

On the spectral theory and dispersive estimates for a discrete Schrödinger equation in one dimension

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Based on the recent work [Komech *et al.*, “Dispersive estimates for 1D discrete Schrödinger and Klein-Gordon equations,” *Appl. Anal.* **85**, 1487 (2006)] for compact potentials, we develop the spectral theory for the one-dimensional discrete Schrödinger operator, $H\phi = (-\Delta + V)\phi = -(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + V_n\phi_n$. We show that under appropriate decay conditions on the general potential (and a nonresonance condition at the spectral edges), the spectrum of H consists of finitely many eigenvalues of finite multiplicities and the essential (absolutely continuous) spectrum, while the resolvent satisfies the limiting absorption principle and the Puiseux expansions near the edges. These properties imply the dispersive estimates $\|e^{itH}P_{\text{a.c.}}(H)\|_{l^2_{\sigma} \rightarrow l^2_{-\sigma}} \lesssim t^{-3/2}$ for any fixed $\sigma > \frac{5}{2}$ and any $t > 0$, where $P_{\text{a.c.}}(H)$ denotes the spectral projection to the absolutely continuous spectrum of H . In addition, based on the scattering theory for the discrete Jost solutions and the previous results by Stefanov and Kevrekidis [“Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein-Gordon equations,” *Nonlinearity* **18**, 1841 (2005)], we find new dispersive estimates $\|e^{itH}P_{\text{a.c.}}(H)\|_{l^1 \rightarrow l^\infty} \lesssim t^{-1/3}$, which are sharp for the discrete Schrödinger operators even for $V=0$. © 2008 American Institute of Physics. [DOI: [10.1063/1.3005597](https://doi.org/10.1063/1.3005597)]

I. INTRODUCTION

We consider a stationary one-dimensional discrete Schrödinger equation in the form

$$H\phi := (-\Delta + V)\phi = \lambda\phi \Leftrightarrow -(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + V_n\phi_n = \lambda\phi_n, \quad (1)$$

where V is a real-valued potential on \mathbb{Z} , $\lambda \in \mathbb{C}$ is a spectral parameter, and ϕ is an eigenfunction in an appropriate space. We will use plain letters V and ϕ to denote sequences $\{V_n\}_{n \in \mathbb{Z}}$ and $\{\phi_n\}_{n \in \mathbb{Z}}$. We will also use standard weighted spaces l^2_σ and l^1_σ on \mathbb{Z} equipped with the norms

$$\|u\|_{l^2_\sigma} = \left(\sum_{n \in \mathbb{Z}} (1+n^2)^\sigma |u_n|^2 \right)^{1/2}, \quad \|u\|_{l^1_\sigma} = \sum_{n \in \mathbb{Z}} (1+n^2)^{\sigma/2} |u_n| \quad (2)$$

for some fixed $\sigma \geq 0$. In what follows, we denote the space of bounded linear operators from $l^2_{\sigma'}$ to l^2_σ by $B(\sigma, \sigma')$ and from l^1 to l^∞ by $B(1, \infty)$.

Our work is motivated by recent advances in analysis of the discrete Schrödinger operators in one dimension. Spectral theory and dispersive estimates in $B(\sigma, -\sigma)$ for H with compact V were considered by Komech *et al.*⁶ by extending the previous work on continuous wave and Schrödinger equations (these results were generalized by Komech *et al.*⁷ for two-dimensional discrete operators). Independently to this work, dispersive estimates in $B(1, \infty)$ for H with $V=0$

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were obtained by Stefanov and Kevrekidis¹² by analyzing integrals with fast oscillations. We shall extend the results of Refs. 6 and 12 to general potentials V under some decay conditions at infinity.

Our ultimate goal in this work is to prove asymptotic stability of single-humped solitons in the discrete one-dimensional nonlinear Schrödinger equations. Asymptotic stability of solitary waves in continuous nonlinear Schrödinger equations was considered in recent works of Cuccagna,¹ Perelman,⁹ Schlag,¹⁰ and others. Although orbital stability of single-humped solitons in the discrete nonlinear Schrödinger equations has been proved long ago by Weinstein,¹⁴ no work has been reported toward the proof of asymptotic stability of single-humped solitons in the long-time evolution. This paper is the first step in this direction. With the help of recent works by Mizumachi⁸ and Cuccagna² for continuous one-dimensional nonlinear Schrödinger equations, we expect our results to be useful for analysis of asymptotic stability of small solitons, but this will be a subject of the forthcoming work.

Let $R(\lambda) = (-\Delta + V - \lambda)^{-1}$ denote the resolvent operator for $H = -\Delta + V$ and $R_0(\lambda) = (-\Delta - \lambda)^{-1}$ denote the free resolvent for $H_0 = -\Delta$. Since the spectrum of H_0 is purely continuous and located on $[0, 4]$, we are particularly interested in the behavior of the resolvent $R(\lambda)$ near the interval $[0, 4]$ on $\text{Im } \lambda = 0$. We will use letter ω to indicate values of λ on the open interval $(0, 4)$ and letter λ to indicate values on $\mathbb{C} \setminus [0, 4]$.

Our article is structured as follows. We review properties of the free resolvent $R_0(\lambda)$ in Sec. II. These properties are used to prove the limiting absorption principle for the resolvent $R(\lambda)$ on $(0, 4)$ in Sec. III, the Puiseux expansions of the resolvent $R(\lambda)$ associated with a generic potential V near the end points 0 and 4 in Sec. IV, and the dispersive estimates on the time evolution of $\dot{u} = iHu$ in Sec. V. Appendices A–D give proofs to technical lemmas used in the main part of the article.

II. PROPERTIES OF THE FREE RESOLVENT

Let $\lambda \in \mathbb{C} \setminus [0, 4]$ and define $\theta = \theta(\lambda)$ to be a unique solution of the transcendental equation

$$2 - 2 \cos \theta = \lambda \quad (3)$$

in the domain $D = \{-\pi \leq \text{Re } \theta \leq \pi, \text{Im } \theta < 0\}$. If $\lambda = \omega \pm i\varepsilon$ and $\varepsilon \downarrow 0$, then $\theta(\lambda) = \theta_{\pm}(\omega, \varepsilon) + i\nu_{\pm}(\omega, \varepsilon)$ satisfies the expansion

$$2 - 2 \cos \theta_{\pm} = \omega + O(\varepsilon^2), \quad \nu_{\pm} = \pm \frac{\varepsilon}{2 \sin \theta_{\pm}} + O(\varepsilon^3).$$

Since $\nu_{\pm} < 0$ for $\varepsilon > 0$, we obtain that roots of $2 - 2 \cos \theta_{\pm} = \omega$ for $\omega \in (0, 4)$ and $\varepsilon = 0$ lie in the intervals $\theta_{+} \in (-\pi, 0)$ and $\theta_{-} \in (0, \pi)$ with the symmetry $\theta_{+} = -\theta_{-}$.

Let ϕ solve the difference equation $(-\Delta - \lambda)\phi = f$ for any $f \in \ell^2(\mathbb{Z})$ and define the free resolvent operator $R_0(\lambda)$ by its solution $\phi = R_0(\lambda)f$. Then, direct substitution shows that $R_0(\lambda)$ is explicitly represented by

$$\phi_n = (R_0(\lambda)f)_n = -\frac{i}{2 \sin \theta} \sum_{m \in \mathbb{Z}} e^{-i\theta|n-m|} f_m. \quad (4)$$

In what follows, we summarize properties of the free resolvent. See Secs. II and III of Ref. 6 for further details.

- (1) Since the sequence $\{e^{-i\theta|n|}\}_{n \in \mathbb{Z}}$ is exponentially decaying as $|n| \rightarrow \infty$ if $\text{Im } \theta < 0$ and ℓ^2_{σ} is closed with respect to convolution for any $\sigma > \frac{1}{2}$, we can see that $R_0(\lambda)$ is defined in $B(\sigma, \sigma)$ for any $\lambda \in \mathbb{C} \setminus [0, 4]$ and $\sigma > \frac{1}{2}$.
- (2) Since the sum of the double-infinite sequence

$$\left\{ \frac{e^{-i\theta_{\pm}|n-m|}}{(1+n^2)^{\sigma}(1+m^2)^{\sigma}} \right\}_{m,n \in \mathbb{Z}}$$

on $m, n \in \mathbb{Z}$ is bounded if $\sigma > \frac{1}{2}$ and $|\sin \theta_{\pm}| > 0$ if $\omega \in (0, 4)$, then

$$R_0^\pm(\omega) = \lim_{\varepsilon \downarrow 0} R_0(\omega \pm i\varepsilon)$$

are Hilbert–Schmidt operators in $B(\sigma, -\sigma)$ for $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$.

- (3) For any $\omega \in (0, 4)$, the operators $R_0^\pm(\omega)$ map $l^1(\mathbb{Z})$ to $l^\infty(\mathbb{Z})$ since

$$\|R_0^\pm(\omega)f\|_{l^\infty} \leq \frac{1}{2|\sin \theta_\pm|} \|f\|_{l^1}$$

and $|\sin \theta_\pm| > 0$ for $\omega \in (0, 4)$.

