J. math. fluid mech. 11 (2009) 60–90
1422-6928/09/010060-31

DOI 10.1007/s00021-007-0248-8

Journal of Mathematical Fluid Mechanics

Incompressible Viscous Fluid Flows in a Thin Spherical Shell

Ranis N. Ibragimov and Dmitry E. Pelinovsky

Communicated by G. Iooss

Abstract. Linearized stability of incompressible viscous fluid flows in a thin spherical shell is studied by using the two-dimensional Navier–Stokes equations on a sphere. The stationary flow on the sphere has two singularities (a sink and a source) at the North and South poles of the sphere. We prove analytically for the linearized Navier–Stokes equations that the stationary flow is asymptotically stable. When the spherical layer is truncated between two symmetrical rings, we study eigenvalues of the linearized equations numerically by using power series solutions and show that the stationary flow remains asymptotically stable for all Reynolds numbers.

Mathematics Subject Classification (2000). 76D05, 76E20, 34B24, 34L16.

Keywords. Navier–Stokes equations on a sphere, associated Legendre equation, asymptotic stability of stationary flow, numerical approximation of eigenvalues.

1. Introduction

The Navier–Stokes (NS) equations for an incompressible viscous fluid are the fundamental governing equations of fluid mechanics. In many cases, exact solutions can be constructed to these equations [10] and spectral and nonlinear stability of these exact solutions can be analyzed [11]. Our work addresses stability of exact solutions for the NS equations in spherical coordinates.

The three-dimensional NS equations in a thin rotating spherical shell describe large-scale atmospheric dynamics that plays an important role in the global climate control and weather prediction [21, 22]. It was rigorously proved by Temam & Ziane [28] that the average of the longitudinal velocity in the radial direction converges to the strong solution of the two-dimensional NS equation on a sphere as the thickness of the spherical shell goes to zero. The latter model has been used in geophysical fluid dynamics since the middle of the last century [19].

The treatment of the geometric singularity in spherical coordinates has for many years been a difficulty in the development of numerical simulations for oceanic and atmospheric flows around the Earth. Blinova [4, 5] represented solutions in the inviscous case by the eigenfunction expansions in spherical harmonics. Vorticity equations were considered by Ben-Yu with the spectral method [3].

More recent work of Furnier et al. [13] applied the spectral-element method to the axis-symmetric solutions (see [17, 23, 29] for other applications of the spectral methods in spherical coordinates). Finite-element approximations of the vector Laplace–Beltrami equation on the sphere were studied by Simonnet [27]. Finally, point vortex motion on a sphere was modeled by ordinary differential equations for vortex centers in Boatto & Cabral [6] and Crowdy [9].

We address the three-dimensional NS equations for an incompressible viscous fluid,

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0, & \mathbf{x}\in\Omega,\ t\in\mathbb{R}_+, \\ \nabla\cdot\mathbf{u} = 0, & \mathbf{x}\in\Omega,\ t\in\mathbb{R}_+, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, & \mathbf{x}\in\Omega, \end{cases} \tag{1.1}$$

in a thin spherical shell $\Omega = \{\mathbf{x}\in\mathbb{R}^3 : 1 < |\mathbf{x}| < 1+\varepsilon\}$ with $\varepsilon\to 0$, subject to the boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad (\nabla\times\mathbf{u})\times\mathbf{n} = \mathbf{0}, \qquad \mathbf{x}\in\partial\Omega. \tag{1.2}$$

Here $\mathbf{u} : \Omega\times\mathbb{R}_+ \mapsto \mathbb{R}^3$ is the velocity vector, $p : \Omega\times\mathbb{R}_+\mapsto\mathbb{R}$ is the ratio of the pressure to constant density, ν is the kinematic viscosity, $\mathbf{n}$ is the normal vector to the boundary $\partial\Omega$ of the spherical shell Ω and $\mathbf{u}_0 : \Omega\mapsto\mathbb{R}^3$ is a given initial condition. Although Coriolis and gravity forces may be dynamically significant in oceanographic applications, our model is considered in a non-rotating reference frame and without external forces.

We employ the spherical coordinates (r,θ,ϕ) with the velocity vector $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_\phi\mathbf{e}_\phi$, where $(\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_\phi)$ are basic orthonormal vectors along the spherical coordinates. For completeness, we write explicitly the three-dimensional NS equations (1.1) in spherical coordinates [1]:

$$\begin{aligned}
&\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\theta^2+u_\phi^2}{r} \\
&= -\frac{\partial p}{\partial r} + \nu\left(\Delta u_r + \frac{2}{r}\frac{\partial u_r}{\partial r} + \frac{2u_r}{r^2}\right), \\
&\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r} \\
&= -\frac{1}{r}\frac{\partial p}{\partial\theta} + \nu\left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right), \\
&\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r u_\phi}{r} + \frac{u_\theta u_\phi\cot\theta}{r} \\
&= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \nu\left(\Delta u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r^2\sin^2\theta}\right), \\
&\qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} = 0,
\end{aligned}$$

where

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

is the Laplacian in spherical coordinates. One can check by direct differentiation that there exists an exact stationary solution to the three-dimensional NS equations in spherical coordinates:

$$u_r = 0, \quad u_\theta = \frac{\alpha}{r\sin\theta}, \quad u_\phi = 0, \quad p = \beta - \frac{\alpha^2}{2r^2\sin^2\theta}, \tag{1.3}$$

where (α, β) are arbitrary parameters. The stationary solution (1.3) describes the flow tangential to a sphere of any given radius r. The stationary flow has two pole singularities at $\theta = 0$ and $\theta = \pi$. The singularities correspond to the source and sink of the velocity vector at the North and South poles of the spherical shell Ω: the fluid is injected at the North pole from an external source and it leaks out at the South pole to an external sink.

In the limit $\varepsilon \to 0$, the non-stationary three-dimensional fluid flow is confined on a sphere S of unit radius parameterized by the polar (latitude) angle θ and azimuthal (longitude) angle ϕ,

$$S = \{(\theta, \phi), \quad 0 \leqslant \theta \leqslant \pi, \ 0 \leqslant \phi < 2\pi\}. \tag{1.4}$$

Since the velocity vector $\mathbf{u}$ and the pressure p in the NS equations (1.1) are coupled together by the incompressibility constraint $\nabla \cdot \mathbf{u} = 0$, it is difficult to analyze the full set of three-dimensional equations. A common approach to simplify the problem is to use the artificial methods such as the pressure stabilization and projections [26]. The error estimate of the pressure stabilization and projection methods is not however mathematically precise. Instead, we shall use the result of Theorem B in [28], which states that provided the function $\mathbf{u}_0(r, \theta, \phi)$ is smooth enough, the strong global solution $\mathbf{u}(r, \theta, \phi, t)$ of the three-dimensional NS equations converges as $\varepsilon \to 0$ to the strong unique global solution $\mathbf{v}(\theta, \phi, t)$ of the two-dimensional NS equations on the sphere, where

$$\mathbf{v}(\theta, \phi, t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_1^{1+\varepsilon} r\mathbf{u}(r, \theta, \phi, t)dr = (0, v_\theta, v_\phi).$$

The vector $\mathbf{v}(\theta, \phi, t)$ is interpreted as the average velocity with respect to the radial coordinate r. (Other applications of the averaged method for the three-dimensional NS equations in cartesian coordinates with a thin layer and various boundary conditions are reviewed in [16].) The averaged two-dimensional NS equations on a sphere S in spherical angles (θ, ϕ) can be written in the form [28]:

$$\frac{\partial v_\theta}{\partial t} + v_\theta\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{\sin\theta}\frac{\partial v_\theta}{\partial\phi} - v_\phi^2\cot\theta = -\frac{\partial p}{\partial\theta} + \nu\left(\Delta_S v_\theta - \frac{v_\theta}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right),$$

$$\begin{aligned}&\frac{\partial v_\phi}{\partial t} + v_\theta \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{\sin\theta}\frac{\partial v_\phi}{\partial \phi} + v_\theta v_\phi \cot\theta \\ &= -\frac{1}{\sin\theta}\frac{\partial p}{\partial \phi} + \nu\left(\Delta_S v_\phi + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{\sin^2\theta}\right),\end{aligned}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi} = 0,$$

where Δ_S is the Laplace–Beltrami operator in spherical angles

$$\Delta_S = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Note that no boundary conditions are specified for the vector $\mathbf{v}(\theta,\phi,t)$ on sphere S, while the initial condition $\mathbf{v}|_{t=0} = \mathbf{v}_0$ on S is not written. For the purposes of our work, we rewrite the two-dimensional NS equations on the sphere S in an equivalent form:

$$\frac{\partial v_\theta}{\partial t} - \frac{v_\phi \omega}{\sin\theta} + \frac{\partial q}{\partial\theta} = \nu\left(\Delta_S v_\theta - \frac{v_\theta}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right), \tag{1.5}$$

$$\frac{\partial v_\phi}{\partial t} + \frac{v_\theta \omega}{\sin\theta} + \frac{1}{\sin\theta}\frac{\partial q}{\partial\phi} = \nu\left(\Delta_S v_\phi + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v_\phi}{\sin^2\theta}\right), \tag{1.6}$$

$$\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{\partial v_\phi}{\partial\phi} = 0, \tag{1.7}$$

where q is a static (stagnation) pressure and ω is the vorticity:

$$q = p + \frac{1}{2}\left(v_\theta^2 + v_\phi^2\right), \qquad \omega = \frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}. \tag{1.8}$$

The stationary solution (1.3) corresponds to the exact stationary solution of the two-dimensional NS equations (1.5)–(1.7) on the unit sphere S:

$$v_\theta = \frac{\alpha}{\sin\theta}, \qquad v_\phi = 0, \qquad q = \beta, \tag{1.9}$$

where (α,β) are arbitrary parameters. Besides analysis of the stationary flow (1.9) on the sphere S, we shall also model it on the truncated domain with no external source and sink singularities at $\theta = 0$ and $\theta = \pi$, e.g. on the spherical layer

$$S_0 = \big\{(\theta,\phi) :\ \theta_0 \le \theta \le \pi - \theta_0,\ 0 \le \phi \le 2\pi\big\}, \tag{1.10}$$

where $0 < \theta_0 < \frac{\pi}{2}$. Without loss of generality, the spherical layer S_0 is truncated symmetrically at the two rings located in the Northern and Southern semi-spheres so that the stationary flow (1.9) is free of pole singularities in S_0. In other words, without dipping into details on how the fluid flow is injected on the sphere and is collected from the sphere in a neighborhood of the North and South poles, we study how the fluid leaks from the Northern semi-sphere to the Southern semi-sphere along the spherical layer (1.10). In this context, the stationary solution

(1.9) is interpreted as the mass conservation law which is obtained by integrating the free divergence condition (1.7).

We are interested in spectral stability of the stationary fluid flow (1.9). In the case of S when the singularities are included, we prove analytically that the linearized NS equations (1.5)–(1.7) about the stationary solution (1.9) are asymptotically stable. In the case of S_0 when the singularities are excluded, the asymptotical stability of the stationary flow can only be proved for the case $\nu = \infty$, that is in the limit of zero Reynolds numbers. By using the power series expansions, we approximate solutions numerically and show that the stationary flow remains asymptotically stable for all Reynolds numbers.

