists in the linear problem (31) for $\lambda>0$. If $D(\lambda)>0$ for small λ, then the function $D(\lambda)$ either has no zeros at all or at least two zeros are present.

Now we analyze the zeros of the function $D(\lambda)$ by considering the asymptotic limit for small λ. To do this we follow the paper by Pego and Weinstein (see the proof of Theorem 1.11 in Ref. 17) and derive the following integral formula for the derivative $D'(\lambda)$:

$$D'(\lambda)=\int_{-\infty}^{+\infty}dx\left[\int_{-\infty}^{x}dx'K(u_s,u_s')\frac{\partial Z(x';\lambda)}{\partial x'}Y(x;\lambda)+D(\lambda)\frac{d\mu(\lambda)}{d\lambda}\right]. \tag{43}$$

The expansions of the functions $Y(x;\lambda)$ and $Z(x;\lambda)$ as a power series for small λ can be easily found from (31) and (36) by taking into account the boundary conditions (26), (35), (37), (39), (40), and (41) [$y_1=1$ for simplicity]. The first terms of these expansions (see Section II (b) in Ref. 17 for details) are given by

$$Y_0(x)=Y(x;\lambda)|_{\lambda=0}=-\frac{1}{\beta\sqrt{v}}\frac{\partial u_s}{\partial x}, \tag{44}$$

$$Z_0(x)=Z(x;\lambda)|_{\lambda=0}=\frac{1}{2\beta v}u_s, \tag{45}$$

$$Y_{0\lambda}(x)=\frac{\partial Y(x;\lambda)}{\partial\lambda}\Big|_{\lambda=0}=\frac{1}{\beta\sqrt{v}}\frac{\partial u_s}{\partial v}+\frac{1}{\beta^2 v}\frac{\partial\beta}{\partial v}\frac{\partial u_s}{\partial x}, \tag{46}$$

$$Z_{0\lambda}(x)=\frac{\partial Z(x;\lambda)}{\partial\lambda}\Big|_{\lambda=0}=\frac{1}{2\beta v}\int_{-\infty}^{x}\frac{\partial u_s'}{\partial v}dx'-\frac{1}{2\beta v^{3/2}}\left[\frac{1}{\beta}\frac{\partial\beta}{\partial v}+\frac{1}{v}\right]u_s. \tag{47}$$

Using these formulas and also (39) and (43) we find that $D(0)=0$,

$$D'(0)=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{x}dx'K(u_s,u_s')Y_0(x)\frac{\partial Z_0(x')}{\partial x'}=-\frac{1}{2\beta^2 v^{3/2}}\int_{-\infty}^{+\infty}dx\,u_s\frac{\partial u_s}{\partial x}=0,$$

$$D''(0)=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{x}dx'K(u_s,u_s')\left[Y_{0\lambda}(x)\frac{\partial Z_0(x')}{\partial x'}+Y_0(x)\frac{\partial Z_{0\lambda}(x')}{\partial x'}\right]=\frac{1}{\beta^2 v^{3/2}}\frac{dP_s}{dv},$$

where $P_s(v)=P[u_s]$ given by (29). It follows from this formula that $D''(0)<0$ if the values of the solitary wave velocity v belongs to the domain

$$\frac{dP_s}{dv}<0. \tag{48}$$

In this case, $D(\lambda)<0$ for small λ whereas $D(\lambda)\to 1$ for large λ. Hence, a steady-state solitary wave is linearly unstable if the momentum of the solitary wave is a decreasing function of its velocity. This criterion is exactly the traditional criterion of solitary wave instability which occurs for other local long-wave evolution equations.[16–18]

In the rest of this section we study transition to instability occurring when $dP_s/dv\approx 0$. In this case extension of $D(\lambda)$ by the Taylor series,

$$D(\lambda)=\tfrac{1}{2}\lambda^2[D''(0)+\tfrac{1}{3}\lambda D'''(0)+O(\lambda^2)], \tag{49}$$

gives an approximate value of λ near the instability onset. To find this value we calculate in Appendix B the value of $D'''(0)$ in the following form:

$$D'''(0)=\frac{3}{2\beta^2 v^{3/2}}\int\int_{-\infty}^{+\infty}K(u_s,u_s')\frac{\partial u_s}{\partial v}\frac{\partial u_s'}{\partial v}dx\,dx'-\frac{3}{\beta^2 v^2}\left[\frac{2}{\beta}\frac{\partial\beta}{\partial v}+\frac{1}{v}\right]\frac{dP_s}{dv}. \tag{50}$$