- (4) The free resolvent $R_0^\pm(\omega)$ diverges near $\omega=0$ and $\omega=4$ because $\sin \theta_\pm$ vanishes in the limit. Without loss of generality, we consider only the case $\omega=0$, where $\theta_+ = \theta_- = 0$. Using the asymptotic expansion

$$\sin \theta(\lambda) = -\sqrt{\lambda - \frac{\lambda^2}{4}} = -\sqrt{\lambda}(1 + O(\lambda)),$$

where the minus sign is chosen to ensure that $\text{Im } \theta < 0$ if $0 < \arg(\lambda) \leq \pi$ near $\lambda=0$, we write a formal Puiseux expansion of the free resolvent in the form

$$(R_0(\lambda)f)_n = \frac{i}{2\sqrt{\lambda}} \sum_{m \in \mathbb{Z}} f_m - \frac{1}{2} \sum_{m \in \mathbb{Z}} |n-m|f_m + r_0(\lambda)f, \tag{5}$$

where $r_0(\lambda)$ is the remainder term and $0 < \arg(\lambda) \leq \pi$. Substituting $\lambda=\omega$ for $R_0^+(\omega)$ and $\lambda = \omega e^{2\pi i}$ for $R_0^-(\omega)$, we obtain, therefore,

$$(R_0^\pm(\omega)f)_n = \pm \frac{i}{2\sqrt{\omega}} \sum_{m \in \mathbb{Z}} f_m - \frac{1}{2} \sum_{m \in \mathbb{Z}} |n-m|f_m + r_0^\pm(\omega)f, \tag{6}$$

where $\omega > 0$ is small. The first two terms in (6) are Hilbert–Schmidt operators in space $B(\sigma, -\sigma)$ for $\sigma > \frac{3}{2}$, while the remainder term $r_0^\pm(\omega)$ is estimated to be of the order of $O(\sqrt{\omega})$ in space $B(\sigma, -\sigma)$ for $\sigma > \frac{5}{2}$. Therefore, if we fix $\sigma > \frac{5}{2}$, we can write (6) in the form

$$R_0^\pm(\omega) = \pm \frac{iR_{-1}}{\sqrt{\omega}} + R_0 + O(\sqrt{\omega}), \tag{7}$$

where

$$(R_{-1}f)_n = \frac{1}{2}(1, f), \quad (R_0f)_n = -\frac{1}{2} \sum_{m \in \mathbb{Z}} |n-m|f_m$$

and $\omega > 0$ is small.

- (5) Due to the symmetry $\theta(\lambda) = -\theta(\bar{\lambda})$ of roots of Eq. (3) for all $\lambda \in \mathbb{C} \setminus [0, 4]$, the following symmetry holds $R_0^-(\omega) = \bar{R}_0^+(\omega)$ for all $\omega \in (0, 4)$. Therefore, it is sufficient to consider only $R_0^+(\omega)$ and drop the superscript from the rest of the article.

III. LIMITING ABSORPTION PRINCIPLE

To study how the resolvent operator $R(\lambda)$, defined for $\lambda \in \mathbb{C} \setminus [0, 4]$, is extended to the interval $\omega \in (0, 4)$, we shall use the standard resolvent properties,

$$R(\lambda) = (I + R_0(\lambda)V)^{-1}R_0(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1}. \tag{8}$$

The second identity is due to the fact that the operators $R_0(\lambda)$ and $R(\lambda)$ are self-adjoint in l^2 for $\lambda \in \mathbb{C} \setminus [0, 4]$. If $V \in l_{2\sigma}^\infty(\mathbb{Z})$, that is,

$$\sup_{n \in \mathbb{Z}} (1 + n^2)^\sigma |V_n| < \infty \tag{9}$$

for any fixed $\sigma > \frac{1}{2}$, then $V: l^2_{-\sigma} \mapsto l^2_\sigma$. We note that

$$\|V\|_{l^2_{2\sigma}} \leq \|V\|_{l^2_{2\sigma}} \leq \|V\|_{l^1_{2\sigma}},$$

so that we will sometimes use condition $V \in l^1_{2\sigma}$ instead of condition (9). Since $R_0(\omega): l^2_\sigma \rightarrow l^2_{-\sigma}$ for every fixed $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$, then $VR_0(\omega)$ is a bounded Hilbert–Schmidt operator in $B(\sigma, \sigma)$ for $\sigma > \frac{1}{2}$. Therefore, the operator $I + VR_0(\omega)$ is invertible in l^2_σ if and only if it has a trivial kernel. We will show that the kernel of $I + VR_0(\omega)$ is indeed trivial for $\omega \in (0, 4)$, which leads to the limiting absorption principle formulated as follows.

Theorem 1: Fix $\sigma > \frac{1}{2}$ and assume that $V \in l^1_{2\sigma}$. The resolvent $R(\lambda) = (-\Delta + V - \lambda)^{-1}$ satisfies

$$\sup_{\varepsilon > 0} \|R(\omega \pm i\varepsilon)\|_{B(\sigma, -\sigma)} < \infty. \tag{10}$$

for any fixed $\omega \in (0, 4)$. As a consequence, there exist $R^\pm(\omega) = \lim_{\varepsilon \downarrow 0} R(\omega \pm i\varepsilon)$ in the norm of $B(\sigma, -\sigma)$.

Proof: By property (5) and identity (8), the symmetry $R^-(\omega) = \bar{R}^+(\omega)$ holds so it is sufficient to consider $R^+(\omega)$ and drop the superscript from the formalism. We will show that

$$[I + VR_0(\omega)]^{-1} \in B(\sigma, \sigma)$$

for any fixed $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$. Since $R_0(\omega) \in B(\sigma, -\sigma)$ by property (2), the proof of the theorem will follow from the second resolvent identity (8). To show that the kernel of $I + VR_0(\omega)$ is trivial in l^2_σ for any fixed $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$, we will assume the opposite and obtain a contradiction.

Let $f \in l^2_\sigma$ be an eigenvector of $I + VR_0(\omega)$. Then, it solves the difference equation

$$f_n - iV_n \sum_{m \in \mathbb{Z}} \frac{e^{-i\theta|n-m|}}{2 \sin \theta} f_m = 0, \quad n \in \mathbb{Z}. \tag{11}$$

Multiplying both sides of (11) by \bar{f}_n/V_n , taking the imaginary part, and summing over $n \in \mathbb{Z}$, we obtain

$$\text{Im} \left[i \sum_{m, n \in \mathbb{Z}} e^{-i\theta|m-n|} f_m \bar{f}_n \right] = \sum_{m, n \in \mathbb{Z}} \cos(\theta(m-n)) f_m \bar{f}_n = 0,$$

whence

$$\left| \sum_{m \in \mathbb{Z}} \cos(\theta m) f_m \right|^2 + \left| \sum_{m \in \mathbb{Z}} \sin(\theta m) f_m \right|^2 = 0. \tag{12}$$

Therefore, the eigenvector f lies in the constrained subspace of l^2_σ of codimension 2,

$$\tilde{l}^2_\sigma = \left\{ f \in l^2_\sigma: \sum_{n \in \mathbb{Z}} \cos(\theta n) f_n = \sum_{n \in \mathbb{Z}} \sin(\theta n) f_n = 0 \right\}. \tag{13}$$

Define an operator $\tilde{R}_0: l^2_\sigma \rightarrow l^2_{-\sigma}$ by

$$(\tilde{R}_0 f)_n = - \sum_{m \in \mathbb{Z}} \frac{\sin(\theta|n-m|)}{2 \sin \theta} f_m.$$

Then, $f + V\tilde{R}_0 f = 0$, which implies that f may be taken to be real-valued, which we assume henceforth. To restrict operator \tilde{R}_0 to the subspace \tilde{l}^2_σ , we introduce (a_1, a_2) as solutions of the algebraic system

$$\begin{cases} a_1 \cos \theta + a_2 \cos(2\theta) = 0 \\ a_1 \sin \theta + a_2 \sin(2\theta) = 1, \end{cases}$$

which is nonsingular, since its determinant equals to $\sin \theta < 0$ for any fixed $\theta \in (-\pi, 0)$. Let \tilde{K} be an operator defined by

$$(\tilde{K}f)_n = V_n \tilde{R}_0 f_n - \left(\sum_m V_m (\tilde{R}_0 f)_m \cos(\theta m) \right) \delta_{n,0} - \left(\sum_m V_m (\tilde{R}_0 f)_m \sin(\theta m) \right) (a_1 \delta_{n,1} + a_2 \delta_{n,2}).$$

Since $V\tilde{R}_0$ is a Hilbert–Schmidt operator in $B(\sigma, \sigma)$ under the condition (9), then \tilde{K} is a compact operator from $\tilde{\mathcal{L}}_\sigma^2$ to $\tilde{\mathcal{L}}_\sigma^2$. If f is an eigenvector of $f + V\tilde{R}_0 f = 0$ and $f \in \tilde{\mathcal{L}}_\sigma^2$, then $f_n + V_n (\tilde{R}_0 f)_n = 0$, and thus

$$\sum_m V_m (\tilde{R}_0 f)_m \cos(\theta m) = - \sum_m f_m \cos(\theta m) = 0,$$

$$\sum_m V_m (\tilde{R}_0 f)_m \sin(\theta m) = - \sum_m f_m \sin(\theta m) = 0,$$

so that $f_n + (\tilde{K}f)_n = f_n + V_n (\tilde{R}_0 f)_n = 0$. Therefore, if f exists, then $-1 \in \text{spec}_{\tilde{\mathcal{L}}_\sigma^2}(\tilde{K})$.

