EXISTENCE AND STABILITY OF STOKES WAVES IN A FLUID OF A FINITE DEPTH VIA A CONFORMAL TRANSFORMATION

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Abstract. We review formulation of the Euler equations for a free surface in conformal variables for a fluid of a finite depth. The main purpose of this work is to incorporate the local and nonlocal constraints required for the derivation of equations of motion in conformal variables. We show that the small-amplitude expansion of Stokes waves satisfies the constraints and allows us to recover the correct expansions in physical coordinates. We also study existence and the spectral stability of locally unique continuations of Stokes waves with respect to their speed to recover the classical result that the instability transition occur at the extremal points of the wave energy, or equivalently, at the extremal points of the horizontal momentum.

1. Introduction

Surface gravity waves on water with finite depth are essential in fluid dynamics, modelling wave behaviour in coastal areas, lab tanks, and large ocean flows. Unlike idealized deep-water waves, finite depth causes complex bottom effects that modify both dispersion and nonlinear interactions between waves. These effects are especially significant near the beach. Understanding the stability of finite-amplitude Stokes waves—periodic travelling waves of constant form—is important for predicting wave breaking and rogue waves.

In deep water, linear stability of two-dimensional Stokes waves has been studied by Longuet-Higgins [1],[2]. Longuet-Higgins suggested that change of stability happen when the phase speed c is maximum with respect to wave height. Numerical computations by Tanaka [3] showed that exchange of stability occur at the stationary point of the total wave energy. In his next paper, Tanaka [4] confirmed this result using two independent numerical methods. He showed that seeming contradictions with earlier bifurcation studies disappear once the explicit form of the unstable eigenvector is taken into account: at the critical steepness the unstable mode becomes linearly dependent on the trivial translational mode.

Saffman [5] provided a serious analytical foundation for these numerical findings. Using Zakharov's Hamiltonian formulation, he proved that superharmonic disturbances exchange stability exactly when the wave energy becomes stationary with respect to wave height. He also explained why bifurcation does not produce a new branch of physically distinct steady waves. The Jordan block associated with zero eigenvalue has algebraic multiplicity four, but geometric multiplicity one corresponding only to a trivial translation. Thus, the extremal point of total energy identified by Saffman coincides with the extremal point of horizontal momentum.

Far fewer analytical and numerical studies exist for Stokes waves on water of finite depth. Early numerical work on solitary waves was performed by Tanaka [6]. Zufiria and Saffman

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[7] extended the energy-momentum criterion to both solitary and periodic waves in finite depth.

Recent years have seen big progress in numerical bifurcation analysis and high-order asymptotic methods. Berti, Maspero, and Ventura [9] conducted a detailed numerical study of the Benjamin–Feir instability. Creedon, Deconinck, and Trichtchenko [11] developed rigorous asymptotic techniques for high-frequency instabilities. Dyachenko and Pelinovsky [15] provided precise analytical proof of instability criterion using Babenko's pseudo-differential equation in conformal variables. They showed that superharmonic disturbances exchange stability exactly at the stationary points of the total wave energy and, equivalently, of the horizontal momentum.

Present work develops systematic conformal framework for Stokes waves on water of finite depth. We include all compatibility and normalization conditions into the Lagrangian formulation. Also, we demonstrate that the small-amplitude Stokes expansion satisfies these conditions, and recover classical expansions in physical variables up to the required order (see Appendices A and B for details). Using this foundation, we derived Babenko's equation in conformal variables and studied the existence and spectral stability of local Stokes waves with respect to speed. In particular, we prove that the classical instability transition occurs precisely at the extremal points of the total wave energy, equivalently at the extremal points of the horizontal momentum.

2. Equations of motion for a fluid of a finite depth

Following the plan, we start with conformal formulation of the water wave problem in finite depth. We work in the space of square–integrable and 2π-periodic functions denoted as $L^2_{\rm per}$ and equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cot \|_{L^2_{\rm per}}$. Functions in $L^2_{\rm per}$ with the first derivative being also square–integrable belong to Sobolev space $H^1_{\rm per}$ equipped with the norm $\| \cdot \|_{H^1_{\rm per}}$..

2.1. Conformal formulation of the water wave problem.

We consider two–dimensional 2π-periodic waves on the surface of a fluid of a finite depth h_0. The free surface is given by $y = \eta(x,t)$, and velocity potential $\varphi(x,y,t)$ satisfies Laplace equation in fluid domain. At free surface, kinematic and dynamic boundary conditions hold.

Let $z = z(u,t)$ be the conformal transformation of the region

$$\mathcal{D}_h = \{u \in \mathbb{C} : -\pi \le {\rm Re}(u) \le \pi, \ -h \le {\rm Im}(u) \le 0\}$$

into the fluid domain, where the free surface corresponds to ${\rm Im}(u) = 0$. The canonical variables for the equations of motion are defined on the free surface:

$$z(u,t) = \xi(u,t) + i\eta(u,t), \quad \varphi(u,t) = \psi(u,t), \quad u \in \mathbb{T} = [-\pi, \pi], \quad t \in \mathbb{R}$$

where we abuse notations for $\eta = \eta(u,t)$, which represents the free surface in the parameteric form: $x = \xi(u,t)$, $y = \eta(u,t)$.

The Dirichlet–to–Neumann operator for solutions of the Laplace equation in $\mathcal{D}_h$ yields a nonlocal operator T_h, whose Fourier symbol is given by

$$(T_h)_n = i \tanh(nh), \qquad n \in \mathbb{Z}.$$

It is obvious that T_h is a bounded, skew-adjont operator in $L^2_{\rm per}$ with the kernel ${\rm Ker}(T_h) = {\rm span}(1)$. The inverse operator T_h^{-1} is only defined on periodic functions with zero mean and we define it by the Fourier symbol

$$(T_h^{-1})_n = \begin{cases} -i\coth(nh), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{cases}$$

It follows from the Cauchy–Riemann equations for $z = z(u,t)$ that the variable $\xi = \xi(u,t)$ is related to the variable η by

$$\eta = h - h_0 + T_h(\xi - u), \tag{1}$$

where the relation between the physical depth h_0, the conformal depth h, and the mean value of η in u follows by integrating (1) over the period:

$$h = h_0 + \bar{\eta}, \quad \bar{\eta} := \frac{1}{2\pi}\int_0^{2\pi} \eta(u,t)\,du, \tag{2}$$

which suggest that both h and $\bar{\eta}$ may depend on time $t \in \mathbb{R}$. Inverting (1) yields

$$\xi - u = T_h^{-1}(\eta + h_0 - h) \quad \Rightarrow \quad \xi_u = 1 + K_h\eta, \tag{3}$$

where $K_h = T_h^{-1}\partial_u$ is a self–adjoint, positive operator on $L^2_{\rm per}$ with the domain $\mathrm{Dom}(K_h) = H^1_{\rm per}$, which is defined on periodic functions by its Fourier symbol

$$(K_h)_n = n\coth(nh), \qquad n \in \mathbb{Z}. \tag{4}$$

In the inversion formula (3), we set the mean-value of the variable part of ξ to 0 by the choice of the end points in the 2π-periodic domain for u.

2.2. **Derivation of equations of motion.** As is known from Appendix A in [16], see also Appendix A in [14], equations of motion in conformal variables are derived from the Lagrangian functional

$$\begin{aligned}\mathcal{L}(\psi,\xi,\eta,h,f,g) = & \int_0^{2\pi} \psi\big(\eta_t\xi_u - \eta_u\xi_t\big)\,du + \frac{1}{2}\int_0^{2\pi} \psi\,T_h\psi_u\,du - \frac{1}{2}\int_0^{2\pi} \eta^2\xi_u\,du \\ & + \int_0^{2\pi} f\big(\eta - h + h_0 - T_h(\xi - u)\big)\,du + g\left(\frac{1}{2\pi}\int_0^{2\pi}\eta du - h + h_0\right),\end{aligned}$$

where f and g is the Lagrange multipliers introduced to preserve the constraints (1) and (2), respectively. Without loss of generality, we can impose the zero-mean constraint on f since the corresponding term is accounted by g. The first term in the Lagrangian represents the kinetic energy of the fluid. The second and third terms represent the negative potential energy of the fluid.

Variations of the Lagrangian $\mathcal{L}$ in f and g recover the constraints (2) and (1). Variations of $\mathcal{L}$ in ξ, η, x, and h yield the following system of equations:

$$\delta_\psi: \quad \eta_t\xi_u - \eta_u\xi_t + T_h\psi_u = 0, \tag{5}$$

$$\delta_\eta: \quad \psi_u\xi_t - \psi_t\xi_u - \eta\xi_u + f + \frac{1}{2\pi}g = 0, \tag{6}$$

$$\delta_\xi: \quad \psi_t\,\eta_u - \psi_u\,\eta_t + \eta\,\eta_u + T_h f = 0, \tag{7}$$

$$\delta_h: \quad \frac{1}{2}\langle\psi,(\partial_h T_h)\,\psi_u\rangle - \langle f,(\partial_h T_h)(\xi - u)\rangle - g = 0, \tag{8}$$

where we have used the zero-mean constraint on f. By taking mean values of (5), (6), and (7), we obtain three conserved quantities that remain invariant under the evolution:

$$\frac{d}{dt}M(\eta) = 0, \qquad M(\eta) = \langle \eta, \xi_u \rangle = \langle \eta, (1 + K_h \eta) \rangle, \tag{9}$$

$$\frac{d}{dt}Q(\psi, \eta) + M(\eta) - g = 0, \qquad Q(\psi, \eta) = \langle \psi, \xi_u \rangle = \langle \psi, (1 + K_h \eta) \rangle, \tag{10}$$

$$\frac{d}{dt}P(\psi, \eta) = 0, \qquad P(\psi, \eta) = -\langle \psi, \eta_u \rangle, \tag{11}$$

where M, P, and Q correspond to the physical quantities of the mass, horizonal, and vertical momentum of the fluid [8]. We can now express Lagrange multipliers f and g from (7) and (8):

$$f = -T_h^{-1}(\psi_t \, \eta_u - \psi_u \, \eta_t + \eta \, \eta_u) \tag{12}$$

and

$$g = \frac{1}{2}\langle \psi, (\partial_h T_h) \, \psi_u \rangle - \langle f, (\partial_h T_h)(\xi - u) \rangle. \tag{13}$$

Substituting these expressions into (6) closes the system of equations (5) and (6) for (ψ, ξ, η, h) augmented with the constraints (2) and (3).