Similar works can be mentioned in connection to our work. The stationary flow (1.9) is related to the so-called Darcy's law on a sphere, which is derived from the NS equations (1.1) without the inertial term $(\mathbf{u} \cdot \nabla)\,\mathbf{u}$ (see Appendix in [25]). Direct separation of variables in spherical coordinates or other curve-linear coordinates is possible for this approximation, after which the exact stationary flow can be found [20]. Unlike this work, we use the *asymptotic* reduction of the three-dimensional NS equations (1.1) to the two-dimensional NS equations on sphere, which have the exact stationary solution (1.9). Another relevant work is the analysis of spectral stability of the inhomogeneous incompressible NS equations with linear density profiles, where a similar treatment of the hypergeometric equations and singular Sturm–Liouville operators was reported [7].

Our paper is structured as follows. Section 2 introduces the linearization of the two-dimensional NS equations (1.5)–(1.7) at the stationary solution (1.9) and discusses boundary conditions for the perturbation vector. Analytical results on location of the spectrum of the linearized problem are reported in Section 3 for symmetry-breaking (ϕ-dependent) perturbations and in Section 4 for symmetry-preserving (ϕ-independent) perturbations. Numerical results on computations of eigenvalues of the linearized problem are described in Section 5 for symmetry-breaking perturbations and in Section 6 for symmetry-preserving perturbations. Section 7 discusses open directions.

2. Linearized equations and separation of variables

Without loss of generality, we consider the stationary solution (1.9) with $\alpha = 1$ and $\beta = 0$. The presence of arbitrary parameters (α, β) introduces time-independent (neutral) modes of the linearized equations, which we account at the end of this section. We consider infinitesimal time-dependent perturbations of the stationary flow with $\alpha = 1$ and $\beta = 0$ in the form

$$v_\theta = \frac{1}{\sin\theta} + U(\theta,\phi)e^{\lambda t}, \qquad v_\phi = V(\theta,\phi)e^{\lambda t}, \qquad q = Q(\theta,\phi)e^{\lambda t}, \tag{2.1}$$

where $\lambda \in \mathbb{C}$ is a parameter. Perturbations with $\mathrm{Re}(\lambda) > 0$ imply spectral instability of the stationary flow. If $\mathrm{Re}(\lambda) < 0$ for all perturbations, the stationary flow is asymptotically stable, while if $\mathrm{Re}(\lambda) = 0$ for some perturbations and $\mathrm{Re}(\lambda) < 0$ for all other perturbations, the stationary flow is stable in the sense of Lyapunov.

By neglecting the quadratic terms of the perturbation, we linearize the NS equations (1.5)–(1.7) with the expansion (2.1) to the form:

$$\lambda U + \frac{\partial Q}{\partial \theta} = \nu\left(\Delta_S U - \frac{U}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial V}{\partial \phi}\right), \tag{2.2}$$

$$\lambda V + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, V) - \frac{\partial U}{\partial \phi}\right) + \frac{1}{\sin\theta}\frac{\partial Q}{\partial \phi} = \nu\left(\Delta_S V + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial U}{\partial\phi} - \frac{V}{\sin^2\theta}\right), \tag{2.3}$$

$$\frac{\partial}{\partial\theta}(\sin\theta\, U) + \frac{\partial V}{\partial\phi} = 0. \tag{2.4}$$

Perturbation terms of the velocity vector must satisfy some boundary conditions in the domains S_0 or S. It is naturally to assume that the velocity vector is periodic with respect to the angle ϕ:

$$U(\theta, \phi + 2\pi) = U(\theta, \phi), \quad V(\theta, \phi + 2\pi) = V(\theta, \phi). \tag{2.5}$$

Therefore, we look for Fourier series solutions of the system (2.2)–(2.4):

$$U(\theta,\phi) = \sum_{k\in\mathbb{Z}} U_k(\theta)e^{ik\phi}, \quad V(\theta,\phi) = \sum_{k\in\mathbb{Z}} V_k(\theta)e^{ik\phi}, \quad Q(\theta,\phi) = \sum_{k\in\mathbb{Z}} Q_k(\theta)e^{ik\phi}. \tag{2.6}$$

We also require that the components (U_k, V_k) of the velocity vector be square integrable in S_0 or S with respect to the spherical weight:

$$\int_{\theta_0}^{\pi-\theta_0} \left(|U_k|^2 + |V_k|^2\right)\sin\theta d\theta < \infty, \tag{2.7}$$

where $0 \le \theta_0 < \pi/2$. When the domain is the truncated spherical shell S_0, we require that components the velocity vector vanish at the regular end points of the domain:

$$U_k(\theta_0) = U_k(\pi - \theta_0) = V_k(\theta_0) = V(\pi - \theta_0) = 0. \tag{2.8}$$

The complete sphere S with the singular end points is then considered in the limit $\theta_0 \to 0$. We require that the components of the vorticity in (1.8) vanish at the singular end points of the domain:

$$\lim_{\theta\to 0} U_k(\theta) = \lim_{\theta\to\pi} U_k(\theta) = \lim_{\theta\to 0}\sin\theta V_k(\theta) = \lim_{\theta\to\pi}\sin\theta V_k(\theta) = 0. \tag{2.9}$$

It will be clear later that separation of variables is different between the cases $k = 0$ and $k \neq 0$. We say that the correction terms with $k = 0$ represent *symmetry-preserving* perturbations of the stationary flow (2.1), while the correction terms with $k \neq 0$ represent *symmetry-breaking* perturbations.

Case $k \neq 0$. It follows from the divergence-free condition (2.4) that one can introduce the stream function $\Psi_k(\theta)$ for the velocity vector (U_k, V_k) as follows:

$$U_k = \frac{ik}{\sin\theta}\Psi_k(\theta), \qquad V_k = -\Psi_k'(\theta). \tag{2.10}$$

The system of linearized equations (2.2)–(2.3) reduces to the coupled ODE system for $\Psi_k(\theta)$ and $Q_k = ikP_k(\theta)$:

$$\frac{d}{d\theta} P_k = \frac{1}{\sin\theta} \left(\nu \Delta_k \Psi_k - \lambda \Psi_k \right), \tag{2.11}$$

$$\frac{k^2}{\sin\theta} P_k = \frac{d}{d\theta} \left(\nu \Delta_k \Psi_k - \lambda \Psi_k \right) - \frac{1}{\sin\theta} \Delta_k \Psi_k, \tag{2.12}$$

where

$$\Delta_k = \frac{d^2}{d\theta^2} + \frac{\cos\theta}{\sin\theta} \frac{d}{d\theta} - \frac{k^2}{\sin^2\theta}. \tag{2.13}$$

Let $\Phi_k = \Delta_k \Psi_k$ be a new variable. Then, the variable P_k can be excluded from the system (2.11)–(2.12), such that the system reduces to a closed second-order ODE:

$$\nu \Delta_k \Phi_k - \frac{\Phi_k'}{\sin\theta} = \lambda \Phi_k. \tag{2.14}$$

Due to the boundary conditions (2.8) and the representation (2.10), the solution $\Psi_k(\theta)$ for the truncated spherical layer S_0 is defined on a closed interval $\theta_0 \le \theta \le \pi - \theta_0$ for $0 < \theta_0 < \pi/2$ subject to the boundary conditions

$$\Psi_k(\theta_0) = \Psi_k'(\theta_0) = \Psi_k(\pi - \theta_0) = \Psi_k'(\pi - \theta_0) = 0. \tag{2.15}$$

Since $\theta = 0$ and $\theta = \pi$ are singular points of the interval $0 \le \theta \le \pi$, the solution $\Psi_k(\theta)$ for the complete sphere S is defined on an open interval $0 < \theta < \pi$ satisfying the boundary conditions from (2.9) and (2.10):

$$\lim_{\theta\to 0} \Psi_k(\theta) = \lim_{\theta\to\pi} \Psi_k(\theta) = \lim_{\theta\to 0} \sin\theta \Psi_k'(\theta) = \lim_{\theta\to\pi} \sin\theta \Psi_k'(\theta) = 0. \tag{2.16}$$

Case $k = 0$. It follows from the divergence-free condition (2.4) that

$$U_0 = \frac{\alpha}{\sin\theta},$$

where $\alpha \in \mathbb{R}$. This solution resembles the neutral eigenmode generated by the arbitrary constant α in the stationary solution (1.9). Since the eigenmode violates the boundary conditions (2.15) on S_0 and has pole singularities on S, we set $\alpha = 0$. In this case, the first equation (2.2) admits a solution $Q_0 = \beta$, where $\beta \in \mathbb{R}$. It is also a neutral eigenmode generated by the arbitrary constant β in the stationary solution (1.9). Since it is a trivial eigenmode (the pressure term is defined with accuracy to an addition of an arbitrary constant), we can set $\beta = 0$.

When $\alpha = \beta = 0$, the representation for $U_0 = Q_0 = 0$ matches the previous representation for U_k and Q_k with $k = 0$. Using the representation (2.10), we introduce $V_0 = -\Psi_0'(\theta)$ and rewrite the second equation (2.3) as follows:

$$\frac{d}{d\theta} \left(\nu \Delta_0 \Psi_0 - \lambda \Psi_0 \right) - \frac{1}{\sin\theta} \Delta_0 \Psi_0 = 0, \tag{2.17}$$

where Δ_0 is defined by (2.13) with $k = 0$. Letting $\Phi_0 = \Delta_0 \Psi_0$ and taking one more derivative in θ, one can convert the non-trivial equation (2.17) to the previous form (2.14) with $k = 0$. Therefore, all solutions of (2.17) are also solutions of (2.14) with $k = 0$, while the converse statement is not true. It follows from (2.8) and (2.9) that the stream function $\Psi_0(\theta)$ satisfies the Neumann boundary conditions

$$\Psi_0'(\theta_0) = \Psi_0'(\pi - \theta_0) = 0 \tag{2.18}$$

in the case of S_0 and the boundary conditions

$$\lim_{\theta \to 0} \sin\theta \Psi_0'(\theta) = \lim_{\theta \to \pi} \sin\theta \Psi_0'(\theta) = 0 \tag{2.19}$$

in the case of S. Stability analysis of the linearized problem (2.14) with $k \neq 0$ is developed separately from that of the linearized equation (2.17) with $k = 0$. Our main results on eigenvalues of the linearized problems (2.14) and (2.17) are summarized in Table 1. The remainder of this article is devoted to the proofs and numerical verifications of results described in Table 1.

Index k	Viscosity ν	Cut-off θ_0	eigenvalues	results
$k \neq 0$	$0 < \nu \leq \infty$	$\theta_0 = 0$	real negative	Proposition 2
$k \neq 0$	$\nu = \infty$	$0 < \theta_0 < \frac{\pi}{2}$	real negative	Propositions 3 and 4
$k \neq 0$	$0 < \nu < \infty$	$0 < \theta_0 < \frac{\pi}{2}$	real or complex	Section 5
$k = 0$	$0 < \nu \leq \infty$	$\theta_0 = 0$	real negative or absent	Proposition 8
$k = 0$	$0 < \nu \leq \infty$	$0 < \theta_0 < \frac{\pi}{2}$	real negative	Propositions 9 and 10
$k = 0$	$0 < \nu < \infty$	$0 < \theta_0 < \frac{\pi}{2}$	real negative	Section 6

Table 1. Summary of main results

3. Stability analysis for $k \neq 0$

We rewrite the ODE (2.14) supplemented with the relation $\Delta_k \Psi_k = \Phi_k$ by using the variable $x = \cos\theta$:

$$L_k \Psi_k = \Phi_k, \qquad L_k \Phi_k + \epsilon \Phi_k' = \mu \Phi_k, \tag{3.1}$$

where $\epsilon = 1/\nu$ is the Reynolds number of the basic flow, $\mu = \lambda/\nu$ is a rescaled eigenvalue, and L_k is the Sturm–Liouville operator for associated Legendre functions

$$L_k = \frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}\right] - \frac{k^2}{1 - x^2}. \tag{3.2}$$

The system (3.1) is defined on the symmetric interval $[-x_0, x_0]$, where $x_0 = \cos\theta_0$. The spherical layer S_0 corresponds to the case $0 < x_0 < 1$, while the complete sphere S corresponds to the limit $x_0 \to 1$. In the latter case, the interval $[-1, 1]$ connects two singular points $x = \pm 1$ of the Sturm–Liouville operator (3.2). The case $\epsilon = 0$ corresponds to the infinitely viscous fluid, while the case $\epsilon = \infty$ corresponds to the inviscous fluid.

