

Global Well-Posedness of the Short-Pulse and Sine–Gordon Equations in Energy Space

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We prove global well-posedness of the short-pulse equation with small initial data in Sobolev space H^2 . Our analysis relies on local well-posedness results of Schäfer and Wayne [15], the correspondence of the short-pulse equation to the sine–Gordon equation in characteristic coordinates, and a number of conserved quantities of the short-pulse equation. We also prove local and global well-posedness of the sine–Gordon equation in an appropriate function space.

Keywords Conserved quantities; Short-pulse equation; Sine–Gordon equation.

Mathematics Subject Classification 35A01; 35Q60; 37K40.

1. Introduction

Short-pulse approximations of nonlinear wave packets in dispersive media were considered recently with various analytical techniques (see, e.g., [5] for a review of results). The previously known model for small-amplitude quasi-harmonic pulses, the nonlinear Schrödinger equation, is replaced in the short-pulse approximation by a new set of nonlinear evolution equations. Among these models, we consider the model derived by Schäfer and Wayne [15] for short pulses in nonlinear Maxwell's equations. Chung et al. [4] justified derivation of this model in the linear case and presented numerical approximations of modulated pulse solutions. This model dubbed as *the short-pulse equation* can be conveniently expressed in the normalized form by

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad (1.1)$$

where $u(x, t) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ and the subscript denotes partial derivatives. In addition to the derivation of the short-pulse equation, the pioneer paper [15] contains two mathematical results. First, non-existence of a smooth traveling wave

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solution was proved in the entire range of the speed parameter. Second, local well-posedness was proven in space H^s for an integer $s \geq 2$, where H^s denotes the standard Sobolev space with the norm

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}} (1 + |k|^2)^s |\hat{f}(k)|^2 dk \right)^{1/2}$$

and $\hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx$ is the Fourier transform of $f(x)$ on a real axis.

The first result was recently extended by Costanzino et al. [6], who proved existence of smooth traveling solutions in the regularized short-pulse equation,

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} + \beta u_{xxxx}, \quad (1.2)$$

for a small positive $\beta > 0$. They have also derived the regularized short-pulse equation (1.2) in the context of the nonlinear Maxwell equation with a high-frequency dispersion. To construct homoclinic solutions with slow and fast motions, the authors of [6] applied the Fenichel theory for singularly perturbed differential equations.

In this paper, we are extending the second result of [15], namely we prove global well-posedness of the short-pulse equation (1.1) in H^2 . The problem of global well-posedness has been studied in a number of recent publications [8, 10, 16] in the context of a similar Ostrovsky equation

$$u_{xt} = u + (u^2)_{xx} + \beta u_{xxxx}, \quad (1.3)$$

which models small-amplitude long waves in a rotating fluid. Clearly, the regularized short-pulse equation (1.2) can be considered as the Ostrovsky equation with a cubic nonlinearity, so that many of the well-posedness results can be equally applied to both (1.2) and (1.3). Varlamov and Liu [16] proved local well-posedness of (1.3) in space $H^s \cap \dot{H}^{-1}$ for $s > \frac{3}{2}$, where \dot{H}^{-1} is defined by the norm

$$\|f\|_{\dot{H}^{-1}} = \left(\int_{\mathbb{R}} \frac{|\hat{f}(k)|^2}{k^2} dk \right)^{1/2}.$$

If $f \in \dot{H}^{-1}$, then the constraint $\hat{f}(0) = 0$ holds, which is written in physical space as $\int_{\mathbb{R}} f(x) dx = 0$. Under the constraint above, we can define an operator $\partial_x^{-1} f$ in any of the equivalent forms

$$\partial_x^{-1} f := \int_{-\infty}^x f(x') dx' = - \int_x^{\infty} f(x') dx' = \frac{1}{2} \left(\int_{-\infty}^x - \int_x^{\infty} \right) f(x') dx',$$

so that $\|f\|_{\dot{H}^{-1}} = \|\partial_x^{-1} f\|_{L^2}$. The space $H^1 \cap \dot{H}^{-1}$ is the energy space of the Ostrovsky equation (1.3), where the power $V(u) = \|u\|_{L^2}^2$ and the energy

$$E(u) = \int_{\mathbb{R}} \left(\beta (\partial_x u)^2 + \frac{1}{2} (\partial_x^{-1} u)^2 - \frac{1}{3} u^3 \right) dx$$

conserve in time t . Using conserved quantities and local existence in $H^1 \cap \dot{H}^{-1}$, Linares and Milanes [10] and Gui and Liu [8] proved global well-posedness of the

Ostrovsky equation (1.3) in the energy space. However, their proof is only valid for $\beta > 0$ and it is not applicable to the short-pulse equation (1.1).

Using local well-posedness results of [15] and conserved quantities of the short-pulse equation (1.1) found by Brunelli [2], we shall prove our main theorem on global well-posedness of the short-pulse equation.

Theorem 1. *Assume that $u_0 \in H^2$ and*

$$\|u'_0\|_{L^2}^2 + \|u''_0\|_{L^2}^2 < 1.$$

Then the short-pulse equation (1.1) admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ satisfying $u(0) = u_0$.