2.3. **Bernoulli constant.** Equations of motion can be extended by using the Bernoulli constant in the choice:

$$\psi(u, t) = \int_0^t B(t) dt + \tilde{\psi}(u, t), \tag{14}$$

where $\tilde{\psi}$ is required to have the zero mean value in u. With the decomposition (14), we redefine the vertical momentum as

$$Q(\tilde{\psi}, \eta) = \langle \tilde{\psi}, \xi_u \rangle = \langle \tilde{\psi}, (1 + K_h \eta) \rangle,$$

compared to (10). The vertical momentum $Q(\tilde{\psi}, \eta)$ is conserved in time if and only if B, $M(\eta)$, and g are related by

$$g = 2\pi B + M(\eta). \tag{15}$$

Combining (3), (12), (13), (14), and (15), we obtain the integral constraint in the form:

$$2\pi B + M(\eta) = \frac{1}{2}\langle \tilde{\psi}, (\partial_h T_h) \, \tilde{\psi}_u \rangle + \langle (\tilde{\psi}_t \, \eta_u - \tilde{\psi}_u \, \eta_t + (B + \eta) \, \eta_u), (\partial_h T_h^{-1}) \eta \rangle, \tag{16}$$

where we have used $\partial_h T_h^{-1} = -T_h^{-1} (\partial_h T_h) T_h^{-1}$ and the skew-adjointness of T_h^{-1}.

It follows from (9) that $M(\eta)$ is conserved in time. However, the Bernoulli constant B may depend on time. The two standard choices for B and $M(\eta)$ are:

- $B = 0$, then $M(\eta) \neq 0$;
- $M(\eta) = 0$, then $B \neq 0$.

The constraint $M(\eta) = 0$ with $B \neq 0$ is more convenient since it implies that the mean value of the surface elevation $\eta = \eta(x, t)$ in the physical coordinate x is zero, which defines the physical depth h_0 uniquely.

Substituting f and g from (12) and (15) into (6) yields

$$\tilde{\psi}_u \xi_t - \tilde{\psi}_t \xi_u - (B + \eta) \xi_u - T_h^{-1}(\tilde{\psi}_t \, \eta_u - \tilde{\psi}_u \, \eta_t + (B + \eta) \, \eta_u) + B + \frac{1}{2\pi} M(\eta) = 0. \tag{17}$$

The system of equations of motion is now closed as system (5), (16), and (17).

2.4. **Travelling wave reduction and Babenko's equation.** Steady travelling waves with constant speed c are expressed by the 2π-periodic profiles

$$\eta(u,t) = \eta(u - ct), \qquad \tilde{\psi}(u,t) = \tilde{\psi}(u - ct), \tag{18}$$

where $\tilde{\psi}$ is defined in (14). The conformal depth h in the domain $\mathcal{D}_h$ and the Bernoulli constant B are independent of time for the traveling waves. It follows from (3) that

$$\xi_t = -cK_h\eta, \quad \xi_u = 1 + K_h\eta.$$

Substutiting (18) into (5) yields

$$\tilde{\psi} = c\,T_h^{-1}\eta.$$

Substituting expressions for ξ and ψ into (17) yields Babenko's equation:

$$(c^2 - 2B)K_h\eta - \eta = \frac{1}{2}K_h\eta^2 + \eta\, K_h\eta - \frac{1}{2\pi}M(\eta). \tag{19}$$

The only variable of (19) is the traveling wave coordinate $u - ct$, which we denote again by u. The constraint (16) can be rewitten in the equivalent form:

$$2\pi B + M(\eta) = \frac{1}{2}\langle (c^2 - 2B)\eta - \eta^2, (\partial_h K_h)\eta\rangle, \tag{20}$$

where we have used that $(\partial_h T_h^{-1}) = -T_h^{-1}(\partial_h T_h)T_h^{-1}$, as well as the skew-adjointness of T_h^{-1} and ∂_u. Profiles of the traveling waves are defined by Babenko's equation (19) closed with the constraint (20).

The system of equations (19) and (20) admits a variational formulation. Define the action functional

$$\begin{aligned}\Lambda_{c,B}(\eta,h) := &\int_0^{2\pi} \left[\frac{1}{2}\,\eta\,(c^2K_h - 1)\eta - \frac{1}{2}\,\eta^2 K_h\eta - B\eta(1 + K_h\eta)\right] du \\ &+ g\left(\frac{1}{2\pi}\int_0^{2\pi} \eta du - h + h_0\right),\end{aligned} \tag{21}$$

where g is the Lagrange multiplier introduced to impose the constraint (2). Since c and B are fixed parameters, variation of $\Lambda_{c,B}(\eta,h)$ in (η,h) yields the system of equton:

$$\begin{aligned}\delta_\eta : &\quad (c^2K_h - 1)\eta - \frac{1}{2}K_h\eta^2 - \eta K_h\eta - B - 2BK_h\eta + \frac{g}{2\pi} = 0, \\ \delta_h : &\quad \frac{1}{2}\langle (c^2 - 2B)\eta - \eta^2, (\partial_h K_h)\eta\rangle - g = 0.\end{aligned}$$

With the formulation of traveling waves, Babenko's equation and together with the integral constraint, we obtained closed system which satisfied local and nonlocal conditions. The small-amplitude expansion of solutions to this system is presented in Appendix A. Having equations of motion and the traveling wave reduction, we now turn to the stability analysis. To this end, it is convenient to work in the traveling coordinate frame. In the next section we reformulate the time-dependent system in this frame, perform linearization around a Stokes wave profile, and derive the corresponding spectral problem. We will give particular attention to the zero eigenvalue and its multiplicity.

3. Time-dependent equations of motion and their linearization

In the previous section, we completed the derivation of the equations of motion and the travelling-wave reduction. We now start the time-dependent formulation and its linearization. Since the Bernoulli constant is useful in the computation of the traveling wave profile η, we use (14) with time-dependent $B = B(t)$. We also recall that conformal depth is time-dependent: $h = h(t)$. Combining (5), (16), and (17), we rewrite the main equations of motion in the form:

$$\eta_t \xi_u - \eta_u \xi_t + T_h \tilde{\psi}_u = 0, \tag{22}$$

$$\tilde{\psi}_u \xi_t - \tilde{\psi}_t \xi_u - (B+\eta)\xi_u - T_h^{-1}(\tilde{\psi}_t\, \eta_u - \tilde{\psi}_u\, \eta_t + (B+\eta)\, \eta_u) + B + \frac{1}{2\pi} M(\eta) = 0, \tag{23}$$

subject to the constraint

$$2\pi B + M(\eta) = \frac{1}{2}\langle \tilde{\psi}, (\partial_h T_h)\, \tilde{\psi}_u \rangle + \langle (\tilde{\psi}_t\, \eta_u - \tilde{\psi}_u\, \eta_t + (B+\eta)\, \eta_u), (\partial_h T_h^{-1})\eta \rangle. \tag{24}$$

The conservation of the mass $M(\eta)$, the horizontal momentum $P(\tilde{\psi}, \eta)$, and the vertical momentum $Q(\tilde{\psi}, \eta)$ follow from (9), (10), and (11). We write these three conserved quantities again:

$$M(\eta) = \langle \eta, (1 + K_h \eta) \rangle, \tag{25}$$

$$P(\tilde{\psi}, \eta) = -\langle \tilde{\psi}, \eta_u \rangle, \tag{26}$$

$$Q(\tilde{\psi}, \eta) = \langle \tilde{\psi}, (1 + K_h \eta) \rangle. \tag{27}$$

The following result gives the energy conservation.

Proposition 3.1. *The energy $H(\tilde{\psi}, \eta)$ given by*

$$H(\tilde{\psi}, \eta) = \langle (\eta^2 + 2B\eta), (1 + K_h \eta) \rangle - \langle \tilde{\psi}, T_h \partial_u \tilde{\psi} \rangle \tag{28}$$

is conserved in time if and only if the integral constraint (24) holds and $B'(t) M(\eta) = 0$.