Using the representation (2.10) and the transformation $x = \cos\theta$ with $\Psi_k'(\theta) = -\sqrt{1-x^2}\Psi_k'(x)$, we rewrite the condition (2.7) as the norm on function space $\mathcal{H}_k$, which is used throughout our work:

$$\|\Psi_k\|^2_{\mathcal{H}_k} = \int_{-x_0}^{x_0} \left[(1-x^2)|\Psi_k'(x)|^2 + \frac{k^2}{1-x^2}|\Psi_k(x)|^2\right] dx < \infty. \tag{3.3}$$

We shall denote $\mathcal{H}_k([-x_0, x_0])$ when $0 < x_0 < 1$ and $\mathcal{H}_k([-1,1])$ when $x_0 = 1$. When $0 < x_0 < 1$, the linearized system (3.1) is defined on function space

$$X_0 = \{\Psi_k \in \mathcal{H}_k([-x_0, x_0]) : \quad \Psi_k(\pm x_0) = \Psi_k'(\pm x_0) = 0\}, \tag{3.4}$$

where the boundary conditions (2.15) are taken into account. When $x_0 = 1$, the linearized system (3.1) is defined in function space

$$X = \left\{\Psi_k \in \mathcal{H}_k([-1,1]) : \quad \lim_{x\to\pm 1}\Psi_k(x) = \lim_{x\to\pm 1}(1-x^2)\Psi_k'(x) = 0\right\}, \tag{3.5}$$

where the boundary conditions (2.16) are taken into account. We note that the boundary conditions in the definition of X are redundant, since the norm (3.3) is finite for $x_0 = 1$ only if the boundary conditions in (3.5) are satisfied. Nevertheless, we write these redundant boundary conditions according to the standard formalism of the singular Sturm–Lioville problems [24].

The Sturm–Liouville operator L_k in (3.2) is self-adjoint with respect to the boundary conditions in X_0 and X, such that $(\Psi_k, L_k\Psi_k) = -\|\Psi_k\|^2_{\mathcal{H}_k} < 0$ is finite and real-valued for $\Psi_k \in \mathcal{H}_k([-x_0, x_0])$, $\Psi_k \neq 0$. Therefore, the kernel of L_k is empty in X_0 and X. Because the smallest eigenvalue of L_k is bounded away from zero, the operator L_k is invertible and range(L_k) is dense in the space of square integrable functions on $[-x_0, x_0]$ for any $0 < x_0 \le 1$. Therefore, as it follows from the first equation of the system (3.1), the component $\Phi \in$ range(L_k) is square integrable on $[-x_0, x_0]$ but it does not satisfy any specific boundary conditions at the end points $x = \pm x_0$.

The eigenvalue problem (3.1) in X_0 and X has two continuous parameters $0 < x_0 \le 1$ and $\epsilon \ge 0$ and one integer parameter $k \in \mathbb{Z}\backslash\{0\}$, while (μ, Ψ_k) is the eigenvalue-eigenfunction pair that defines spectral stability of the stationary flow. The following results characterize the spectrum of the eigenvalue problem in the cases: (i) $x_0 = 1$ and $\epsilon \ge 0$; (ii) $0 < x_0 < 1$ and $\epsilon = 0$; and (iii) in the limit $x_0 \to 1$ when $\epsilon = 0$. Based on these results, we prove the following theorem:

Theorem 1. *When $x_0 = 1$ and $\epsilon \ge 0$ or $0 < x_0 \le 1$ and $\epsilon = 0$, the stationary flow* (1.9) *is asymptotically stable with respect to symmetry-breaking perturbations in the sense that the spectrum of the linearized problem* (3.1) *in X_0 or X consists of a set of isolated eigenvalues μ of finite multiplicities and $\mu \in \mathbb{R}_-$ is bounded away from zero.*

The proof of theorem consists of the proofs of three individual propositions.

Proposition 2. *A complete spectrum of the eigenvalue problem* (3.1) *with* $x_0 = 1$ *and* $\epsilon \geq 0$ *in* X *consists of simple isolated eigenvalues at* $\mu = \mu_n$,

$$\mu_n = -s_n(s_n + 1), \qquad s_n = \sigma + n, \tag{3.6}$$

where $\sigma = \sqrt{k^2 + \epsilon^2/4} > 0$ *and* $n \geq 0$ *is integer.*

Proof. Let $\mu = -s(s+1)$ and

$$\Phi_k(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \varphi(x). \tag{3.7}$$

The second equation of the system (3.1) transforms to the associated Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d\varphi}{dx}\right] - \frac{\sigma^2}{1-x^2}\varphi + s(s+1)\varphi = 0, \qquad -1 < x < 1, \tag{3.8}$$

where $\sigma = \sqrt{k^2 + \epsilon^2/4} > 0$. Since the linear ODE (3.8) has no singular points on $-1 < x < 1$, there exists a set of two linearly independent and twice continuously differentiable solutions in any compact subset of $(-1, 1)$ [8]. Singularity analysis of the ODE (3.8) as $x \to \pm 1$ shows that the solution $\varphi(x)$ either have a singular (unbounded) behavior like $(1 \mp x)^{-\sigma/2}$ as $x \to \pm 1$ or a regular (vanishing) behavior like $(1 \mp x)^{\sigma/2}$ as $x \to \pm 1$.

Let $\varphi(x)$ be a regular solution of (3.8) on $x \in [-1, 1]$, such that $\varphi(x) \sim (1 \mp x)^{\sigma/2}$ and $\Phi_k(x) \sim (1 \mp x)^{\pm\epsilon/4+\sigma/2}$ as $x \to \pm 1$. Since the Sturm–Liouville operator L_k is invertible on $\Phi_k \in L^2([-1, 1])$ for $k \neq 0$, the first equation of the system (3.1) admits a solution $\Psi_k(x)$ that behaves like $(1 \mp x)^{1\pm\epsilon/4+\sigma/2}$ as $x \to \pm 1$. Since $\pm\epsilon + \sqrt{\epsilon^2 + 4k^2} \geq 0$ for any $k \in \mathbb{Z}$ and $\epsilon \geq 0$, the function $\Phi_k(x)$ is bounded and square integrable on $[-1, 1]$, while the function $\Psi_k(x)$ belongs to the function space X in (3.5). Therefore, if $\varphi(x)$ is a regular solution of (3.8), then $\Psi_k(x)$ is an eigenfunction of the eigenvalue problem (3.1) in X.

Let $\varphi(x)$ be a singular solution of (3.8), such that $\varphi(x) \sim (1 \mp x)^{-\sigma/2}$ and $\Phi_k(x) \sim (1 \mp x)^{\pm\epsilon/4-\sigma/2}$ in at least one limit $x \to \pm 1$. Since $\pm\epsilon - \sqrt{\epsilon^2 + 4k^2} \leq -2|k| \leq -2$ for $k \neq 0$ and $\epsilon \geq 0$, the function Φ_k is not square integrable on $[-1, 1]$ and hence $\Psi_k(x)$ can not be in X. By Theorem 10 on p. 1441 in [12], the essential spectrum of the formally self-adjoint operator (3.8) is void. Therefore, the complete spectrum of the linearized system (3.1) in X consists of isolated eigenvalues μ, which correspond to *regular* solutions $\varphi(x)$ of the associated Legendre equation (3.8).

Let $\varphi(x)$ be a regular solution of (3.8) and write $\varphi(x) = (1-x^2)^{\sigma/2}F(x)$, where $F(x)$ is bounded as $x \to \pm 1$. This substitution transforms the associated Legendre equation (3.8) to the hypergeometric equation

$$z(1-z)F''(z) + (\gamma - (\alpha + \beta + 1)z)\,F'(z) - \alpha\beta F(z) = 0, \tag{3.9}$$

where

$$z = \frac{1-x}{2}, \quad \alpha = \sigma - s, \quad \beta = \sigma + s + 1, \quad \gamma = \sigma + 1. \tag{3.10}$$

The only solution of the ODE (3.9) which is bounded as $x \to 1$ ($z \to 0$) is the hypergeometric function $F(z; \alpha, \beta, \gamma)$, which admits the power series at $z = 0$ (see 9.100 on p. 995 in [14]):

$$F(z; \alpha, \beta, \gamma) = 1 + \frac{\alpha\beta}{\gamma 1!} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!} z^2 + \cdots . \tag{3.11}$$

It follows from properties 9.101–9.102 on p. 995 in [14] that the hypergeometric series (3.11) with $\alpha + \beta - \gamma = \sigma > 0$ converges for $|z| < 1$ and diverges as $z \to 1$, unless it is truncated into a polynomial. The latter case is the only case when the solution of the ODE (3.9) is bounded in both limits $x \to 1$ ($z \to 0$) and $x \to -1$ ($z \to 1$). The truncation occurs when either $\alpha = -n$ or $\beta = -m$ with non-negative integers n and m. The two cases are in fact equivalent to each other since $\mu = -s(s+1) = (\alpha - \sigma)(\beta - \sigma)$ and $\alpha + \beta = 1 + 2\sigma$. Let $\alpha = -n$, such that $s = \sigma + n$, $\beta = 2\sigma + n + 1$ and $\gamma = \sigma + 1$. In this case, the function $F(z; -n, n+1+2\sigma, 1+\sigma) \equiv F_n(x)$ is a polynomial of degree n, e.g.

$$F_0 = 1, \quad F_1 = x, \quad F_2 = \frac{(2\sigma+3)x^2 - 1}{2(1+\sigma)}, \quad F_3 = \frac{(5+2\sigma)x^3 - 3x}{2(1+\sigma)}, \tag{3.12}$$

while the simple eigenvalues $\mu = \mu_n$ are given by the expression (3.6). When $\sigma = 0$, polynomials $F_n(x)$ coincide with the Legendre polynomials $P_n(x)$ given by identity 8.91 on p. 973 of [14]. □

Proposition 3. *A complete spectrum of the eigenvalue problem* (3.1) *with* $0 < x_0 < 1$ *and* $\epsilon = 0$ *in* X_0 *consists of isolated eigenvalues* μ*, which are* (i) *real and strictly negative and* (ii) *either simple or double with linearly independent eigenfunctions.*

Proof. We first show that no zero eigenvalue $\mu = 0$ exists in the eigenvalue problem (3.1) with $0 < x_0 < 1$ and $\epsilon = 0$ in X_0. Let $\Psi_k(x)$ be a $C^4([-x_0, x_0])$ solution of the fourth-order ODE $L_k^2 \Psi_k = 0$ in function space X_0. Then,

$$\begin{aligned}(\Psi_k, L_k^2 \Psi_k) &= (1-x^2)\left[\Psi_k (L_k\Psi_k)' - \Psi_k'(L_k\Psi_k)\right]\Big|_{x=-x_0}^{x=x_0} + (L_k\Psi_k, L_k\Psi_k) \\ &= (L_k\Psi_k, L_k\Psi_k).\end{aligned}$$

Therefore, $\Psi_k(x)$ is in fact the solution of the second-order ODE $L_k\Psi_k = 0$. The boundary conditions in X_0 admit the only solution $\Psi_k(x) \equiv 0$, such that the eigenvalue problem (3.1) contains no eigenvalue $\mu = 0$ in X_0.

