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On quadratic eigenvalue problems arising in stability of discrete vortices

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ABSTRACT

We develop a count of unstable eigenvalues in a finite-dimensional quadratic eigenvalue problem arising in the context of stability of discrete vortices in a multi-dimensional discrete nonlinear Schrödinger equation [D.E. Pelinovsky, P.G. Kevrekidis, D.J. Frantzeskakis, Persistence and stability of discrete vortices in nonlinear Schrödinger lattices, Physica D 212 (2005) 20–53]. The count is based on the Pontryagin Invariant Subspace Theorem and the parameter continuation arguments. Another application of the method is given in the context of front–pulse solutions of neuron networks with piecewise constant nonlinear functions [D.E. Pelinovsky, V.G.Yakhno, Generation of collective–activity structures in a homogeneous neuron-like medium, Int. J. Bifurcation and Chaos 6 (1996) 81–87].

1. Introduction

We address quadratic eigenvalue problems arising in the context of stability of discrete vortices in multi-dimensional discrete nonlinear Schrödinger equations, see [5,7] for details. The Lyapunov–Schmidt reduction method is applied for continuation of a limiting vortex configuration from the anti-continuum limit when the coupling constant between lattice nodes is small. Since lattice equations

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linearized at the limiting vortex configuration admit a non-empty finite-dimensional null space, the Lyapunov–Schmidt reduction method results in a finite-dimensional eigenvalue problem.

The eigenvalue problem associated with the Lyapunov–Schmidt reductions at odd orders was found in the form

$$M_{2k+1}\mathbf{c} = \frac{1}{2}\gamma^2\mathbf{c}, \quad k \in \mathbb{N},$$

where M_{2k+1} is a symmetric matrix in $\mathbb{R}^n$ and γ is the spectral parameter, which determines the time evolution of the perturbed discrete vortex. Since $\sigma(M_{2k+1}) \in \mathbb{R}$, all positive eigenvalues of M_{2k+1} result in an unstable time evolution with $\gamma \in \mathbb{R}$, while all negative eigenvalues result in a neutrally stable time evolution with $\gamma \in i\mathbb{R}$. Since the corresponding eigenvector gives negative values of $\langle M_{2k+1}\mathbf{c}, \mathbf{c}\rangle$, the Krein signature of eigenvalues $\gamma \in i\mathbb{R}$ is negative such that these eigenvalues may bifurcate to an unstable domain if the vortex configuration is continued beyond the anti-continuum limit [7]. This count of eigenvalues bifurcating from the zero eigenvalue agrees with the standard results in the Lyapunov–Schmidt reduction method for solitary waves [3,4].

On the other hand, the eigenvalue problem associated with the Lyapunov–Schmidt reductions at even orders was found in the form of a quadratic eigenvalue problem

$$M_{2k}\mathbf{c} = \gamma L_{2k}\mathbf{c} + \frac{1}{2}\gamma^2\mathbf{c}, \quad k \in \mathbb{N},$$

where M_{2k} is a symmetric matrix in $\mathbb{R}^n$ and L_{2k} is an antisymmetric matrix in $\mathbb{R}^n$. Particular examples of the quadratic eigenvalue problem were considered in [7], e.g.

$$M_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$L_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Using a simple transformation $M = 2M_{2k}$, $L = 2iL_{2k}$, and $\lambda = i\gamma$, the quadratic eigenvalue problem is reduced to the form

$$P(\lambda)\mathbf{c} = \left(\lambda^2 I + \lambda L + M\right)\mathbf{c} = \mathbf{0}, \tag{1}$$

where $M^T = M$ and $L^T = \bar{L} = -L$ are Hermitian matrices in $\mathbb{C}^n$, and I is an identity matrix in $\mathbb{C}^n$. We note that M has real-valued coefficients and L has purely imaginary coefficients.

Our main goal is to study the number of unstable eigenvalues λ with $\operatorname{Im}\lambda > 0$ in connection to the number of positive and negative eigenvalues of M. This count is useful to analytically prove the numerical results of [7], which are found to be different from the standard count of eigenvalues in the Lyapunov–Schmidt reduction method for solitary waves [4].

If L and M commute, then their eigenvectors are the same and the quadratic eigenvalue problem (1) is diagonalized into n quadratic equations

$$\lambda^2 + \nu_j\lambda + \mu_j = 0, \quad j = 1, 2, \ldots, n,$$

where $\{\nu_j\}_{j=1}^n$ and $\{\mu_j\}_{j=1}^n$ are eigenvalues of L and M. Therefore, the unstable eigenvalues of the quadratic eigenvalue problem can be counted in this case from the number of positive and negative eigenvalues of M using the quadratic equations above. However, matrices L and M do not commute generally and neither do they in the explicit example above.

On the other hand, setting $\mathbf{d} = -\lambda \mathbf{c}$, the quadratic eigenvalue problem (1) can be rewritten as the generalized eigenvalue problem

$$\begin{bmatrix} M & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} = \lambda \begin{bmatrix} -L & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix} \tag{2}$$

for two self-adjoint matrix operators in $\mathbb{C}^{2n}$. Therefore, the count of unstable eigenvalues of the quadratic eigenvalue problem is related to the count of unstable eigenvalues in the generalized eigenvalue problem. This approach for a different quadratic eigenvalue problem was undertaken in [2]. A general count of unstable eigenvalues of the generalized eigenvalue problem was studied in [1]. Both works [1,2] rely on the Pontryagin Invariant Subspace Theorem and the parameter continuation arguments. In this article, we will follow this approach to obtain the count of unstable eigenvalues for the particular quadratic eigenvalue problem (1) arising in stability of discrete vortices. A general spectral theory of polynomial operator pencils can be found in the book [6].