As a result, we find the same scenario of the instability onset as that described for local evolution equations (see Refs. 17 and 18). A real positive eigenvalue λ leading to the linearized instability of a solitary wave emerges from the origin when the parameters of a solitary wave pass through the marginal stability curve given by $dP_s/dv=0$ into the instability domain (48). The approximate value for λ can be found from (49) and (50) as

$$\lambda=-\frac{2}{K_s}\frac{dP_s}{dv}, \tag{51}$$

where

$$K_s(v)=\int\int_{-\infty}^{+\infty}K(u_s,u_s')\frac{\partial u_s}{\partial v}\frac{\partial u_s'}{\partial v}dx\,dx'>0. \tag{52}$$

The positiveness of $K_s(v)$ follows from the fact that the integral kernel $K(A,A')$ is always positive (see Fig. 1).

We would like to mention that the same results on transition to instabilities can also be obtained by a different asymptotic technique applied for analysis of the type II instability bifurcation (see the classification and review in Ref. 19). This bifurcation emerges from the origin where the continuous spectrum coexists with a localized neutral eigenmode. Then, the type II bifurcation technique (see Ref. 19) allows us to re-derive the same approximation (51) and (52) by expanding solutions to (31) in a power series of λ and applying a solvability condition subject to the condition that the function $Y(x;\lambda)$ is exponentially decaying at infinity. This solvability condition takes the form

$$\widetilde{D}(\lambda)=\lambda\int_{-\infty}^{+\infty}u_s Y(x;\lambda)dx=0. \tag{53}$$

The difference between the two approaches described above is the fact that the function $\widetilde{D}(\lambda)$ is not an analytic function in λ while $D(\lambda)$ is. Therefore, the former function has only a local sense defined for a *localized* eigenfunction $Y(x;\lambda)$. As a result, the bifurcation approach enables us to obtain only local characteristics of the type II instability bifurcation while the approach based on analytical properties of Evans's function provides not only local but also global criteria for instability of solitary waves in the model under consideration.

V. NONLINEAR ANALYSIS OF SOLITARY WAVE INSTABILITY

Here we assume that a solitary wave is weakly unstable, i.e., the derivative dP_s/dv is small but negative and the instability criterion (48) holds. In this case, the asymptotic multi-scale expansion method allows us to study the nonlinear quasi-adiabatic dynamics of these unstable solitary waves.[18,19] To do this, we introduce a slow evolution time $T=\mu t$, where $\mu\ll 1$, and expand solutions to (20) as an asymptotic series,

$$A=u_s(x-X_s)+\mu u_1(x-X_s,T)+\mu^2 u_2(x-X_s,T) +O(\mu^3), \tag{54}$$

where $u_s(x)$ is the profile of a solitary wave satisfying (24), X_s is the coordinate of an effective solitary wave orbit,

$$X_s=\frac{1}{\mu}\int_0^T v(T')dT',$$

and $v=v(T)$ is the varying solitary wave speed. We bring all terms of the asymptotic expansion into a balance by demanding that

$$\frac{dP_s}{dv}=O(\mu) \tag{55}$$

for the range of the speed's variation under the consideration.

Substitution of (54) into (20) reduces the nonlinear equation to a set of linear inhomogeneous equations for u_1, u_2, etc. In the first-order approximation, we find

$$\left[\frac{\partial^2}{\partial x^2}-v+f'(u_s)\right]u_1=-\frac{dv}{dT}\int_{+\infty}^{x}K(u_s,u_s')\frac{\partial u_s'}{\partial v}dx'. \tag{56}$$

This equation has no exponentially divergent terms at $O(1)$ provided the condition (55) holds. However, integration of (56) reveals that it still possesses a shelf, i.e., a nonlocalized part in the asymptotic limit $x\to-\infty$, where

$$u^-=\lim_{x\to-\infty}u_1(x,T)=-\frac{1}{v}\frac{dM_s}{dv}\frac{dv}{dT}, \tag{57}$$

where $M_s(v)=M[u_s]$ given by (28). In the opposite limit, the correction u_1 is exponentially localized according to the radiation condition (9).