Now, we shall approximate the potential V by the compactly supported potential V^N with the entries $V_n^N = \sum_{j=-N+1}^{N-1} V_j \delta_{n,j}$. Let \tilde{K}^N be the compact operator obtained from the operator \tilde{K} when V is replaced by V^N . If $-1 \in \text{spec}_{\tilde{\mathcal{L}}_\sigma^2}(\tilde{K})$, then, by Lemma A in Appendix A, there exists a subsequence of eigenvalues $-(a_{N_j} + ib_{N_j})$ of the operators \tilde{K}^{N_j} with eigenvectors $f^{N_j} \in \tilde{\mathcal{L}}_\sigma^2$, so that $\lim_{j \rightarrow \infty} (a_{N_j} + ib_{N_j}) = 1$ and $\lim_{j \rightarrow \infty} \|f^{N_j} - f\|_{\tilde{\mathcal{L}}_\sigma^2} = 0$. For simplicity, we drop the subscript j from N_j . More precisely, the eigenvectors satisfy

$$\begin{aligned} (a_N + ib_N) f_n^N + V_n^N (\tilde{R}_0 f^N)_n &= \left(\sum_m V_m^N (\tilde{R}_0 f^N)_m \cos(\theta m) \right) \delta_{n,0} \\ &+ \left(\sum_m V_m^N (\tilde{R}_0 f^N)_m \sin(\theta m) \right) (a_1 \delta_{n,1} + a_2 \delta_{n,2}). \end{aligned} \tag{14}$$

Equation (14) implies that the support of f^N is finite.

Define the discrete Fourier transform $\mathcal{F}: \ell^2(\mathbb{Z}) \rightarrow L^2[0, 2\pi]$ by

$$u = \{u_n\} \leftrightarrow \hat{u}(\xi) = \mathcal{F}(u) = \sum_{n=-\infty}^{\infty} u_n e^{in\xi}.$$

Since f^N has a compact support, then $\hat{f}^N(\xi) = \mathcal{F}(f^N)$ is a trigonometric polynomial. Since f^N belongs to $\tilde{\mathcal{L}}_\sigma^2$, it satisfies the two constraints in (13), which implies that $\hat{f}^N(\theta) = \hat{f}^N(-\theta) = 0$. Define a sequence ψ^N via the inverse Fourier transform of

$$\hat{\psi}^N(\xi) = \frac{\hat{f}^N(\xi)}{2 - 2 \cos \xi - \omega}.$$

Since the denominator is equal to zero exactly at $\xi = \pm \theta$, which are also among zeros of the numerator, and since $\hat{f}^N(\xi)$ is a trigonometric polynomial in ξ , we conclude that $\hat{\psi}^N(\xi)$ is a trigonometric polynomial as well. Therefore, ψ^N has a compact support. By definition, ψ^N is found from equation $(-\Delta - \omega)\psi^N = f^N$, which is equivalent to the equation

$$-\psi_{n+1}^N - \psi_{n-1}^N + (2 - \omega)\psi_n^N + \frac{V_n}{a_N + ib_N}\psi_n^N = \alpha_1 \delta_{n,0} + \alpha_2(a_1 \delta_{n,1} + a_2 \delta_{n,2}), \tag{15}$$

where

$$\alpha_1 = \frac{1}{a_N + ib_N} \sum_m V_m^N (\tilde{R}_0 f^N)_m \cos(\theta m),$$

$$\alpha_2 = \frac{1}{a_N + ib_N} \sum_m V_m^N (\tilde{R}_0 f^N)_m \sin(\theta m).$$

The only compact support solution of (15) has a nonzero value at $n=1$. Therefore, the eigenvector f^N has a compact support at $n=\{0, 1, 2\}$. By $\lim_{N \rightarrow \infty} \|f^N - f\|_{l^2_\sigma} = 0$, we conclude that the support of f is also at the three nodes $n=\{0, 1, 2\}$. Therefore, the function ψ , defined by a solution of $(-\Delta - \omega)\psi = f$, is also compactly supported at $n=1$. However, ψ is also a solution of $(-\Delta + V - \omega)\psi = 0$, and the only compact support solution of this equation is $\psi \equiv 0$. Hence $f \equiv 0$, and we obtain a contradiction. This contradiction implies, of course, that $I + VR_0(\omega)$ is an invertible operator on l^2_σ as claimed. \square

IV. PUISEUX EXPANSIONS AT THE SPECTRAL EDGES

The free resolvent $R_0(\omega)$ has a singular behavior as $\omega \downarrow 0$, as follows from expansion (6). Recall that the superscripts are dropped for $R_0(\omega)$ and $R(\omega)$. We will show that the resolvent operator $R(\omega)$ has a regular behavior in the same limit provided that V is a generic potential in the following sense.

Definition 1: $V \in l^1_1$ is called a generic potential if no solution ψ_0 of equation $(-\Delta + V)\psi_0 = 0$ exists in $l^2_{-\sigma}$ for $\frac{1}{2} < \sigma \leq \frac{3}{2}$.

Remark 1: We show in Appendix B that solutions of $(-\Delta + V)\psi_0 = 0$ always belong to $l^2_{-\sigma}$ for $\sigma > \frac{3}{2}$.

Since $R_0(\omega): l^2_\sigma \mapsto l^2_{-\sigma}$ for any fixed $\omega \in (0, 4)$ and $\sigma > \frac{1}{2}$ and since $V: l^2_{-\sigma} \mapsto l^2_\sigma$ if $V \in l^1_{2\sigma}$, $T(\omega) = I + R_0(\omega)V$ is a bounded Hilbert–Schmidt operator in $B(-\sigma, -\sigma)$ for $\sigma > \frac{1}{2}$. Since $R_0(\omega)$ is represented by expansion (7) for $\sigma > \frac{5}{2}$, we obtain

$$T(\omega) = \frac{iT_{-1}}{\sqrt{\omega}} + T_0 + O(\sqrt{\omega}),$$

where

$$(T_{-1}f)_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} V_m f_m, \quad (T_0f)_n = f_n - \frac{1}{2} \sum_{m \in \mathbb{Z}} |n - m| V_m f_m$$

and $\omega > 0$ is small. We shall denote $\tilde{T}(\omega) = T(\omega) - iT_{-1}/\sqrt{\omega} = T_0 + O(\sqrt{\omega})$. We will show that the operator $\tilde{T}(\omega)$ is invertible near $\omega=0$ if no solution u_0 of equation $T_0u_0 = 0$ exists in $l^2_{-\sigma}$ for any $\sigma > \frac{3}{2}$ and $\langle V, T_0^{-1}1 \rangle \neq 0$, where angular brackets denote inner products in l^2 and 1 is the vector with $1_n = 1$ for all $n \in \mathbb{Z}$. Lemma B in Appendix B shows that this condition is equivalent to the condition that V is a generic potential of Definition 1. Puisseux expansion of the resolvent near $\omega=0$ is defined in the following theorem.

Theorem 2: Fix $\sigma > \frac{5}{2}$ and assume that $V \in l^1_{2\sigma}$ is generic in the sense that no solution u_0 of equation $T_0u_0 = 0$ exists in $l^2_{-\sigma}$ for any $\sigma > \frac{3}{2}$ and $\langle V, T_0^{-1}1 \rangle \neq 0$. The resolvent $R(\omega)$, defined for $\omega \in (0, 4)$, has the expansion

$$R(\omega) = R(0) + O(\sqrt{\omega}) \tag{16}$$

in the norm of $B(\sigma, -\sigma)$ for sufficiently small $\omega > 0$.