When $\mu \neq 0$ and $\epsilon = 0$, the system (3.1) admits a general solution in the form

$$\Psi_k(x) = \frac{\phi(x)}{\mu} + \psi(x), \qquad \Phi_k(x) = \phi(x),$$

where $\psi(x)$ and $\phi(x)$ are general solutions of the homogeneous second-order ODEs

$$L_k\psi = 0, \qquad L_k\phi = \mu\phi.$$

Since the operator L_k is invariant with respect to the inversion symmetry $x \mapsto -x$, each homogeneous second-order ODE has linearly independent symmetric (even) and anti-symmetric (odd) solutions denoted by subscripts + and − respectively. Therefore, we obtain the decomposition

$$\begin{aligned} \Psi_k(x) &= d_+\frac{\phi_+(x)}{\mu} + c_+\psi_+(x) + d_-\frac{\phi_-(x)}{\mu} + c_-\psi_-(x), \\ \Phi_k(x) &= d_+\phi_+(x) + d_-\phi_-(x), \end{aligned}$$

where (c_+, c_-, d_+, d_-) are constants and the functions $\phi_\pm(x)$ and $\psi_\pm(x)$ are uniquely normalized by the initial values at $x = 0$ (e.g. $\phi_+(0) = 1$, $\phi'_+(0) = 0$ and $\phi_-(0) = 0$, $\phi'_-(0) = 1$). We note that either $\psi_\pm(x_0) \neq 0$ or $\psi'_\pm(x_0) \neq 0$ (since $\psi_\pm(x) \equiv 0$ otherwise). By using the boundary conditions in (3.4), we decompose the boundary-value problems into two uncoupled systems with

$$d_\pm\phi_\pm(x_0) + \mu c_\pm\psi_\pm(x_0) = 0, \qquad d_\pm\phi'_\pm(x_0) + \mu c_\pm\psi'_\pm(x_0) = 0,$$

so that a non-zero solution for (c_+, c_-, d_+, d_-) exists provided

$$\phi'_\pm(x_0)\psi_\pm(x_0) = \phi_\pm(x_0)\psi'_\pm(x_0).$$

Since the functions $\psi_\pm(x)$ are independent of μ, we have thus obtained that the functions $\phi_\pm(x)$ solve the *closed* eigenvalue problem

$$L_k\phi_\pm = \mu\phi_\pm, \qquad -x_0 \le x \le x_0, \tag{3.13}$$

defined on the function space

$$\begin{aligned} H_0 = \big\{\phi_\pm \in \mathcal{H}_k([-x_0, x_0]) :\ & \psi_\pm(x_0)\phi'_\pm(x_0) - \psi'_\pm(x_0)\phi_\pm(x_0) = 0, \\ & \phi_\pm(-x) = \pm\phi_\pm(x)\big\}. \end{aligned} \tag{3.14}$$

The μ-independent boundary values in (3.14) are Robin boundary conditions when $\psi_\pm(x_0)$ and $\psi'_\pm(x_0)$ are both non-zero, Dirichlet boundary conditions when $\psi_\pm(x_0) = 0$ and Neumann boundary conditions when $\psi'_\pm(x_0) = 0$. The associated Legendre operator L_k is self-adjoint in H_0 with respect to any of these boundary conditions [24]. Therefore, all eigenvalues μ of the eigenvalue problem (3.13) in H_0 are real-valued and isolated, while the corresponding eigenfunctions $\phi_\pm(x)$ are real-valued. Moreover, all eigenvalues of (3.13) are simple since the Wronskian of any two solutions of (3.13) with boundary conditions in (3.14) is zero. Since $\Psi_k \in X_0$ and $\Phi_k \in H_0$, we obtain that

$$(\phi, \phi) = (L_k\Psi_k, \phi) = (\Psi_k, L_k\phi) = \mu(\Psi_k, \phi) = (\phi, \phi) + \mu(\psi, \phi),$$

so that $(\psi, \phi) = 0$ for $\mu \neq 0$. By using the above identity, we obtain that

$$\frac{1}{\mu}(\phi, \phi) = (\Psi_k, \phi) = (\Psi_k, L_k\Psi_k) = -\|\Psi_k\|^2_{\mathcal{H}_k} < 0, \tag{3.15}$$

so that $\mu < 0$ for *each* eigenvalue with $\phi \neq 0$. By construction, eigenvalues are at most double. The case of double eigenvalues corresponds to the situation when the eigenvalue problems (3.13)–(3.14) admit two linearly independent (even and odd) eigenfunctions for the same value of μ. □

Proposition 4. *Let* $\{\mu_n\}_{n\geq 0}$ *be isolated eigenvalues of the eigenvalue problem* (3.1) *in* X_0 *with* $0 < x_0 < 1$ *and* $\epsilon = 0$ *ordered as*

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_n \geq \cdots .$$

Then,

$$\lim_{x_0 \to 1} \mu_n = -s_n(s_n + 1), \qquad s_n = |k| + n,$$

where $n \geq 0$.

Proof. Consider even and odd solutions of the second-order ODE $L_k \psi_\pm = 0$ in the limit $x_0 \to 1$. Since the kernel of L_k admits no eigenfunctions in $\mathcal{H}_k$ for $k \neq 0$ and $0 < x_0 \leq 1$, the solutions $\psi_\pm(x_0)$ must diverge as $x_0 \to 1$. Singularity analysis as $x \to \pm 1$ suggests that the solution $\psi_\pm(x)$ grows like $(1 \mp x)^{-|k|/2}$ as $x \to \pm 1$, such that $\lim_{x_0 \to 1} \psi_\pm(x_0)/\psi'_\pm(x_0) = 0$. Therefore, eigenfunctions $\phi_\pm(x)$ of the auxiliary eigenvalue problem (3.13) for $0 < x_0 < 1$ satisfy in the limit $x_0 \to 1$ the singular eigenvalue problem

$$L_k \phi_\pm = \mu \phi_\pm, \qquad -1 < x < 1 \tag{3.16}$$

defined on the function space

$$H = \left\{ \phi_\pm \in \mathcal{H}_k([-1,1]) : \quad \lim_{x \to \pm 1} \phi_\pm(x) = \lim_{x \to \pm 1} (1 - x^2)\phi'_\pm(x) = 0 \right\}. \tag{3.17}$$

Again, the boundary conditions in H are redundant due to convergence of the integral in $\mathcal{H}_k([-1,1])$. A complete spectrum of the eigenvalue problem (3.16)–(3.17) is constructed in the proof of Proposition 2: eigenvalues are given by (3.6) with $\epsilon = 0$ and eigenfunctions are $\phi_\pm(x) = (1 - x^2)^{|k|/2} F_n(x)$, where $F_n(x)$ are associated Legendre polynomials (3.12) with $\sigma = |k|$. Convergence and uniqueness of continuations from eigenvalues of (3.13) in H_0 for $x_0 < 1$ to eigenvalues of (3.16) in H for $x_0 = 1$ is proved in two steps. Theorem 5.3 of [2] guarantees convergence and uniqueness of continuations from the singular Sturm–Liouville problem (3.16) in H to the regular Dirichlet problem for the Sturm–Liouville operator (3.13) on $-x_0 \leq x \leq x_0$. The Dirichlet problem is generally different from the Robin boundary-value problem in H_0 by the terms $\psi_\pm(x_0)\phi'_\pm(\pm x_0)/\psi'_\pm(x_0)$ in the boundary conditions in H_0. However, these terms are small in the limit $x_0 \to 1$. Unique continuation of simple eigenvalues of the Dirichlet problem to the simple eigenvalues of the Robin problem (separately for $\phi_+(x)$ and $\phi_-(x)$) follows by standard perturbation theory of eigenvalues of self-adjoint Sturm–Liouville operators in Lemma VIII 1.24 of [18]. □

Remark 5. Theorem 1 does not cover the case $0 < x_0 < 1$ and $\epsilon > 0$. Eigenvalues of the linearized problem (3.1) in this case will be computed in Section 5 numerically.

4. Stability analysis for $k = 0$

We rewrite the ODE (2.17) supplemented with the relation $\Delta_0 \Psi_0 = \Phi_0$ in the variable $x = \cos\theta$:

$$L_0 \Psi_0 = \Phi_0, \qquad \Phi_0' + \frac{\epsilon}{1 - x^2} \Phi_0 = \mu \Psi_0'. \tag{4.1}$$

where L_0 is the Sturm–Liouville operator for Legendre functions

$$L_0 = \frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} \right]. \tag{4.2}$$

Incorporating the boundary conditions (2.18) and (2.19) in new variables, we introduce the function spaces X_0 and X for the eigenvalue problem (4.1). When $0 < x_0 < 1$, the function space X_0 is

$$X_0 = \left\{ \Psi_0 \in \mathcal{H}_0([-x_0, x_0]) : \quad \Psi_0'(\pm x_0) = 0 \right\}. \tag{4.3}$$

When $x_0 = 1$, the function space X is

$$X = \left\{ \Psi_0 \in \mathcal{H}_0([-1, 1]) : \quad \lim_{x \to \pm 1} (1 - x^2) \Psi_0'(x) = 0 \right\}, \tag{4.4}$$

where the boundary conditions are redundant due to convergence of the integral in $\mathcal{H}_0([-1, 1])$. No boundary conditions on $\Psi_0(x)$ are set at $x = \pm x_0$. Moreover, the system (4.1) defines the function $\Psi_0(x)$ up to an arbitrary additive constant. Therefore, the constant function $\Psi_0(x) \equiv \text{const}$ is always an eigenfunction of the system (4.1) with $\Phi_0(x) \equiv 0$.

Lemma 6. *The eigenvalue $\mu = 0$ of the linearized system* (4.1) *in either X_0 or X is algebraically simple.*

Proof. Integrating the first equation in the system (4.1) on $x \in [-x_0, x_0]$ for $\Psi_0(x)$ in either X_0 or X, we obtain the Fredholm Alternative condition

$$\int_{-x_0}^{x_0} \Phi_0(x) dx = 0, \tag{4.5}$$

where $0 < x_0 \leq 1$. Integrating the second equation in the system (4.1), we obtain a general solution for $\mu = 0$:

$$\Phi_0 = c_0 \left(\frac{1 - x}{1 + x} \right)^{\epsilon/2},$$

where c_0 is constant. Since $\Phi_0(x)$ does not satisfy the Fredholm Alternative condition (4.5), we have to set $c_0 = 0$. Then, $\Psi_0(x)$ satisfies the second-order ODE $L_0\Psi_0 = 0$, which admits only one eigenfunction $\Psi_0(x) \equiv \text{const}$ in either X_0 or X. Similarly one can prove that the Jordan block of the zero eigenvalue with the eigenfunction $\Psi_0(x) \equiv \text{const}$ and $\Phi_0(x) \equiv 0$ is of the length one. $\square$

We will extend results of Section 3 to the linearized problem (4.1) with $\mu \neq 0$ in X_0 and X. Neglecting the only zero eigenvalue $\mu = 0$ with the trivial eigenfunction $\Psi_0(x) \equiv \text{const}$, we prove the following theorem.