Proof. To derive the expression for $\frac{d}{dt} H(\tilde{\psi}, \eta)$, we multiply (22) by $\tilde{\psi}_t$, multiply (23) by η_t, integrate both in u over the period, and add them together. We get

$$\int_0^{2\pi} \left[\tilde{\psi}_t T_h \partial_u \tilde{\psi} + (\eta_t \tilde{\psi}_u - \tilde{\psi}_t \eta_u)\xi_t - (B+\eta)\eta_t \xi_u - B\eta_t K_h \eta - \frac{1}{2}\eta_t K_h \eta^2 \right] du$$
$$+ \int_0^{2\pi} \eta_t T_h^{-1}(\eta_t \tilde{\psi}_u - \tilde{\psi}_t \eta_u) du + \frac{1}{2\pi} \int_0^{2\pi} (M(\eta) + 2\pi B)\eta_t du = 0.$$

We recall from (2) and (3) that

$$\xi_t = T_h^{-1} \eta_t + h'(t)(\partial_h T_h^{-1})\eta, \quad \xi_u = 1 + K_h \eta$$

as well as $h'(t) = \frac{1}{2\pi}\int_0^{2\pi} \eta_t du$. By using skew-adjointness of T_h^{-1} and self-adjoinness of $T_h \partial_u$ and K_h, we rewrite the previous equation in the equivalent form:

$$h'(t) \left[2\pi B + M(\eta) + \langle (\eta_t \tilde{\psi}_u - \tilde{\psi}_t \eta_u), (\partial_h T_h^{-1})\eta \rangle \right] + B'(t) M(\eta)$$
$$= \frac{1}{2}\frac{d}{dt} H(\tilde{\psi}, \eta) - \frac{1}{2} h'(t) \left[\langle (\eta^2 + 2B\eta), (\partial_h K_h)\eta \rangle - \langle \tilde{\psi}, (\partial_h T_h)\tilde{\psi}_u \rangle \right].$$

Since $K_h = T_h^{-1}\partial_u$, integration by parts yields that $\frac{d}{dt} H(\tilde{\psi}, \eta) = 0$ if and only if the integral constraint (24) is satisfied for any $h'(t)$ and $B'(t) M(\eta) = 0$. $\square$

3.1. **Reformulation in the traveling wave coordinate.** Passing to the traveling wave coordinate with the transformation

$$\eta(u,t) = \eta(u-ct,t), \qquad \tilde{\psi}(u,t) = \tilde{\psi}(u-ct,t), \tag{29}$$

and assuming that $h = h(t)$, we obtain from (3) that

$$\xi_t = -cK_h\eta + T_h^{-1}\eta_t + h'(t)(\partial_h T_h^{-1})\eta, \quad \xi_u = 1 + K_h\eta. \tag{30}$$

Furthermore, written in the traveling frame as in (29), we relate η and $\tilde{\psi}$ by

$$\tilde{\psi} = cT_h^{-1}\eta + \zeta, \tag{31}$$

which yields

$$\tilde{\psi}_t = cT_h^{-1}\eta_t + \zeta_t + ch'(t)(\partial_h T_h^{-1})\eta, \quad \tilde{\psi}_u = cK_h\eta + \zeta_u. \tag{32}$$

Equations (22) and (23) after the transformations (29), (30), and (32) are rewritten in the explicit form:

$$(1+K_h\eta)\eta_t - \eta_u T_h^{-1}\eta_t = -T_h\zeta_u + h'(t)\eta_u(\partial_h T_h^{-1})\eta, \tag{33}$$

$$\begin{aligned}(1+K_h\eta)\zeta_t + T_h^{-1}(\eta_u\zeta_t) + 2cT_h^{-1}\eta_t - \zeta_u(T_h^{-1}\eta_t) - T_h^{-1}(\zeta_u\eta_t) = S_h(\eta)\\ +\frac{1}{2\pi}M(\eta) + h'(t)\zeta_u(\partial_h T_h^{-1})\eta - ch'(t)(\partial_h T_h^{-1})\eta,\end{aligned} \tag{34}$$

where $S_h(\eta) := (c^2-2B)K_h\eta - \eta - \eta K_h\eta - \frac{1}{2}K_h\eta^2$ expresses Babenko's equation (19). After expansion, integration by parts, and using (33), the constraint (24) is rewritten in the form:

$$\begin{aligned}2\pi B + M(\eta) = \frac{1}{2}\langle (c^2-2B)\eta - \eta^2, (\partial_h K_h)\,\eta\rangle \\ + \frac{1}{2}\langle \zeta, (\partial_h T_h)\,\zeta_u\rangle + \langle (\zeta_t\eta_u - \zeta_u\eta_t + c\eta_t), (\partial_h T_h^{-1})\,\eta\rangle.\end{aligned} \tag{35}$$

The mass conservation $M(\eta)$ is still written in the form (25), but the other three conserved quantities (26), (27), and (28) are rewritten with the help of (31) in the equivalent form:

$$P(\tilde{\psi},\eta) = c\langle K_h\eta,\eta\rangle - \langle \zeta, \eta_u\rangle, \tag{36}$$

$$Q(\tilde{\psi},\eta) = \langle \zeta, (1+K_h\eta)\rangle, \tag{37}$$

$$H(\tilde{\psi},\eta) = c^2\langle K_h\eta,\eta\rangle + \langle \eta^2, (1+K_h\eta)\rangle + 2BM(\eta) - 2c\langle\zeta,\eta_u\rangle - \langle \zeta, T_h\partial_u\zeta\rangle, \tag{38}$$

where we used the fact that $T_h^{-1}\eta$ is orthogonal to $(1+K_h\eta)$ due to the skew-adjoint property of T_h^{-1} and ∂_u. By Proposition 3.1, the integral constraint (35) is *equivalent* to the conservation of $H(\tilde{\psi},\eta)$ if $h'(t) \neq 0$ and $B'(t)M(\eta) = 0$. Note that $H(\tilde{\psi},\eta)$ at $\zeta = 0$ does not give the action functional (21), for which the principal part is given by $c^2\langle K_h\eta,\eta\rangle - \langle \eta^2, (1+K_h\eta)\rangle - 2BM(\eta)$ divided by 2.

3.2. **Linearized equation of motion.** Next, we linearize equations (33), (34), and (35) near the profile $(\eta, 0)$ satisfying (19) and (20) with the perturbation (v, w). Due to the relation (2), we linearize $h(t)$ near the constant value $h = h_0 + \bar{\eta}$ with the perturbation $\bar{v}(t)$, where

$$\bar{\eta} = \frac{1}{2\pi}\int_0^{2\pi} \eta du, \quad \bar{v}(t) = \frac{1}{2\pi}\int_0^{2\pi} v(u,t)du. \tag{39}$$

We further linearize the Bernoulli constant B by the perturbation δB and the mean value $M(\eta) = \langle \eta, (1+K_h\eta)\rangle$ by the perturbation δM. Linearization of the system (33) and (34) yields the linearized equations of motion:

$$\mathcal{M}_h v_t = -T_h \partial_u w + \bar{v}' \eta' (\partial_h T_h^{-1})\eta, \tag{40}$$

$$\mathcal{M}_h^* w_t + 2c T_h^{-1} v_t = \mathcal{L}_h v + \bar{v} \partial_h S_h(\eta) - 2(\delta B) K_h \eta + \frac{1}{2\pi}\delta M - c\bar{v}'(\partial_h T_h^{-1})\eta, \tag{41}$$

where we have introduced the linearized operators

$$\begin{aligned} \mathcal{M}_h &= (1+K_h\eta) - \eta' T_h^{-1}, \\ \mathcal{M}_h^* &= (1+K_h\eta) + T_h^{-1}(\eta' \cdot), \\ \mathcal{L}_h &= (c^2-2B)K_h - 1 - K_h\eta - \eta K_h - K_h(\eta\cdot), \end{aligned}$$

and the derivative of coefficients of Babenko's equation (19) in h,

$$\partial_h S_h(\eta) = (c^2-2B)(\partial_h K_h)\eta - \eta(\partial_h K_h)\eta - \frac{1}{2}(\partial_h K_h)\eta^2.$$

The linearized constraint (35) is rewritten in the form

$$\begin{aligned} 2\pi(\delta B)\left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] + \delta M &= \langle v, \partial_h S_h(\eta)\rangle + \frac{1}{2}\bar{v}\langle (c^2-2B)\eta - \eta^2, (\partial_h^2 K_h)\,\eta\rangle \\ &\quad + \langle w_t, \eta'(\partial_h T_h^{-1})\,\eta\rangle + c\langle v_t, (\partial_h T_h^{-1})\eta\rangle. \end{aligned} \tag{42}$$

Proposition 3.2. *The linearized corrections to mass (25), horizontal momentum (36), vertical momentum (37) are given by*

$$\delta M = \langle (1+2K_h\eta), v\rangle + \bar{v}\langle \eta, (\partial_h K_h)\eta\rangle, \tag{43}$$

$$\delta P = 2c\langle K_h\eta, v\rangle - \langle \eta', w\rangle + c\bar{v}\langle \eta, (\partial_h K_h)\eta\rangle, \tag{44}$$

$$\delta Q = \langle (1+K_h\eta), w\rangle. \tag{45}$$

If $(\delta B)M(\eta) = 0$, then the linearized correction to energy (38) is given by $\delta H = 2c\delta P$. If (v,w) is a suitable solution of the linearized system (40)–(41), then δM, δP, and δQ are independent of time t.

Proof. Projection of (40) to 1 yields

$$\frac{d}{dt}\left[\langle (1+2K_h\eta), v\rangle + \bar{v}\langle \eta, (\partial_h K_h)\eta\rangle\right] = 0.$$

where we have used $T_h\partial_u 1 = 0$, $\mathcal{M}_h^* 1 = (1+2K_h\eta)$, and integration by parts. Hence, δM in (43) is constant in time t.

Projection of (41) to η' yields

$$\frac{d}{dt}\left[\langle \eta', w\rangle - 2c\langle K_h\eta, v\rangle - c\bar{v}\langle \eta, (\partial_h K_h)\eta\rangle\right] = 0,$$

where we have used $\mathcal{L}_h\eta' = 0$, $\mathcal{M}_h\eta' = \eta'$, and integration by parts. We have also used $\langle \eta', \partial_h S_h(\eta)\rangle = 0$ and $\langle \eta', K_h\eta\rangle = 0$ because η is even and η' is odd in u. Hence, δP in (44) is constant in time t.

Projection of (41) to 1 yields

$$\frac{d}{dt}\langle (1+K_h\eta), w\rangle = -\langle (1+2K_h\eta), v\rangle + \bar{v}\langle 1, \partial_h S_h(\eta)\rangle + \delta M = 0,$$

where we have used $\mathcal{M}_h 1 = 1 + K_h\eta$, $\mathcal{L}_h 1 = -(1+2K_h\eta)$, and integration by parts. Since $\langle 1, \partial_h S_h(\eta)\rangle = -\langle \eta, (\partial_h K_h)\eta\rangle$, the right-hand side is zero due to (43). Hence, δQ in (45) is constant in time t.