Next, we proceed to the second-order approximation, where the linear inhomogeneous equation has the form

$$\left[\frac{\partial^2}{\partial x^2}-v+f'(u_s)\right]u_2=-\frac{1}{2}f''(u_s)u_1^2-\int_{+\infty}^{x}\left[K(u_s,u_s')\frac{\partial u_1'}{\partial T}+\frac{\partial K(u_s,u_s')}{\partial u_s}u_1\frac{\partial u_s'}{\partial v}\frac{dv}{dT}+\frac{\partial K(u_s,u_s')}{\partial u_s'}u_1'\frac{\partial u_s'}{\partial v}\frac{dv}{dT}\right]dx'. \tag{58}$$

The solvability condition both to (56) and (58) has the form of the conservation law,

$$\frac{1}{\mu}\frac{dP_0}{dT}=-\frac{1}{2}v(u^-)^2, \tag{59}$$

where P_0 is the momentum of a localized wave given by

$$P_0=P_s(v)+\mu\int_{-\infty}^{+\infty}u_s u_1 dx+O(\mu^2).$$

Using simple algebra (see, e.g., Ref. 18) we transform this equation to the form

$$P_0=P_s(v)+\frac{1}{2}\mu K_s(v)\frac{dv}{dT}+O(\mu^2), \tag{60}$$

where $K_s(v)$ is given by (52). As a result, at the leading order of the asymptotic multi-scale expansion method we arrive to the effective dynamical equation for the parameter $v(T)$ of the solitary wave speed,

$$\frac{1}{\mu}\frac{dP_s}{dv}\frac{dv}{dT}+\frac{d}{dT}\left[\frac{1}{2}K_s(v)\frac{dv}{dT}\right]=-\frac{1}{2v}\left(\frac{dM_s}{dv}\right)^2\left(\frac{dv}{dT}\right)^2. \tag{61}$$

It should be noted that the same form of effective dynamical equation has been derived for a local generalized KdV equation[18] but with the coefficient $K_s(v)$ given by

$$K_s=\left(\frac{dM_s}{dv}\right)^2.$$

As a result, in the local case, (61) has a first integral which is conservation of the nonlinear field energy given by (30). In the nonlocal case governed by (20), the energy functional (30) is no longer constant. Indeed, by reconstructing the energy H_0 of a localized wave through the variational principle for solitary waves (see Ref. 19) we find that $H_0=H_s(v)+vP_s(v)-vP_0$, where $H_s(v)=H[u_s]$, satisfies

$$\frac{1}{\mu}\frac{dH_0}{dT}=-\frac{1}{2}\left(\frac{dv}{dT}\right)^2\int\int_{-\infty}^{+\infty}[K(u_s,u_s')-K(u_s,0)K(0,u_s')]\frac{\partial u_s}{\partial v}\frac{\partial u_s'}{\partial v}dxdx', \tag{62}$$

where the right-hand side is obviously nonzero.

The changes of the momentum P_0 of a localized wave described by (59) are induced by radiating waves $u_{\rm rad} = \mu u^-(\mu x, \mu^3 t)$ generated by the varying solitary wave according to (57). On the other hand, it is obvious from (30) that the small-amplitude long-scale radiation takes away the $O(\mu^4)$ part of the energy H_0 of a localized wave field. However, the leading-order equations (59) and (62) are effective to the order of $O(\mu^2)$ and already in this asymptotic order the value of H_0 is not conserved. Hence, the change of the energy H_0 is unlikely to be induced by radiating waves but it indicates that the energy functional (30) is not conserved for the nonlocal generalized KdV equation with the asymmetric integral operator (20). (We recall that a nonlocal equation with a symmetric operator discussed in Appendix A conserves the energy.)