Our main motivation of interest in global well-posedness of the short-pulse equation (1.1) begins in the recent discovery of exact modulated pulse solutions, which were modeled numerically in [4]. The exact solutions were found by Sakovich and Sakovich [13], who also showed in [12, 14] integrability of the short-pulse equation (1.1) and its equivalence to the sine–Gordon equation in characteristic coordinates

$$w_{yt} = \sin(w), \tag{1.4}$$

where $w(y, t) : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ is a new function in a new independent variable y . According to the results in [14], there exists a local transformation from solutions of the sine–Gordon equation (1.4) to solutions of the short-pulse equation (1.1) given by $u(x, t) = w_t(y(x, t), t)$, where $y = y(x, t)$ is found by inverting the function $x = x(y, t)$ from solutions of the system of partial differential equations

$$\frac{\partial x}{\partial y} = \cos(w), \quad \frac{\partial x}{\partial t} = -\frac{1}{2}(w_t)^2. \tag{1.5}$$

The compatibility of system (1.5) results in the sine–Gordon equation (1.4). The function $x = x(y, t)$ is invertible with the inverse $y = y(x, t)$ for a fixed $t \in \mathbb{R}$ if $w(\cdot, t)$ belongs to the space

$$H_c^s = \left\{ w \in H^s : \|w\|_{L^\infty} \leq w_c < \frac{\pi}{2}, \quad s > \frac{1}{2} \right\}. \tag{1.6}$$

To understand the long-term dynamics of perturbations near the exact modulated pulse solutions (which are dubbed as *breathers* in the context of the sine–Gordon equation), we need to prove local and global existence of solutions of the sine–Gordon equation (1.4) in H_c^s for $s \geq 1$. If the constraint in H_c^s can be kept globally in time, these results would imply that a solution of the short-pulse equation (1.1) starting with small data $u_0 \in H^2$ will remain bounded in H^2 for all $t \in \mathbb{R}_+$. Together with integrability of the short-pulse equation (1.1), these results may suggest orbital or asymptotic stability of the modulated pulse solutions. The latter problem is, however, beyond the scope of the present paper.

The sine–Gordon equation in characteristic coordinates was considered long ago by Kaup and Newell [9] using formal applications of the stationary phase method.

Local well-posedness of this equation is a non-trivial problem due to the presence of the constraint

$$\int_{\mathbb{R}} \sin(w(y, t)) dy = 0, \quad \forall t \in \mathbb{R}_+, \quad (1.7)$$

which gives a necessary but not sufficient condition that the solution $w(\cdot, t)$ stays in H_c^s for all $t \in \mathbb{R}_+$. Our treatment of this equation is rigorous and we shall prove that the sine–Gordon equation (1.4) is locally well-posed in space $H^s \cap \dot{H}^{-1}$ for an integer $s \geq 1$ in the new variable $q = \sin(w)$. Global well-posedness is proved in $H^1 \cap \dot{H}^{-1}$ with the help of three conserved quantities of the sine–Gordon equation (1.4). The result can be extended in $H^s \cap \dot{H}^{-1}$ for an integer $s \geq 2$ if more conserved quantities are incorporated into analysis.

The sine–Gordon equation in the laboratory coordinates

$$w_{\tau\tau} - w_{\xi\xi} = \sin(w)$$

is known to be locally well-posed in a weaker space L^p for $p \geq 2$, see Appendix B of Buckingham and Miller [3]. Similarly to this work, our analysis is also based on the method of Picard iterations to prove local well-posedness of the sine–Gordon equation in characteristic coordinates (1.4). These results provide an alternative proof of the local well-posedness theorem for the short-pulse equation (1.1), thanks to the transformation (1.5). Our treatment of the problem is, however, simpler than the original method of [15], where modified Picard iterations were constructed using solutions of quasi-linear hyperbolic equations along the characteristics. Moreover, additional properties of local solutions to the two equations are identified via the transformation (1.5) and these properties are found to be useful in rigorous treatment of conserved quantities for the two equations.

The paper is organized as follows. Section 2 deals with local well-posedness of the sine–Gordon equation (1.4). Correspondence of local solutions is established in Section 3. Section 4 gives the proof of Theorem 1 on global well-posedness of the short-pulse equation (1.1). Global well-posedness of the sine–Gordon equation (1.4) is proven in Section 5.

2. Local Well-Posedness of the Sine–Gordon Equation (1.4)

We shall consider solutions $w(y, t)$ of the sine–Gordon equation (1.4) vanishing at infinity $|y| \rightarrow \infty$ for $t \geq 0$. Therefore, we eliminate kink solutions of the sine–Gordon equation, which connect different equilibrium states between multiples of 2π at different infinities. Our reasoning is that the kink solutions lead to non-invertible functions $x = x(y, t)$ with respect to y in the transformation (1.5) and give multi-valued loop solutions of the short-pulse equation (1.1) (see [13] for details). Not only we are considering localized solutions $w(\cdot, t)$ in space H^s for $s > \frac{1}{2}$, we need to control

$$\|w(\cdot, t)\|_{L^\infty} \leq w_c < \frac{\pi}{2},$$

to ensure that the transformation (1.5) is invertible at least for $t \in [0, T] \subset \mathbb{R}_+$ for some $T > 0$. Thus, we need to prove that the sine–Gordon equation (1.4) admits a

local solution $w(\cdot, t)$ in space $C([0, T], H_c^s)$ for some $T > 0$, where H_c^s is defined by (1.6).

To incorporate the constraint (1.7) for solutions of the sine–Gordon equation (1.4), we introduce a new variable $q := \sin(w)$, so that the constraint (1.7) becomes a linear constraint

$$\int_{\mathbb{R}} q(y, t) dy = 0. \quad (2.1)$$

The sine–Gordon equation (1.4) transforms to the evolution equation

$$q_t = \sqrt{1 - q^2} \partial_y^{-1} q, \quad (2.2)$$

where the operator ∂_y^{-1} acts on an element of H^s under the constraint (2.1), so that

$$\partial_y^{-1} q := \int_{-\infty}^y q(y', t) dy' = - \int_y^{\infty} q(y', t) dy' = \frac{1}{2} \left(\int_{-\infty}^y - \int_y^{\infty} \right) q(y', t) dy'.$$

Let us introduce the nonlinear function

$$f(q) := 1 - \sqrt{1 - q^2} = \frac{q^2}{1 + \sqrt{1 - q^2}} \quad (2.3)$$

and write the Cauchy problem for equation (2.2) in the equivalent form

$$\begin{cases} q_t = (1 - f(q)) \partial_y^{-1} q, \\ q|_{t=0} = q_0. \end{cases} \quad (2.4)$$

The nonlinear function $f(q)$ is squeezed by the quadratic functions

$$\forall |q| \leq 1 : \quad \frac{q^2}{2} \leq f(q) \leq q^2,$$

which allows us to interpret the term $f(q) \partial_y^{-1} q$ as a nonlinear perturbation to the linear evolution induced by $\partial_y^{-1} q$.