The structure of this article is as follows. Section 2 develops a general formalism of quadratic eigenvalue problems and gives a general count of eigenvalues in the particular problem (1). A more specific class of matrices M and L is considered in Section 3, where more details in the count of unstable eigenvalues are obtained with the parameter continuation arguments. Another application of the method is reported in Section 4 in the context of stability of front–pulse solutions in neuron networks with piecewise constant nonlinear functions [8].

2. General formalism

Let M and L be bounded, invertible, self-adjoint operators acting in some Hilbert space X with inner product $\langle \cdot, \cdot \rangle$. Define an operator-valued function $P(\lambda) = \lambda^2 I + \lambda L + M$, called the quadratic operator pencil. The following abstract definitions characterize eigenvalues of the quadratic eigenvalue problem $P(\lambda)\mathbf{c} = \mathbf{0}$ in X.

Definition 1. A point $\lambda_0 \in \mathbb{C}$ is said to be a regular point of the operator pencil $P(\lambda)$ if 0 is a regular point of the operator $P(\lambda_0)$.

Definition 2. A nontrivial vector $\mathbf{c}_0 \in X$ is an eigenvector of the operator pencil $P(\lambda)$ for an eigenvalue $\lambda_0 \in \mathbb{C}$ if $P(\lambda_0)\mathbf{c}_0 = \mathbf{0}$.

Definition 3. Vectors $\{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\} \in X$ form a Jordan chain of the generalized eigenvectors associated with eigenvalue $\lambda_0 \in \mathbb{C}$ if

$$\sum_{j=0}^{k} \frac{1}{j!} \frac{d^j}{d\lambda^j} P(\lambda_0) \mathbf{c}_{k-j} = \mathbf{0}$$

for $k = 0, 1, \ldots, m$. If $m = 0$, the eigenvalue λ_0 is called simple.

To apply the spectral theory of a self-adjoint operator acting in a Pontryagin space, we use the following factorization of the matrix pencil $P(\lambda)$:

$$T = \begin{bmatrix} 0 & -I \\ M & -L \end{bmatrix},$$

which follows from the generalized eigenvalue problem (2). We represent the Hilbert space for operator T as $X_2 = X \times X$ and equip it with inner product $(\cdot, \cdot)$.

Lemma 1. *Eigenvalues of the operator pencil $P(\lambda)$ in X are equivalent to eigenvalues of operator T in X_2.*

Proof. If λ is a simple eigenvalue of the operator T, then there exists vector $[\mathbf{c}, \mathbf{d}] \in X_2$ such that

$$-\mathbf{d} = \lambda \mathbf{c}, \qquad M\mathbf{c} - L\mathbf{d} = \lambda \mathbf{d}$$

or, after a substitution, $P(\lambda)\mathbf{c} = \mathbf{0}$. In the opposite direction, if λ is a simple eigenvalue of the operator pencil $P(\lambda)$ and $\mathbf{c} \in X$ is the corresponding eigenvector, then λ and $[\mathbf{c}, -\lambda\mathbf{c}] \in X_2$ form an eigenvalue–eigenvector pair of the operator T. Using the same but longer computations, one can show a relation between Jordan blocks of the operator pencil $P(\lambda)$ and those of the operator T. $\square$

Define operator J as

$$J = \begin{bmatrix} M & 0 \\ 0 & -I \end{bmatrix}$$

Operator T is considered in Pontryagin space equipped with the indefinite inner product generated by the quadratic form $[\cdot, \cdot] = (J\cdot, \cdot)$.

Proposition 1. *The matrix T is J-symmetric with respect to $[\cdot, \cdot] = (J\cdot, \cdot)$.*

Proof. The statement is proved by straightforward computations

$$\forall f, g \in X_2: \quad [Tf, g] = (JTf, g) = (Jf, Tg) = [f, Tg],$$

where we can use that M and L are self-adjoint with respect to $\langle \cdot, \cdot \rangle$. $\square$

Definition 4. The subspace X_- (X_+) of X_2 is called non-positive (non-negative) with respect to the indefinite inner product if for any vector $x \in X_-$ $(x \in X_+)$, it holds that $[x, x] \leqslant 0$ $([x, x] \geqslant 0)$. A finite-dimensional sign-definite subspace is called maximal if it is not a part of a higher-dimensional sign-definite subspace.

Theorem 1 (Pontryagin, 1944). *Let J be a bounded invertible self-adjoint operator in X_2 with a finite-dimensional positive (negative) invariant subspace of dimension κ. Let T be a J-symmetric operator with respect to $[\cdot, \cdot] = (J\cdot, \cdot)$. There exists a maximal non-negative (non-positive) subspace of X_2, which is invariant under T and has the dimension κ.*

Proof. See [1,2] for a restored proof of this theorem. $\square$

As an application of the Pontryagin Theorem, we can now formulate and prove the main result on the count of unstable eigenvalues of the quadratic eigenvalue problem (1).

Theorem 2. *Let $\dim(X) = n$, $M = M^T$ be a real-valued matrix with n_M negative and $n - n_M$ positive eigenvalues, and $L^T = \bar{L} = -L$ be a matrix with purely imaginary elements. Then,*

$$n - n_M = N_i + 2N_c + 2N_r^+, \tag{3}$$

$$n + n_M = N_i + 2N_c + 2N_r^-, \tag{4}$$

where N_i is dimension of the maximal invariant subspace of T associated with eigenvalues in

$$\mathbb{C}_i^+ = \{\lambda \in \mathbb{C}: \ \mathrm{Re}\,\lambda = 0, \ \mathrm{Im}\,\lambda > 0\}$$

N_c is dimension of the maximal invariant subspace of T associated with eigenvalues in

$$\mathbb{C}_I = \{\lambda \in \mathbb{C}: \ \mathrm{Re}\,\lambda > 0, \ \mathrm{Im}\,\lambda > 0\}$$

and N_r^+ (N_r^-) is dimension of the maximal invariant subspace of T associated with eigenvalues in

$$\mathbb{C}_r^+ = \{\lambda \in \mathbb{C}: \ \mathrm{Re}\,\lambda > 0, \ \mathrm{Im}\,\lambda = 0\},$$

such that $(Jx, x) \geqslant 0$ $((Jx, x) \leqslant 0)$ for all eigenvectors of T in the invariant subspaces.