The absence of the third conserved functional makes difficult a direct application of a formal energetic stability theory for solitary wave solutions of (20). However, the effective dynamical equation (61) allows us not only to immediately recover the results of the linear analysis [see (51)] but also predict nonlinear regimes of the solitary wave instability. The detailed discussion of these regimes was done in our previous papers (see Refs. 18 and 19). Here we mention only the basic fact. Whereas a solitary wave is unstable and the dependence $P_s(v)$ is a decreasing function, the growth of the solitary wave parameter v cannot be prevented at the nonlinear stage of the wave instability and this leads to blow-up of a steady-state solitary wave and its transformation into strongly nonlinear structures of the nonlinear wave field. The sequences of this transformation in application to the dynamics of internal waves are discussed in the next section.

VI. DISCUSSION AND APPLICATIONS

In this section we consider useful applications of the nonlocal evolution equation (20) for some particular stratifications. It follows from (18) and (19) that the kernel of the integral term $K(A,A')$ is completely defined by the structure of an internal mode (12) supported by the uniform stratification while the nonlinear function $f(A)$ depends crucially on the deviation $M(z)$ of the stratification profile. Usually the function $M(z)$ is approximated by a finite polynomial in z. Taking this into account, we expand the nonlinear function $f(A)$ as a Taylor series with respect to A,

$$f(A) = \sum_{j=1}^{\infty} f_j\, A^j, \tag{63}$$

where the coefficients f_j are given by

$$f_1 = \frac{2}{c^2}\int_0^1 W^2(z) M(z)\,dz,$$

$$f_2 = -\frac{2}{c^2}\int_0^1 W^3(z) M'(z)\,dz + \frac{\sigma N_0^3}{3c}[1-(-1)^n],$$

$$f_{j+1} = (-1)^j \frac{2}{c^2 j!}\int_0^1 W^{j+2}(z) M^{(j)}(z)\,dz, \quad j=2,3,\ldots .$$

We note that the linear term can be excluded from (63) by renormalizing the limiting speed c and henceforth we will not keep this term in all subsequent calculations.

Applying now a linear approximation, $M(z) = m_1 z$, where m_1 is constant, we find that the nonlinear function $f(A)$ reduces solely to the quadratic nonlinearity with the coefficient

$$f_2 = \frac{\sigma N_0^3}{3c}\left(1 - \frac{4m_1}{\sigma N_0^4}\right)[1-(-1)^n].$$

Therefore, the quadratic nonlinear function exists only for the internal modes with odd n and its sign depends on the coefficient m_1. The solitary wave solutions to (24) are just the well-known KdV soliton given by

$$u_s = \frac{3v}{2f_2}\,\mathrm{sech}^2\left[\frac{\sqrt{v}}{2}x\right]. \tag{64}$$

It follows from this analysis that for $m_1 < \frac{1}{4}\sigma N_0^4$ the solitary waves for the internal modes with odd n are the elevation waves ($f_2 > 0$) including the solitary waves supported solely by the non-Boussinesq terms ($m_1 = 0$) (cf. with formulas (50a,b) in Ref. 6). On the other hand, for $m_1 > \frac{1}{4}\sigma N_0^4$ the solitary waves are depression waves ($f_2 < 0$) including the case of linear $R(z)$ when $m_1 = \sigma N_0^4$ (see Ref. 10 and the original papers by Miles[20] and Weidman[21]).

For the modes with even n and in the critical case $m_1 \approx \frac{1}{4}\sigma N_0^4$ the linear approximation of the deviation $M(z)$ is not adequate because the quadratic nonlinearity term in (63) vanishes. Therefore, in these cases one should use a quadratic approximation, $M(z) = m_1 z + m_2 z^2$, for which the expansion (63) produces the cubic nonlinear function $f(A)$ with the coefficients

$$f_2 = \frac{\sigma N_0^3}{3c}\left(1 - \frac{4m_1}{\sigma N_0^4}\right)[1-(-1)^n] + \frac{8m_2}{3cN_0}(-1)^n,$$

$$f_3 = \frac{3m_2}{4c^2}.$$

Therefore, for the case $m_2 > 0$ (the stratification profile is concave upward) the nonlinear function in the nonlocal KdV equation (20) is of focusing type which supports the existence of two different solitary wave solutions described by

$$u_s = \frac{3v}{\alpha_\pm \cosh[\sqrt{v}x] + f_2}, \quad \alpha_\pm = \pm f_2\sqrt{1 + \frac{9f_3 v}{2f_2^2}}. \tag{65}$$

For the case $m_2 < 0$ (the stratification profile is concave downward) there is only one branch of solitary wave solutions to (24) given by (65) for α_+.