Proof: We will first fix $\sigma > \frac{3}{2}$ and show that $\tilde{T}(\omega) \in B(-\sigma, -\sigma)$ is invertible for any fixed small $\omega > 0$, provided that the potential $V \in l^1_{2\sigma}$ is generic. Let $u = u^\perp + c(\omega)V$, where $c(\omega)$ is a coefficient and u^\perp satisfies the orthogonal projection $\langle V, u^\perp \rangle = 0$. For any $f \in l^2_{-\sigma}$, equation $T(\omega)u = f$ is equivalent to

$$\tilde{T}(\omega)u^\perp + c(\omega)T(\omega)V = f. \tag{17}$$

Since V is decaying as $|n| \rightarrow \infty$, we have $u^\perp \in l^2_{-\sigma}$ if and only if $u \in l^2_{-\sigma}$ for any fixed $\sigma > \frac{1}{2}$. Since $\tilde{T}(\omega) - T_0 \rightarrow 0$ as $\omega \downarrow 0$ in $B(-\sigma, -\sigma)$ for $\sigma > \frac{3}{2}$, the operator $\tilde{T}(\omega)$ is invertible if T_0 is invertible. Under the condition (9), we have

$$\sum_{m,n \in \mathbb{Z}} \frac{|n-m|^2 V_m^2 (1+m^2)^\sigma}{(1+n^2)^\sigma} \leq C \sum_{m,n \in \mathbb{Z}} \frac{|n-m|^2}{(1+m^2)^\sigma (1+n^2)^\sigma} < \infty$$

for some $C > 0$ and any $\sigma > \frac{1}{2}$. Therefore, $\tilde{T}_0 = T_0 - I$ is a Hilbert-Schmidt operator in $B(-\sigma, -\sigma)$ for any fixed $\sigma > \frac{3}{2}$, such that T_0 is invertible if and only if the kernel of T_0 is empty in l^2_σ for $\sigma > \frac{3}{2}$, which is a condition that V is a generic potential.

Since $\tilde{T}(\omega)$ is invertible for sufficiently small $\omega > 0$, a unique solution of (17) is

$$u^\perp = [\tilde{T}(\omega)]^{-1}(f - c(\omega)T(\omega)V) = -cV + [\tilde{T}(\omega)]^{-1}\left(f - \frac{ic(\omega)}{\sqrt{\omega}}T_{-1}V\right).$$

To find uniquely the coefficient $c(\omega)$ in the decomposition $u = u^\perp + c(\omega)V$, we let $S(\omega) = [\tilde{T}(\omega)]^{-1}$ and define the adjoint operators $[S(\omega)]^*$ and $[\tilde{T}(\omega)]^*$ as bounded maps in $B(\sigma, \sigma)$ for any fixed $\sigma > \frac{3}{2}$. Since $\|V\|_{l^2_{2\sigma}} \leq \|V\|_{l^1_{2\sigma}}$ and $V \in l^1_{2\sigma}$, then $W = S^*V \in l^2_\sigma$ for any $\sigma > \frac{3}{2}$.

Let us now fix $\sigma > \frac{5}{2}$ and represent $\tilde{T}(\omega)$ by $T_0 + O(\sqrt{\omega})$. Then, $S(\omega) = S_0 + O(\sqrt{\omega})$, where $S_0 = T_0^{-1}$. Using the inner products in l^2 , we obtain $\langle W, \tilde{T}(\omega)u^\perp \rangle = \langle V, u^\perp \rangle = 0$, such that

$$c(\omega) = \frac{\langle W, f \rangle}{\langle W, T(\omega)V \rangle} = \frac{\langle V, S(\omega)f \rangle}{\langle V, S(\omega)T(\omega)V \rangle},$$

provided that $\langle V, S(\omega)T(\omega)V \rangle = |V|_{l^2}^2 \left(1 + \frac{i}{2\sqrt{\omega}} \langle V, S(\omega)1 \rangle\right) \neq 0$ for sufficiently small $\omega > 0$. Since $S(\omega) - S_0 \rightarrow 0$ as $\omega \downarrow 0$ in $B(\sigma, \sigma)$ for $\sigma > \frac{3}{2}$, this condition is satisfied if $\langle V, S_0 1 \rangle = \langle V, T_0^{-1} 1 \rangle \neq 0$, which is true for generic potentials V .

The first resolvent identity (8) implies that if $T(\omega)u = f$ for some $f \in l^2_{-\sigma}$ and $f = R_0(\omega)\psi$ for some $\psi \in l^2_\sigma$, then $u = R(\omega)\psi$ for a fixed $\omega \in (0, 4)$. We shall now finish the proof of theorem by computing the limit $\omega \downarrow 0$ in the following chain of identities:

$$R(\omega)\psi = u = u^\perp + c(\omega)V = [\tilde{T}(\omega)]^{-1}\left(R_0^+(\omega)\psi - \frac{ic(\omega)}{\sqrt{\omega}}T_{-1}V\right), \tag{18}$$

where

$$c(\omega) = \frac{\langle V, S(\omega)R_0(\omega)\psi \rangle}{\langle V, S(\omega)T(\omega)V \rangle} = \frac{\frac{i}{2\sqrt{\omega}}\langle V, S(\omega)1 \rangle \langle 1, \psi \rangle + \langle V, S(\omega)\tilde{R}_0(\omega)\psi \rangle}{\|V\|_{l^2}^2 \left(\frac{i}{2\sqrt{\omega}}\langle V, S(\omega)1 \rangle + 1\right)},$$

where $\tilde{R}_0(\omega) = R_0(\omega) - iR_{-1}/\sqrt{\omega}$. Therefore, $\lim_{\omega \downarrow 0} c(\omega) = c(0)$ exists and the singular term of (18) is canceled because of the explicit computation

$$\frac{i}{2\sqrt{\omega}} \sum_{m \in \mathbb{Z}} \psi_m - \frac{ic(0)}{\sqrt{\omega}} T_{-1}V = \frac{i}{2\sqrt{\omega}} \left[\langle 1, \psi \rangle - \frac{\langle 1, \psi \rangle \|V\|_{l^2}^2}{\|V\|_{l^2}^2} \right] = 0.$$

As a result, the expansion (16) is proven with

$$R(0)\psi = S_0R_0\psi + \left(\frac{\langle 1, \psi \rangle}{\langle V, S_01 \rangle} - \frac{\langle V, S_0R_0\psi \rangle}{\langle V, S_01 \rangle} \right) S_01,$$

where we have used again that $\langle V, S_01 \rangle \neq 0$ for generic potentials. □

Remark 2: Not only Theorem 2 generalizes Theorems 5.1 and 6.1 in Ref. 6 from compact to spatially decaying potentials V but also the class of generic potentials V is defined more precisely compared to Definition 5.1 in Ref. 6. In addition, the values of σ can be taken for $\sigma > \frac{5}{2}$ compared to $\sigma > \frac{7}{2}$ in Ref. 6, since no terms of $O(\sqrt{\omega})$ in the expansions of $T(\omega)$ and $S(\omega)$ are used to obtain the leading order term of $R(0)$.

V. DISPERSIVE ESTIMATES

Using the previous analysis of the resolvent operator $R(\omega)$, we switch our focus to the discussion of the dispersive estimates for the time-dependent discrete Schrödinger equation $u_t = iHu$ with initial data $u(0) = u_0$ in an appropriate function space. We have two types of dispersive estimates. The first one describes decay of the semigroup e^{iHt} acting on the weighted l^2 spaces, and it is an extension of Theorem 7.1 in Ref. 6. The second, more delicate estimate describes decay of the semigroup that maps l^1 into l^∞ , and it is an extension of the dispersive estimate of the free resolvent in Ref. 12.

Let P_j denote projections on the eigenspaces corresponding to the eigenvalues $\omega_j \in \mathbb{R} \setminus [0, 4]$ of the self-adjoint operator H . We shall prove that the discrete spectrum is finite dimensional, such that j can be enumerated from $j=1$ to $j=n < \infty$. By the spectral theory, projection to the essential (absolutely continuous) spectrum of H is defined by $P_{a.c.}(H) = I - \sum_{j=1}^n P_j$.

Lemma 1: Fix $\sigma > \frac{5}{2}$ and assume that $V \in l^1_{2\sigma}$ is generic in the sense of Definition 1. The discrete spectrum of H is finite dimensional and located in the two segments $(\omega_{\min}, 0) \cup (4, \omega_{\max})$, where

$$\omega_{\min} = \min_{n \in \mathbb{Z}} \{0, V_n\}, \quad \omega_{\max} = \max_{n \in \mathbb{Z}} \{4, 4 + V_n\}.$$

Proof: Since H is self-adjoint in $l^2(\mathbb{Z})$, eigenvalues of the discrete spectrum are all located on \mathbb{R} . By Theorem 1, no embedded eigenvalues may occur in $(0, 4)$ if $\sigma > \frac{1}{2}$. By Theorem 2, no eigenvalues are located at 0 and 4 if $\sigma > \frac{5}{2}$ and V is a generic potential in the sense of Definition 1. The upper and lower bounds on the location of eigenvalues follow from the fact that $0 \leq (\phi, -\Delta\phi) \leq 4\|\phi\|_{l^2}^2$. Since the resolvent operator $R(\lambda)$ is a meromorphic function on $\lambda \in \mathbb{R} \setminus [0, 4]$ with bounded limits in $B(\sigma, -\sigma)$ for $\sigma > \frac{5}{2}$ as $\lambda \rightarrow 0$ and $\lambda \rightarrow 4$, it has a finite number of poles in the compact domain $(\omega_{\min}, 0) \cup (4, \omega_{\max})$. Therefore, the discrete spectrum of H is finite dimensional. □

Remark 3: Isolated eigenvalues of the discrete Schrödinger equation outside $[0, 4]$ can be supported by the potential V with the range in $[0, 4]$. Appendix in Ref. 6 gives examples of such eigenvalues for compact potentials supported at one or two nodes with any nonzero values of V . The main result of Ref. 5 states that if $V \neq 0$ with some decay at infinity, H has always spectrum outside $[0, 4]$.