Proof. According to the Pontryagin Theorem, we need to count eigenvalues of operator T whose eigenvectors lie in the non-negative and non-positive invariant subspaces of T. To simplify the count, we assume that all eigenvalues are simple. (A more general application of the Pontryagin theorem for multiple eigenvalues and semi-bounded differential operators is considered in [1].) We note that if $x = [\mathbf{c}, \mathbf{d}]$ is an eigenvector of T for an eigenvalue λ, then

$$[x, x] = (Jx, x) = \langle M\mathbf{c}, \mathbf{c}\rangle - |\lambda|^2 \langle \mathbf{c}, \mathbf{c}\rangle. \tag{5}$$

On the other hand, constructing quadratic forms for an eigenvalue of $P(\lambda)$ with an eigenvector $\mathbf{c}$, we obtain a quadratic equation for λ,

$$\langle M\mathbf{c}, \mathbf{c}\rangle + \lambda\langle L\mathbf{c}, \mathbf{c}\rangle + \lambda^2 \langle \mathbf{c}, \mathbf{c}\rangle = 0, \tag{6}$$

all coefficients of which are real-valued. Since M is invertible, no zero eigenvalues of T exist. Three cases of non-zero eigenvalues of T are described as follows:

- If $\operatorname{Re}\lambda = 0$, then $\langle L\mathbf{c}, \mathbf{c}\rangle = 0$ and $(Jx, x) = 0$.
- If $\operatorname{Re}\lambda \neq 0$ and $\operatorname{Im}\lambda \neq 0$, then $0 \neq |\langle L\mathbf{c}, \mathbf{c}\rangle|^2 < 4\langle \mathbf{c}, \mathbf{c}\rangle\langle M\mathbf{c}, \mathbf{c}\rangle$ and $(Jx, x) = 0$.
- If $\operatorname{Im}\lambda = 0$, then $|\langle L\mathbf{c}, \mathbf{c}\rangle|^2 \geqslant 4\langle \mathbf{c}, \mathbf{c}\rangle\langle M\mathbf{c}, \mathbf{c}\rangle$ and

  $$(Jx, x) = \langle M\mathbf{c}, \mathbf{c}\rangle - \lambda^2\langle \mathbf{c}, \mathbf{c}\rangle = -\lambda\langle (L + 2\lambda)\mathbf{c}, \mathbf{c}\rangle. \tag{7}$$

  If the eigenvalue $\lambda \in \mathbb{R}$ is simple and $\lambda \neq 0$, then $(Jx, x) \neq 0$.

Eigenvalues of $P(\lambda)$ have two symmetries:

- If λ is an eigenvalue of $P(\lambda)$ with the eigenvector $\mathbf{c}$, then $-\bar{\lambda}$ is also an eigenvalue of $P(\lambda)$ with the eigenvector $\bar{\mathbf{c}}$.
- If λ is an eigenvalue of $P(\lambda)$ with the eigenvector $\mathbf{c}$, then $-\lambda$ is also an eigenvalue of $P(\lambda)$ with the eigenvector $\hat{\mathbf{c}}$ such that $P^T(\lambda)\hat{\mathbf{c}} = P(-\lambda)\hat{\mathbf{c}} = \mathbf{0}$.

The first statement follows from the fact that M is real-valued and L is purely imaginary, such that the complex conjugation of $P(\lambda)\mathbf{c} = (\lambda^2 I + \lambda L + M)\mathbf{c} = \mathbf{0}$ gives $((\bar{\lambda}^2 I - \bar{\lambda}L + M)\bar{\mathbf{c}} = P(-\bar{\lambda})\bar{\mathbf{c}} = \mathbf{0}$. The second statement follows from the equality $\det P^T(\lambda) = \det P(\lambda)$, such that if there exists $\mathbf{c} \in \operatorname{Null} P(\lambda)$, then there exists $\hat{\mathbf{c}} \in \operatorname{Null} P^T(\lambda)$ with $P^T(\lambda) = P(-\lambda)$ since $M^T = M$ and $L^T = -L$.

With the above properties of quadratic forms and symmetries of eigenvalues, we develop count of eigenvalues λ associated with non-negative and non-positive invariant subspaces of X under T.

- The symmetries imply that there exists a pair of eigenvalues $\pm\lambda \in i\mathbb{R}$ associated with real-valued eigenvectors $\mathbf{c}$ and $\hat{\mathbf{c}}$. According to the count of complex eigenvalues [1], N_i appears both in non-positive and non-negative subspaces with respect to (Jx, x).
- The symmetries imply that there exists a quartet of complex eigenvalues $\pm\lambda, \pm\bar{\lambda} \in \mathbb{C}$ associated with complex-valued eigenvectors $\mathbf{c}, \bar{\mathbf{c}}, \hat{\mathbf{c}}, \hat{\bar{\mathbf{c}}}$. According to the count of complex eigenvalues [1], $2N_c$ appears both in non-positive and non-negative subspaces with respect to (Jx, x).
- The symmetries imply that there exists a pair of eigenvalues $\pm\lambda \in \mathbb{R}$ associated with eigenvectors $\mathbf{c}$ and $\hat{\mathbf{c}} = \bar{\mathbf{c}}$. Both eigenvalues have the same sign of (Jx, x). Therefore, if $(Jx, x) > 0$, then $2N_r^+$ appears in (3), while if $(Jx, x) < 0$, then $2N_r^-$ appears in (4). (The case $(Jx, x) = 0$ is excluded if the real eigenvalue is simple.)