To analyze the Cauchy problem for the nonlinear time evolution (2.4), we obtain information on the fundamental solution of the underlying linear problem

$$\begin{cases} Q_t = \partial_y^{-1} Q, \\ Q|_{t=0} = Q_0. \end{cases} \quad (2.5)$$

Let us denote $L = \partial_y^{-1}$ and $Q(t) = e^{tL} Q_0$. The solution operator e^{tL} is a norm-preserving map from H^s to H^s for any $s \geq 0$ in the sense of

$$\|Q(t)\|_{H^s} = \|e^{tL} Q_0\|_{H^s} = \|Q_0\|_{H^s}, \quad \forall t \in \mathbb{R}, \quad (2.6)$$

which follows from the Fourier transform

$$\mathcal{F}(e^{tL}) = e^{-\frac{it}{k}}, \quad k \in \mathbb{R},$$

involving a bounded oscillatory integral with a singular behavior as $k \rightarrow 0$. The solution operator e^{tL} can be represented by the convolution integrals involving Bessel's function J_0 of the first kind.

Lemma 1. *Let $K_t(y)$ and $J_t(y)$ be defined by*

$$K_t(y) := \sqrt{\frac{t}{y}} J_0'(2\sqrt{ty}), \quad J_t(y) := J_0(2\sqrt{ty}), \quad \forall t, y \in \mathbb{R}_+.$$

There exists $C > 0$ such that for all $t \in \mathbb{R}_+$

$$\|K_t\|_{L^\infty} \leq Ct, \quad \|K_t\|_{L^2} \leq Ct^{1/2}, \quad \|J_t\|_{L^\infty} \leq 1,$$

whereas $K_t \notin L^1$, $J_t \notin L^1$, and $J_t \notin L^2$.

Proof. These properties follow from properties of Bessel function $J_0(z)$, see, e.g., [7]. \square

It follows by direct substitution that the linear Cauchy problem (2.5) has a solution in the form

$$Q(y, t) = Q_0(y) + \int_y^\infty K_t(y' - y) Q_0(y') dy', \quad (y, t) \in \mathbb{R} \times \mathbb{R}_+, \quad (2.7)$$

which is bounded if $Q_0 \in L^\infty \cap L^1$. In addition to $Q(y, t)$, we shall also consider an integral of $Q(y, t)$ in y , given by

$$P(y, t) := - \int_y^\infty Q(y', t) dy' = - \int_y^\infty J_t(y' - y) Q_0(y') dy'. \quad (2.8)$$

The local well-posedness analysis is based on the integral equation

$$q(t) = Q(t) - \int_0^t e^{(t-t')L} f(q(t')) p(t') dt', \quad (2.9)$$

which follows from Duhamel's principle for the nonlinear Cauchy problem (2.4). Here

$$q(t) := q(y, t), \quad p(t) := p(y, t) = - \int_y^\infty q(y', t) dy',$$

$Q(t) = e^{tL} q_0$ is the solution of the linear problem (2.5) with $Q_0 = q_0$, and $f(q)$ is defined by (2.3). We shall work in the space X_c^s given by

$$X_c^s = \left\{ q \in H^s \cap \dot{H}^{-1} : \|q\|_{L^\infty} \leq q_c < 1 \right\}, \quad s > \frac{1}{2}. \quad (2.10)$$

Since $p_y = q$, it is clear that $p \in H^{s+1}$ if $p \in L^2$ and $q \in H^s$. We need to show that the vector field of the integral equation (2.9) is a Lipschitz map in the vector space $X^s := H^s \cap \dot{H}^{-1}$ equipped with the norm

$$\|q\|_{X^s} := \|q\|_{H^s} + \|p\|_{L^2}$$

for any $t \in [0, T]$ and it is a contraction operator for a sufficiently small $T > 0$. This construction gives local well-posedness of the sine–Gordon equation (1.4).

Theorem 2. *Assume that $q_0 \in X_c^s$, $s > \frac{1}{2}$. There exist a $T > 0$ such that the Cauchy problem (2.4) admits a unique local solution $q(t) \in C([0, T], X_c^s)$ satisfying $q(0) = q_0$. Moreover, the solution $q(t)$ depends continuously on initial data q_0 .*

Proof. Fix $s > \frac{1}{2}$, $q_c \in (0, 1)$, and $\delta \in (0, C_s^{-1})$, where constant $C_s > 0$ gives the upper bound of the Banach algebra property

$$\forall f, g \in H^s : \|fg\|_{H^s} \leq C_s \|f\|_{H^s} \|g\|_{H^s}. \quad (2.11)$$

We assume that $\|q_0\|_{X^s} \leq \alpha\delta$ and $\|q_0\|_{L^\infty} \leq \alpha q_c$ for a fixed $\alpha \in (0, 1)$ and prove that there exists a $T > 0$ such that $q(t) \in C([0, T], X_c^s)$ is a solution of the Cauchy problem (2.4) such that $q(0) = q_0$, $\|q(t)\|_{X^s} \leq \delta$, and $\|q(t)\|_{L^\infty} \leq q_c$ for all $t \in [0, T]$. To do so, we find bounds on the supremum of $\|q(t)\|_{H^s}$, $\|q(t)\|_{\dot{H}^{-1}}$, and $\|q(t)\|_{L^\infty}$ on $[0, T]$ from the solution of the integral equation (2.9).