Finally, we check the energy conservation. Expanding the energy $H(\tilde{\psi},\eta)$ in (38) gives the linear correction δH to $H(\tilde{\psi},\eta)$ in the form:

$$\begin{aligned}
\delta H &= 2\langle c^2 K_h\eta + \eta(1+K_h\eta) + \frac{1}{2}K_h\eta^2, v\rangle + 2B(\delta M) + 2(\delta B)M(\eta) \\
&\quad + \bar{v}\left[c^2\langle \eta, (\partial_h K_h)\eta\rangle + \langle \eta^2, (\partial_h K_h)\eta\rangle\right] - 2c\langle w, \eta'\rangle \\
&= 4c^2\langle K_h\eta, v\rangle + 2(\delta B)M(\eta) - 2c\langle w,\eta'\rangle \\
&\quad + \bar{v}\left[c^2\langle \eta, (\partial_h K_h)\eta\rangle + \langle \eta^2, (\partial_h K_h)\eta\rangle + 2B\langle \eta, (\partial_h K_h)\eta\rangle + 4\pi B + 2M(\eta)\right] \\
&= 2c\delta P + 2(\delta B)M(\eta),
\end{aligned}$$

where we have used Babenko's equation (19), the integral constraint (20), the mean-value definition (39), and the first-order corrections (43) and (44). If either $\delta B = 0$ or $M(\eta) = 0$, then $\delta H = 2c\delta P$ is constant in time t. $\square$

Proposition 3.2 shows that the linearized equations of motion (40) and (41) inherit conservation of $M(\eta)$, $P(\tilde{\psi},\eta)$, $Q(\tilde{\psi},\eta)$, and $H(\tilde{\psi},\eta)$ of the nonlinear equations (22) and (23). Since the constraint (35) is equivalent to the energy conservation by Proposition 3.1, the linearized constraint (42) is equivalent to the time conservation of the second variation of the energy $H(\tilde{\psi},\eta)$ at the traveling wave solution with the profile $(cT_h^{-1}\eta,\eta)$.

3.3. **Spectral problem.**

Separating variables as

$$v(t,u) = e^{\lambda t}v(u), \quad w(t,u) = e^{\lambda t}w(u), \quad \bar{v}(t) = e^{\lambda t}\bar{v}, \quad (\delta B) = e^{\lambda t}b,$$

we rewrite the linearized system of equations (40) and (41) in the form:

$$-T_h\partial_u w = \lambda\left(\mathcal{M}_h v - \bar{v}\eta'(\partial_h T_h^{-1})\eta\right), \tag{46}$$

$$\mathcal{L}_h v + \bar{v}\partial_h S_h(\eta) - 2bK_h\eta = \lambda\left(\mathcal{M}_h^* w + 2cT_h^{-1}v + c\bar{v}(\partial_h T_h^{-1})\eta\right), \tag{47}$$

where we have set $\delta M = 0$ for $\lambda \neq 0$ due to the conservation of δM in time. The linear constraint (42) with $\delta M = 0$ is rewritten in the form

$$\begin{aligned}
2\pi b\left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] &= \langle v, \partial_h S_h(\eta)\rangle + \frac{1}{2}\bar{v}\langle (c^2-2B)\eta - \eta^2, (\partial_h^2 K_h)\,\eta\rangle \\
&\quad + \lambda\left(\langle w, \eta'(\partial_h T_h^{-1})\,\eta\rangle + c\langle v, (\partial_h T_h^{-1})\eta\rangle\right).
\end{aligned} \tag{48}$$

Proposition 3.3. *Let $(v,w) \in H^1_{\rm per} \times H^1_{\rm per}$ be an eigenfunction of the spectral problem (46) and (47) for $\lambda \neq 0$. Then, $\delta M = \delta P = \delta Q = 0$ in (43), (44), and (45). Moreover, the second variation of the energy $H(\tilde{\psi},\eta)$ at the traveling wave solution with the profile $\tilde{\psi} = cT_h^{-1}\eta$ and η is zero.*

Proof. Projection of (46) to 1 yields for $\lambda \neq 0$,

$$\langle (1+2K_h\eta), v\rangle + \bar{v}\langle \eta, (\partial_h K_h)\eta\rangle = 0, \tag{49}$$

which is the same as $\delta M = 0$ in (43). This was already used in the derivation of (47).

Projection of (47) to η' yields for $\lambda \neq 0$,

$$2c\langle K_h\eta, v\rangle - \langle \eta', w\rangle + c\bar{v}\langle \eta, (\partial_h K_h)\eta\rangle = 0, \tag{50}$$

which is the same as $\delta P = 0$ in (44). Eliminating $\bar{v}$ in (50) by using (49) yields a shorter orthogonality condition:

$$\langle \eta', w \rangle + c\langle 1, v \rangle = 0, \quad \text{or} \quad 2\pi c\bar{v} + \langle \eta', w \rangle = 0.$$

Projection of (47) to 1 yields for $\lambda \neq 0$,

$$\langle (1 + K_h\eta), w \rangle = 0, \tag{51}$$

which is the same as $\delta Q = 0$ in (45).

Finally, we show that the linear constraint (48) is equivalent to the zero second variation of the energy $H(\tilde{\psi}, \eta)$ at the traveling wave solution with the profile $\tilde{\psi} = cT_h^{-1}\eta$ and η. Taking the inner product of (46) with w and (47) with v, we obtain

$$\begin{aligned} \lambda\langle w, \mathcal{M}_h v \rangle &= -\langle T_h\partial_u w, w \rangle + \lambda\bar{v}\langle w, \eta'(\partial_h T_h^{-1})\,\eta \rangle \\ &= \langle \mathcal{L}_h v, v \rangle + \bar{v}\langle v, \partial_h S_h(\eta) \rangle - 2b\langle v, K_h\eta \rangle - c\bar{v}\lambda\langle v, (\partial_h T_h^{-1})\eta \rangle, \end{aligned}$$

where we have used $\langle v, T_h^{-1}v \rangle = 0$ due to skew-adjointness of T_h^{-1}. This yields

$$\lambda\bar{v}\left(\langle w, \eta'(\partial_h T_h^{-1})\,\eta \rangle + c\langle v, (\partial_h T_h^{-1})\eta \rangle\right) = \langle T_h\partial_u w, w \rangle + \langle \mathcal{L}_h v, v \rangle + \bar{v}\langle v, \partial_h S_h(\eta) \rangle - 2b\langle v, K_h\eta \rangle,$$

which allows to rewrite (48) in the equivalent form:

$$\begin{aligned} 2\pi b\bar{v} = {}& \langle T_h\partial_u w, w \rangle + \langle \mathcal{L}_h v, v \rangle + 2\bar{v}\langle v, \partial_h S_h(\eta) \rangle - 2b\langle v, K_h\eta \rangle \\ &+ \frac{1}{2}\bar{v}^2\left((c^2 - 2B)\langle \eta, (\partial_h^2 K_h)\,\eta \rangle - \langle \eta^2, (\partial_h^2 K_h)\,\eta \rangle\right) - b\bar{v}\langle \eta, (\partial_h K_h)\eta \rangle. \end{aligned}$$

This constraint is expressed in terms of the quadratic power of the perturbation term, which coincide with the second variation of the energy being zero if $\lambda \neq 0$. □

The linearized system which we studied above, preserves the conservation laws of the original nonlinear problem. The second variation of the energy is conserved when the integral constraint satisfies. Now, we can study spectral stability. In the following section we study the eigenvalue problem and determine multiplicity of the zero eigenvalue. We will show that the change in algebraic multiplicity occurs precisely when the horizontal momentum becomes stationary with respect to the wave speed, which provides the desired stability criterion.

4. Stability bifurcations at $\lambda = 0$

We begin by examining the spectral analysis of the zero eigenvalue. After deriving the linearized system and its spectral problem earlier, we now concentrate on the multiplicity of $= 0$. To understand potential stability exchanges, we first describe the traveling wave profiles in terms of parameters, then analyze the linearized spectral problem at the zero eigenvalue.

4.1. **Characterization of the profile η in Babenko's equation (19).** For uniqueness of the definition of solutions of Babenko's equation (19), we impose the zero mean-value constraint $M(\eta) = 0$ for the profile $\eta \in C^\infty_{\text{per}}$.

First, we neglect the constraint (2) and interpret the profile $\eta \in C^\infty_{\text{per}}$ and the Bernoulli constant B as functions of two independent parameters (c, h) in a certain region $\Omega \subset \mathbb{R}^2$ of the parameter plane for (c, h). Practical computations of Ω can only be obtained numerically.

Proposition 4.1. *Let $\eta \in C^\infty_{\rm per}$ be a solution of Babenko's equation (19) with $M(\eta) = 0$ for $(c,h) \in \Omega$ and assume that*

$$1 + \frac{1}{2\pi}\langle \eta, \partial_h K_h \eta\rangle \neq 0. \tag{52}$$

Then, the profile $\eta \in C^\infty_{\rm per}$ and the Bernoulli constant $B = B(c,h)$ are smoothly continued in (c,h) and $\partial_c B \neq c$.

Proof. We introduce $\gamma^2 = c^2 - 2B$ and rewrite (19) with $M(\eta) = 0$ in the explicit form:

$$\gamma^2 K_h \eta - \eta = \frac{1}{2} K_h \eta^2 + \eta K_h \eta. \tag{53}$$

Solution $\eta \in C^\infty_{\rm per}$ of (53) depends on the two independent parameters (γ, h), that is, $\eta = \eta(u;\gamma,h)$. The integral constraint (20) with $M(\eta) = 0$ yields

$$B = \frac{1}{4\pi}\langle \gamma^2 \eta - \eta^2, (\partial_h K_h)\eta\rangle, \tag{54}$$

which is uniquely defined as a function of (γ, h), that is, $B = B(\gamma,h)$.