Adding the counts for all simple eigenvalues of $P(\lambda)$, we finish the proof of the theorem. □

Corollary 1. *If* $\operatorname{Im}\lambda \neq 0$, *then*

$$\frac{|\langle L\mathbf{c}, \mathbf{c}\rangle|}{2\|\mathbf{c}\|^2} < |\lambda| \leqslant \|M\|,$$

where the lower bound makes sense only if $\operatorname{Re}(\lambda) \neq 0$.

Proof. The upper bound follows from the fact that if $\mathrm{Im}\,\lambda \neq 0$, then $(Jx, x) = \langle M\mathbf{c}, \mathbf{c}\rangle - |\lambda|^2\langle \mathbf{c}, \mathbf{c}\rangle = 0$. The lower bound follows from (5) and (6) since $|\langle L\mathbf{c}, \mathbf{c}\rangle|^2 < 4\langle \mathbf{c}, \mathbf{c}\rangle\langle M\mathbf{c}, \mathbf{c}\rangle = 4|\lambda|^2\|\mathbf{c}\|^4$. $\square$

Example 1. We shall consider the quadratic eigenvalue problem (1) with matrices M_2 and L_2 given in Section 1. It is easy to compute

$$\sigma(M_2) = \{-2, 0, 0, 1, 1, 1, 1, 2\},$$
$$\sigma(iL_2) = \{-2, -\sqrt{2}, -\sqrt{2}, 0, 0, \sqrt{2}, \sqrt{2}, 2\}$$

and to check that the null space of M_2 and L_2 coincide. Therefore, the quadratic eigenvalue problem (1) has a quadruple zero eigenvalue, while its non-zero eigenvalues are defined in an orthogonal complement of $\mathrm{Null}(M_2) = \mathrm{Null}(iL_2)$, denoted as X with $n = \dim(X) = 6$. Since $n_M = 1$, the count of Theorem 2 gives

$$N_i + 2N_c + 2N_r^+ = 5, \qquad N_i + 2N_c + 2N_r^- = 7. \tag{8}$$

Explicit computation of eigenvalues of the quadratic eigenvalue problem in [7] shows that $N_i = 1, N_c = 0, N_r^+ = 2$ and $N_r^- = 3$ with a pair $\{\pm i\sqrt{\sqrt{80} - 8}\}$, a quadruple pair $\{\pm\sqrt{2}\}$ and a pair $\{\pm\sqrt{\sqrt{80} + 8}\}$. To justify this count, we shall look into a detailed structure of the matrices M and L.

3. Parameter continuations and instability bifurcations

Following to the main example in [7], we shall consider a particular form of M and L in the quadratic eigenvalue problem (1). We set

$$M = \frac{1}{4}L^2 - aR, \tag{9}$$

where R is a positive operator in X and a is a parameter. The operator $P(\lambda)$ is factorized by $P(\lambda) = \left(\frac{1}{2}L + \lambda I\right)^2 - aR$. By Lemma 31.1 in [6, p. 169], the spectrum of a hyperbolic pencil $P(\lambda)$ is real and the eigenvalues of T have equal algebraic and geometric multiplicities. Therefore, all eigenvalues λ of the quadratic eigenvalue problem $P(\lambda)\mathbf{c} = \mathbf{0}$ are real-valued for $a \leqslant 0$. We shall hence consider continuations of eigenvalues with respect to parameter $a > 0$ and characterize the onset of unstable eigenvalues λ.

For small values of a, we have the following perturbation result.

Lemma 2. *There exists $a_0 > 0$, such that for any $a \in (0, a_0)$, the spectrum of the quadratic eigenvalue problem*

$$\left(\frac{1}{2}L + \lambda I\right)^2 \mathbf{c} = aR\mathbf{c} \tag{10}$$

is real and the corresponding eigenvalues of T have equal algebraic and geometric multiplicities. If λ is a simple positive eigenvalue with a positive (negative) sign of (Jx, x), then it decreases (increases) with $a > 0$.

Proof. If $a = 0$, all eigenvalues of the quadratic problem (1) are real-valued and have even multiplicities. Because L is self-adjoint, an invariant subspace of L for a particular eigenvalue $\lambda_0 \in \mathbb{R}_+$ is spanned by a complete set of linearly independent eigenvectors. Let us pick up a particular eigenvector $\mathbf{c}_0$. Since $\langle (L + 2\lambda_0)\mathbf{c}_0, \mathbf{c}_0\rangle = 0$, we have $(Jx, x) = 0$ for the corresponding eigenvalue. If $\mathbf{c}_0 \in \mathrm{Null}(R)$, the double eigenvalue is preserved at λ_0 for any $a > 0$. If $\mathbf{c}_0 \notin \mathrm{Null}(R)$, the double eigenvalue splits into two simple eigenvalues, according to the perturbation theory for a double root of the quadratic equation (6):

$$(\lambda - \lambda_0)^2 = a\frac{\langle R\mathbf{c}_0, \mathbf{c}_0\rangle}{\langle \mathbf{c}_0, \mathbf{c}_0\rangle} + \mathrm{O}(a^2) > 0 \quad \text{for } a > 0,$$

since R is positive and $\mathbf{c}_0 \notin \mathrm{Null}(R)$.