By the triangle inequality, the norm-preserving property (2.6), and the Banach algebra property of H^s , we obtain

$$\begin{aligned} \|q(t)\|_{H^s} &\leq \|Q(t)\|_{H^s} + \int_0^t \|e^{(t-t')L} f(q(t')) p(t')\|_{H^s} dt' \\ &\leq \|q_0\|_{H^s} + C_s \int_0^t \|f(q(t'))\|_{H^s} \|p(t')\|_{H^s} dt'. \end{aligned}$$

To deal with nonlinear function $f(q)$, we expand it in the Taylor series

$$\forall |q| < 1 : f(q) = 1 - \sqrt{1 - q^2} = \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!2^n} q^{2n},$$

which involves only positive coefficients. By invoking again the Banach algebra property, we obtain

$$\begin{aligned} \forall C_s \|q\|_{H^s} < 1 : \|f(q)\|_{H^s} &\leq \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!2^n} C_s^{2n-1} \|q\|_{H^s}^{2n} \\ &= \frac{1 - \sqrt{1 - C_s^2 \|q\|_{H^s}^2}}{C_s} \leq C_s \|q\|_{H^s}^2, \end{aligned}$$

thanks to the representation (2.3). As a result, we have

$$\|q(t)\|_{H^s} \leq \|q_0\|_{H^s} + C_s^2 \int_0^t \|q(t')\|_{X^s}^3 dt'. \quad (2.12)$$

To estimate the L^2 norm of $p(t)$, we use the integral representation

$$p(t) = P(t) - \int_0^t L e^{(t-t')L} f(q(t')) p(t') dt',$$

where $P(t) = LQ(t)$ is defined by solution of the same linear problem (2.5) with initial data $P(0) = p_0$. By the triangle inequality and the norm-preservation property, we obtain

$$\begin{aligned} \|p(t)\|_{L^2} &\leq \|P(t)\|_{L^2} + \int_0^t \|Le^{(t-t')L} f(q(t'))p(t')\|_{L^2} dt', \\ &\leq \|p_0\|_{L^2} + \int_0^t \|Le^{(t-t')L} f(q(t'))p(t')\|_{L^2} dt'. \end{aligned}$$

The norm preservation (2.6) is not useful for the second term since $Lf(q(t))p(t)$ may not be in L^2 . On the other hand, using formula (2.8), we write

$$Le^{(t-t')L} f(q(t'))p(t') = - \int_y^\infty J_{t-t'}(y' - y) f(q(y', t')) p(y', t') dy'. \quad (2.13)$$

We shall use the Hausdorff-Young's inequality

$$\|f \star g\|_{L^p} \leq \|f\|_{L^q} \|g\|_{L^r}, \quad (2.14)$$

where p, q, r are related by the constraint $q^{-1} + r^{-1} = 1 + p^{-1}$ and the star denotes convolution operator $f \star g = \int_{\mathbb{R}} f(y') g(y - y') dy'$. Using inequality (2.14) and Lemma 1, we obtain

$$\|Le^{(t-t')L} f(q(t'))p(t')\|_{L^2} \leq \|J_{t-t'}\|_{L^\infty} \|f(q(t'))p(t')\|_{L^{2/3}} \leq \|f(q(t'))p(t')\|_{L^{2/3}}.$$

Using the Hölder inequality, we obtain

$$\|f(q(t))p(t)\|_{L^1} \leq \|f(q(t))\|_{L^\rho} \|p(t)\|_{L^r},$$

with $\rho^{-1} + r^{-1} = 1$, so that

$$\|Le^{(t-t')L} f(q(t'))p(t')\|_{L^2} \leq \|f(q(t'))\|_{L^{2\rho/3}} \|p(t')\|_{L^{2r/3}}.$$

If we choose $r = 3$, then we have $\rho = \frac{3}{2}$ and $\|f(q)\|_{L^1} \leq \|q\|_{L^2}^2$. As a result, we conclude that

$$\|p(t)\|_{L^2} \leq \|p_0\|_{L^2} + \int_0^t \|q(t')\|_{X^s}^3 dt'. \quad (2.15)$$

Finally, we estimate the L^∞ norm of $q(t)$ from the integral equation (2.9). We obtain

$$\|q(t)\|_{L^\infty} \leq \|Q(t)\|_{L^\infty} + \int_0^t \|e^{(t-t')L} f(q(t'))p(t')\|_{L^\infty} dt'.$$

By Lemma 1, the convolution formula (2.7), and the Hausdorff-Young's inequality (2.14), the free term is estimated by

$$\|Q(t)\|_{L^\infty} \leq \|q_0\|_{L^\infty} + \|K_t\|_{L^2} \|q_0\|_{L^2} \leq \|q_0\|_{L^\infty} + C_1 t^{1/2} \|q_0\|_{X^s},$$

for some $C_1 > 0$. The nonlinear term is estimated by

$$\begin{aligned} \|e^{(t-t')L}f(q(t'))p(t')\|_{L^\infty} &\leq \|f(q(t'))p(t')\|_{L^\infty} + \|K_{t-t'}\|_{L^\infty}\|f(q(t'))p(t')\|_{L^1} \\ &\leq \|q(t')\|_{L^\infty}^2\|p(t')\|_{L^\infty} + \|K_{t-t'}\|_{L^\infty}\|q(t')\|_{L^\infty}\|q(t')\|_{L^2}\|p(t')\|_{L^2} \\ &\leq C_2(1 + (t - t'))\|q(t')\|_{X_c^s}^3, \end{aligned}$$

for some $C_2 > 0$. As a result, we conclude that

$$\|q(t)\|_{L^\infty} \leq \|q_0\|_{L^\infty} + C_1 t^{1/2}\|q_0\|_{X_c^s} + C_2 \int_0^t (1 + (t - t'))\|q(t')\|_{X_c^s}^3 dt'. \quad (2.16)$$