By the implicit function theorem for $c^2 = \gamma^2 + 2B(\gamma,h)$, $B = B(\gamma,h)$ and $\eta = \eta(u;\gamma,h)$ are smoothly continued in (c,h) if $\gamma + \partial_\gamma B(\gamma,h) \neq 0$, which is equivalent to (52). Assuming (52), we invert $c^2 = \gamma^2 + 2B(\gamma,h)$ and define $\gamma = \gamma(c,h)$ locally uniquely for $(c,h) \in \Omega$. Substituting $\gamma = \gamma(c,h)$ to $B = B(\gamma,h)$ and $\eta = \eta(u;\gamma,h)$, we can overload notations and express them as $B = B(c,h)$ and $\eta = \eta(u;c,h)$.

In view of new notations, we have $c^2 = \gamma^2 + 2B(c,h)$ so that the solvability condition (52) is equivalent to $\partial_c B \neq c$ by the implicit function theorem. □

Assuming (52), we are allowed by Proposition 4.1 to differentiate (19) with respect to u, h, and c. This yields three linear equations:

$$\mathcal{L}_h \eta' = 0, \tag{55}$$

$$\mathcal{L}_h \partial_h \eta = -\partial_h S_h(\eta) + 2(\partial_h B) K_h \eta, \tag{56}$$

$$\mathcal{L}_h \partial_c \eta = -2c K_h \eta + 2(\partial_c B) K_h \eta, \tag{57}$$

where η' stands for the derivative of $\eta = \eta(u;c,h)$ with respect to u and we have used that $B = B(c,h)$, while preserving the normalization $M(\eta) = 0$. Since $\partial_c B \neq c$ by Proposition 4.1, the inhomogeneous term in (57) is not identically equal to 0.

Furthermore, derivative of the integral constraint (20) in h and c yields

$$2\pi(\partial_h B)\left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] = \langle \partial_h \eta, \partial_h S_h(\eta)\rangle + \frac{1}{2}\langle (c^2 - 2B)\eta - \eta^2, (\partial_h^2 K_h)\,\eta\rangle \tag{58}$$

and

$$2\pi(\partial_c B)\left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] = \langle \partial_c \eta, \partial_h S_h(\eta)\rangle + c\langle \eta, (\partial_h K_h)\,\eta\rangle. \tag{59}$$

Next, we add the constraint (2) to the locally unique smooth solutions of Proposition 4.1.

Proposition 4.2. *Let $\eta \in C^\infty_{\rm per}$ be a solution of Babenko's equation (19) with $M(\eta) = 0$ and $B = B(c,h)$ defined for $(c,h) \in \Omega$ under the condition (52). Assuming*

$$1 - \frac{1}{2\pi}\int_0^{2\pi} \partial_h \eta(u;c,h)\,du \neq 0, \tag{60}$$

there exists a locally unique solution $h = h(c)$ of (2) for fixed $h_0 > 0$ such that $\eta = \eta(u; c, h(c))$, $B = B(c, h(c))$, and $h = h(c)$ are smoothly continued in c.

Proof. The result follows by the implicit function theorem applied to the constraint (2) for a fixed $h_0 > 0$. □

4.2. **Geometric multiplicity of $\lambda = 0$.** We prove the following result.

Proposition 4.3. *Assume the non-degeneracy condition* $\mathrm{Ker}(\mathcal{L}_h) = \mathrm{span}(\eta')$ *and the solvability conditions (52) and (60). Then, the geometric multiplicity of $\lambda = 0$ in the spectral problem (46)–(47) closed with the integral constraint (48) is equal to two.*

Proof. Setting $\lambda = 0$ in the spectral problem (46)–(47) yields two nontrivial solutions:

$$\text{either} \quad (v, w) = (0, 1) \quad \text{or} \quad (v, w) = (\mathfrak{v}, 0),$$

where $\mathfrak{v} \in H^1_{\mathrm{per}}$ is a general solution of the linear equation

$$\mathcal{L}_h \mathfrak{v} + \bar{v} \partial_h S_h(\eta) - 2b K_h \eta = 0. \tag{61}$$

Both $\bar{v}$ and b are related to the solution $\mathfrak{v}$ of (61) by $\bar{v} = \frac{1}{2\pi}\langle 1, \mathfrak{v}\rangle$ and

$$2\pi \left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] b = \langle \mathfrak{v}, \partial_h S_h(\eta)\rangle + \frac{1}{2}\bar{v}\langle (c^2 - 2B)\eta - \eta^2, (\partial_h^2 K_h)\eta\rangle, \tag{62}$$

where (62) follows from (48) for $\lambda = 0$.

Due to different parity in u, $\partial_h S(\eta)$ and $K_h \eta$ are orthogonal to η'. If $\mathrm{Ker}(\mathcal{L}_h) = \mathrm{span}(\eta')$, then $\partial_h S(\eta)$ and $K_h \eta$ belong to the range of $\mathcal{L}_h$, so that there exists an inhomegeneous solution $\mathfrak{v} \in H^1_{\mathrm{per}}$ of (61) for any $\bar{v}, b \in \mathbb{R}$. It follows from (55), (56), and (57) that a general solution of (61) is given by

$$\mathfrak{v} = \alpha_1 \eta' + \alpha_2 \partial_h \eta + \alpha_3 \partial_c \eta, \tag{63}$$

for arbitrary $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with the correspondence:

$$\alpha_2 = \bar{v}, \quad \alpha_3 = \frac{b - \bar{v}(\partial_h B)}{(\partial_c B) - c}. \tag{64}$$

If (52) is assumed, then Proposition 4.1 implies that $\partial_c B \neq c$ so that α_3 is well-defined. If (60) is assumed, then Proposition 4.2 implies that

$$(1 - \overline{\partial_h \eta}) h'(c) - \overline{\partial_c \eta} = 0. \tag{65}$$

We will now show that $\alpha_2 = \alpha_3 = 0$, which implies that the geometric multiplicity of $\lambda = 0$ is equal to two. By taking the mean value of (63), we obtain

$$\bar{v} = \alpha_2 \overline{\partial_h \eta} + \alpha_3 \overline{\partial_c \eta}.$$

Due to (64) and (65), we use the constraint (60) and obtain

$$\bar{v} = \frac{b - \bar{v}\partial_h B}{(\partial_c B) - c} h'(c). \tag{66}$$

Substituting (63) into (62) yields

$$2\pi \left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] b = \alpha_2 \langle \partial_h \eta, \partial_h S_h(\eta)\rangle + \alpha_3 \langle \partial_c \eta, \partial_h S_h(\eta)\rangle + \frac{1}{2}\bar{v}\langle (c^2 - 2B)\eta - \eta^2, (\partial_h^2 K_h)\eta\rangle.$$

Using (58) and (64), we obtain

$$2\pi\left[1+\frac{1}{2\pi}\langle\eta,(\partial_h K_h)\eta\rangle\right](b-\bar{v}(\partial_h B))=\alpha_3\langle\partial_c\eta,\partial_h S_h(\eta)\rangle,$$

or equivalently,

$$2\pi\left[1+\frac{1}{2\pi}\langle\eta,(\partial_h K_h)\eta\rangle\right]\alpha_3(\partial_c B-c)=\alpha_3\langle\partial_c\eta,\partial_h S_h(\eta)\rangle.$$

Eliminating $(\partial_c B)$ by using (59) yields $-2\pi c\alpha_3=0$, which implies that $\alpha_3=0$. Hence, $b=\bar{v}\partial_h B$, which yields $\bar{v}=0$ from (66), which is equivalent to $\alpha_2=0$. □

4.3. **Algebraic multiplicity of $\lambda = 0$.** We prove the following result.

Proposition 4.4. *Assume the non-degeneracy condition* $\mathrm{Ker}(\mathcal{L}_h)=\mathrm{span}(\eta')$ *and the solvability conditions (52) and (60). Then, the algebraic multiplicity of $\lambda=0$ in the spectral problem (46)–(47) closed with the integral constraint (48) is at least three.*

Proof. We differentiate (46)–(47) in λ at $\lambda=0$ and substitute

$$v=\alpha\eta',\quad w=\beta, \tag{67}$$

with arbitary α and β, for which $\bar{v}=0$ and $b=0$. This yields the linear inhomogeneous system of equations:

$$-T_h\partial_u w_1=\alpha\mathcal{M}_h\eta', \tag{68}$$

$$\mathcal{L}_h v_1+\bar{v}_1\partial_h S_h(\eta)-2b_1K_h\eta=\beta\mathcal{M}_h^*1+2c\alpha T_h^{-1}\eta', \tag{69}$$

where $\bar{v}_1=\frac{1}{2\pi}\langle 1,v_1\rangle$ and b_1 is the derivative of b in λ at $\lambda=0$. The system (68)–(69) is closed with the linear constraint, which is obtained from differentiating (48) in $\lambda=0$ at $\lambda=0$:

$$\begin{aligned}2\pi b_1\left[1+\frac{1}{2\pi}\langle\eta,(\partial_h K_h)\eta\rangle\right]&=\langle v_1,\partial_h S_h(\eta)\rangle+\frac{1}{2}\bar{v}_1\langle(c^2-2B)\eta-\eta^2,(\partial_h^2K_h)\,\eta\rangle\\&\quad+(\beta+c\alpha)\langle\eta',(\partial_h T_h^{-1})\eta\rangle,\end{aligned} \tag{70}$$

where the last projection is generally nonzero since both $(\partial_h T_h^{-1})\eta$ and η' are spatially odd.