Let λ_a be a simple positive eigenvalue for $a > 0$ with the eigenvector $\mathbf{c}_a$, such that

$$\lim_{a\to 0} \lambda_a = \lambda_0, \quad \lim_{a\to 0} \mathbf{c}_a = \mathbf{c}_0.$$

If λ_a is simple, then $(Jx_a, x_a) \neq 0$ for the corresponding eigenvector. By the perturbation theory for simple eigenvalues, we obtain

$$\frac{d\lambda_a}{da} = \frac{\langle R\mathbf{c}_a, \mathbf{c}_a\rangle}{\langle (L + 2\lambda_a)\mathbf{c}_a, \mathbf{c}_a\rangle}, \quad \text{for } a > 0.$$

If $\lambda_a > 0$ for small $a > 0$, it follows from (7) and the positivity of R that the eigenvalue with a positive sign of (Jx_a, x_a) decreases with a and the eigenvalue with a negative sign of (Jx_a, x_a) increases with a. □

In what follows, we assume again that L is a Hermitian matrix with purely imaginary elements. Therefore, real eigenvalues of L are symmetric about the origin. Thus, n is even and we can consider only positive eigenvalues. Since positive eigenvalues of the quadratic problem (10) for $a > 0$ move to each other and have opposite signature (Jx, x), we may expect instability bifurcations for $a > a_0$ with appearance of many complex unstable eigenvalues λ. To be precise, if $n_M = 0$ and all $n/2$ pairs of double eigenvalues are distinct for $a = 0$, at most $n/2 - 1$ eigenvalues may coalesce and split into complex domain upon continuation in $a > 0$, in agreement with the counts (3) and (4) resulting in $N_i = 0$, $N_c \leqslant n/2 - 1$ and $N_r^+ = N_r^- \geqslant 1$ under the conditions above. Examples 2, 3, and 4 of the particular matrices L and R show, however, a surprisedly low number of unstable eigenvalues. This property is explained by the decomposition of X into R-invariant orthogonal subspaces. To accommodate this property into the count of unstable eigenvalues, we will need the following elementary result.

Proposition 2. *Assume that there exists a splitting $X = X_1 \oplus X_2$ with $X_2 = X_1^\perp$ uniformly in a, such that*

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}.$$

Then, the quadratic eigenvalue problem (1) decomposes into two problems

$$P_1(\lambda_1)\mathbf{c}_1 = \left(\lambda^2 I + \lambda L_1 + M_1\right)\mathbf{c}_1 = \mathbf{0},$$
$$P_2(\lambda)\mathbf{c}_2 = \left(\lambda^2 I + \lambda L_2 + M_2\right)\mathbf{c}_2 = \mathbf{0},$$

associated with the following matrices T and J:

$$T = \begin{bmatrix} 0 & -I & 0 & 0 \\ M_1 & -L_1 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & M_2 & -L_2 \end{bmatrix}, \quad J = \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & M_2 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}.$$

No instability bifurcation occurs in parameter continuation in a if the coalescent eigenvalues correspond to different blocks of matrices T and J.

Proof. The proof follows by direct substitutions. □

When eigenvalues of the quadratic problem (10) are continued in $a \geqslant a_0$, the instability bifurcations may occur due to two reasons:

1. Real eigenvalues of opposite signatures (Jx, x) associated with the same subspace in the a-uniform decomposition of M and L coalesce at $a = a_*$ and split off the real axis for $a > a_*$.
2. Real eigenvalues associated with the same subspace in the a-uniform decomposition of M and L coalesce at the origin at $a = a_*$ and split off the real axis for $a > a_*$ when M has eigenvalues passing the origin at $a = a_*$ from positive to negative values.

In the first case, the left-hand-side in the counts (3) and (4) remains unchanged, but the right-hand-side leads to a decrease of N_r^+ and N_r^- with the corresponding increase in N_c.

In the second case, the left-hand-side in the counts (3) and (4) is decreased and increased, respectively. If the multiplicity k of the zero eigenvalue of M is odd, there exists at least one eigenvalue pair in N_i after the crossing of the zero eigenvalue. Generally, at most k eigenvalues may bifurcate in $N_i + 2N_c$ after the crossing.

Example 2. We shall complete the count of eigenvalues in Example 1. Indeed, matrix M can be represented in the form (9) with $a = 2$ and $R = \eta_0 \oplus \eta_0^T$, where $\eta_0 = [0, 1, 0, -1, 0, 1, 0, -1]^T$ and the outer product is used. It is clear by Lemma 2 that

$$a = 0: \quad N_i = N_c = 0, \quad N_r^+ = N_r^- = 3,$$

which remains valid for small $a > 0$. Because $n_M = 1$ for $a = 2$, the only negative eigenvalue of M has to give $N_i = 1$ due to the bifurcation of type 2 at $a = a_* \in (0, 2)$. As a result, the number N_r^+ is reduced by 1 for $a > a_*$. On the other hand, the matrices M and L are block-diagonalized simultaneously to the form

$$M = \begin{bmatrix} 4-2a & 2a & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a & 4-2a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

and

$$L = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\sqrt{2} \end{bmatrix},$$

which shows that all bifurcations of type 1 do not occur in parameter continuations for $a > 0$. As a result, the count (8) becomes more precise:

$$a = 2: \quad N_i = 1, \quad N_c = 0, \quad N_r^+ = 2, \quad N_r^- = 3,$$

exactly according to the numerical data in Example 1. By Proposition 2, the quadruple pair of eigenvalues $\lambda = \pm\sqrt{2}$ with $(Jx, x) = 0$ and $N_r^+ = N_r^- = 2$ persists because the two eigenvectors of L for the double eigenvalue $2\sqrt{2}$ are located in the null space of $R = (L^2/4 - M)/a$.