The results on the dispersive estimates for the one-dimensional discrete Schrödinger equation are described in the following two theorems.

Theorem 3: Fix $\sigma > \frac{5}{2}$ and assume that $V \in l^1_{2\sigma}$ is generic in the sense of Definition 1. Then, there exists a constant C depending on V , so that

$$\|e^{iHt} P_{a.c.}(H)\|_{l^2_\sigma \rightarrow l^2_{-\sigma}} \leq Ct^{-3/2} \tag{19}$$

for any $t > 0$.

Proof: The proof of (19) is standard and it follows the outline in Ref. 6. By the Cauchy formula in $B(\sigma, -\sigma)$, we obtain

$$e^{itH} P_{\text{a.c.}}(H) = \frac{1}{2\pi i} \int_0^4 e^{it\omega} [R^+(\omega) - R^-(\omega)] d\omega = \frac{1}{\pi} \int_0^4 e^{it\omega} \text{Im } R(\omega) d\omega, \tag{20}$$

where we have dropped the superscript for $R^+(\omega)$. By the representation of the perturbed resolvent in Theorem 2 [in particular, by the fact that $R(0)$ is real], we have

$$\|R(\omega)\|_{B(\sigma, -\sigma)} \leq C, \quad \|\text{Im } R(\omega)\|_{B(\sigma, -\sigma)} \leq C\omega^{1/2} \tag{21}$$

and

$$\left\| \frac{d^j}{d\omega^j} R(\omega) \right\|_{B(\sigma, -\sigma)} \leq C\omega^{1/2-j}, \quad j = 1, 2 \tag{22}$$

for some $C > 0$. Introduce smooth cutoff functions $\chi_1, \chi_2 \in C_0^\infty$, so that $\chi_1 + \chi_2 = 1$ for all $\omega \in [0, 4]$, while $\text{supp}(\chi_1) \subset [0, 3]$ and $\text{supp}(\chi_2) \subset [1, 4]$. Write

$$e^{itH} P_{\text{a.c.}}(H) = \frac{1}{2\pi i} \int_0^3 e^{it\omega} \chi_1(\omega) \text{Im } R(\omega) d\omega + \frac{1}{2\pi i} \int_1^4 e^{it\omega} \chi_2(\omega) \text{Im } R(\omega) d\omega. \tag{23}$$

To each of the two terms, one can apply Lemma C from Appendix C. Note that the conditions on the function $F(\omega) = \text{Im } R(\omega) \chi_1(\omega) \in B(\sigma, -\sigma)$ are satisfied because of the bounds (21) and (22). \square

Theorem 4: Fix $\sigma > \frac{5}{2}$ and assume that $V \in l_\sigma^1$ is generic in the sense of Definition 1. Then, there exists a constant C depending on V , so that

$$\|e^{itH} P_{\text{a.c.}}(H)\|_{l^1 \rightarrow l^\infty} \leq Ct^{-1/3} \tag{24}$$

for any $t > 0$.

To prove Theorem 4, we develop scattering theory for fundamental solutions of the discrete Schrödinger equation, following the works of Weder¹³ and Goldberg and Schlag³ for the continuous Schrödinger equation. Let ψ^\pm be two linearly independent solutions of

$$\psi_{n+1} + \psi_{n-1} + (\omega - 2)\psi_n = V_n \psi_n, \tag{25}$$

according to the boundary conditions $|\psi_n^\pm - e^{\mp in\theta}| \rightarrow 0$ as $n \rightarrow \pm \infty$, where θ is a root of

$$2 - 2 \cos \theta = \omega \tag{26}$$

for $\omega \in [0, 4]$. Since the solutions depend on θ , we may use $\psi^\pm(\theta)$ instead of ψ^\pm . The Green function representations of the two solutions are

$$\psi_n^+(\theta) = e^{-in\theta} - \frac{i}{2 \sin \theta} \sum_{m=n}^\infty (e^{i\theta(m-n)} - e^{-i\theta(m-n)}) V_m \psi_m^+(\theta),$$

$$\psi_n^-(\theta) = e^{in\theta} - \frac{i}{2 \sin \theta} \sum_{m=-\infty}^n (e^{i\theta(n-m)} - e^{-i\theta(n-m)}) V_m \psi_m^-(\theta).$$

Let $\psi_n^\pm(\theta) = e^{\mp in\theta} f_n^\pm(\theta)$ for all $n \in \mathbb{Z}$. Writing the Green function representation for $f^+(\theta)$ in the form

$$f_n^+(\theta) = 1 - \frac{i}{2 \sin \theta} \sum_{m=n}^{\infty} (1 - e^{-2i\theta(m-n)}) V_m f_m^+(\theta) \tag{27}$$

and using the formula

$$\left| \frac{1 - e^{-2i\theta x}}{2i \sin \theta} \right| \leq C_0 |x|, \quad \forall \theta \in [-\theta_0, \theta_0], \quad \forall x \in \mathbb{R},$$

for some $C_0 > 0$ and fixed $0 < \theta_0 < \pi/2$, it follows from the Neumann series that the sequence $\{f_n^+(\theta)\}_{N_0}^{\infty}$ is uniformly bounded in l^{∞} on $[-\theta_0, \theta_0]$ if $V \in l_1^1$, where N_0 is defined as the smallest integer, for which

$$C_0 \sum_{k=1}^{\infty} k |V_{N_0+k}| < 1.$$

Moreover, the sequence $\{f_n^+(\theta)\}_{N_0}^{\infty}$ is analytically continued in the strip $\Sigma_0 = \{-\theta_0 \leq \text{Re } \theta \leq \theta_0, \text{ Im } \theta \geq 0\}$, such that $\partial_{\theta} f^+(\theta)$ exists in the interior of Σ_0 . By taking the derivative of (27) in θ , we obtain

$$\partial_{\theta} f_n^+(\theta) = \sum_{m=n}^{\infty} \partial_{\theta} \left(\frac{1 - e^{-2i\theta(m-n)}}{2i \sin \theta} \right) V_m f_m^+(\theta) - \frac{i}{2 \sin \theta} \sum_{m=n}^{\infty} (1 - e^{-2i\theta(m-n)}) V_m \partial_{\theta} f_m^+(\theta).$$

By the same argument, it follows from the Neumann series that the sequence $\{\partial_{\theta} f_n^+(\theta)\}_{N_0}^{\infty}$ is uniformly bounded in l^{∞} on $[-\theta_0, \theta_0]$ if $V \in l_2^1$. Therefore, if $V \in l_2^1$, then

$$\sup_{\theta \in [-\theta_0, \theta_0]} (\|\partial_{\theta} f^+(\theta)\|_{l^{\infty}(\mathbb{Z}_+, \infty)} + \|\partial_{\theta} f^+(\theta)\|_{l^{\infty}(\mathbb{Z}_0, \infty)}) < \infty.$$

If $N_0 > 0$, then the above bound can be extended in $l^{\infty}(\mathbb{Z}_+)$ since the finite sequence $\{f_n^+(\theta)\}_{n=0}^{N_0}$ satisfies a second-order difference equation with analytic coefficients and analytic boundary values $f_{N_0}^+(\theta)$ and $f_{N_0+1}^+(\theta)$ in the strip Σ_0 . Similar estimates hold for $\{f_n^-(\theta)\}_{n \in \mathbb{Z}_-}$. Thus, if $V \in l_2^1(\mathbb{Z})$, then $\{f_n^{\pm}(\theta)\}_{n \in \mathbb{Z}_{\pm}}$ are analytic in the strip Σ_0 and there exist uniform bounds

$$F_{\pm} = \sup_{\theta \in [-\theta_0, \theta_0]} (\|f^{\pm}(\theta)\|_{l^{\infty}(\mathbb{Z}_{\pm})} + \|\partial_{\theta} f^{\pm}(\theta)\|_{l^{\infty}(\mathbb{Z}_{\pm})}) < \infty. \tag{28}$$