This analysis can be extended for higher-order approximations and, in a general case, the concave upward stratification profiles $N^2(z) = N_0^2 + m_{2k} z^{2k}$ with $m_{2k} > 0$ generates the focusing-type nonlinear functions $f(A) = f_{2k+1} A^{2k+1}$ with $f_{2k+1} > 0$. Furthermore, because the instability criterion (48) for solitary waves in the nonlocal equation (20) coincides with the local case, we apply a standard analysis (see Refs. 16 and 17) to conclude that long internal solitary waves are unstable for $k \geqslant 2$ and this instability leads to focusing-induced growth of the solitary wave amplitude according to the dynamical equation (61) (see Ref. 18). Recalling now that the solitary wave amplitudes are limited by the condition (16) we come to the hypothesis that this growth of solitary wave amplitudes leads to breakdown of the smooth wave profiles and formation of vortex structures in a nearly uniformly stratified fluid. Indeed, it was shown by Derzho and Grimshaw[14] that the streamlines become ambiguous for slightly supercritical wave amplitudes and the steady-state vortex-type solutions bifurcate from the sharp corners. Thus, we support the idea that the blow-up of the solitary waves that occurs as a result of buoyancy destabilization effects provides a route to vortex structure formation in a stratified fluid.

If the buoyancy frequency decreases because of the deviation $M(z)$ with $m_{2k}<0$, then the effective nonlocal KdV equation (20) contains a defocusing-type nonlinearity and this leads generally to stabilization of finite-amplitude solitary waves. In this case, we expect that the steady-state solitary wave solutions play the same fundamental role in an evolution problem as do stable solitons in other nonlinear evolution equations. A detailed analysis of the evolution of localized initial perturbations in the framework of the nonlocal model (20) as well as in the primitive Euler equations is beyond the scope of our paper.

ACKNOWLEDGMENTS

The authors appreciate stimulating discussions with O. Derzho and M. Pavlov. The work described in this paper was supported in part by ARC Grant No. A8923061.

APPENDIX A: LAGRANGIAN REPRESENTATION FOR A SYMMETRIC NONLOCAL EQUATION

In order to find the Lagrangian representation for a nonlocal evolution equation we consider instead of the asymmetric nonlocal equation (20), the following related equation, where the nonlocal operator is in a symmetric form:

$$\frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty} \right) K(A,A') \frac{\partial A'}{\partial t} dx' + \frac{\partial^2 A}{\partial x^2} + f(A) = 0. \tag{A1}$$

Using a potential representation, $A=\theta_x$, and differentiating (A1) we transform this equation to

$$K(\theta_x,\theta_x)\frac{\partial^2 \theta}{\partial x \partial t} + \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty} \right) \frac{\partial K(\theta_x,\theta'_{x'})}{\partial \theta_x} \frac{\partial^2 \theta}{\partial x^2} \frac{\partial^2 \theta'}{\partial x' \partial t} dx' + \frac{\partial^4 \theta}{\partial x^4} + f'(\theta_x)\frac{\partial^2 \theta}{\partial x^2} = 0. \tag{A2}$$

This equation can be obtained by a variation of the action $S=\int_0^t L dt'$ with the following Lagrangian,

$$L = \int_{-\infty}^{+\infty} dx \left[\frac{1}{4}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty} \right) \Pi(\theta_x,\theta'_{x'}) \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta'}{\partial x' \partial t} dx' - \frac{1}{2}\left(\frac{\partial^2 \theta}{\partial x^2} \right)^2 + \int_0^{\theta_x} f(u) du \right], \tag{A3}$$

where $\Pi(A,A')$ is a new integral kernel which is related to $K(A,A')$ according to the following formula:

$$K(A,A') = \Pi(A,A') + \frac{1}{2}\left[A \frac{\partial \Pi(A,A')}{\partial A} + A' \frac{\partial \Pi(A,A')}{\partial A'} \right]. \tag{A4}$$