Theorem 7. *The stationary flow* (1.9) *is asymptotically stable with respect to symmetry-preserving perturbations in the sense that all eigenvalues μ (excluding the trivial zero) of the linearized problem* (4.1) *with $0 < x_0 \leq 1$ and $\epsilon \geq 0$ in X_0 or X are real and strictly negative.*

In order to develop analysis of eigenvalues for $\mu \neq 0$, we shall use two equivalent reformulations of the third-order ODE system (4.1) as the second-order eigenvalue problems associated with formally self-adjoint operators. In the first reformulation, we exclude $\mu\Psi_0'(x)$ from the system (4.1) and find a closed equation for $\Phi_0(x)$,

$$L_0\Phi_0 + \epsilon\Phi_0' = \mu\Phi_0. \tag{4.6}$$

By introducing new dependent variable $\varphi(x)$ via

$$\Phi_0(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \varphi(x), \tag{4.7}$$

the linearized equation (4.6) is transformed to the self-adjoint form given by the associated Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d\varphi}{dx}\right] - \frac{\epsilon^2}{4(1-x^2)}\varphi = \mu\varphi, \qquad -x_0 < x < x_0. \tag{4.8}$$

By using the second equation of the system (4.1), we obtain the first-order ODE for the function $\Psi_0(x)$:

$$\mu\Psi_0'(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \left(\frac{d\varphi}{dx} + \frac{\epsilon}{2(1-x^2)}\varphi\right). \tag{4.9}$$

While the linearized equation (4.6) coincides with the second equation of the system (3.1) for $k = 0$, the present role of this equation is different. In order to find $\Psi_0(x)$ from its solution $\Phi_0(x)$, we can solve the first-order ODE (4.9) in either X_0 or X with $\mu \neq 0$. Therefore, as opposed to the case $k \neq 0$, we do not have to solve the first equation of the system (4.1) and the Fredholm Alternative condition (4.5) can be ignored in this approach.

In the second reformulation of the third-order ODE system (4.1), we introduce a new dependent variable $\chi(x)$ via

$$\Psi_0'(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \frac{\chi(x)}{\sqrt{1-x^2}}. \tag{4.10}$$

By using the first equation of the system (4.1), we express the function $\Phi_0(x)$ in terms of $\chi(x)$:

$$\Phi_0(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \left(\sqrt{1-x^2}\frac{d\chi}{dx} - \frac{\epsilon+2x}{2\sqrt{1-x^2}}\chi\right). \tag{4.11}$$

The second equation of the system (4.1) transforms then to the self-adjoint form:

$$\frac{d}{dx}\left[(1-x^2)\frac{d\chi}{dx}\right] - \frac{\epsilon^2+4+4\epsilon x}{4(1-x^2)}\chi = \mu\chi, \qquad -x_0 < x < x_0. \tag{4.12}$$

Although the second-order ODE (4.12) is more complicated than the associated Legendre equation (4.8), the eigenfunction $\chi(x)$ is related to the function $\Psi_0(x)$ better than the eigenfunction $\varphi(x)$. In particular, when $x_0 = 1$ and $\Psi_0 \in X$, the eigenfunction $\chi(x)$ satisfies the conditions:

$$\int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{\epsilon/2} |\chi|^2(x)dx < \infty, \qquad \lim_{x\to\pm1} \left(\frac{1-x}{1+x}\right)^{\epsilon/4} \sqrt{1-x^2}\chi(x) = 0. \tag{4.13}$$

When $0 < x_0 < 1$ and $\Psi_0 \in X_0$, the eigenfunction $\chi(x)$ is a classical solution of the second-order ODE (4.12) on $[-x_0, x_0]$ with the Dirichlet boundary conditions $\chi(\pm x_0) = 0$. There exists a pair of Darboux–Backlund transformations between the Sturm–Liouville problems (4.8) and (4.12):

$$\varphi(x) \;=\; \sqrt{1-x^2}\chi'(x) - \frac{\epsilon+2x}{2\sqrt{1-x^2}}\chi(x), \tag{4.14}$$

$$\mu\chi(x) \;=\; \sqrt{1-x^2}\varphi'(x) + \frac{\epsilon}{2\sqrt{1-x^2}}\varphi(x), \tag{4.15}$$

where $\mu \neq 0$ is assumed. By the Friedrichs' theorems (see, e.g. Theorem 10 on p. 1441 or Theorem 67 on p. 1501 of [12]), the essential spectrum of the formally self-adjoint operators (4.8) and (4.12) is void. Therefore, the spectrum of these operators consists of a sequence of isolated eigenvalues of finite multiplicities, which we categorize in three individual propositions.

Proposition 8. *A complete spectrum of the eigenvalue problem* (4.1) *with* $x_0 = 1$ *and* $0 \le \epsilon < 2$ *in* X *consists of simple isolated eigenvalues at* $\mu = \mu_n$*, where*

$$\mu_n = -n(n+1), \qquad n \ge 0. \tag{4.16}$$

No non-zero eigenvalues of the eigenvalue problem (4.1) *with* $x_0 = 1$ *and* $\epsilon \ge 2$ *exists in* X*.*

Proof. Let $\mu = -s(s+1) \neq 0$ and $\varphi(x) = (1-x^2)^{\epsilon/4}F(x)$ and consider the associated Legendre equation (4.8) with $x_0 = 1$. Then, the function $F(x)$ satisfies the hypergeometric equation (3.9) under parametrization (3.10) with $\sigma = \epsilon/2$. In order to identify solutions $F(x)$ of the hypergeometric equations in the function space $\Psi_0 \in X$, we shall rewrite the relation (4.9) as follows:

$$-s(s+1)\Psi_0'(x) = (1-x)^{\epsilon/2}\left(F'(x) + \frac{\epsilon}{2(1+x)}F(x)\right). \tag{4.17}$$

Also recall that $\Phi_0(x) = (1-x)^{\epsilon/2}F(x)$. When $\epsilon = 0$, we find that $\Phi_0(x) = F(x)$ and $\Psi_0(x) = -\frac{1}{s(s+1)}F(x) + \text{const}$, such that $\Psi_0 \in X$ if and only if $F(x) \in X$. The only set of eigenfunctions of the Legendre equation (4.8) with $\epsilon = 0$ in X is the set of Legendre polynomials $F = P_n(x)$ for $s = n$ with $n \geq 0$ (see 8.91 on p. 973 in [14]). This set corresponds to the eigenvalues (4.16). Although the zero eigenvalue ($s = n = 0$) is excluded from the approach above, it is still added to the spectrum by Lemma 6.

When $\epsilon > 0$, the eigenfunction $\Psi_0(x)$ belongs to X only if $F(x)$ has a regular behavior as $x = 1$ ($z = 0$). The only solution of the hypergeometric equation (3.9) which is bounded as $x \to 1$ is the hypergeometric function $F(z;\alpha,\beta,\gamma)$. (Indeed, by 9.153 on p. 1001 of [14], the other linearly independent solution $F(x)$ has a singular behavior like $F(x) \sim (1-x)^{-\epsilon/2}$ as $x \to 1$, which results in the divergence $\Psi_0'(x) \sim (1-x)^{-1}$ as $x \to 1$, such that $\Psi_0 \notin X$.) By the identity 9.131 on p. 998 of [14], the hypergeometric function $F(z;\alpha,\beta,\gamma)$ admits the following behavior at the other singular point $x = -1$ ($z = 1$):

$$\begin{aligned} &F(z;\alpha,\beta,\gamma) \\ &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(1-z;\alpha,\beta,\alpha+\beta-\gamma+1) \\ &\quad + (1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(1-z;\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1), \end{aligned} \tag{4.18}$$

where $\Gamma(z)$ is the Gamma function and

$$z = \frac{1-x}{2}, \quad \alpha = \frac{\epsilon}{2} - s, \quad \beta = \frac{\epsilon}{2} + s + 1, \quad \gamma = \frac{\epsilon}{2} + 1.$$

Since $\alpha+\beta-\gamma+1 = \gamma$ and $\gamma-\alpha-\beta+1 = 1-\frac{\epsilon}{2}$, the relation (4.18) produces an unbounded function $F(1-z;\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1)$ for $\epsilon = 2m$ with $m \geq 1$ (the hypergeometric function $F(z;\alpha,\beta,\gamma)$ diverges for $\gamma = -n$ with $n \geq 0$ integer).

It follows from (4.17) that the first term in (4.18) leads to the singular behavior $\Psi_0'(x) \sim (1+x)^{-1}$ as $x \to -1$ ($z \to 1$) if $\epsilon \neq 0$, while the second term in (4.18) leads to the singular behavior $\Psi_0'(x) \sim (1+x)^{-\epsilon/2}$ as $x \to -1$ ($z \to 1$) if $\mu \neq 0$. The eigenfunction $\Psi_0(x)$ belongs to X if and only if $\epsilon < 2$ and the first term in (4.18) is removed. The last constraint is achieved when $\gamma - \alpha = 1 + s = -n$ or $\gamma - \beta = -s = -m$ with integers $n, m \geq 0$. Both choices define the same set of eigenvalues (4.16) in the parametrization $\mu = -s(s+1)$. Using another identity 9.131 on p. 998 of [14],

$$F(z;\alpha,\beta,\gamma) = (1-z)^{\gamma-\alpha-\beta}F(z;\gamma-\alpha,\gamma-\beta,\gamma), \tag{4.19}$$

we set $s = -1 - n$ with $n \geq 1$, such that

$$F\left(z; \frac{\epsilon}{2} + 1 + n, \frac{\epsilon}{2} - n, \frac{\epsilon}{2} + 1\right) = (1-z)^{-\epsilon/2} F\left(z; -n, n+1, \frac{\epsilon}{2} + 1\right),$$

where $F(z; -n, n+1, 1+\epsilon/2) \equiv \tilde{F}_n(x)$ is a polynomial of degree n, e.g.

$$\tilde{F}_0 = 1, \quad \tilde{F}_1 = \frac{x+\sigma}{1+\sigma}, \quad \tilde{F}_2 = \frac{3x^2 + 3\sigma x + \sigma^2 - 1}{(1+\sigma)(2+\sigma)},$$

$$\tilde{F}_3 = \frac{15x^3 + 15\sigma x^2 + (6\sigma^2 - 9)x + \sigma(\sigma^2 - 4)}{(1+\sigma)(2+\sigma)(3+\sigma)},$$

with $\sigma = \epsilon/2$. When $\epsilon = 0$ ($\sigma = 0$), polynomials $\tilde{F}_n$ coincide with Legendre polynomials $P_n(x)$ in identity 8.91 on p. 973 of [14]. The zero eigenvalue ($n = 0$) is excluded from the construction but added to the spectrum by Lemma 6. When $\epsilon < 2$, the resulting eigenfunction $\Psi_0(x)$ belongs to X. Since $\Psi_0'(x) \sim (1+x)^{-\epsilon/2}$ as $x \to -1$, the resulting eigenfunction $\Psi_0(x)$ does not belong to X for $\epsilon \geq 2$.