Solvability condition for projection of (68) to $\mathrm{Ker}(T_h\partial_u)=\mathrm{span}(1)$ is satisfied for arbitary α and β since

$$\alpha\langle 1,\mathcal{M}_h\eta'\rangle=\alpha\langle(1+2K_h\eta),\eta'\rangle=0.$$

Solvability condition for projection of (69) to $\mathrm{Ker}(\mathcal{L}_h)=\mathrm{span}(\eta')$ is satisfied for arbitary α and β since

$$\langle\eta',\beta\mathcal{M}_h^*1+2c\alpha T_h^{-1}\eta'\rangle=\beta\langle\eta',1\rangle+2c\alpha\langle\eta',K_h\eta\rangle=0.$$

Finally, projection of (69) to 1 yields the constraint:

$$-\langle(1+2K_h\eta),v_1\rangle-\bar{v}_1\langle\eta,(\partial_h K_h)\eta\rangle=2\pi\beta$$

since $\mathcal{L}_h1=-(1+2K_h\eta)$ and $\mathcal{M}_h1=1+2K_h\eta$. Since the left-hand side vanishes due to the orthogonality condition (49), it is required to set $\beta=0$.

Since $\beta = 0$, we only have a one-parameter family of generalized eigenvectors $(v_1, w_1) \in H^1_{\rm per} \times H^1_{\rm per}$ from solutions of the linear inhomogeneous system (68)–(69) for arbitrary α. Due to (57), we can obtain the exact solution of (68)–(69) with $\beta = 0$ in the form:

$$v_1 = \bar{v}_1 \partial_h \eta + \frac{b_1 - \bar{v}_1 \partial_h B + c\alpha}{(\partial_c B) - c} \partial_c \eta, \quad w_1 = -\alpha T_h^{-1} \eta.$$

By taking the mean value and using (65), we obtain similarly to (66),

$$\bar{v}_1 = \frac{b_1 - \bar{v}_1 \partial_h B + c\alpha}{(\partial_c B) - c} h'(c).$$

By using (70) with $\beta = 0$, we obtain

$$\begin{aligned} 2\pi b_1 \left[1 + \frac{1}{2\pi} \langle \eta, (\partial_h K_h) \eta \rangle \right] &= \frac{b_1 - \bar{v}_1 \partial_h B + c\alpha}{(\partial_c B) - c} \langle \partial_c v, \partial_h S_h(\eta) \rangle - c\alpha \langle \eta, (\partial_h K_h) \eta \rangle \\ &\quad + \bar{v}_1 \langle \partial_h \eta, \partial_h S_h(\eta) \rangle + \frac{1}{2} \bar{v}_1 \langle (c^2 - 2B)\eta - \eta^2, (\partial_h^2 K_h) \, \eta \rangle. \end{aligned}$$

By using (58) and (59), we obtain

$$-2\pi c (b_1 - \bar{v}_1 \partial_h B + \alpha \partial_c B) = 0,$$

which yields $\bar{v} = -\alpha h'(c)$ and $b_1 = -\alpha(\partial_c B + h'(c) \partial_h B)$. Hence, we obtain the exact solution to (68)–(69) for arbitrary α:

$$v_1 = -\alpha(\partial_c \eta + h'(c) \partial_h \eta), \quad w_1 = -\alpha T_h^{-1} \eta. \tag{71}$$

This gives the only generalized eigenvector in (71) to the two eigenvectors in (67), suggesting the Jordan block structure of one simple and one double blocks for $\lambda = 0$. □

4.4. **Threshold criterion on the triple algebraic multiplicity of $\lambda = 0$.** First, we express partial derivatives of the vertical momentum $\mathcal{P}(c, h) = c\langle K_h \eta, \eta \rangle$ computed at the two-parameter profile $\eta = \eta(u; c, h)$ satisfying (19) with $M(\eta) = 0$ and $B = B(c, h)$, according to Proposition 4.1.

Proposition 4.5. *Let $\eta \in C^\infty_{\rm per}$ satisfy (19) with $M(\eta) = 0$ and $B = B(c, h)$ under the solvability condition (52). Then, $\mathcal{P}(c, h) = c\langle K_h \eta, \eta \rangle$ satisfies*

$$\frac{\partial \mathcal{P}}{\partial c} = \langle K_h \eta, \eta \rangle - c \langle 1, \partial_c \eta \rangle, \quad \frac{\partial \mathcal{P}}{\partial h} = -c \langle 1, \partial_h \eta \rangle. \tag{72}$$

Proof. We have by differentiating $M(\eta) = 0$ in c and h that

$$\frac{\partial}{\partial c} M(\eta) = \langle (1 + 2K_h \eta), \partial_c \eta \rangle = 0, \quad \frac{\partial}{\partial h} M(\eta) = \langle (1 + 2K_h \eta), \partial_h \eta \rangle + \langle \eta, (\partial_h K_h) \eta \rangle = 0. \tag{73}$$

Then, we obtain

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial c} &= \langle K_h \eta, \eta \rangle + 2c \langle K_h \eta, \partial_c \eta \rangle \\ &= \langle K_h \eta, \eta \rangle + c \langle (1 + 2K_h \eta), \partial_c \eta \rangle - c \langle 1, \partial_c \eta \rangle \\ &= \langle K_h \eta, \eta \rangle - c \langle 1, \partial_c \eta \rangle \end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathcal{P}}{\partial h} &= 2c\langle K_h\eta, \partial_h\eta\rangle + c\langle \eta, (\partial_h K_h)\eta\rangle \\ &= c\langle (1+2K_h\eta), \partial_h\eta\rangle + c\langle \eta, (\partial_h K_h)\eta\rangle - c\langle 1, \partial_h\eta\rangle \\ &= -c\langle 1, \partial_h\eta\rangle,\end{aligned}$$

which yields the result. ☐

Assuming the constraint (60), we define the 1-parameter family of solutions in Proposition 4.2. By using (65), we obtain

$$\begin{aligned}\mathcal{P}'(c) &= \frac{\partial \mathcal{P}}{\partial c} + h'(c)\frac{\partial \mathcal{P}}{\partial h} \\ &= \langle K_h\eta, \eta\rangle - c\langle 1, \partial_c\eta + h'(c)\partial_h\eta\rangle \\ &= \langle K_h\eta, \eta\rangle - 2\pi c h'(c).\end{aligned}$$

This quantity gives the threshold criterion on the triple algebraic multiplicity of $\lambda = 0$.

Proposition 4.6. *Assume the non-degeneracy condition* $\mathrm{Ker}(\mathcal{L}_h) = \mathrm{span}(\eta')$ *and the solvability conditions (52) and (60). Then, the algebraic multiplicity of* $\lambda = 0$ *in the spectral problem (46)–(47) closed with the integral constraint (48) is triple if and only if* $\mathcal{P}'(c) \neq 0$.

Proof. We construct a Jordan chain of generalized eigenvectors which starts with the translational mode $\eta' \in \mathrm{Ker}(\mathcal{L}_h)$ for $\alpha = 1$ and $\beta = 0$. Thus, we select the only solution of (61) as

$$\mathfrak{v} = \eta', \quad w = 0,$$

for which $\bar{v} = 0$ and $b = 0$ by Proposition 4.3. Solving (68), (69), and (70) with $\alpha = 1$ and $\beta = 0$, we obtain the following solution

$$v_1 = -\partial_c\eta - h'(c)\partial_h\eta, \quad w_1 = -T_h^{-1}\eta,$$

for which $\bar{v}_1 = -h'(c)$ and $b_1 = -\partial_c B - h'(c)\partial_h B$ by Proposition 4.4. We now consider solutions of the second Jordan block from the system

$$-T_h\partial_u w_2 = -\mathcal{M}_h(\partial_c\eta + h'(c)\partial_h\eta) + h'(c)\eta'(\partial_h T_h^{-1})\eta, \tag{74}$$

$$\mathcal{L}_h v_2 + \overline{v}_2\partial_h S_h(\eta) - 2b_2 K_h\eta = -\mathcal{M}_h^* T_h^{-1}\eta - 2cT_h^{-1}(\partial_c\eta + h'(c)\partial_h\eta) - ch'(c)(\partial_h T_h^{-1})\eta, \tag{75}$$

closed with the integral constraint

$$2\pi b_2\left[1 + \frac{1}{2\pi}\langle \eta, (\partial_h K_h)\eta\rangle\right] = \langle v_2, \partial_h S_h(\eta)\rangle + \frac{1}{2}\overline{v}_2\langle (c^2 - 2B)\eta - \eta^2, (\partial_h^2 K_h)\,\eta\rangle, \tag{76}$$

where we have used the parity argument on η.

Solvability condition for projection of (74) to $\mathrm{Ker}(T_h\partial_u) = \mathrm{span}(1)$ is satisfied since

$$\begin{aligned}&\langle 1, -\mathcal{M}_h(\partial_c\eta + h'(c)\partial_h\eta) + h'(c)\eta'(\partial_h T_h^{-1})\eta\rangle \\ &= -\langle (1+2K_h\eta), (\partial_c\eta + h'(c)\partial_h\eta)\rangle - h'(c)\langle \eta, (\partial_h K_h)\eta\rangle,\end{aligned}$$

which is 0 due to (73).

Solvability condition for projection of (75) to $\mathrm{Ker}(\mathcal{L}_h) = \mathrm{span}(\eta')$ is given by

$$\begin{aligned} 0 &= \langle \eta', -\mathcal{M}_h^* T_h^{-1}\eta - 2cT_h^{-1}(\partial_c \eta + h'(c)\partial_h \eta) - ch'(c)(\partial_h T_h^{-1})\eta\rangle \\ &= \langle K_h \eta, \eta\rangle - 2\pi c h'(c) \\ &= \mathcal{P}'(c), \end{aligned}$$

where we have used (72) in Proposition 4.5. If $\mathcal{P}'(c) \neq 0$, the solvability condition is not satisfied and no second generalized eigenvector satisfying (74), (75), and (76) exists.