Let us define the discrete Wronskian

$$W[\psi^+, \psi^-] = \psi_n^+ \psi_{n+1}^- - \psi_{n+1}^+ \psi_n^- \equiv W(\theta), \tag{29}$$

which is independent of $n \in \mathbb{Z}$ and analytic in Σ_0 . The discrete Green function for the resolvent operators $R^{\pm}(\omega)$ has the kernel

$$[R^{\pm}(\omega)]_{n,m} = \frac{1}{W(\theta_{\pm})} \begin{cases} \psi_n^+(\theta_{\pm}) \psi_m^-(\theta_{\pm}) & \text{for } n \geq m \\ \psi_m^+(\theta_{\pm}) \psi_n^-(\theta_{\pm}) & \text{for } n < m, \end{cases}$$

where $\theta_- = -\theta_+$ and $\theta_{\pm} \in [0, \pi]$ for $\omega \in [0, 4]$ (see Sec. II). Using (20), we represent $e^{itH} P_{\text{a.c.}}(H)$ by its kernel for $n < m$,

$$\begin{aligned} [e^{itH} P_{\text{a.c.}}(H)]_{n,m} &= \frac{1}{2\pi i} \int_0^4 e^{it\omega} \left[\frac{\psi_m^+(\theta_+) \psi_n^-(\theta_+)}{W(\theta_+)} - \frac{\psi_m^+(\theta_-) \psi_n^-(\theta_-)}{W(\theta_-)} \right] d\omega \\ &= \frac{i}{\pi} \int_{-\pi}^{\pi} e^{it(2-2\cos \theta)} \frac{\psi_m^+(\theta) \psi_n^-(\theta)}{W(\theta)} \sin \theta d\theta, \end{aligned} \tag{30}$$

where we have unfolded the branch points $\omega=0$ and $\omega=4$ by using the transformation (26). If V

is a generic potential in the sense of Definition 1, then Appendix B shows that the two solutions $\psi^\pm(0)$ are linearly independent, such that $W(0) \neq 0$ (the point $\theta=0$ corresponds to $\omega=0$). A similar analysis applies to the points $\theta = \pm \pi$ which correspond to $\omega=4$.

Let $\chi_0, \chi \in C_0^\infty: \chi_0(\theta) + \chi(\theta) = 1$ for all $\theta \in [-\pi, \pi]$, so that $\text{supp } \chi_0 \subset [-\theta_0, \theta_0] \cup (-\pi, -\pi + \theta_0) \cup (\pi - \theta_0, \pi)$ and $\text{supp } \chi \subset [\theta_0/2, \pi - \theta_0/2] \cup [-\pi + \theta_0/2, -\theta_0/2]$, where $0 < \theta_0 \leq \pi/4$. Here the value θ_0 is the same number, which is used in the bounds (28). (If the original number $\theta_0 > \pi/4$, we reassign it to be $\theta_0 = \pi/4$.) It is important for our argument that the support of χ stays away (by a fixed number $\theta_0/2!$) from both 0 and π .

We can now formulate and prove two technical lemmas needed for the proof of Theorem 4.

Lemma 2: Assume $V \in l_2^1$ and $W(0) \neq 0$. Then, there exists $C > 0$ such that

$$\sup_{n < 0 < m} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi_0(\theta) \frac{\psi_m^+(\theta) \psi_n^-(\theta) \sin \theta}{W(\theta)} d\theta \right| \leq Ct^{-1/2} \tag{31}$$

for any $t > 0$.

Proof: The proof is different for regions $n < 0 < m$, $0 < n < m$, and $n < m < 0$. In the case $n < 0 < m$, we write

$$\sup_{n < 0 < m} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi_0(\theta) \frac{\psi_m^+(\theta) \psi_n^-(\theta)}{W(\theta)} \sin \theta d\theta \right| = \sup_{n < 0 < m} \left| \int_{-\theta_0}^{\theta_0} e^{it(2-2 \cos \theta)} e^{i(n-m)\theta} g_{n,m}(\theta) d\theta \right|,$$

where $g_{n,m}(\theta) = \sin \theta \chi_0(\theta) f_m^+(\theta) f_n^-(\theta) / W(\theta)$. Since $f_n^+(\theta)$ and $f_m^-(\theta)$ are continuously differentiable on $[-\theta_0, \theta_0]$ and satisfy the uniform bounds (28) if $V \in l_2^1$ and since $|W(\theta)| \geq W_0 > 0$ if $W(0) \neq 0$, the dispersive estimate (31) follows by Lemma D of Appendix D.

In the case $0 < n < m$, the above estimate is not sufficient since $f_n^-(\theta)$ grows linearly as $n \rightarrow \infty$. Therefore, we use the scattering theory for fundamental solutions of (25) and represent

$$\psi^-(\theta) = a(\theta) \psi^+(\theta) + b(\theta) \psi^+(-\theta),$$

where

$$a(\theta) = \frac{W[\psi^-(\theta), \psi^+(-\theta)]}{2i \sin \theta}, \quad b(\theta) = \frac{W[\psi^-(\theta), \psi^+(\theta)]}{-2i \sin \theta} = \frac{W(\theta)}{2i \sin \theta},$$

and the discrete Wronskian is defined by (29). As a result, we write

$$\begin{aligned} & \sup_{0 < n < m} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi_0(\theta) \frac{\psi_m^+(\theta) \psi_n^-(\theta)}{W(\theta)} \sin \theta d\theta \right| \\ & \leq \frac{1}{2} \sup_{0 < n < m} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} e^{i-(n+m)\theta} \chi_0(\theta) \frac{f_m^+(\theta) f_n^+(\theta) W[\psi^-(\theta), \psi^+(-\theta)]}{W(\theta)} d\theta \right| \\ & \quad + \frac{1}{2} \sup_{0 < n < m} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} e^{-i(m-n)\theta} \chi_0(\theta) f_m^+(\theta) f_n^+(-\theta) d\theta \right|. \end{aligned}$$

Each term here is estimated by the bound (38) of Lemma D for an appropriate function $g(\theta)$. The last case $n < m < 0$ is estimated similarly to the case $0 < n < m$ by using the scattering theory for $\psi^+(\theta)$ in terms of $\psi^-(\theta)$ and $\psi^-(-\theta)$. \square

It remains to treat the case, when the cutoff χ is placed on $[\theta_0/2, \pi - \theta_0/2] \cup [-\pi + \theta_0/2, -\theta_0/2]$, where $0 < \theta_0 \leq \pi/4$ is a fixed number. Using the original representation (20), we need to estimate the operator norm of

$$I_V = \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi(\theta) \operatorname{Im} R(2-2 \cos \theta) \sin \theta d\theta$$

in $B(1, \infty)$. This estimate is given by the following lemma. Clearly, Lemma 2 and Lemma 3 imply Theorem 4.

Lemma 3: Fix $\sigma > \frac{5}{2}$ and assume that $V \in l^1_{\sigma}$. Then, there exists $C > 0$ such that

$$\left\| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi(\theta) \operatorname{Im} R(2-2 \cos \theta) \sin \theta d\theta \right\|_{B(1, \infty)} \leq Ct^{-1/3} \tag{32}$$

for any $t > 0$.

Proof: We start by recalling the finite Born series,

$$R(\omega) = R_0(\omega) - R_0(\omega)VR_0(\omega) + R_0(\omega)VR(\omega)VR_0(\omega),$$

which follows by iterating the resolvent identity (8). We can write $I_V = I^1 - I_V^2 + I_V^3$, where

$$I^1 = \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi(\theta) \operatorname{Im} R_0(2-2 \cos \theta) \sin \theta d\theta,$$

$$I_V^2 = \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi(\theta) \operatorname{Im} R_0(2-2 \cos \theta) VR_0(2-2 \cos \theta) \sin \theta d\theta,$$

$$I_V^3 = \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi(\theta) \operatorname{Im} R_0(2-2 \cos \theta) G_V(\theta) R_0(2-2 \cos \theta) \sin \theta d\theta,$$

where $G_V(\theta) := VR(2-2 \cos \theta)V$. For I^1 , we observe that this is, in fact, a solution of the free Schrödinger equation and can be written as

$$(I^1 f)_n = -\frac{1}{2} \sum_{m \in \mathbb{Z}} f_m \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi(\theta) \cos((n-m)\theta) d\theta.$$

Clearly,

$$\|I^1\|_{B(1, \infty)} \leq \frac{1}{2} \sup_{n \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \chi(\theta) e^{in\theta} d\theta \right| \leq \sup_{a \in \mathbb{R}} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta - a\theta)} \chi(\theta) d\theta \right|.$$

The last expression has been shown in Theorem 3 of Ref. 12 to decay like $t^{-1/3}$ and this dispersive estimate is sharp. The argument relies on the van der Corput lemma formulated in Appendix D. Indeed, if $h(\theta) = 2-2 \cos(\theta) - a\theta$, then $h'(\theta_1) = h''(\theta_1) = 0$ and $h'''(\theta_1) = 4$ for $a=2$ and $\theta_1 = \pi/2$, such that the van der Corput lemma can be applied with $k=3$ to produce $t^{-1/3}$ decay.

