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Universal Power Law for the Energy Spectrum of Breaking Riemann Waves¶

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The universal power law for the spectrum of one-dimensional breaking Riemann waves is justified for the simple wave equation. The spectrum of spatial amplitudes at the breaking time $t = t_b$ has an asymptotic decay of $k^{-4/3}$, with corresponding energy spectrum decaying as $k^{-8/3}$. This spectrum is formed by the singularity of the form $(x - x_b)^{1/3}$ in the wave shape at the breaking time. This result remains valid for arbitrary nonlinear wave speed. In addition, we demonstrate numerically that the universal power law is observed for long time in the range of small wavenumbers if small dissipation or dispersion is taken into account in the viscous Burgers or Korteweg–de Vries equations.

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1. INTRODUCTION

One-dimensional traveling nonlinear waves in non-dissipative systems are called Riemann or simple waves; their dynamics is well studied in various physical media, e.g., acoustic waves in the compressible fluids and gases [1, 2], surface and internal waves in oceans [3–7], tidal and tsunami waves in rivers [8, 9], ion and magnetic sound waves in plasmas [10], electromagnetic waves in transmission lines [11], and optical tsunami in fiber optics [12]. In homogeneous and stationary media, the Riemann waves continuously deform and transform to the shock waves yielding breaking in a finite time.

In nonlinear acoustics where the wave intensity is not very high, the nonlinear deformation of the Riemann wave has been studied in many details since this stage occurs during many wavelengths [13–16]. Corresponding nonlinear evolution equation is a so-called simple wave equation

$$u_t + V(u)u_x = 0, \tag{1}$$

where u is a wavefunction, and $V(u)$ is a characteristic local velocity of the various points of the wave shape. If $V'(u) > 0$ for all u (that is, if V is invertible), Eq. (1) can be written for the local velocity:

$$V_t + VV_x = 0, \tag{2}$$

which is equivalent to the inviscid Burgers equation

$$v_t + vv_x = 0, \quad v := V(u). \tag{3}$$

The Cauchy problem for Eq. (3) starts with the initial condition given by a smooth function that decays to zero at infinity:

$$v(x, 0) = F(x), \quad \lim_{|x| \to \infty} F(x) = 0, \tag{4}$$

and results in the implicit solution called a simple or Riemann wave:

$$v(x, t) = F[x - tv(x, t)]. \tag{5}$$

We are interested in the asymptotic behavior of these solutions near the breaking time $t = t_b$. The appearance of the singularity in the wave shape yields a power law in the Fourier spectrum of wave turbulence [17]. It was argued earlier in [18] that for simple (Riemann) waves this singularity is of the form $v - v_b \sim (x_b - x)^{1/2}$; this corresponds to the spectrum of spatial amplitudes decaying at the breaking time as $k^{-3/2}$, with corresponding energy spectrum decaying as k^{-3}.

However, we will show that this assumption is in fact incorrect and the spectrum of spatial amplitudes decays at the breaking time as $k^{-4/3}$, with corresponding energy spectrum decaying as $k^{-8/3}$, because the simple waves develop singularities of the form $v - v_b \sim (x - x_b)^{1/3}$. This analytical result confirms earlier numerical observations in [19, 20].

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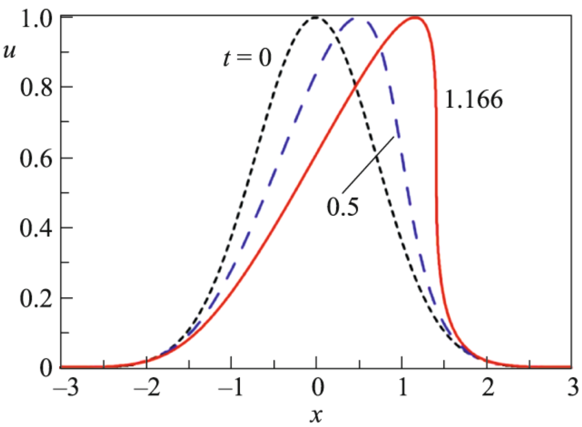

Fig. 1. Deformation of the Riemann wave for the case of quadratic nonlinearity and initial pulse of Gaussian shape $F(x) = \exp(-x^2)$. The wave shape at the moments of time $t = 0, 0.5, 1.166$ is shown as small dashed, long dashed, and bold lines, respectively.