We shall now consider a more interesting pattern of eigenvalues for examples of L and R in the form

$$(L)_{j,k} = 2i\left(\delta_{k,j+1} - \delta_{k,j-1}\right) \mathrm{mod}(n), \quad 1 \leqslant j, k \leqslant n$$

and

$$R = \eta_1 \oplus \eta_1^T + \eta_2 \oplus \eta_2^T + \eta_3 \oplus \eta_3^T + \eta_4 \oplus \eta_4^T,$$

where $n = 4m$ for a fixed integer $m \geqslant 3$ and the column-vectors $\{\eta_j\}_{j=1}^4$ have the elements for $1 \leqslant k \leqslant n$:

$$(\eta_1)_k = \delta_{k,2} - \delta_{k,4m}, \quad (\eta_2)_k = \delta_{k,m+2} - \delta_{k,m},$$
$$(\eta_3)_k = \delta_{k,2m+2} - \delta_{k,2m}, \quad (\eta_4)_k = \delta_{k,3m+2} - \delta_{k,3m}.$$

Table 1
Eigenvalues and their multiplicities for $m = 3$.

M at $a = 0$	M at $a = 2$	T at $a = 0$	T at $a = 2$
4, "2"	3, "2"	± 2, "2"	± 3.69, "1"
3, "4"	2.56, "2"	± 1.73, "4"	± 3.21, "2"
1, "4"	1, "2"	± 1, "4"	± 2.25, "1"
0, "2"	0, "2"	0, "4"	± 1, "2"
	-1, "2"		$\pm 0.28 \pm 0.63\,\mathrm{i}$, "1"
	-1.56, "2"		0, "4"
			$\pm 0.54\,\mathrm{i}$, "2"

It is clear that the matrix R has rank 4. Although we consider continuation of matrix M in $a > 0$, the value of a needed for applications in [7] is $a = 2$.

Example 3. For the simplest case $m = 3$, the two-dimensional null spaces of M and L coincide, such that $n = \dim(X) = 10$ in the orthogonal complement of $\mathrm{Null}(M)$. By Lemma 2, we have

$$a = 0: \quad N_i = N_c = 0, \quad N_r^+ = N_r^- = 5,$$

which remains valid for small $a > 0$. Since $n_M = 4$ for $a = 2$ (see Table 1), the count of Theorem 2 gives

$$N_i + 2N_c + 2N_r^+ = 6, \qquad N_i + 2N_c + 2N_r^- = 14.$$

Fig. 1 shows eigenvalues of M (left) and real and imaginary parts of the eigenvalues of the quadratic problem (1) (right) versus parameter a on $[0, 2]$. Different colors correspond to different blocks in the block-diagonal representations of M and L. Although eigenvalues of opposite Krein signatures of (Jx, x) coalesce when $a > 0$ is increased, all coalescences for $a < 1$ do not result in unstable eigenvalues, since the coalescent eigenvalues correspond to different blocks of the matrices M and L. There are two crossings of positive eigenvalues of M through the origin at $a = 1$ and $a = 1.5$. Both crossing involve double eigenvalues of multiplicity $k = 2$. In the first case, the crossing does not lead to instability bifurcations since they correspond to different blocks of the matrices M and L. After the crossing, the number N_r^+ is reduced by 1 and the number N_r^- is increased by 1. In the second case, the crossing involves eigenvalues of the same block and results in bifurcation of type 2 with $N_i = 2$. The number N_r^+ is reduced by 2 after the bifurcation. Additionally, the real eigenvalues of the same block and opposite Krein signature (Jx, x) coalesce at $a \approx 1.2$ and lead to the bifurcation of type 1 with $N_c = 1$.

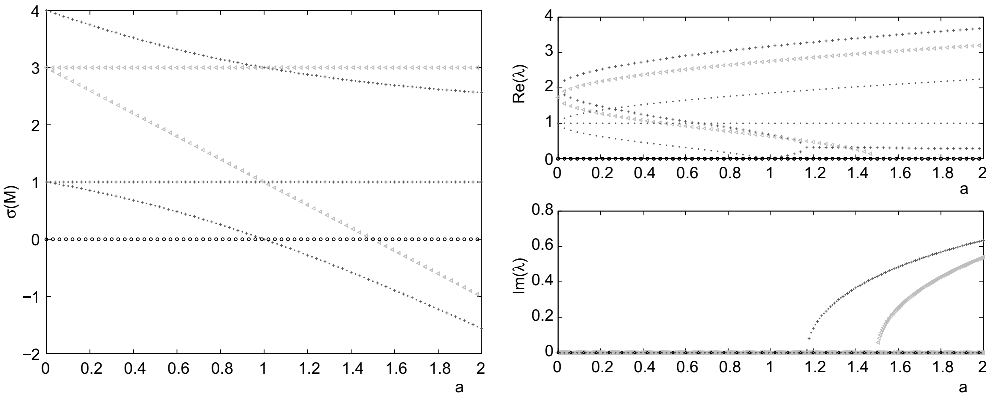

Fig. 1. Eigenvalues of M (left) and real and imaginary parts of eigenvalues of the quadratic problem (1) (right) versus parameter a on $[0, 2]$ for the case $m = 3$.

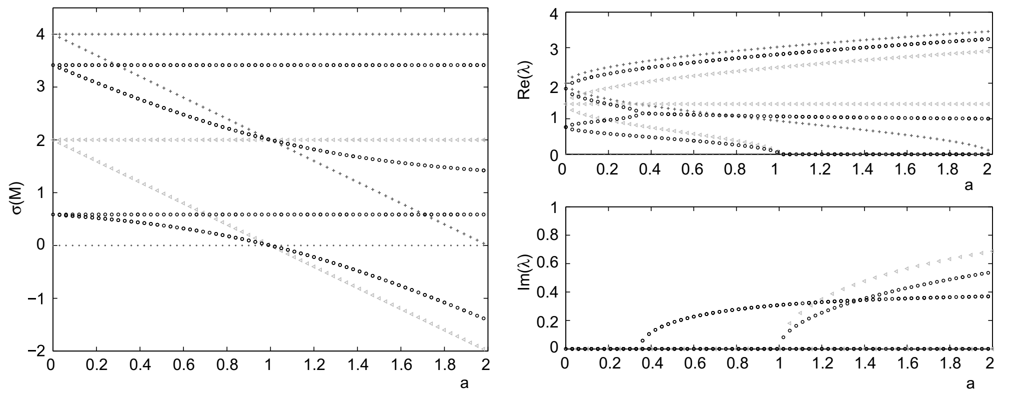

Fig. 2. Same as Fig. 1 but for the case $m = 4$.