Proceeding further with I_V^2 , we have

$$(I_V^2 f)_n = \sum_{m, l \in \mathbb{Z}} V_m f_l \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \cos(\theta(|n-m| + |m-l|)) \frac{\chi(\theta)}{4 \sin(\theta)} d\theta,$$

such that

$$\begin{aligned} \|I_V^2\|_{B(1,\infty)} &\leq \sup_{n,l \in \mathbb{Z}} \left| \sum_{m \in \mathbb{Z}} V_m \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta) - i\theta(|n-m|+|m-l|)} \frac{\chi(\theta)}{4 \sin(\theta)} d\theta \right| \\ &\leq \|V\|_{l^1} \sup_{a \in \mathbb{R}} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta - a\theta)} \frac{\chi(\theta)}{4 \sin(\theta)} d\theta \right|. \end{aligned}$$

Thus, we can apply again the van der Corput lemma with

$$h(\theta) = 2 - 2 \cos \theta - a\theta, \quad g(\theta) = \frac{\chi(\theta)}{4 \sin \theta}.$$

Since χ is supported away from $0, -\pi, \pi$, the function $g(\theta)$ is smooth and vanishes in a neighborhood of the end points $-\pi, \pi$. On the other hand, the function $h(\theta)$ is the same as in the estimate I^1 .

Finally, we deal with I_V^3 . We claim first that for all $\sigma > 5/2$,

$$\sup_{\theta \in [-\pi, \pi]} \sum_m |G_m(\theta)| + \left| \frac{d}{d\theta} G_m(\theta) \right| \leq C \|V\|_{l^2_\sigma}^2 \|f\|_{l^1}. \tag{33}$$

We will work with the derivative only, since the estimates for $G_m(\theta)$ are similar. We have

$$\frac{d}{d\theta} G_m(\theta) = \left(V \frac{d}{d\theta} [R(2 - 2 \cos \theta)] V f \right)_m = 2V_m \sin \theta R'(2 - 2 \cos \theta) [Vf]_m.$$

By (22) for $j=1$, we obtain for every $\theta \in [-\pi, \pi]$,

$$\sum_m \left| \frac{d}{d\theta} G_m(\theta) \right| \leq C |\sin \theta| \|V\|_{l^2_\sigma} \|R'(2 - 2 \cos \theta) Vf\|_{l^2_{-\sigma}} \leq C \|V\|_{l^2_\sigma} \frac{|\sin \theta|}{\sqrt{2 - 2 \cos \theta}} \|Vf\|_{l^2_\sigma} \leq C \|V\|_{l^2_\sigma}^2 \|f\|_{l^1}.$$

This finishes the proof of the claim (33). Thus, we write

$$(I_V^3 f)_n = \sum_{l \in \mathbb{Z}} f_l \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} \cos(\theta(|n-m| + |m-l|)) \frac{\chi(\theta) G_m(\theta)}{4 \sin(\theta)} d\theta,$$

such that

$$\|I_V^3\|_{B(1,\infty)} \leq C \sup_{l \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} e^{it(2-2 \cos \theta)} e^{-i\theta(|n-m|+|m-l|)} \frac{\chi(\theta) G_m(\theta)}{4 \sin(\theta)} d\theta \right|.$$

We write

$$h(\theta) = 2 - 2 \cos \theta - \theta a_{t,n,m,l}, \quad g_m(\theta) = \frac{\chi(\theta)}{\sin \theta} G_m^+(\theta),$$

where $a_{t,n,m,l} = (|n-m| + |m-l|)/t$. Our aim is to estimate

$$\sum_{m \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} e^{ih(\theta)} g_m(\theta) d\theta \right|,$$

where $g_m(\theta)$ vanishes in a neighborhood of the end points $-\pi, \pi$ and $h(\theta)$ has the property

$$\max(|h'(\theta)|, |h''(\theta)|, |h'''(\theta)|) \geq 1,$$

as discussed earlier. This is valid for every fixed t, n, m, l . We can therefore apply the van der Corput lemma from Appendix D with either $k=1, 2, 3$. In the worst possible scenario, that is, $k=3$, we obtain

$$\sum_{m \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} e^{i\theta m} g_m(\theta) d\theta \right| \leq C t^{-1/3} \sum_{m \in \mathbb{Z}} \left| \int_{-\pi}^{\pi} \frac{d}{d\theta} g_m(\theta) \right| d\theta \leq C t^{-1/3} \|V\|_{\sigma}^2 \|f\|_{L^1},$$

where the last inequality follows from (33). This finishes the proof of Lemma 3. □

Appendix A: Approximation of compact operators

Here we will prove a lemma, which is used in the proof of Theorem 1.

Lemma A: *Let X be a Banach space and $\{K_n\}_{n \in \mathbb{N}}: X \rightarrow X$ be a sequence of compact operators, such that $\lim_{n \rightarrow \infty} \|K_n - K\|_{B(X,X)} = 0$ for some $K: X \rightarrow X$. Then for every $\lambda \neq 0$, such that $\lambda \in \sigma(K)$ with an eigenvector $f \neq 0$, such that $Kf = \lambda f$, there exists a subsequence $\{\lambda_{n_j}\}_{j \in \mathbb{N}}$ of eigenvalues with eigenvectors $\{f_{n_j}\}_{j \in \mathbb{N}}$, such that $K_{n_j} f_{n_j} = \lambda_{n_j} f_{n_j}$, such that $\lim_{j \rightarrow \infty} \lambda_{n_j} = \lambda$ and $\lim_{j \rightarrow \infty} \|f_{n_j} - f\|_{B(X,X)} = 0$.*

Proof: First, we show the existence of a subsequence of eigenvalues λ_{n_j} of K_{n_j} that converges to eigenvalue λ of K . Then, we construct eigenvectors f_{n_j} . Assume the contrary, that is, there exists $\varepsilon_0 > 0$, so that

$$0 < \varepsilon_0 \leq \limsup_{n \rightarrow \infty} \text{dist}(\sigma(K_n), \lambda).$$

By the functional calculus, there exists a subsequence $\{K_{n_j}\}_{j \in \mathbb{N}}$, such that

$$\|(K_{n_j} - \lambda)^{-1}\|_{B(X,X)} \leq 2\varepsilon_0^{-1}.$$

Pick any eigenvector f for the eigenvalue λ , such that $Kf = \lambda f$ and $\|f\|_X = 1$. Let $g_j = (K_{n_j} - \lambda)f = (K_{n_j} - K)f$. Clearly, $\|g_j\|_X \leq \|K_{n_j} - K\|_{B(X,X)} \rightarrow 0$. On the other hand,

$$1 = \|f\|_X = \|(K_{n_j} - \lambda)^{-1} g_j\|_X \leq 2\varepsilon_0^{-1} \|g_j\|_X \rightarrow 0.$$

A contradiction arises, whence there is a subsequence $\{\lambda_{n_j}\}_{j \in \mathbb{N}}$ which converges to λ . Pick eigenvectors f_{n_j} , such that $K_{n_j} f_{n_j} = \lambda_{n_j} f_{n_j}$ and $\|f_{n_j}\|_X = 1$. Since K is compact, it follows that $K f_{n_j}$ will have a convergent subsequence, call it again f_{n_j} . Let $g := \lim_{j \rightarrow \infty} K f_{n_j}$. We have

$$\begin{aligned} \|g - \lambda f_{n_j}\|_X &\leq |\lambda - \lambda_{n_j}| + \|K f_{n_j} - g\|_X + \|(K - K_{n_j}) f_{n_j}\|_X \\ &\leq |\lambda - \lambda_{n_j}| + \|K f_{n_j} - g\|_X + \|(K - K_{n_j})\|_{B(X,X)} \rightarrow 0. \end{aligned}$$

Thus, the subsequence $\{f_{n_j}\}_{j \in \mathbb{N}}$ converges to $f := g/\lambda$ in $B(X, X)$ norm if $\lambda \neq 0$. Also, $\lambda f = g = \lim_{j \rightarrow \infty} K f_{n_j} = K f$. □

Appendix B: Conditions on generic potentials

Let us consider the difference equation $(-\Delta + V)\psi = 0$ or

$$\psi_{n+1} + \psi_{n-1} = (2 + V_n)\psi_n, \quad n \in \mathbb{Z}. \tag{34}$$

Two fundamental solutions of (34) are defined by the discrete Green function in the form

$$\begin{aligned} \psi_n^+ &= 1 - \sum_{m=n}^{\infty} (n-m)V_m \psi_m^+, \\ \psi_n^- &= 1 + \sum_{m=-\infty}^n (n-m)V_m \psi_m^-. \end{aligned}$$

It is straightforward to check that the discrete Wronskian,

$$W[\psi^+, \psi^-] = \psi_{n+1}^- \psi_n^+ - \psi_{n+1}^+ \psi_n^-$$

is constant on $n \in \mathbb{Z}$. Therefore, the Green function representation of ψ^+ and ψ^- immediately implies that $\sum_{m \in \mathbb{Z}} V_m \psi_m^+ = \sum_{m \in \mathbb{Z}} V_m \psi_m^-$ or simply $\langle V, \psi^+ \rangle = \langle V, \psi^- \rangle$. If $V \in l_1^1$, then

$$\lim_{n \rightarrow +\infty} \psi_n^+ = \lim_{n \rightarrow -\infty} \psi_n^- = 1$$

and

$$- \lim_{n \rightarrow -\infty} \frac{\psi_n^+}{n} = \lim_{n \rightarrow +\infty} \frac{\psi_n^-}{n} = \langle V, \psi^+ \rangle = \langle V, \psi^- \rangle.$$

It follows by this construction that the solution of $(-\Delta + V)\psi_0 = 0$ spanned by the fundamental solutions ψ^+ and ψ^- always exists in $l_{-\sigma}^2$ for $\sigma > \frac{3}{2}$ (Remark 1). We can now prove the equivalence of conditions in Definition 1 and the conditions in Theorem 2.