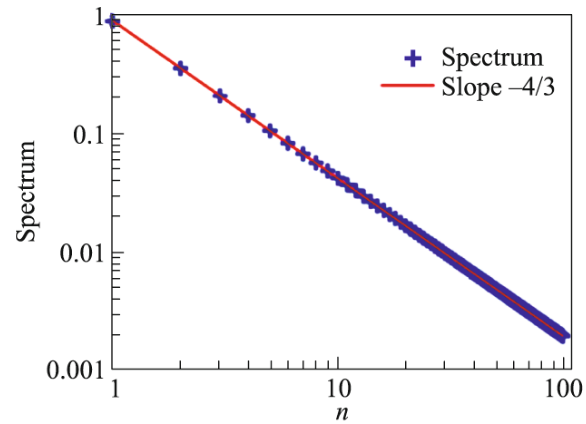

Fig. 2. (Color online) Amplitude spectrum of Riemann waves for the solution (8) at the breaking time $t_b = 1$ is shown by blue crosses. The power rate $n^{-4/3}$ is shown as the red solid line. Wavenumber n (horizontal axis) and the amplitude spectrum $|A_n|$ (vertical axis) are shown in logarithmical coordinates.

We note that the wave field in the vicinity of the breaking time was also reviewed in a number of earlier works [21–24]. Although the result $v - v_b \sim (x - x_b)^{1/3}$ has appeared in these works in different physical contexts, the authors of these works did not study the Fourier spectrum of the breaking Riemann wave, which was studied only recently in numerical works [19, 20].

Let us now illustrate the wave breaking and the main result with examples. First, it is easy to see that the wave steepness v_x increases on the wave front where F' is negative:

$$v_x = \frac{F'(\zeta)}{1 + tF'(\zeta)}, \quad \zeta = x - tv(x, t). \tag{6}$$

The breaking time is computed explicitly as

$$t_b = \frac{1}{\max_x[-F'(x)]} = \frac{-1}{\min_x[F'(x)]}. \tag{7}$$

This process is illustrated in Fig. 1 for an initial pulse of the Gaussian shape $F(x) = \exp(-x^2)$.The shock is formed at the point $x_b = 2^{-1/2}$ and $v_b = \exp(-1/2)$ at the moment of time $t_b = 2^{-1/2}\exp(1/2) \approx 1.166$. At the breaking time the wave shape contains singularity on its front (i.e., its steepness becomes infinite) and the solution (5) is not valid anymore.

If $f(x) = \sin x$, an analytical solution for the Fourier spectrum of the simple waves (5) is known. Corresponding spectrum is known as the Bessel–Fubini spectrum and is given in [14]:

$$v(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{nt} J_n(nt)\sin(nx), \tag{8}$$

where J_n is Bessel function of the first order with integer n and the breaking time is $t_b = 1$. The amplitude of the Fourier spectrum at the breaking time reads

$$|A_n| = \frac{2}{n} J_n(n) \tag{9}$$

and is distributed close to the rate of $n^{-4/3}$ (see Fig. 2).

We shall now prove that the power rate of the Fourier amplitude spectrum at the time of wave breaking is $k^{-4/3}$ for a simple (Riemann) wave supported by an arbitrary smooth initial pulse and an arbitrary local velocity $V(u)$. Moreover, the same rate remains valid in the range of small wavenumbers if small dissipation or dispersion is added in the framework of the viscous Burgers or Korteweg–de Vries equations.