Using (2.12), (2.15), and (2.16), we can see that there exists $T = T(s, q_c, \delta, \alpha) > 0$ such that the vector field of the integral equation (2.9) is a closed map of a finite-radius ball in X_c^s to itself. The value of $T > 0$ satisfies the bounds

$$\begin{aligned} \alpha + T(1 + C_s^2)\delta^2 &\leq 1, \\ \alpha q_c + C_1 T^{1/2}\alpha\delta + \frac{1}{2}C_2 T(T + 2)\delta^3 &\leq q_c. \end{aligned}$$

Moreover, since $f(q)$ behaves like a quadratic function, a similar analysis shows that the map is Lipschitz with respect to q and it is a contraction if $T > 0$ is sufficiently small. Existence of a unique fixed point of the integral equation (2.9) in a complete space $C([0, T], X_c^s)$ follows by the contraction mapping principle (see, e.g., [17]). \square

Corollary 1. *Under the conditions of Theorem 2, we actually have $q(t) \in C([0, T], X_c^s) \cap C^1([0, T], H^s)$.*

Proof. The proof is based on the identity $q_t = \sqrt{1 - q^2}p$ and the fact that $p(t) \in C([0, T], H^{s+1})$. \square

Remark 1. Existence of a unique solution can be proved more easily in a weaker space

$$\tilde{X}_c^s = \{q \in H^s, p \in L^\infty : \|q\|_{L^\infty} \leq q_c < 1\}, \quad s > \frac{1}{2},$$

provided that $q_0 \in X_c^s$. Since $p \in H^1$ if $q, p \in L^2$, X_c^s is continuously embedded to \tilde{X}_c^s . The space X_c^s turns out to be more suitable if we are to use conserved quantities of the sine–Gordon equation.

3. Correspondence Between Short-Pulse and Sine–Gordon Equations

We start by stating the local well-posedness theorem for the short-pulse equation (1.1) from [15].

Theorem 3 ([15]). *Assume that $u_0 \in H^2$. There exists a $T > 0$ such that the short-pulse equation (1.1) admits a unique solution*

$$u(t) \in C([0, T], H^2) \cap C^1([0, T], H^1)$$

satisfying $u(0) = u_0$. Furthermore, the solution $u(t)$ depends continuously on u_0 .

We can now compare the results following from Theorem 2 with Theorem 3. Using the transformation (1.5) and setting $q = \sin(w)$ and $p_y = q$, we have

$$\begin{aligned} u &= w_t = \frac{q_t}{\sqrt{1-q^2}} = p, \\ u_x &= \frac{w_{ty}}{\cos(w)} = \tan(w) = \frac{p_y}{\sqrt{1-q^2}}, \\ u_{xx} &= \frac{w_y}{\cos^3 w} = \frac{p_{yy}}{(1-q^2)^2}. \end{aligned} \tag{3.1}$$

If $q(t) \in X_c^1$ for all $t \in [0, T]$, there exists a uniform bound $q_c \in (0, 1)$ such that $\|q(t)\|_{L^\infty} \leq q_c$ for all $t \in [0, T]$. As a result, the H^2 norm on u in x is equivalent to the H^2 norm on p in y and the H^1 norm on q in y , since $p_y = q$. The following lemma summarizes on the correspondence.

Lemma 2. *Assume that $\|q\|_{L^\infty} \leq q_c < 1$ and consider transformations (1.5) and (3.1). There exist $C^\pm > 0$ such that*

$$\forall u, p \in H^2(\mathbb{R}) : C^- \|p(t)\|_{H^2} \leq \|u(t)\|_{H^2} \leq C^+ \|p(t)\|_{H^2}.$$

Proof. The proof is given by direct computations, e.g.

$$\begin{aligned} \sqrt{1-q_c^2} \|p\|_{L^2}^2 &\leq \|u\|_{L^2}^2 \leq \|p\|_{L^2}^2, \\ \|\partial_y p\|_{L^2}^2 &\leq \|\partial_x u\|_{L^2}^2 \leq \frac{1}{\sqrt{1-q_c^2}} \|\partial_y p\|_{L^2}^2, \\ \|\partial_y^2 p\|_{L^2}^2 &\leq \|\partial_x^2 u\|_{L^2}^2 \leq \frac{1}{\sqrt{(1-q_c^2)^7}} \|\partial_y^2 p\|_{L^2}^2, \end{aligned}$$

where $q_c < 1$ by the assumption of the lemma. □

Combining Theorems 2 and 3, we obtain a more precise result on local well-posedness of the short-pulse and sine-Gordon equations.

Theorem 4. *Let $q(t) \in C([0, T_1], X_c^1) \cap C^1([0, T], H^1)$ be a solution of the sine-Gordon equation in Theorem 2 and Corollary 1 and $T_1 > 0$. Let $u(t) \in C([0, T_2], H^2) \cap C^1([0, T_2], H^1)$ be a solution of the short-pulse equation in Theorem 3 for some $T_2 > 0$. Let q_0, p_0 and u_0 be related by the transformations (1.5) and (3.1). Then, in fact, $p(t) \in C^1([0, T], H^2)$ and $u(t) \in C^1([0, T], H^2)$ for $T = \min(T_1, T_2)$, where $p_y = q$.*

Proof. If $q(t) \in X_c^1$ on $[0, T_1]$, then the bound $\|q(t)\|_{L^\infty} \leq q_c$ holds on $[0, T_1]$ for some $q_c \in (0, 1)$. By Lemma 2 and Corollary 1, if $p(t) \in C([0, T_1], H^2)$ then $u(t) \in C([0, T_1], H^2)$ and if $q(t) \in C^1([0, T_1], H^1)$ then $u_x \in C^1([0, T_1], H^1)$. The first assertion recovers the result of Theorem 3, while if $T = \min(T_1, T_2)$, the second assertion combining with $u(t) \in C^1([0, T], H^1)$ from Theorem 3 implies that $u(t) \in C^1([0, T], H^2)$.