We shall prove that no non-zero eigenvalues exist in X for $\epsilon \geq 2$. Using the identity (4.19), we transform the solution $F(x)$ to the equivalent form $F(x) = (1+x)^{-\epsilon/2}\tilde{F}(x)$, where $\tilde{F}(x)$ satisfies the hypergeometric equation (3.9) with new parameters

$$z = \frac{1-x}{2}, \quad \tilde{\alpha} = \gamma - \alpha = 1 + s, \quad \tilde{\beta} = \gamma - \beta = -s, \quad \tilde{\gamma} = \gamma = \frac{\epsilon}{2} + 1.$$

Up to a constant factor, $\tilde{F}(x)$ is represented by the hypergeometric function $F(z; 1+s, -s, 1+\epsilon/2)$. It follows from the ODE (4.17) that the eigenfunction $\Psi_0(x)$ is related to $\tilde{F}(x)$ by

$$-s(s+1)\Psi_0'(x) = \left(\frac{1-x}{1+x}\right)^{\epsilon/2} \tilde{F}'(x).$$

Since $\tilde{\alpha} + \tilde{\beta} - \tilde{\gamma} = -\epsilon/2 < 0$ for $\epsilon > 0$, the hypergeometric series for the function $F(z; 1+s, -s, 1+\epsilon/2)$ converges absolutely on the entire interval $x \in [-1, 1]$ ($z \in [0, 1]$) (see identity 9.102 on p. 995 of [14]). Therefore, the value of $\tilde{F}'(-1)$ is well-defined. We shall prove that $\tilde{F}'(-1) \neq 0$ for any $s \neq 0$ and $\epsilon \geq 2$. It follows from the hypergeometric equation (3.9) with $(\alpha, \beta, \gamma) = (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$ at $z = 1$ that

$$s(s+1)\tilde{F}(-1) + \left(\frac{\epsilon}{2} - 1\right)\tilde{F}'(-1) = 0.$$

If $\tilde{F}'(-1) = 0$, then $\tilde{F}(-1) = 0$ for any $s \neq 0$ and $\epsilon \geq 2$, and the only regular solution of the hypergeometric equation (3.9) is $\tilde{F}(x) \equiv 0$. Therefore, $\tilde{F}'(-1) \neq 0$, and therefore, $\Psi_0 \notin X$ for $\epsilon \geq 2$. $\square$

Proposition 9. *A complete spectrum of the eigenvalue problem* (4.1) *with* $0 < x_0 < 1$ *and* $\epsilon \geq 0$ *in* X_0 *consists of simple isolated eigenvalues* μ *with* $\mu \in \mathbb{R}_-$.

Proof. When $\Psi_0 \in X_0$, the eigenfunction $\varphi(x)$ of the associated Legendre equation (4.8) satisfies the Robin boundary conditions

$$2(1-x_0^2)\varphi'(\pm x_0) + \epsilon\varphi(\pm x_0) = 0,$$

while the eigenfunction $\chi(x)$ of the second-order ODE (4.12) satisfies the Dirichlet boundary conditions $\chi(\pm x_0) = 0$. Each eigenvalue problem is self-adjoint with respect to these boundary conditions [24]. Therefore, all eigenvalues μ of the regular boundary-value problems are real-valued and isolated. Moreover, these eigenvalues are negative due to the Green identity [24]:

$$\mu \int_{-x_0}^{x_0} \varphi^2(x)dx = -\int_{-x_0}^{x_0} (1-x^2)\left(\varphi'(x)\right)^2 dx - \frac{\epsilon^2}{4}\int_{-x_0}^{x_0} \frac{\varphi^2(x)}{1-x^2}dx < 0. \quad (4.20)$$

These eigenvalues are also simple, since the Wronskian of any two solutions with the Robin or Dirichlet boundary conditions is zero. □

Proposition 10. *Let $\{\mu_n\}_{n\geq 0}$ be isolated simple eigenvalues of the eigenvalue problem* (4.12) *with $0 < x_0 < 1$ and Dirichlet boundary conditions $\chi(\pm x_0) = 0$ ordered as*

$$0 > \mu_0 > \mu_1 > \cdots > \mu_n > \cdots.$$

Then, $\lim_{x_0\to 1} \mu_n = -s_n(s_n+1)$, where

$$s_n = 1+n, \text{ for } 0 \leq \epsilon \leq 2 \qquad \text{and} \qquad s_n = \frac{\epsilon}{2} + n, \text{ for } \epsilon \geq 2 \quad (4.21)$$

with $n \geq 0$.

Proof. Singularity analysis of the second-order ODE (4.12) shows that the solution $\chi(x)$ behaves as

$$\chi \to c_1^+(1-x)^{(\epsilon+2)/4} + c_2^+(1-x)^{-(\epsilon+2)/4}, \quad \text{as } x \to 1$$

and

$$\chi \to c_1^-(1+x)^{(\epsilon-2)/4} + c_2^-(1+x)^{-(\epsilon-2)/4}, \quad \text{as } x \to -1.$$

The ODE (4.12) admits a bounded (regular) solution $\chi(x)$ on $[-1,1]$ if and only if the singular components are removed. This leads to the constraints $c_2^+ = c_1^- = 0$ for $0 \leq \epsilon < 2$ and $c_2^+ = c_2^- = 0$ for $\epsilon > 2$. It is explained in Proposition 8 that the conditions $c_2^+ = c_1^- = 0$ for $0 \leq \epsilon < 2$ are equivalent to $s = m$ with $m \geq 0$, when the first term in the relation (4.18) is removed and the hypergeometric function $F(z;\tilde{\alpha},\tilde{\beta},\tilde{\gamma})$ is a polynomial. Note that the zero eigenvalue $s = 0$ ($m = 0$) of the problem (4.8) is excluded from the spectrum of the problem (4.12), such that $s = s_n = 1+n$ with $n \geq 0$. On the other hand, the conditions $c_2^+ = c_2^- = 0$ for $\epsilon > 2$ are equivalent to $s = s_n = \epsilon/2 + n$ with $n \geq 0$, when the second term in the relation (4.18) is removed and the hypergeometric function $F(z;\alpha,\beta,\gamma)$ is a polynomial. Note that the first Darboux–Backlund transformation (4.14) implies that if $c_2^+ = 0$, then

$$\varphi \to c_1^+(1-x)^{\epsilon/4}, \quad \text{as } x \to 1$$

and

$$\varphi \to c_1^-(1+x)^{\epsilon/4} + c_2^-(1+x)^{-\epsilon/4}, \quad \text{as } x \to -1.$$

Recall that $\varphi(x) = (1-x^2)^{\epsilon/4}F(x)$. When $c_1^- = 0$ ($0 \le \epsilon < 2$), $F(x)$ is singular like $F(x) \to (1+x)^{-\epsilon/2}$ as $x \to -1$ in accordance with the relation (4.19). When $c_2^- = 0$ ($\epsilon > 2$), $F(x)$ is bounded as $x \to -1$. The marginal case $\epsilon = 2$ corresponds to the case when $\chi(x)$ has one bounded and one logarithmically growing components as $x \to -1$. The logarithmic growth is excluded if $s_n = 1 + n$ with $n \ge 1$, which is the intersection between the two spectra (4.21) at $\epsilon = 2$.

When $s = s_n$ and $\epsilon \ne 2$, the eigenfunction $\chi(x)$ of the formally self-adjoint problem (4.12) satisfies the Dirichlet boundary conditions $\lim_{x\to\pm 1}\chi(x) = 0$. When $\epsilon = 2$, the eigenfunction $\chi(x)$ is bounded at $x = -1$ and zero at $x = 1$. In either case, convergence and uniqueness of continuations from eigenvalues of the regular Dirichlet problem (4.12) with $x_0 < 1$ to eigenvalues of the singular boundary-value problem (4.12) with $x_0 = 1$ is proved by Theorem 5.3 of [2]. □

Remark 11. Bounded (for $\epsilon = 2$) and decaying (for $\epsilon > 2$) eigenfunctions $\chi(x)$ of the self-adjoint problem (4.12) with $x_0 = 1$ for eigenvalues $\mu = -s_n(s_n + 1)$ with $s_n = \epsilon/2 + n$ violate the conditions (4.13). Indeed, one can check that the limit in (4.13) as $x \to -1$ is non-zero (proportional to c_1^-) and the integral in (4.13) hence diverges for $\epsilon \ge 2$. Therefore, the eigenvalues of the self-adjoint problem (4.12) for $\epsilon \ge 2$ do not correspond to eigenvalues of the original problem (4.1) in space $\Psi_0 \in X$, in agreement with Proposition 8.

Remark 12. Theorem 7 covers the entire parameter domain $0 < x_0 \le 1$ and $\epsilon \ge 0$. However, there is an interesting problem of convergence of eigenvalues of the associated Legendre equation (4.8) in the limit $x_0 \to 1$. While the eigenvalues with $0 < x_0 < 1$ are expected to converge to the eigenvalues in (4.16) for $0 \le \epsilon < 2$, no eigenvalues with the eigenfunctions $\Psi \in X$ exist for $\epsilon \ge 2$. Convergence of eigenvalues of the linearized problem (4.1) as $x_0 \to 1$ will be computed in Section 6 numerically.

5. Numerical computations of eigenvalues for $k \ne 0$

In order to illustrate distribution of eigenvalues in Propositions 2, 3 and 4 and to investigate eigenvalues in the domain $0 < x_0 < 1$ and $\epsilon > 0$ in Remark 5, we develop a numerical method based on power series expansions. Since $x = 0$ is an ordinary point and $x = \pm 1$ are regular singular points of system (3.1), the expansions of the functions $\Psi_k(x)$ and $\Phi_k(x)$ in powers of x converge uniformly and absolutely for $|x| < 1$. The numerical method is based on truncation of the power series.

Let $\mu \in \mathbb{C}$ be parameterized by $\mu = -s(s+1)$, $s \in \mathbb{C}$. Due to the symmetry, it is sufficient to consider the domain $\{s \in \mathbb{C} : \operatorname{Re}(s) \ge -\frac{1}{2}\}$. The stability domain

$\mathrm{Re}(\mu) < 0$ corresponds to the domain

$$\left\{ s \in \mathbb{C} : \quad |\mathrm{Im}(s)| < \sqrt{\mathrm{Re}(s)(\mathrm{Re}(s)+1)}, \;\; \mathrm{Re}(s) > 0 \right\}. \tag{5.1}$$

Consider the power series with separated even and odd terms:

$$\Psi_k(x) = \sum_{m\geq 0} c_m x^{2m} + \sum_{m \geq 0} d_m x^{2m+1}, \tag{5.2}$$

$$\Phi_k(x) = \sum_{m\geq 0} a_m x^{2m} + \sum_{m \geq 0} b_m x^{2m+1}, \tag{5.3}$$

where the starting coefficients (a_0, b_0, c_0, d_0) are parameters. Substituting (5.3) into the second equation of the system (3.1) we find that (a_1, b_1) are defined by

$$a_1 = \frac{(k^2 - s(1+s))a_0 - \epsilon b_0}{2}, \tag{5.4}$$

$$b_1 = \frac{(k^2 + 2 - s(s+1))b_0 - \epsilon 2 a_1}{6}, \tag{5.5}$$

while the coefficients $\{a_m, b_m\}_{m\geq 2}$ are defined by the two-step recurrence equations:

$$a_{m+2} = \frac{(k^2 - s(s+1) + 2(2m+2)^2)a_{m+1} + (s(s+1) - 2m(2m+1))a_m}{(2m+4)(2m+3)} \\ \frac{-\epsilon(2m+3)b_{m+1} + \epsilon(2m+1)b_m}{(2m+4)(2m+3)}, \tag{5.6}$$

$$b_{m+2} = \frac{(k^2 - s(s+1) + 2(2m+3)^2)b_{m+1} + (s(s+1) - (2m+2)(2m+1))b_m}{(2m+5)(2m+4)} \\ \frac{-\epsilon(2m+4)a_{m+2} + \epsilon(2m+2)a_{m+1}}{(2m+5)(2m+4)}. \tag{5.7}$$

We note that the initial equations (5.4)–(5.5) follow from the recurrence equations (5.6)–(5.7) for $m = -1$ with $a_{-1} = b_{-1} = 0$.