2. POWER LAW OF WAVE BREAKING

Using the method of characteristics, we write the inviscid Burgers equation (3) as the system of two ordinary differential equations

$$\frac{dx}{dt} = v(x, t), \quad \frac{dv}{dt} = 0, \tag{10}$$

therefore, each point on the wave shape moves with velocity proportional to the magnitude of v. The solution is now written in the parametric form:

$$u(t) = F(\zeta), \quad x(t) = \zeta + tF(\zeta). \tag{11}$$

Let ζ_b be the global minimum of F' (which always exists since F is smooth and decays to zero at infinity). We assume that the minimum is not degenerate, hence, $F''(\zeta_b) = 0$ and $F'''(\zeta_b) > 0$. Let t_b be the time of breaking defined by Eq. (7) such that $1 + t_bF'(\zeta_b) = 0$. The wave breaks at the point $x_b = \zeta_b + t_bF(\zeta_b)$ and $v_b = F(\zeta_b)$. Using the decomposition $\zeta = \zeta_b + \eta$ and

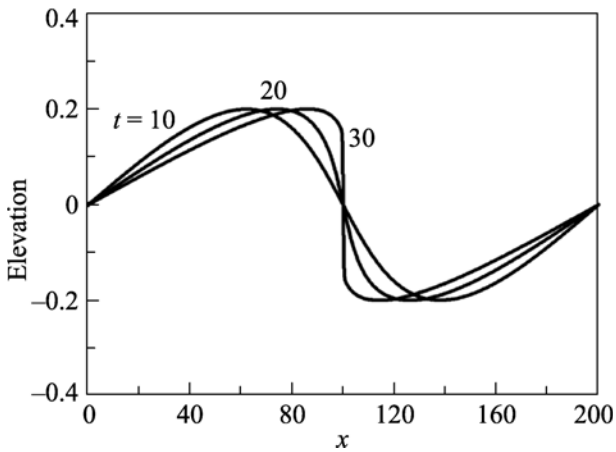

Fig. 3. Solutions of the Burgers equation (16) with $\nu = 0.1$ and initial data $u(x, 0) = S_0 \sin x$ in physical space, for different values of time $t = 10$, 20, 30.

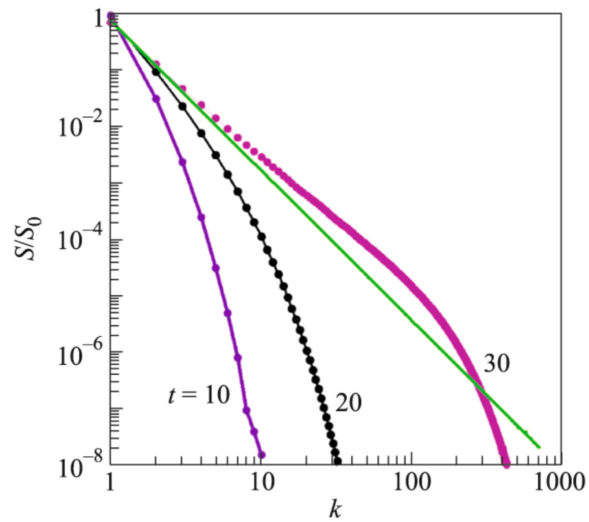

Fig. 4. Solutions of the Burgers equation (16) as in Fig. 3 but in Fourier space: the normalized energy spectrum S/S_0 is shown versus wavenumbers k, with axes in logarithmic coordinates. The universal power law $k^{-8/3}$ is shown by green solid line.

expanding the exact solution (11) into Taylor series, we obtain at the time of breaking:

$$\begin{aligned} x &= \zeta_b + \eta + t_b F(\zeta_b + \eta) \\ &= x_b + \frac{1}{6} t_b F'''(\zeta_b)\eta^3 + \mathbb{O}(\eta^4) \end{aligned} \quad (12)$$

and

$$\begin{aligned} v(x, t_b) &= F(\zeta_b + \eta) \\ &= v_b + F'(\zeta_b)\eta + \mathbb{O}(\eta^2). \end{aligned} \quad (13)$$