If $\mathcal{P}'(c) = 0$, then we can solve (74), (75), and (76) without problems. Indeed, projection of (75) to 1 specifies uniquely $\bar{v}_2$ from

$$\langle (1 + 2K_h\eta), v_2\rangle + \bar{v}_2 \langle \eta, (\partial_h K_h)\eta\rangle = 0.$$

Finally, the integral constraint (76) specifies uniquely b_2 under the constraint (52). This gives a solution for v_2 and w_2, where v_2 is odd in u and w_2 is even in u. □

5. Conclusion

In this work, we developed a systematic conformal formulation for Stokes waves on water of finite depth. The central technical achievement of the paper is the consistent incorporation of both the local compatibility condition, which follows from the Cauchy–Riemann equations, and the nonlocal constraint on the conformal depth into the Lagrangian formulation via suitable Lagrange multipliers. This approach yields a closed system of equations of motion in conformal variables and leads to a well-posed Babenko equation supplemented by the corresponding integral constraint.

Within this framework, we carried out a spectral stability analysis of traveling wave solutions with respect to their speed. We proved that superharmonic disturbances exchange stability precisely at the extremal points of the total wave energy, or equivalently, at the extremal points of the horizontal momentum. The zero eigenvalue of the linearized operator has geometric multiplicity two and algebraic multiplicity at least three. Moreover, the algebraic multiplicity is exactly three if and only if the derivative of the horizontal momentum with respect to the wave speed is nonzero. These results recover the classical energy-momentum criterion in the finite-depth case and place it on a conformal foundation.

The formulation developed in this paper provides a solid basis for further analytical and numerical investigation of Stokes waves in finite depth. In particular, it opens the possibility of studying high-frequency instabilities, performing numerical continuation of solution branches. These results place the classical energy-momentum stability criterion on a foundation within the conformal formulation for water waves of finite depth.

Appendix A. Small-amplitude expansion of the wave profile

In order to verify the consistency of the conformal formulation developed in the main text, we obtain the small–amplitude (Stokes) expansion of the traveling wave profile η from Babenko's equation (19) and the integral constraint (20). The calculations are carried out in conformal variables and then compared with known results from the literature.

A.1. **Spectral properties of K_h.** For each depth $h > 0$, the operator K_h is defined by its action on Fourier modes (4). Applied to the cosine modes, it gives

$$K_h \cos(nu) = n \coth(nh) \cos(nu), \qquad n \in \mathbb{N}. \tag{77}$$

Since the eigenfunctions $\{\cos(nu)\}_{n\geq 1}$ form an orthogonal basis for the space of even 2π–periodic functions, the operator K_h is diagonalized by this Fourier basis. Because the eigenvalues of K_h given by $n\coth(nh)$ are real for all $n\in\mathbb{N}$ and $h>0$, the operator is self–adjoint on $L^2_{\rm per}$:

$$\langle f,(K_h g)\rangle = \langle (K_h f), g\rangle, \qquad f,g\in L^2_{\rm per}.$$

Differentiating (77) with respect to h yields

$$(\partial_h K_h)\cos(nu) = -\frac{n^2}{\sinh^2(nh)}\cos(nu), \qquad n\in\mathbb{N}. \tag{78}$$

Thus $(\partial_h K_h)$ is again diagonal in the Fourier cosine basis, with real negative eigenvalues, so that $\partial_h K_h$ is again self–adjoint on $L^2_{\rm per}$:

$$\langle f,(\partial_h K_h)g\rangle = \langle g,(\partial_h K_h)f\rangle, \quad f,g\in L^2_{\rm per}.$$

For further use, we record the standard trigonometric identities

$$\cos(u)\cos(u) = \frac{1}{2}\left(1+\cos 2u\right), \tag{79}$$

$$\cos(u)\cos(2u) = \frac{1}{2}\left(\cos u+\cos 3u\right), \tag{80}$$

which we use in the computations of the small-amplitude expansion.

A.2. **Stokes expansion of the wave profile.** Steady periodic gravity waves are symmetric with respect to the crest, so the wave profile $\eta(u)$ is an *even* function in $L^2_{\rm per}$. Thus Fourier expansion contains only cosine harmonics:

$$\eta(u;h,a) = \sum_{n=0}^{\infty}\eta^{(n)}(h,a)\cos(nu). \tag{81}$$

In the small–amplitude construction, we define

$$\eta(u;h,a) = a\cos(u) + a^2\left[A_0(h)+A_2(h)\cos(2u)\right] + a^3 A_3(h)\cos(3u) + \mathcal{O}(a^4), \tag{82}$$

$$c^2(h,a) = c_0^2(h) + a^2 c_2(h) + O(a^4), \tag{83}$$

and

$$B = a^2 B_2(h) + \mathcal{O}(a^4), \tag{84}$$

where the coefficients $B_2(h)$, $c_0(h)$, $c_2(h)$, $A_0(h)$, $A_2(h)$, $A_3(h)$ are computed by substituting expansions (82), (83), and (84) into (19) and (20). We note on comparison between (81) and (82) that

$$\left\{\begin{array}{l} \eta^{(0)}(h,a) = a^2 A_0(h) + \mathcal{O}(a^4), \\ \eta^{(1)}(h,a) = a, \\ \eta^{(2)}(h,a) = a^2 A_2(h) + \mathcal{O}(a^4), \\ \eta^{(3)}(h,a) = a^3 A_3(h) + \mathcal{O}(a^5). \end{array}\right.$$

Due to arbitrary normalization of the amplitude parameter $a\in\mathbb{R}$, we do not use correction term in $\eta^{(1)}(h,a)$. We also note that even powers of a enter even Fourier cosine modes and odd powers of a enter odd Fourier cosine modes.

A.3. **Order $\mathcal{O}(a)$: expression for $c_0^2(h)$.** At order $\mathcal{O}(a)$, Babenko's equation (19) reduces to a linear eigenvalue problem for the fundamental mode $\eta_1(u) = \cos u$:

$$(c_0^2 K_h - 1)\eta_1 = 0. \tag{85}$$

Using the diagonal action of K_h on cosine modes, given by (77) for $n = 1$, we obtain

$$c_0^2(h)\,\coth(h) = 1,$$

which yields the leading–order dispersion relation

$$c_0^2(h) = \tanh(h). \tag{86}$$

A.4. **Order $\mathcal{O}(a^2)$: expression for $A_0(h)$, $A_2(h)$, and $B_2(h)$.** At order $\mathcal{O}(a^2)$, Babenko's equation (19) reduces to the linear inhomogeneous equation

$$(c_0^2 K_h - 1)\eta_2 + \frac{1}{2\pi}\langle 1, \eta_2\rangle = \frac{1}{2}K_h\eta_1^2 + \eta_1 K_h \eta_1 - \frac{1}{2\pi}\langle \eta_1, K_h\eta_1\rangle, \tag{87}$$

where we have expanded $M(\eta)$ as follows:

$$M(\eta) = \langle \eta, (1 + K_h\eta)\rangle = a^2\left(\langle 1, \eta_2\rangle + \langle \eta_1, K_h\eta_1\rangle\right) + \mathcal{O}(a^4). \tag{88}$$

By using (77) and (79), we obtain

$$\frac{1}{2}K_h\eta_1^2 + \eta_1 K_h\eta_1 = \frac{1}{2}\coth(2h)\cos 2u + \frac{1}{2}\coth(h)(1 + \cos 2u).$$

Since

$$\frac{1}{2\pi}\langle \eta_1, K_h\eta_1\rangle = \frac{1}{2}\coth(h),$$

the right-hand side of (87) contains only projection to $\cos 2u$:

$$\frac{1}{2}K_h\eta_1^2 + \eta_1 K_h\eta_1 - \frac{1}{2\pi}\langle \eta_1, K_h\eta_1\rangle = \frac{1}{2}[\coth(2h) + \coth(h)]\cos 2u.$$

Solving (87) with $\eta_2(u;h) = A_0(h) + A_2(h)\cos 2u$ yields arbitrary $A_0(h)$ and uniquely defined $A_2(h)$ by

$$A_2(h) = \frac{\coth(2h) + \coth(h)}{2(2\tanh(h)\coth(2h) - 1)} = \frac{c_0^4(h) + 3}{4\,c_0^6(h)}, \tag{89}$$

where we transformed (89) by using the identities:

$$\coth(h) = \frac{1}{c_0^2(h)}, \quad \coth(2h) = \frac{1 + c_0^4(h)}{2c_0^2(h)}.$$

Although $A_0(h)$ is not defined by (87) obtained from Babenko's equation (19), we get a relation between $A_0(h)$ and $B_2(h)$ from the integral constraint (20) at the order of $\mathcal{O}(a^2)$. By using (78), we obtain

$$(\partial_h K_h)\eta = -a\,\operatorname{csch}^2(h)\cos u - 4a^2 A_2\operatorname{csch}^2(2h)\cos 2u + \mathcal{O}(a^3),$$

from which we derive

$$\frac{c^2}{2}\langle \eta, (\partial_h K_h)\eta\rangle = -\frac{\pi}{2}c_0^2\operatorname{csch}^2(h)a^2 + \mathcal{O}(a^4). \tag{90}$$

Collecting terms of the order of $\mathcal{O}(a^2)$ in (20), (84), and (88) yields the relation between $A_0(h)$ and $B_2(h)$

$$A_0(h) + B_2(h) = -\frac{1}{2}\coth(h) - \frac{1}{2\sinh(2h)}, \tag{91}$$

where we have used (86).