Substituting (5.2) into the first equation of the system (3.1) we find that the coefficients $\{c_m, d_m\}_{m\geq 2}$ are defined from the coefficients $\{a_m, b_m\}_{m\geq 0}$ by the two-step recurrence equations:

$$c_{m+2} = \frac{(k^2 + 2(2m+2)^2)c_{m+1} - 2m(2m+1)c_m + a_{m+1} - a_m}{(2m+4)(2m+3)}, \tag{5.8}$$

$$d_{m+2} = \frac{(k^2 + 2(2m+3)^2)d_{m+1} - (2m+2)(2m+1)d_m + b_{m+1} - b_m}{(2m+5)(2m+4)}. \tag{5.9}$$

The initial equations for (c_1, d_1) follow from the recurrence equations (5.8)–(5.9) for $m = -1$ with $a_{-1} = b_{-1} = c_{-1} = d_{-1} = 0$.

The boundary conditions in (3.4) lead to the equations

$$\sum_{m\geq 0} c_m x_0^{2m} = 0, \qquad \sum_{m\geq 0} d_m x_0^{2m} = 0,$$
$$\sum_{m\geq 0} (2m) c_m x_0^{2m} = 0, \qquad \sum_{m\geq 0} (2m+1) d_m x_0^{2m} = 0. \tag{5.10}$$

There exists a linear map from $(a_0, b_0, c_0, d_0) \in \mathbb{C}^4$ parametrized by $s \in \mathbb{C}$ to the sequence $\{a_m, b_m, c_m, d_m\}_{m\in\mathbb{N}}$. Therefore, the boundary conditions (5.10) are equivalent to the homogeneous system $A_k(s)\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (a_0, b_0, c_0, d_0)^T \in \mathbb{C}^4$ and $A_k(s)$ is a 4-by-4 matrix computed from the entries of (5.10). The matrix $A_k(s)$ depends on $s \in \mathbb{C}$ and $k \in \mathbb{N}$, as well as parameters x_0 and ϵ. If the power series are truncated at the M-th term, the matrix $A_k(s)$ depends also on M. Eigenvalues $\mu = -s(s+1)$ of system (3.1) in (3.4) are *equivalent* to roots s of the determinant equation

$$F_k(s; x_0, \epsilon, M) = \det(A_k(s)). \tag{5.11}$$

Numerical results of computations of roots of the function $F_k(s; x_0, \epsilon, M)$ are shown on Figures 1–5. Figure 1 show first few roots s of $F_k(s; x_0, \epsilon, M)$ with $k = 1, 3, 5$ versus x_0 for $\epsilon = 0$ and $M = 150$. In agreement with Proposition 4, the roots converge as $x \to 1$ to the values $s_n = \sigma + n$ with $\sigma = |k|$ and $n \geq 0$. We can see that the convergence is excellent for $k = 3$ and $k = 5$ but it is worse for $k = 1$ in the sense that the roots at $x_0 = 0.99$ are still far from the values s_n. This feature is explained by the decay of the eigenfunctions (Φ_k, Ψ_k) of system (3.1) on the interval $[-1, 1]$. Indeed, it follows from Proposition 2 that $\Phi_k \sim (1 - x^2)^{\sigma/2}$ and $\Psi_k \sim (1-x^2)^{1+\sigma/2}$ as $x \to \pm 1$ for $\epsilon = 0$ and $|k| \geq 1$. Therefore, the derivative of $\Phi_k(x)$ is bounded as $x \to \pm 1$ for $|k| \geq 2$ and unbounded for $|k| = 1$. In the latter case, the power series expansions (5.2)–(5.3) diverge in the limit $x_0 \to 1$ and the numerical approximations are not accurate for x_0 close to 1.

Figure 2 shows first few roots s with $k = 1, 5$ versus M for $\epsilon = 0$ and $x_0 = 0.9$. We can see that the roots quickly converge to constant values, which are taken as approximations of real roots when $M = 150$ in the remainder of the figures. The numerical error for large values of M consists of three sources: truncation of the power series (5.2)–(5.3), root finding algorithms for roots of $F_k(s)$ in (5.11), and rounding entries of the matrix $A_k(s)$ when a number x_0 with $x_0 < 1$ is evaluated at x_0^m with large exponent m. While the first two sources can be reduced to any desired degree, the last source represents an irremovable obstacle on getting accurate approximations when M gets large.

Figure 3 shows the first six roots s versus k for $\epsilon = 0$, $x_0 = 0.9$, and $M = 150$. We observe two properties from this figure: the values of s becomes larger for larger values of k (e.g. the eigenvalues μ becomes more and more negative) and the roots s approach to the integer values for larger values of k even when $x_0 = 0.9$ is different from $x_0 = 1$.

Figure 4 show the first few roots s with $k = 1, 3$ versus ϵ for $x_0 = 0.9$ and $M = 150$. Although the roots are real for small values of ϵ in agreement to

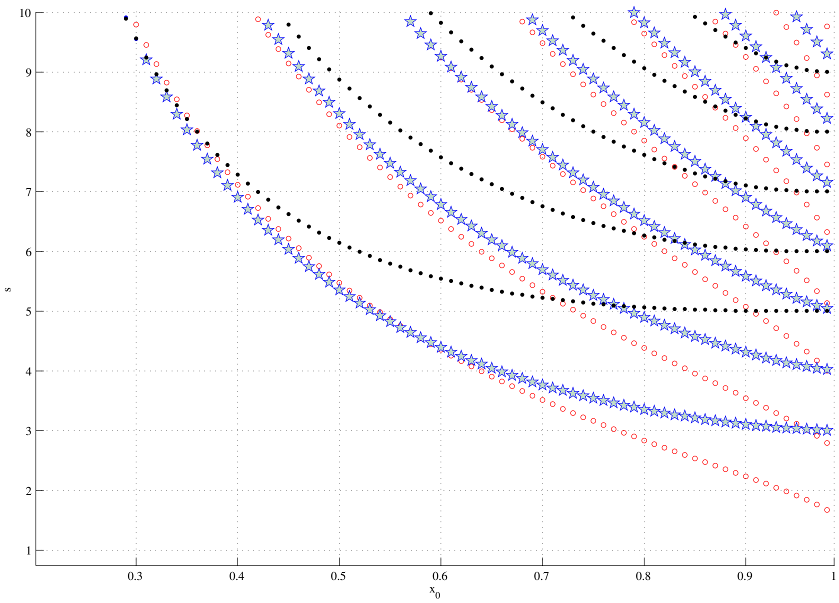

FIG. 1. First few roots s of $F_k(s; x_0, \epsilon, M)$ versus x_0 for $\epsilon = 0$ and $M = 150$: $k = 1$ (circles), $k = 3$ (stars) and $k = 5$ (dots)

Proposition 3, they coalesce for larger values of ϵ. After two roots merge, they split into complex domain and complex values of s are not shown on Figure 4. It is seen from this figure that the roots with larger values of k coalesce for larger values of ϵ.

Figure 5 shows the spectrum of complex roots s with $k = 1, 3$ for $x_0 = 0.9$, $M = 150$, and different values of $0 \leq \epsilon \leq 12$. The boundary of the stability domain (5.1) is shown by the dotted curve. We can see that roots s remain in the stability domain after they bifurcate off the real axes.

6. Numerical computations of eigenvalues for $k = 0$

In order to illustrate convergence of eigenvalues in Propositions 8, 9, 10 and Remark 12, we approximate eigenvalues of system (4.1) with power series solutions explained in Section 5. The solution for $\Psi_0(x)$ and $\Phi_0(x)$ is represented by the power series (5.2)–(5.3), where the starting coefficients (a_0, b_0, c_0, d_0) are parameters. It follows from the ODE (4.6) that the set $\{a_m, b_m\}_{m\in\mathbb{N}}$ is defined by the

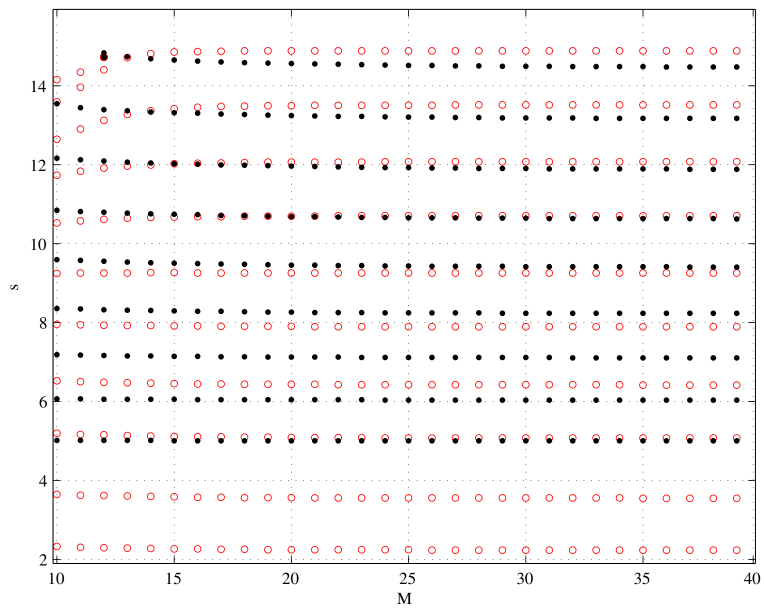

Fig. 2. Convergence of roots s versus M for $\epsilon = 0$ and $x_0 = 0.9$: $k = 1$ (circles) and $k = 5$ (dots)

one-step recurrence equations:

$$a_{m+1} = \frac{(2m-s)(2m+1+s)a_m - \epsilon(2m+1)b_m}{(2m+2)(2m+1)}, \tag{6.1}$$

$$b_{m+1} = \frac{(2m+1-s)(2m+2+s)b_m - \epsilon(2m+2)a_{m+1}}{(2m+3)(2m+2)}. \tag{6.2}$$

It follows from the first equation of system (4.1) that the set $\{c_m, d_m\}_{m\in\mathbb{N}}$ is defined from the set $\{a_m, b_m\}_{m\in\mathbb{N}}$ by the one-step recurrence equations:

$$c_{m+1} = \frac{(2m)(2m+1)c_m + a_m}{(2m+2)(2m+1)}, \tag{6.3}$$

$$d_{m+1} = \frac{(2m+1)(2m+2)d_m + b_m}{(2m+3)(2m+2)}. \tag{6.4}$$

Finally, it follows from the second equation of system (4.1) that there exist two initial equations:

$$\begin{aligned} b_0 + \epsilon a_0 &= -s(s+1)d_0, \\ 2a_1 + \epsilon b_0 &= -2s(s+1)c_1 \end{aligned}$$