Solving (12) for a unique small real root of η, we obtain an explicit relation between u and x at the time of breaking:

$$\eta \sim (x - x_b)^{1/3}, \quad v - v_b \sim (x - x_b)^{1/3}. \quad (14)$$

Therefore, the wave profile changes near the breaking point at the breaking time as $(x - x_b)^{1/3}$, contrary to the behavior $(x - x_b)^{1/2}$ suggested in [18]. The behavior (14) leads to the power spectrum with slope $-4/3$ as was established numerically in [19, 20]. The energy spectrum in this case has a slope of $-8/3$.

If $F'''(\zeta_b) = 0$, then $F^{(4)}(\zeta_b) = 0$ since F is smooth and ζ_b is the point of minimum of F'. If $F^{(5)}(\zeta_b) \neq 0$ (in which case $F^{(5)}(\zeta_b) > 0$), then the modification of the previous analysis shows that the wave profile changes near the breaking point at the breaking time as $(x - x_b)^{1/5}$, leading to the power spectrum with a slope of $-6/5$. We can continue this analysis if ζ_b is a degenerate minimum of a higher order.

Note that the above analysis holds for a general nonlinear evolution equation (1) under the assumption that $V'(u) > 0$ (that is, when V is invertible). In this case, if $F'''(\zeta_b) \neq 0$, the wave field $u(x, t_b)$ of the nonlinear evolution equation (1) changes according to the behavior (14), or explicitly, as

$$\begin{aligned} u(x, t_b) &\approx V^{-1}[v_b + \alpha(x - x_b)^{1/3}] \\ &\approx u_b + \frac{\alpha}{V'(u_b)}(x - x_b)^{1/3}, \end{aligned} \quad (15)$$

where V^{-1} is the inverse function to V, $u_b = V^{-1}(v_b)$, and $\alpha \neq 0$ is a numerical coefficient. Thus, we conclude that the above universal behavior extends to a general nonlinear evolution equation (1) and a general initial data (4) under some restrictive assumptions that are physically relevant.

3. SMALL DISSIPATION AND DISPERSION EFFECTS

The simple wave equation (1) is only valid before the moment of breaking; the study of the wave field evolution at the later times is usually conducted by including effects of dissipation or dispersion. Corresponding terms added to the right hand side of (1) produce different types of equations such as the viscous Burgers equation

$$u_t + 6uu_x = \nu u_{xx}, \quad \nu > 0 \quad (16)$$

and the Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0. \quad (17)$$

In both cases, we consider the initial-value problem starting with initial data $u(x, 0) = S_0 \sin x$.

Taking into account dissipative and dispersive effects will inevitably change the wave spectrum for

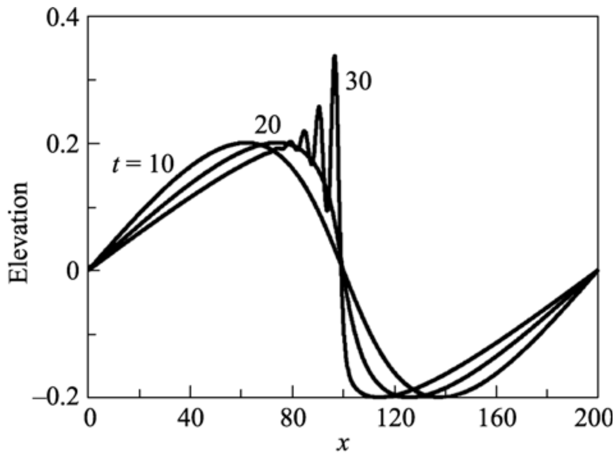

Fig. 5. Solutions of the KdV equation (17) with initial data $u(x, 0) = S_0\sin x$ in physical space, for different values of time $t = 10, 20, 30$.