In the opposite direction, by Lemma 2, if $u(t) \in C([0, T_2], H^2) \cap C^1([0, T_2], H^1)$, then $p(t) \in C([0, T], H^2) \cap C^1([0, T], H^1)$ for $T = \min(T_1, T_2)$. Combining with $q(t) \in C^1([0, T], H^1)$ from Corollary 1, we obtain that $p(t) \in C^1([0, T], H^2)$. □

Remark 2. Theorem 4 shows that the results on the sine–Gordon equation (1.4) allow us to control the C^1 property of $\|\partial_x^2 u\|_{L^2}$ in the short-pulse equation (1.1), while the results on the short-pulse equation (1.1) allow us to control the C^1 property of $\|p\|_{L^2}$ in the sine–Gordon equation (1.4). This duality turns out to be useful for rigorous treatment of the conserved quantities for each of the two equations.

Remark 3. If $u(t)$ is a solution of the short-pulse equation in Theorem 3, then for all $t \in (0, T)$, we have the zero-mass constraint

$$\begin{aligned} \int_{\mathbb{R}} u dx &= \int_{\mathbb{R}} w_t \cos(w) dy = \frac{d}{dt} \int_{\mathbb{R}} \sin(w) dy = \frac{d}{dt} \int_{\mathbb{R}} q dy \\ &= \frac{d}{dt} \left[\lim_{y \rightarrow \infty} p(y, t) - \lim_{y \rightarrow -\infty} p(y, t) \right] = 0, \end{aligned}$$

thanks to the fact that $p \in C^1([0, T], H^2)$ from Theorem 4. We note that the initial data $u_0 \in H^2$ does not have to satisfy the zero-mass constraint $\int_{\mathbb{R}} u_0 dx = 0$, in which case $\int_{\mathbb{R}} u(x, t) dx$ jumps from a nonzero value to zero instantaneously for any $t > 0$. See Ablowitz and Villaroel [1] for analysis of a similar problem in the context of the Kadomtsev–Petviashvili equation.

4. Global Well-Posedness of the Short-Pulse Equation (1.1)

It follows from the method of Picard iterations that the existence time $T > 0$ in Theorems 3 and 4 is inverse proportional to the norm $\|u_0\|_{H^s}$ of the initial data. To prove Theorem 1, we need to control the norm $\|u(T)\|_{H^2}$ by a T -independent constant. This constant will be found from the values of conserved quantities of the short-pulse equation. Formal computations of an infinite hierarchy of conserved quantities were reported by Brunelli [2]. Using Theorem 4, we shall make a rigorous use of the conserved quantities.

Lemma 3. *Let $u(t) \in C^1([0, T], H^2)$ be a solution of the short-pulse equation (1.1). The following integral quantities are constant on $[0, T]$:*

$$\begin{aligned} H_{-1} &= \int_{\mathbb{R}} u^2 dx, \\ H_0 &= \int_{\mathbb{R}} \left(\sqrt{1 + u_x^2} - 1 \right) dx = \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx, \\ H_1 &= \int_{\mathbb{R}} \sqrt{1 + u_x^2} \left[\partial_x \left(\frac{u_x}{\sqrt{1 + u_x^2}} \right) \right]^2 dx = \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx. \end{aligned}$$

Proof. We shall write the balance equations for the densities of H_{-1} , H_0 , and H_1 :

$$\begin{aligned} \partial_t(u^2) &= \partial_x \left(v^2 + \frac{1}{4} u^4 \right), \\ \partial_t \left(\sqrt{1 + u_x^2} - 1 \right) &= \frac{1}{2} \partial_x \left(u^2 \sqrt{1 + u_x^2} \right), \\ \partial_t \left(\frac{u_{xx}^2}{\sqrt{(1 + u_x^2)^5}} \right) &= \partial_x \left(\frac{2u_x^2}{\sqrt{1 + u_x^2}} - \frac{u^2 u_{xx}^2}{2\sqrt{(1 + u_x^2)^5}} \right), \end{aligned}$$

where $v = \partial_x^{-1}u = u_t - \frac{1}{2}u^2u_x$ thanks to the short-pulse equation (1.1). If $u(t) \in C^1([0, T], H^2)$, then $v(t) \in C([0, T], H^1)$. By Sobolev's embedding, we have $v(t), u(t), u_x(t) \in L^\infty$ and $v(t), u(t), u_x(t) \rightarrow 0$ as $|x| \rightarrow \infty$ for any $t \in [0, T]$. Integrating the first two balance equations on \mathbb{R} , we confirm conservation of H_{-1} and H_0 . To prove conservation of H_1 , we need to show that $uu_{xx} \rightarrow 0$ as $|x| \rightarrow \infty$ for any $t \in [0, T]$. Using (1.1) and (1.5), we obtain

$$\frac{1}{2}uu_{xx} - u_x^2 = \frac{u_{xt}}{u} - 1 = \tan^2(w) = \frac{q^2}{1 - q^2},$$

where $u_x \rightarrow 0$ as $|x| \rightarrow \infty$ and $q = q(y, t)$. By Theorem 4, $q(t) \in C^1([0, T], H^1)$ and $\|q(t)\|_{L^\infty} \leq q_c < 1$ for any $t \in [0, T]$. Therefore,

$$\frac{\partial x}{\partial y} = \cos(w) = \sqrt{1 - q^2} \geq \sqrt{1 - q_c^2} > 0,$$

for any $t \in [0, T]$, so that the limits $y \rightarrow \pm\infty$ correspond to the limits $x \rightarrow \pm\infty$. Furthermore, since $q \rightarrow 0$ as $|y| \rightarrow \infty$, we have $uu_{xx} \rightarrow 0$ as $|x| \rightarrow \infty$. \square

We can now prove Theorem 1.