Lemma B: Let $V \in l_1^1$. The two conditions are equivalent.

- (1) No solution ψ_0 of equation $(-\Delta + V)\psi_0 = 0$ exists in $l_{-\sigma}^2$ for $\frac{1}{2} < \sigma \leq \frac{3}{2}$
- (2) No solution u_0 of equation $T_0 u_0 = 0$ exists in $l_{-\sigma}^2$ for any $\sigma > \frac{3}{2}$ and $\langle V, T_0^{-1} 1 \rangle \neq 0$.

Proof: Condition 1 is equivalent to the constraint $\langle V, \psi^+ \rangle \neq 0$. Indeed, if $\langle V, \psi^+ \rangle = 0$, then $\psi^+ \in l^\infty(\mathbb{Z})$, and thus $\psi^+ \in l_{-\sigma}^2$ for $\sigma > \frac{1}{2}$. If $\langle V, \psi^+ \rangle \neq 0$, then no solution ψ of Eq. (34) exists in $l_{-\sigma}^2$ for $\frac{1}{2} < \sigma \leq \frac{3}{2}$.

Let u be a solution of $T_0 u = 1$, which can be rewritten in the explicit form

$$u_n = 1 + \frac{1}{2} \sum_{m=-\infty}^n (n-m)V_m u_m - \frac{1}{2} \sum_{m=n}^{\infty} (n-m)V_m u_m. \tag{35}$$

Direct computations show that u solves the same difference equation (34) and

$$\lim_{n \rightarrow +\infty} \frac{u_n}{n} = - \lim_{n \rightarrow -\infty} \frac{u_n}{n} = \langle V, u \rangle.$$

Therefore, $u = c(\psi^+ + \psi^-)$ with $c \neq 0$ and the constraint $\langle V, u \rangle = \langle V, T_0^{-1} 1 \rangle \neq 0$ is equivalent to the constraint $\langle V, \psi^+ \rangle \neq 0$ that is condition 1.

Assume now that there exists a solution of equation $T_0 u_0 = 0$ in $l_{-\sigma}^2$ for $\sigma > \frac{3}{2}$. This function is a solution of the same equation (35) but without the constant term on the right-hand side. Therefore, u_0 satisfies (34) and u_0 is linearly independent of u , which is another solution of (34). Multiplying Eq. (35) by $V_n(u_0)_n$ and summing over $n \in \mathbb{Z}$, we obtain

$$\langle V, u_0 \rangle = \sum_{m \in \mathbb{Z}} V_m u_m \left[(u_0)_m - \frac{1}{2} \sum_{n \in \mathbb{Z}} |m-n| V_n (u_0)_n \right] = 0.$$

Therefore, $u_0 \in l_\sigma^2$ for $\sigma > \frac{1}{2}$ for exponentially decaying potentials V , that is, there exists ψ_0 of Eq. (34) in l_σ^2 for $\sigma > \frac{1}{2}$. In the opposite direction, if there exists ψ_0 of Eq. (34) in l_σ^2 for $\sigma > \frac{1}{2}$, then $\psi_0 = c\psi^+$ with $c \neq 0$ and, since $\langle V, \psi^+ \rangle = 0$ in this case, one can choose $c = \frac{1}{2} \sum_{m \in \mathbb{Z}} m V_m (\psi_0)_m$ so that ψ_0 solves $T_0 \psi_0 = 0$, that is, there exists $u_0 = \psi_0$. Therefore, the existence of ψ_0 is equivalent to the existence of u_0 . Combining both results together, we have established the equivalence of conditions (1) and (2). \square

Appendix C: Jensen–Kato lemma

A general lemma to estimate the oscillatory integrals is provided in Ref. 4. Here, we formulate and prove a simplified version, which is used in the proof of Theorem 3.

Lemma C: Let \mathcal{B} be a Banach space, so that for $F \in C^2(0, a; \mathcal{B})$, $F(0) = F(a) = 0$, $\|F'(\omega)\|_{\mathcal{B}}$

$\leq C\omega^{-1/2}$, and $\|F''(\omega)\|_{\mathcal{B}} \leq C\omega^{-3/2}$ as $\omega \rightarrow 0$. Then for every $t > 1$,

$$\left\| \int_0^a e^{i\omega w} F(\omega) d\omega \right\|_{\mathcal{B}} \leq C|t|^{-3/2}. \tag{36}$$

Proof: Take an unit element b^* in the dual space \mathcal{B}^* . Then, it is clearly enough to show

$$\left| \int_0^a e^{i\omega w} b^*[F(\omega)] d\omega \right| \leq C|t|^{-3/2},$$

where C is independent of \mathcal{B}^* . Thus, (36) follows from its scalar version, where $\tilde{F}(\omega) := b^*[F(\omega)]$, since the estimates for F carry over \tilde{F} . That is without loss of generality, we may assume that $\mathcal{B} = \mathbb{R}$.

Next, an integration by parts yields

$$\int_0^a e^{i\omega w} F(\omega) d\omega = \frac{i}{t} \int_0^a e^{i\omega w} F'(\omega) d\omega = \frac{i}{t} \left[\int_0^{\min(1/|t|, a)} e^{i\omega w} F'(\omega) d\omega + \int_{\min(1/|t|, a)}^a e^{i\omega w} F'(\omega) d\omega \right].$$

For the first integral, estimate by absolute value and $|F'(\omega)| \leq C|\omega|^{-1/2}$,

$$C \frac{1}{|t|} \int_0^{\min(1/|t|, a)} |\omega|^{-1/2} d\omega \leq 2C|t|^{-3/2}.$$

For the second integral, perform one more integration by parts and $|F''(\omega)| \leq C\omega^{-3/2}$,

$$\frac{1}{|t|} \left| \int_{\min(1/|t|, a)}^a e^{i\omega w} F'(\omega) d\omega \right| \leq \frac{C}{|t|^2} \left[|F'(\min(1/|t|, a))| + \int_{\min(1/|t|, a)}^a |\omega|^{-3/2} d\omega \right] \leq C_1|t|^{-3/2}.$$

These bounds complete the proof of the lemma. □

Appendix D: Estimation of oscillatory integrals

The van der Corput lemma is stated as a corollary on p. 334 in Ref. 11.

Van der Corput Lemma: *Suppose ϕ is a real-valued function, smooth in (a, b) , so that $|\phi^{(k)}(x)| \geq 1$ for some integer k . (If $k=1$, we will have to also assume that $\phi'(x)$ is monotonic). Then,*

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right]. \tag{37}$$

Here we will prove a lemma which is used in the proof of Lemma 2. This lemma is basically a corollary of the van der Corput lemma.

Lemma D: *Assume that the function $g(\theta)$ is continuously differentiable in $[-\theta_0, \theta_0]$ for certain $0 < \theta_0 < \pi/4$ and $\sup_{-\theta_0 \leq \theta \leq \theta_0} [|g'(\theta)| + |g(\theta)|] \leq C$. Then*

$$\sup_{a \in \mathbb{R}} \left| \int e^{it(2-2\cos\theta-a\theta)} \chi_0(\theta) g(\theta) d\theta \right| \leq Ct^{-1/2} \tag{38}$$

for any $t > 0$.

Proof: To use the van der Corput lemma, we need to check that for a fixed parameter a and on the support of the function χ_0 , the phase function $h(\theta) = 2 - 2\cos\theta - a\theta$ satisfies the condition that $\max(|h'(\theta)|, |h''(\theta)|) \geq 1$ for every fixed θ . Assuming this claim and observing that $\theta \rightarrow h'(\theta) = 4\sin\theta - a$ is a monotonic function, we may apply the van der Corput lemma with either $k=1$ or $k=2$, which gives us (38). Thus, compute $h'(\theta) = 4\sin\theta - a$ and $h''(\theta) = 4\cos\theta$, whence

$$(h')^2 + (h'')^2 = 16 - 8a \sin \theta + a^2 \geq 8,$$

where the last inequality is a consequence of $|\sin \theta| \leq 1/\sqrt{2}$ in the interval under consideration. Therefore, $\max(|h'(\theta)|, |h''(\theta)|) \geq 2\sqrt{2} > 1$. \square

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