To prove this result, one needs to introduce an infinitesimal perturbation $\theta \to \theta + \delta\theta$ and transform the first variation of the action S to the form containing $\delta\theta$ by means of integration by parts and changing the limit of integration according to the rule

$$\int_{-\infty}^{+\infty} dx \left(\int_{-\infty}^{x} - \int_{x}^{+\infty} \right) dx' = - \int_{-\infty}^{+\infty} dx' \left(\int_{-\infty}^{x'} - \int_{x'}^{+\infty} \right) dx.$$

The integral kernel $\Pi(A,A')$ defined by the relation (A4) can be found explicitly in the form

$$\Pi = 2 \int_0^1 \frac{\partial z}{\partial A} \frac{\partial z'}{\partial A'} d\zeta. \tag{A5}$$

This result can be shown from (18), (A4), and (A5) by integrating by parts and using the relation following from (15):

$$A \frac{\partial^2 z}{\partial A^2} = \frac{\partial}{\partial \zeta}\left[(z-\zeta) \frac{\partial z}{\partial A} \right].$$

The new integral kernel $\Pi(A, A')$ has the same basic properties (22) and (23) as $K(A,A')$. Moreover, it replaces $K(A,A')$ in the nonlocal evolution equation (20) derived by Warn[11] for description of solitary Rossby waves supported by a weak shear. Hence, we conclude that this integral kernel $\Pi(A,A')$ plays a fundamental role in analysis of the nonlocal evolution equations of the KdV type.

APPENDIX B: EVALUATION OF $D'''(0)$

First, we express the derivative $D'''(0)$ from (43) under the condition $D(0)=D'(0)=0$,

$$D'''(0) = \int_{-\infty}^{+\infty} dx \left[\int_{-\infty}^{x} dx' K(u_s,u'_s) \left(\frac{\partial Z_0(x')}{\partial x'} Y_{0\lambda\lambda}(x) + 2\frac{\partial Z_{0\lambda}(x')}{\partial x'} Y_{0\lambda}(x) + \frac{\partial Z_{0\lambda\lambda}(x')}{\partial x'} Y_0(x) \right) + \frac{1}{2v} D''(0) \right]. \tag{B1}$$

Here the function $Y_{0\lambda\lambda}(x)$ is the second derivative of $Y(x;\lambda)$ at $\lambda=0$ and so is $Z_{0\lambda\lambda}(0)$. For example, we show how to deal with the first integral term in (B1). We use the following linear inhomogeneous equation for $Y_{0\lambda\lambda}$,

$$\left[\frac{\partial^2}{\partial x^2} - v + f'(u_s)\right] Y_{0\lambda\lambda}(x) = -2\int_{+\infty}^{x} K(u_s, u_s') Y_{0\lambda}(x') dx', \tag{B2}$$

and substitute (45) and (46) to transform the first integral term in (B1) as follows:

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{x} dx' K(u_s, u_s') \frac{\partial Z_0(x')}{\partial x'} Y_{0\lambda\lambda}(x)$$

$$= \frac{1}{2\beta v} \int_{-\infty}^{+\infty} dx u_s(x) Y_{0\lambda\lambda}(x)$$

$$= -\frac{1}{\beta v} \int_{-\infty}^{+\infty} dx \frac{\partial u_s}{\partial v} \int_{+\infty}^{x} dx' K(u_s, u_s') Y_{0\lambda}(x')$$

$$- \frac{1}{4\beta^2 v^3} \frac{\partial P_s}{\partial v}$$

$$= \frac{1}{2\beta^2 v^{3/2}} \int \int_{-\infty}^{+\infty} dx dx' K(u_s, u_s') \frac{\partial u_s}{\partial v} \frac{\partial u_s'}{\partial v}$$

$$+ \frac{1}{2v} D''(0) \int_{-\infty}^{0} dx - \frac{2}{\beta^3 v^2} \frac{\partial \beta}{\partial v} \frac{dP_s}{dv} - \frac{1}{4\beta^2 v^3} \frac{\partial P_s}{\partial v}.$$

By virtue of similar calculations we evaluate the other integral terms in (B1) and finally arrive at the expression (50).

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