Proof of Theorem 1. The values of H_{-1}, H_0 and H_1 computed at initial data $u_0 \in H^2$ are bounded by

$$\begin{aligned} H_{-1} &= \int_{\mathbb{R}} u^2 dx \leq \|u_0\|_{H^2}^2, \\ H_0 &= \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx \leq \frac{1}{2} \|u_0\|_{H^2}^2, \\ H_1 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx \leq \|u_0\|_{H^2}^2. \end{aligned}$$

Note that if $\|u'_0\|_{L^2}^2 + \|u''_0\|_{L^2}^2 < 1$, then $2H_0 + H_1 < 1$. By Lemma 3, these quantities remain constant on $[0, T]$. We will show that the quantities H_0 and H_1 give an upper bound for H^1 norm of the variable

$$\tilde{q} = \frac{u_x}{\sqrt{1 + u_x^2}}. \tag{4.1}$$

Note that $\tilde{q}(x, t) = q(y, t) = \sin(w(y, t))$, where $x = x(y, t)$ is defined by the transformation (1.5). To control $\|\tilde{q}\|_{H^1}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \tilde{q}^2 dx &= \int_{\mathbb{R}} \frac{u_x^2}{1 + u_x^2} dx = \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} \frac{1 + \sqrt{1 + u_x^2}}{1 + u_x^2} dx \\ &\leq 2 \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx = 2H_0 \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \tilde{q}_x^2 dx &= \int_{\mathbb{R}} \left[\partial_x \left(\frac{u_x}{\sqrt{1+u_x^2}} \right) \right]^2 dx \\ &\leq \int_{\mathbb{R}} \sqrt{1+u_x^2} \left[\partial_x \left(\frac{u_x}{\sqrt{1+u_x^2}} \right) \right]^2 dx = H_1. \end{aligned}$$

If $u(t) \in C([0, T], H^2)$, then $q(t) \in C([0, T], H^1)$ and $\tilde{q}(t)$ satisfies the T -independent bound

$$\|\tilde{q}(t)\|_{H^1} \leq \sqrt{H_1 + 2H_0} < 1, \quad \forall t \in [0, T].$$

Thanks to Sobolev's embedding $\|\tilde{q}\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|\tilde{q}\|_{H^1}$, we have $\|\tilde{q}(t)\|_{L^\infty} \leq q_c := \frac{1}{\sqrt{2}} \sqrt{H_1 + 2H_0} < 1, \forall t \in [0, T]$. Inverting the map (4.1), we obtain

$$u_x = \frac{\tilde{q}}{\sqrt{1 - \tilde{q}^2}}.$$

Since H^1 is a Banach algebra with $C_1 = 1$ in the bound (2.11) (see, e.g., [11]), we expand the map into Taylor series with positive coefficients to obtain

$$\|u_x\|_{H^1} \leq \frac{\|\tilde{q}\|_{H^1}}{\sqrt{1 - \|\tilde{q}\|_{H^1}^2}},$$

which results in the T -independent bound

$$\|u(T)\|_{H^2} \leq \left(H_{-1} + \frac{H_1 + 2H_0}{1 - (H_1 + 2H_0)} \right)^{1/2}.$$

This bound allows us to choose a constant time step T_0 such that the solution $u(T_0)$ can be continued on the interval $[T_0, 2T_0]$ in space $C([T_0, 2T_0], H^2)$ using the same Theorems 3 and 4. Continuing the solution with a uniform time step $T_0 > 0$, we obtain global existence of solutions in space $u(t) \in C(\mathbb{R}_+, H^2)$, which completes the proof of Theorem 1. \square

The sufficient condition for global well-posedness of Theorem 1 can be sharpened thanks to the scaling invariance of the short-pulse equation (1.1). Let $\alpha \in \mathbb{R}_+$ be an arbitrary parameter. If $u(x, t)$ is a solution of (1.1), then $U(X, T)$ is also a solution of (1.1) with

$$X = \alpha x, \quad T = \alpha^{-1}t, \quad U(X, T) = \alpha u(x, t). \quad (4.2)$$

The scaling invariance yields the following transformation for conserved quantities

$$\begin{aligned} \tilde{H}_0 &= \int_{\mathbb{R}} \left(\sqrt{1 + U_X^2} - 1 \right) dX = \alpha \int_{\mathbb{R}} \left(\sqrt{1 + u_x^2} - 1 \right) dx = \alpha H_0, \\ \tilde{H}_1 &= \int_{\mathbb{R}} \frac{U_{XX}^2}{(1 + U_X^2)^{5/2}} dX = \alpha^{-1} \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx = \alpha^{-1} H_1. \end{aligned}$$

Therefore, for a given $u_0 \in H^2$, we obtain a family of initial data $U_0 \in H^2$ satisfying

$$\phi(\alpha) = 2\tilde{H}_0 + \tilde{H}_1 = 2\alpha H_0 + \alpha^{-1} H_1.$$

Function $\phi(\alpha)$ achieves its minimum of $2\sqrt{2H_0H_1}$ at $\alpha = \sqrt{\frac{H_2}{2H_1}}$, so that

$$\forall \alpha \in \mathbb{R}_+ : \phi(\alpha) \geq 2\sqrt{2H_0H_1}.$$

Using the scaling invariance property, we obtain the following corollary to Theorem 1.