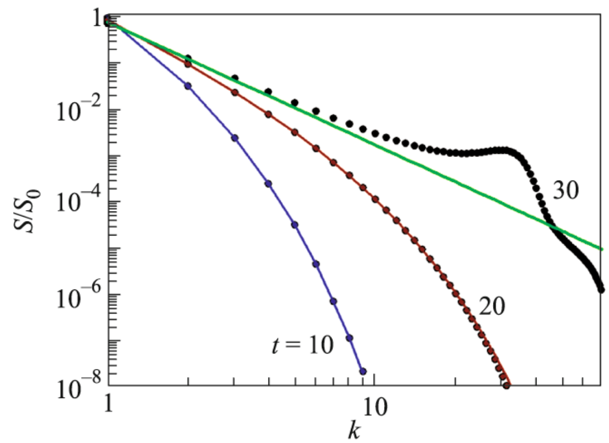

Fig. 6. Solutions of the KdV equation (17) as in Fig. 5 but in Fourier space: the normalized energy spectrum S/S_0 is shown versus wavenumbers k, with axes in logarithmic coordinates. The universal power law $k^{-8/3}$ is shown by green solid line.

large wavenumbers. For instance, it is well-known that shock waves in the viscous Burgers equation (16) have spectral density that decays as k^{-2} for large wavenumbers [2].

Nevertheless, our numerical simulations of the viscous Burgers equation (16) demonstrate that universal power $k^{-8/3}$ of the energy spectrum of breaking Riemann waves is clearly visible in the range of small wavenumbers at least for the evolution times $t \sim t_b \approx 25.5$, where t_b is the breaking time in the inviscid Burgers equation. For longer times, dissipative or dispersive effects become fully developed and drift the energy spectrum away from the power $k^{-8/3}$.

Figures 3 and 4 illustrate solutions of the viscous Burgers equation (16) in physical and Fourier space consequently. The boundary between the spectral asymptotics of $k^{-8/3}$ and k^{-2} can be roughly found from the balance between the terms $6uu_x$ and νu_{xx} in the viscous Burgers equation. If a is the wave amplitude, then the boundary is found at $k \sim 6a/\nu$. If a decreases, then the boundary moves towards the low frequency part of the Fourier spectrum. Note that the fast decay of the energy spectrum for large wavenumbers on Fig. 4 is due to numerical effects.

Similar results of numerical simulations for the Korteweg–de Vries equation (17) are shown in Fig. 5 and 6.

It is clearly visible that although wave breaking is absent in the KdV equation (17), the universal power $k^{-8/3}$ appear in the energy spectrum for small wavenumbers for $t \sim t_b \approx 25.5$.

We also mention results of the numerical simulations of the reduced Ostrovsky equation [25]

$$(\eta_t + \eta\eta_x)_x = \gamma\eta, \quad \gamma > 0, \tag{18}$$

starting with the same initial data $u(x, 0) = S_0\sin x$. A similar effect is observed, namely, the universal power $k^{-8/3}$ appear in the energy spectrum for large wavenumbers regardless of the rotation parameter γ and the initial wave amplitude S_0. The universal behavior is now observed in the range of large wavenumbers because the dispersion term in the reduced Ostrovsky equation (18) affects the wave dispersion for small wavenumbers.

4. SUMMARY

We have justified the universal power law $k^{-8/3}$ in the energy spectrum of one-dimensional breaking Riemann waves in the context of the simple wave equation (1) with smooth initial data (4). This result remains valid for arbitrary nonlinear wave speed provided that the wave speed is an invertible function of the wave amplitude. In addition, we have demonstrated that the same power law is observed for long times in the range of small wavenumbers in the context of the viscous Burgers equations (16) and Korteweg–de Vries equations (17). This universal power law also occurs in other nonlinear evolution equations that reduce to the simple wave equation in the dissipationless and dispersionless limit.

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