Table 2
Eigenvalues and their multiplicities for $m = 4$.

M at $a = 0$	M at $a = 2$	T at $a = 0$	T at $a = 2$
4, "2"	4, "1"	$\pm$2, "2"	$\pm$3.46, "1"
3.4, "4"	3.4, "2"	$\pm$1.85, "4"	$\pm$3.25, "2"
2, "4"	2, "3"	$\pm$1.4, "4"	$\pm$2.97, "1"
0.58, "4"	1.4, "2"	$\pm$0.76, "4"	$\pm$1.41, "2"
0, "2"	0.58, "2"	0, "4"	$\pm 1 \pm 0.37\,\mathrm{i}$, "2"
	0, "3"		0, "6"
	$-$1.4, "2"		$\pm 0.68\,\mathrm{i}$, "1"
	$-$2, "1"		$\pm 0.54\,\mathrm{i}$, "2"

Both numbers N_r^+ and N_r^- are reduced by 1 after the bifurcation. Thus, we obtain the exact count of eigenvalues by

$$a = 2: \quad N_i = 2, \quad N_c = 1, \quad N_r^+ = 1, \quad N_r^- = 5.$$

Eigenvalues and their multiplicities for $a = 0$ and $a = 2$ are summarized in Table 1.

Example 4. For the case $m = 4$, matrices M and L have again the same two-dimensional null space, so that $n = 14$. We have again

$$a = 0: \quad N_i = N_c = 0, \quad N_r^+ = N_r^- = 7,$$

which remains valid for small $a > 0$. Fig. 2 shows again eigenvalues of M (left) and real and imaginary parts of the eigenvalues of the quadratic problem (1) (right) versus parameter a on $[0, 2]$, where different colors correspond to different blocks in the block-diagonal representations of M and L. Eigenvalues and their multiplicities for $a = 0$ and $a = 2$ are summarized in Table 2. We can see that the case $a = 2$ is a bifurcation since M has zero eigenvalue of multiplicity 3. Therefore, we shall count eigenvalues at $a = 2 - \delta$ for any small $\delta > 0$. Since $n_M = 3$ for $a = 2 - \delta$, the count of Theorem 2 gives

$$N_i + 2N_c + 2N_r^+ = 11, \qquad N_i + 2N_c + 2N_r^- = 17.$$

There is only one bifurcation of type 2 at $a = 1$, when the zero eigenvalue has multiplicity $k = 3$. Because of the block-diagonal decomposition of M and L, all real eigenvalues crossing zero become imaginary, resulting in $N_i = 3$ and a decrease of the number N_r^+ by 3. Additionally, there exists a bifurcation of type 1 at $a \approx 0.36$ in the same block, resulting in a double quartet of complex eigenvalues with $N_c = 2$ and in a decrease in numbers N_r^+ and N_r^- by 2. Therefore, we obtain the exact count of eigenvalues by

$$a = 2 - \delta: \quad N_i = 3, \quad N_c = 2, \quad N_r^+ = 2, \quad N_r^- = 5.$$

Of course, the case $a = 2$ is bifurcation and, therefore, the count will change for $a = 2 + \delta$.

4. Application to front–pulse solutions

To show the generality of our method for quadratic eigenvalue problems, we consider a different example of $P(\lambda)$ arising in the stability analysis of front–pulse solutions in neuron networks with piecewise constant nonlinear functions [8]. By using a projection algorithm for a system of integral-differential equations, the authors of [8] derived the quadratic eigenvalue problem in the form

$$P(\lambda)\mathbf{c} = \left(\lambda^2 I + \lambda L + M\right)\mathbf{c} = \mathbf{0}, \tag{11}$$

where $M^T = M$ and $L^T = L$ are real-valued matrices in $\mathbb{R}^n$ and n is the number of front transitions in the front–pulse solution. Because of the translational symmetry, matrix M has always a nontrivial null space. When n is odd, the solution resembles a front from one stable equilibrium to another one with $(n-1)/2$ interior pulses. When n is even, the solution resembles a bound state of $n/2$ pulses. Instability of front–pulse solutions in the time evolution of the system of integral–differential equations corresponds to the case when the quadratic problem (11) has eigenvalues with $\mathrm{Re}\,\lambda > 0$. If all eigenvalues have $\mathrm{Re}\,\lambda < 0$, we say that front–pulse solutions are asymptotically stable. The case of eigenvalues with $\mathrm{Re}\,\lambda = 0$ is interpreted as the instability bifurcation of front–pulse solutions.

We can now formulate and prove the main result on the count of eigenvalues in the quadratic eigenvalue problem (11).

Theorem 3. *Let $M = M^T$ be a real-valued matrix with a simple zero, n_M negative and $n - n_M - 1$ positive eigenvalues and $L^T = L$ be a real-valued matrix. Assume that $L\mathbf{c}_0 = \lambda_0\mathbf{c}_0$ with $\lambda_0 \neq 0$ for $\mathbf{c}_0 \in \mathrm{Null}(M)$. Then,*

$$n - n_M - 1 = N_+ + N_r^+, \tag{12}$$

$$n + n_M = N_- + N_r^-, \tag{13}$$

where N_+ $(N_- = N_+)$ is the dimension of the invariant subspace of T associated with eigenvalues in the upper (lower) half-plane and N_r^+ (N_r^-) is dimension of the maximal invariant subspace of T associated with real non-zero eigenvalues, such that $(Jx, x) \geqslant 0$ $((Jx, x) \leqslant 0)$ for all eigenvectors of T in the invariant subspaces.