Corollary 2. *Assume that $u_0 \in H^2$ and $2\sqrt{2H_0H_1} < 1$. Then the short-pulse equation (1.1) admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ satisfying $u(0) = u_0$.*

Proof. If $u_0 \in H^2$ satisfies $2\sqrt{2H_0H_1} < 1$, there exists $\alpha \in \mathbb{R}_+$ such that the corresponding $U_0 \in H^2$ satisfies $2\tilde{H}_0 + \tilde{H}_1 < 1$. By Theorem 1, the corresponding solution $U(T) \in C(\mathbb{R}_+, H^2)$, so that $u(t) \in C(\mathbb{R}_+, H^2)$ by the scaling transformation (4.2). \square

5. Global Well-Posedness of the Sine–Gordon Equation (1.4)

The sine–Gordon equation (1.4) has an infinite set of conserved quantities similarly to the short-pulse equation (1.1). These conserved quantities can be enumerated by the order $j \geq 0$ in the term $(\partial_y^j w)^2$ involving the highest spatial derivative. We will use only the first two conserved quantities,

$$E_0 = \int_{\mathbb{R}} (1 - \cos(w)) dy, \quad E_1 = \int_{\mathbb{R}} w_y^2 dy,$$

the existence of which follows formally from the balance equations

$$\partial_t (1 - \cos(w)) = \partial_y \left(\frac{1}{2} w_t^2 \right), \quad \partial_t \left(\frac{1}{2} w_y^2 \right) = \partial_y (1 - \cos(w)).$$

Additionally, the sine–Gordon equation (1.4) has another infinite set of conserved quantities involving trigonometric functions of w and their integrals enumerated by $j \leq 0$ in the term $(\partial_y^j w)^2$. Besides E_0 , we need only one conserved quantity of this set,

$$E_{-1} = \int_{\mathbb{R}} \cos(w) w_t^2 dy,$$

existence of which follows formally from the balance equation

$$\partial_t (\cos(w) w_t^2) = \partial_y \left(w_{tt}^2 - \frac{1}{4} w_t^4 \right).$$

Using the transformation $q = \sin(w)$, we rewrite the conserved quantities in the equivalent form

$$E_{-1} = \int_{\mathbb{R}} \sqrt{1 - q^2} p^2 dy, \quad E_0 = \int_{\mathbb{R}} f(q) dy, \quad E_1 = \int_{\mathbb{R}} \frac{q_y^2}{1 - q^2} dy, \quad (5.1)$$

where $p = \partial_y^{-1} q$ and $f(q)$ is defined by (2.3). The balance equations are rewritten in the corresponding forms

$$\partial_t f(q) = \partial_y \left(\frac{1}{2} p^2 \right), \quad \partial_t \left(\frac{q_y^2}{1 - q^2} \right) = \partial_y f(q)$$

and

$$\partial_t \left(\sqrt{1 - q^2} p^2 \right) = \partial_y \left(p_t^2 - \frac{1}{4} p^4 \right).$$

We shall check if E_1 , E_0 , and E_{-1} are time conserved quantities for the Cauchy problem (2.4). Global well-posedness in H^2 follows from analysis of the three conserved quantities.

Lemma 4. *Let $p(t) \in C^1([0, T], H^2)$ be the solution of the Cauchy problem (2.4) and $q(t) = \partial_y p(t)$. Then, E_1 , E_0 , and E_{-1} are constant on $[0, T]$.*

Proof. By Sobolev's embedding for $p(t) \in C^1([0, T], H^2)$, we have $q(t)$, $p(t)$, $p_t(t) \rightarrow 0$ as $|y| \rightarrow \infty$. Therefore, conservation of E_1 , E_0 , and E_{-1} follows by integrating the balance equations on $y \in \mathbb{R}$ for the local solutions. \square

Theorem 5. *Assume that $q_0 \in X_c^1$ and $2E_0 + E_1 < 1$. Then there exist a unique global solution $q(t) \in C(\mathbb{R}_+, X_c^1)$ of the Cauchy problem (2.4) satisfying $q(0) = q_0$.*

Proof. The values of E_{-1} , E_0 , E_1 computed at initial data $q_0 \in X_c^1$ are bounded by

$$E_{-1} \leq \|p_0\|_{L^2}^2, \quad |E_0| \leq \|q_0\|_{L^2}^2, \quad E_1 \leq \frac{1}{1 - \|q_0\|_{L^\infty}^2} \|q_0'\|_{L^2}^2,$$

where the constraint $\|q_0\|_{L^\infty} \leq q_c < 1$ is used. By Lemma 4, if $q(t) \in C^1([0, T], X_c^1)$ is a solution constructed in Theorems 2 and 4 for a fixed $T > 0$, the values of quantities E_{-1} , E_0 , E_1 are constant on $[0, T]$. Therefore, we only need to bound the norm $\|q(t)\|_{X^1} = \|q(t)\|_{H^1} + \|p(t)\|_{L^2}$ by a combination of E_{-1} , E_0 , E_1 . This bound is obtained from the following estimates

$$E_{-1} \geq \|p(t)\|_{L^2}^2 \sqrt{1 - \|q(t)\|_{L^\infty}^2}, \quad E_0 \geq \frac{1}{2} \|q(t)\|_{L^2}^2, \quad E_1 \geq \|\partial_y q(t)\|_{L^2}^2, \quad \forall t \in [0, T].$$

By Sobolev's embedding and the bounds above, we have

$$\|q(t)\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|q(t)\|_{H^1} \leq q_c := \frac{1}{\sqrt{2}} \sqrt{E_1 + 2E_0} < 1,$$

since $E_1 + 2E_0 < 1$ thanks to the assumption of the theorem. As a result, we obtain the bound

$$\|q(t)\|_{X^1} \leq \sqrt{E_1 + 2E_0} + \sqrt{\frac{E_{-1}}{\sqrt{1 - q_c^2}}}, \quad \forall t \in [0, T].$$

The time step $T > 0$ depends on $\|q_0\|_{X^1}$. Since the above norm is bounded by the T -independent constant on $[0, T]$, one can choose a non-zero time step $T_0 > 0$ such that the solution $q(t)$ can be continued on the interval $[T_0, 2T_0]$ using the same Theorems 2 and 4. Continuing the solution with a uniform time step T_0 , we obtain global existence of solutions $q(t) \in C(\mathbb{R}_+, X_c^1)$. \square

Remark 4. Theorem 5 is very similar to Theorem 1 thanks to correspondence between the two equations in Lemma 2. In particular, it follows directly that $H_1 = E_1$, $H_0 = E_0$ and $H_{-1} = E_{-1}$.

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