A.5. **Order $\mathcal{O}(a^3)$: expressions for $c_2(h)$ and $A_3(h)$.** At order $\mathcal{O}(a^3)$, Babenko's equation (19) reduces to the linear inhomogeneous equation

$$(c_0^2 K_h - 1)\eta_3 + c_2 K_h \eta_1 = K_h \eta_1 \eta_2 + \eta_1 K_h \eta_2 + \eta_2 K_h \eta_1 + 2B_2 K_h \eta_1, \tag{92}$$

where we have used the expansion (88). The right–hand side of (92) contains only projections to $\cos u$ and $\cos 3u$, which have zero mean. By using (77) and (80), we obtain

$$\begin{aligned}
K_h \eta_1 \eta_2 &= \frac{1}{2}(2A_0 + A_2)\coth(h)\cos u + \frac{3}{2}A_2 \coth(3h)\cos 3u,\\
\eta_2 K_h \eta_1 &= \frac{1}{2}(2A_0 + A_2)\coth(h)\cos u + \frac{1}{2}A_2 \coth(h)\cos 3u,\\
\eta_1 K_h \eta_2 &= A_2 \coth(2h)\cos u + A_2 \coth(2h)\cos 3u,
\end{aligned}$$

which yields

$$\begin{aligned}
&K_h \eta_1 \eta_2 + \eta_2 K_h \eta_1 + \eta_1 K_h \eta_2\\
&\quad = [(2A_0 + A_2)\coth(h) + A_2 \coth(2h)]\cos u + \frac{1}{2}A_2[\coth(h) + 2\coth(2h) + 3\coth(3h)]\cos 3u.
\end{aligned}$$

The Fourier mode $\cos u$ is a solution of the homogeneous equation (85). To eliminate non-periodic solutions of (92), we remove it by the choice of c_2:

$$c_2(h) = 2B_2(h) + 2A_0(h) + A_2(h) + A_2(h)\,\frac{\coth(2h)}{\coth h}.$$

By using (89) and (91), we obtain

$$\begin{aligned}
c_2(h) &= -\frac{1}{\sinh(2h)} - \frac{1}{\tanh h} + \frac{(3 + \tanh^2 h)^2}{8\tanh^3 h}\\
&= \frac{9 - 6\tanh^2 h + 5\tanh^4 h}{8\tanh^3 h}.
\end{aligned} \tag{93}$$

Solving (92) with $\eta_3(u;h) = A_3(h)\cos 3u$, see (77) and (86) we obtain

$$A_3 = \frac{1}{2}A_2 \frac{\coth(h) + 2\coth(2h) + 3\coth(3h)}{3\tanh(h)\coth(3h) - 1}$$

Using

$$\coth(3h) = \frac{\coth(h)\coth(2h) + 1}{\coth(2h) + \coth(h)} = \frac{1 + 3c_0^4}{c_0^2(3 + c_0^4)},$$

we get

$$\coth(h) + 2\coth(2h) + 3\coth(3h) = \frac{c_0^8 + 14c_0^4 + 9}{c_0^2(3 + c_0^4)}$$

and

$$3\tanh(h)\coth(3h) - 1 = \frac{8c_0^4}{3 + c_0^4}.$$

which yields

$$A_3(h) = \frac{\big(c_0^4(h) + 3\big)\big(c_0^8(h) + 14c_0^4(h) + 9\big)}{64\, c_0^{12}(h)}.$$

The integral constraint (20) does not have nonzero terms at the order of $\mathcal{O}(a^3)$ due to (84), (88), (90), and

$$-\frac{1}{2}\langle \eta^2, (\partial_h K_h)\eta \rangle = \mathcal{O}(a^4).$$

This completes the construction of the leading-order expansions (82), (83), and (84) up to the order of $\mathcal{O}(a^4)$ from Babenko's equation (19) and the integral constraint (20).

Appendix B. Comparison with Stokes expansions in physical depth

For additional verification, we compare the outcome of our computations in Appendix A with the Stokes expansions used in recent works [9], [11], [17], where they were obtained by three different analytical methods.

B.1. **Comparison with** [9]**.** We set the Bernoulli constant B to 0 so that $B_2(h) = 0$ in (91) and

$$A_0(h) = -\frac{1}{2}\coth(h) - \frac{1}{2\sinh(2h)}.$$

Computing the mean value of the surface elevation in physical coordinate x yields by the chain rule $dx = (1 + K_h\eta)du$, see (3),

$$\eta^{[0]} := \frac{1}{2\pi}M(\eta) = a^2\left[A_0(h) + \frac{1}{2}\coth(h)\right] + \mathcal{O}(a^4). \tag{94}$$

This coincides with the correction $\eta_2^{[0]}$ in eq. (2.6) of [9] given by

$$\eta_2^{[0]} = A_0(h) + \frac{1}{2}\coth(h) = -\frac{1}{2\sinh(2h)} = \frac{c_0^4 - 1}{4c_0^2},$$

due to (91) and $c_0^2 = \tanh(h)$. By using (3), we get

$$\xi - u = a\coth(h)\sin u + a^2\coth(2h)\sin 2u + \mathcal{O}(a^3),$$

which is inverted as

$$u = x - a\coth(h)\sin x + \mathcal{O}(a^2).$$

The Stokes expansion (82) is rewritten as

$$\eta = a\cos(x) + a^2\left(\eta_2^{[0]} + \eta_2^{[2]}\cos 2x\right) + \mathcal{O}(a^3),$$

where

$$\eta_2^{[2]} = A_2(h) - \frac{1}{2}\coth(h) = \frac{3 - c_0^4}{4c_0^6},$$

in agreement with [9].

Using (2), we obtain

$$h = h_0 + a^2 A_0(h) + \mathcal{O}(a^4), \tag{95}$$

in terms of the physical depth h_0. By using the expansion (83) with the correction terms (86) and (93), we express the speed c in terms of the physical depth as

$$\begin{aligned}
c^2 &= c_0^2(h) + a^2 c_2(h) + \mathcal{O}(a^4) \\
&= \tanh(h) + a^2 \frac{9 - 6\tanh^2(h) + 5\tanh^4(h)}{8\tanh^3(h)} + \mathcal{O}(a^4) \\
&= \tanh(h_0) + a^2 \left[\operatorname{sech}^2(h_0) A_0(h_0) + \frac{9 - 6\tanh^2(h_0) + 5\tanh^4(h_0)}{8\tanh^3(h_0)} \right] + \mathcal{O}(a^4).
\end{aligned}$$

from which we recover eq. (2.8) in [9]:

$$\begin{aligned}
c &= c_0(h_0) + \frac{a^2}{2c_0(h_0)} \left[(1 - c_0^4)\eta_2^{[0]} - \frac{1}{2}\coth(h_0)\operatorname{sech}^2(h_0) + \frac{9 - 6\tanh^2(h_0) + 5\tanh^4(h_0)}{8\tanh^3(h_0)} \right] + \mathcal{O}(a^4) \\
&= c_0(h_0) + \frac{a^2}{2c_0(h_0)} \left[(1 - c_0^4)\eta_2^{[0]} + \frac{9 - 10\tanh^2(h_0) + 9\tanh^4(h_0)}{8\tanh^3(h_0)} \right] + \mathcal{O}(a^4).
\end{aligned}$$

B.2. **Comparison with** [11]**.** We set $M(\eta) = 0$ to normalize the physical depth h_0 by enforcing the mean value of the surface elevation in physical coordinate x to zero, $\bar{\eta} = 0$. This defines uniquely from (91) and (94):

$$A_0(h) = -\frac{1}{2}\coth(h), \quad B_2(h) = -\frac{1}{2\sinh(2h)}.$$

The expansion (95) yields now

$$h = h_0 - \frac{1}{2} a^2 \coth(h) + \mathcal{O}(a^4),$$

from which we derive

$$c = c_0(h_0) + \frac{a^2}{2c_0(h_0)} \left[\frac{9 - 10\tanh^2(h_0) + 9\tanh^4(h_0)}{8\tanh^3(h_0)} \right] + \mathcal{O}(a^4), \tag{96}$$

which is different from (A.2b) in [11] since

$$\begin{aligned}
\frac{6 + 2\cosh(2h_0) + \cosh(4h_0)}{8\sinh^3(h_0)\cosh(h_0)} &= \frac{9 + 12\sinh^2(h_0) + 8\sinh^4(h_0)}{8\tanh^3(h_0)\cosh^4(h_0)} \\
&= \frac{9 - 6\tanh^2(h_0) + 5\tanh^4(h_0)}{8\tanh^3(h_0)} \\
&\neq \frac{9 - 10\tanh^2(h_0) + 9\tanh^4(h_0)}{8\tanh^3(h_0)}.
\end{aligned}$$

In addition, the expansions (A.1) and (A.2) in [11] shows that the Bernoulli constant B is also set to 0, which is impossible in our solutions. However, expansion (A.3b) with (A.4e) shows that the speed c is not defined by the expansion (A.2) and the velocity potential ψ has a linear growth in x. Renormalizing the speed appropriately, we recover the correct expansion:

$$\begin{aligned} c &= c_0(h_0) + a^2 \left[\frac{9 - 10\tanh^2(h_0) + 9\tanh^4(h_0)}{16\tanh^4(h_0)} - \frac{1}{4\sinh^2(h_0)} \right] + \mathcal{O}(a^4) \\ &= c_0(h_0) + \frac{a^2}{2c_0(h_0)} \left[\frac{9 - 10\tanh^2(h_0) + 9\tanh^4(h_0)}{8\tanh^3(h_0)} - \frac{1}{2}\coth(h_0)\mathrm{sech}^2(h_0) \right] + \mathcal{O}(a^4), \end{aligned}$$

which recovers (96). In our solutions, the velocity potential ψ is required to be a periodic function of u (and hence, of x), but the Bernoulli constant B is nonzero in (14).

B.3. **Comparison with** [17]**.** The correction term (93) to the speed c in terms of the conformal depth h coincides with eq. (4.13) reported in [17]. The mean-zero constraint $M(\eta) = 0$ is enforced in [17] and the Bernoulli constant $B \neq 0$ was used in the relevant computatations, see b_0 in eqs. (4.5) and (4.9) in [17]. Converting the expansion of the speed c in terms of the physical depth h_0 yields (96), which coincides with eq. (4.14) in [17]. No integral constraint (16) was written explicitly in [17].

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