Proof. The only symmetry on eigenvalues of (11) is due to the fact that M and L are real-valued matrices. As a result, if λ is an eigenvalue of $P(\lambda)$ with the eigenvector $\mathbf{c}$, then $\bar{\lambda}$ is also an eigenvalue of $P(\lambda)$ with the eigenvector $\bar{\mathbf{c}}$. Under the condition that $L\mathbf{c}_0 = \lambda_0\mathbf{c}_0$ with $\lambda_0 \neq 0$ for $\mathbf{c}_0 \in \mathrm{Null}(M)$, operator T has a simple zero eigenvalue with the eigenvector $(\mathbf{c}_0, \mathbf{0})$. In addition, the quadratic problem (11) has a real eigenvalue $\lambda = -\lambda_0$ in the count N_r^- since

$$(Jx_0, x_0) = \langle M\mathbf{c_0}, \mathbf{c}_0\rangle - |\lambda_0|^2\langle \mathbf{c}_0, \mathbf{c}_0\rangle = -\lambda_0^2\|\mathbf{c}_0\|^2 < 0.$$

Let P_0 be an orthogonal projection to the complement of $\mathrm{Null}(M)$. Then,

$$T = \begin{bmatrix} 0 & -I \\ P_0MP_0 & P_0LP_0 \end{bmatrix}, \quad J = \begin{bmatrix} P_0MP_0 & 0 \\ 0 & -I \end{bmatrix} \tag{14}$$

satisfy conditions of Theorem 1 with $n - n_M - 1$ positive eigenvalues and $n + n_M - 1$ negative eigenvalues of J. (The number $n + n_M$ is reduced by one because the identity matrix I in representation (14) acts in R^{n-1} after the orthogonal projection P_0.) The rest of the proof coincides with the proof of Theorem 2. The count (13) is increased by one because of the real eigenvalue $\lambda = -\lambda_0$ with $(Jx_0, x_0) < 0$. $\square$

Since the stability boundary $\mathrm{Re}\,\lambda = 0$ separate eigenvalues in all numbers $N_\pm$ and $N_r^\pm$, we have to conclude that Theorem 3 is not useful in the context of stability analysis. For instance, if $n = 1$, then $n_M = 0$, $N_+ = N_- = N_r^+ = 0$ and $N_r^- = 1$ but the real eigenvalue $\lambda = -\lambda_0$ can be either positive or negative depending on the value of (scalar) L. A more useful conclusion on stability of front–pulse solutions is formulated in [8, p. 85] without a proof and is proved here for consistency.

Theorem 4. *Under the conditions of Theorem 3, the following properties are true:*

1. *The front–pulse solutions are stable only if P_0MP_0 is positive.*
2. *The front–pulse solutions are asymptotically stable if L and P_0MP_0 are positive.*
3. *There may exist at most $n-1$ Andronov–Hopf instability bifurcations with $\mathrm{Re}\,\lambda = 0$ and $\mathrm{Im}\,\lambda \neq 0$ and at most one real bifurcation with $\lambda = 0$ if P_0MP_0 is positive.*

Proof. To prove the first statement, we will show that there exist at least n_M real positive eigenvalues if $n_M \geqslant 1$. Indeed, if $n_M \geqslant 1$, and $L = 0$, there exist $2n_M$ real eigenvalues of the quadratic problem (11) in N_r^- with $\langle M\mathbf{c}_j, \mathbf{c}_j\rangle < 0$ for the corresponding eigenvectors. Let us replace L by aL and consider parameter continuation from $a = 0$ to $a = 1$. If $a = 0$, the $2n_M$ eigenvalues form n_M symmetric pairs of real eigenvalues $\lambda = \pm\langle M\mathbf{c}_j, \mathbf{c}_j\rangle/\langle \mathbf{c}_j, \mathbf{c}_j\rangle$ for $j = 1, 2, \ldots, n_M$. Each eigenvalue has a negative Krein signature, since

$$(Jx_j, x_j) = \langle M\mathbf{c}_j, \mathbf{c}_j\rangle - \lambda_j^2\langle \mathbf{c}_j, \mathbf{c}_j\rangle < 0, \quad j = 1, 2, \ldots, n_M.$$

If M is fixed and a is increased from 0 to 1, pairs of real eigenvalues move along real axis but may not cross the origin (since P_0MP_0 has empty kernel) and may not bifurcate due to coalescence with other eigenvalues (since (Jx_j, x_j) is negative for the corresponding eigenvectors). As a result, at least n_M eigenvalues remain positive if $n_M \geqslant 1$, so that the front–pulse solution is stable only if $n_M = 0$.

To prove the second and third statements, we consider quadratic forms associated to the quadratic eigenvalue problem (11):

$$\lambda^2\langle \mathbf{c}', \mathbf{c}'\rangle + \lambda\langle L\mathbf{c}', \mathbf{c}'\rangle + \langle M\mathbf{c}', \mathbf{c}'\rangle = 0, \tag{15}$$

where $\mathbf{c}' = P_0\mathbf{c}$. If P_0LP_0 and P_0MP_0 are positive, all roots of the quadratic equation (15) have $\mathrm{Re}\,\lambda < 0$. If P_0MP_0 is positive, then eigenvalues with $\mathrm{Re}\,\lambda = 0$ must have $\mathrm{Im}\,\lambda \neq 0$, that is all instability bifurcations are of the Andronov–Hopf type, except for a possible bifurcation of the real eigenvalue $\lambda = -\lambda_0$, where λ_0 is defined in Theorem 3. The counts (12) and (13) with $n_M = 0$ show that there may be at most $n-1$ pairs of eigenvalues with $\mathrm{Re}\,\lambda = 0$ and $\mathrm{Im}\,\lambda \neq 0$ in $N_+ = N_-$. $\square$

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