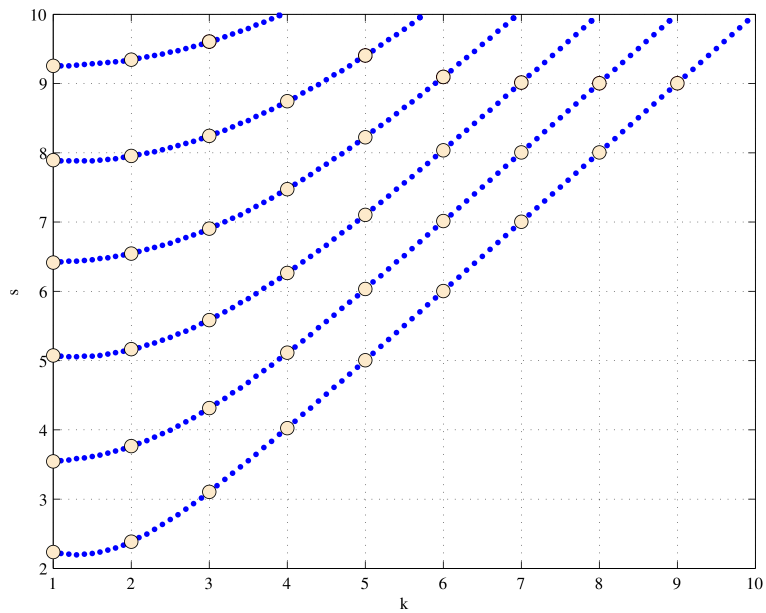

FIG. 3. First six roots s versus k for $\epsilon = 0$, $x_0 = 0.9$, and $M = 150$

in addition to system (6.1)–(6.2). When $s \neq 0$, we can solve the initial equations as

$$b_0 = -\epsilon a_0 - s(s+1)d_0, \qquad c_1 = \frac{a_0}{2},$$

such that the only independent parameters are (a_0, d_0). We also note that the parameter c_0 is trivial since $\Psi_0(x)$ is defined up to the addition of an arbitrary constant.

The boundary conditions in (4.3) lead to the equations:

$$\sum_{m \geq 0} (2m) c_m x_0^{2m} = 0, \qquad \sum_{m \geq 0} (2m+1) d_m x_0^{2m} = 0. \tag{6.5}$$

There exists a linear map from $(a_0, d_0) \in \mathbb{C}^2$ parameterized by $s \in \mathbb{C}$ to the sequence $\{a_m, b_m, c_m, d_m\}_{m \in \mathbb{N}}$. Therefore, the boundary conditions (6.5) are equivalent to the homogeneous system $A_0(s)\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (a_0, d_0)^T \in \mathbb{C}^2$ and $A_0(s)$ is a 2-by-2 matrix which depends on $s \in \mathbb{C}$, parameters x_0 and ϵ, and integer M for truncation of power series. Eigenvalues $\mu = -s(s+1)$ of the system (4.1) in (4.3) are *equivalent* to roots s of the determinant equation

$$F_0(s; x_0, \epsilon, M) = \det(A_0(s)). \tag{6.6}$$

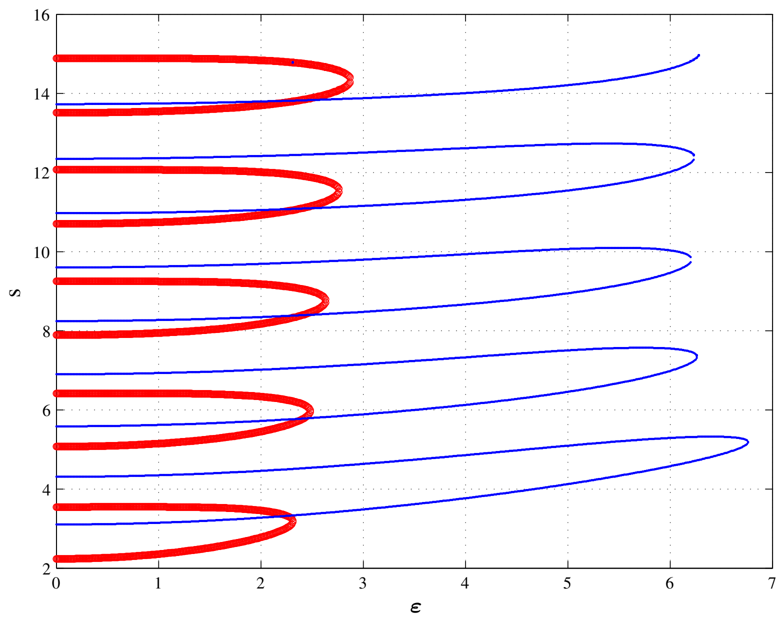

FIG. 4. First few roots s versus ϵ for $x_0 = 0.9$ and $M = 150$: $k = 1$ (bolded curve) and $k = 3$ (thin curve)

Figure 6 represents the first ten eigenvalues s versus x_0 for $\epsilon = 1$ and $M = 100$. In agreement with Proposition 8, the roots converge to the integer values in the limit $x_0 \to 1$. Since the convergence of power series becomes slower with M for $x_0 \neq 1$, there is a gap between the last numerical value of x_0 and the value $x_0 = 1$. We also note that the numerical accuracy of the limiting eigenvalues (4.16) becomes worse for larger eigenvalues.

Figure 7 represents the first ten eigenvalues s versus ϵ for $x_0 = 0.9$ and $M = 100$. It is obvious that the eigenvalues remain real in agreement with Proposition 9.

Figure 8 represents the first seven eigenvalues s versus x_0 for $\epsilon = 4$ and two values of $M = 100$ (dashed) and $M = 1000$ (solid). In agreement with Proposition 10, the roots converge to their limiting values $s_n = \epsilon/2 + n$ which are not eigenvalues of the problem (4.1) in space (4.4). We also note limitations of the numerical method based on truncations of the power series. True limits can only be recovered if too many terms of the power series are taken into accounts which leads to long computational time and large round-off errors of numerical computations. The effects of slow convergence and truncations of power series lead to coalescence of real eigenvalues and their splitting to the complex plane, which is not observed if the values of M are large enough.

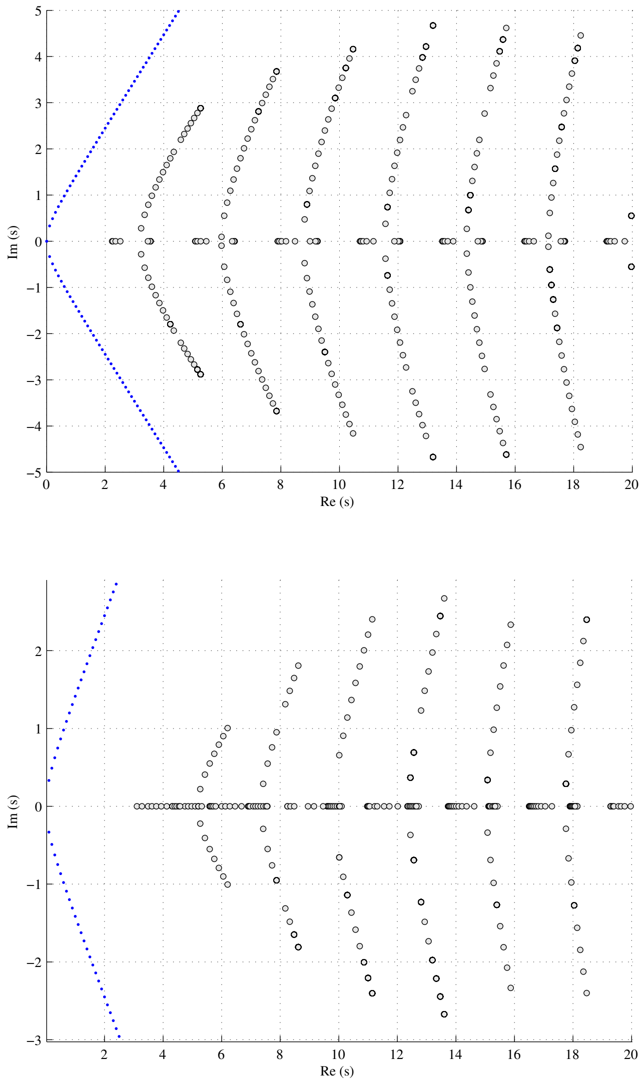

FIG. 5. Complex roots s for $k = 1$ (top) and $k = 3$ (bottom), $x_0 = 0.9$ and $M = 150$ when parameter ϵ transverses in the interval $0 \leq \epsilon \leq 12$. The dotted curve shows the boundary of the stability domain (5.1).

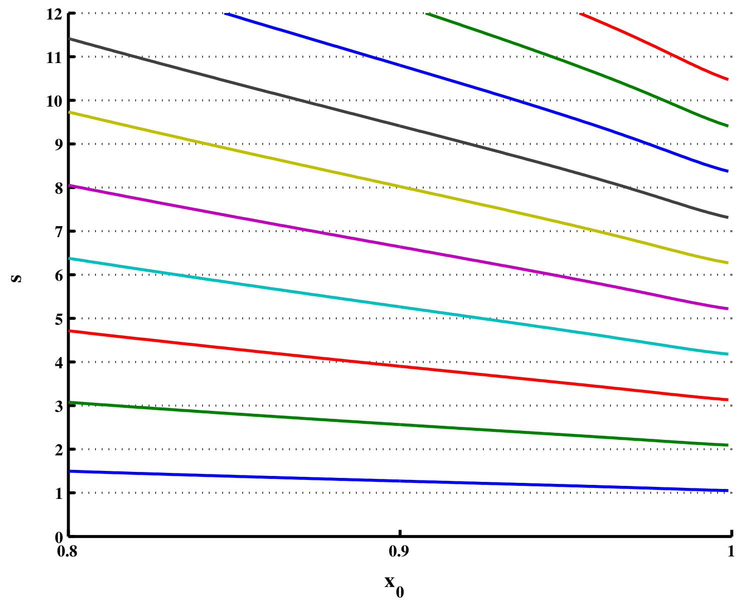

FIG. 6. First ten eigenvalues of the problem (4.1) for $\epsilon = 1$ and $M = 100$

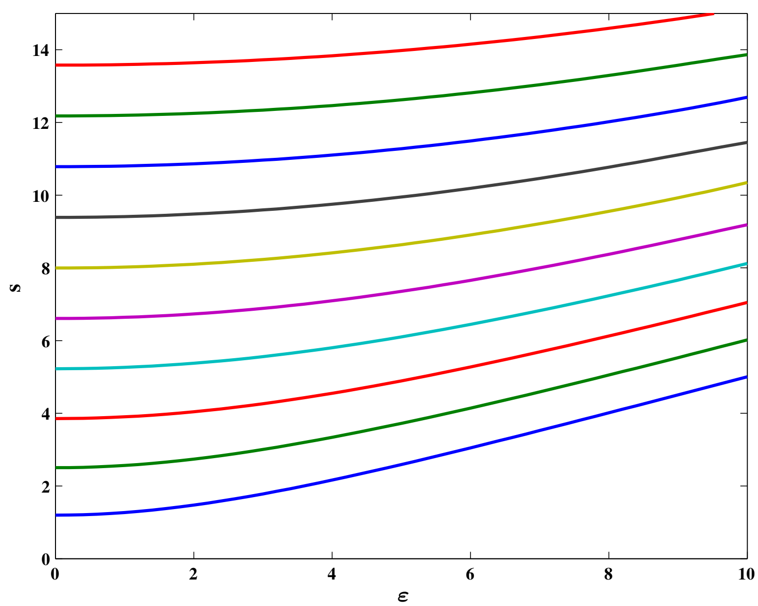

FIG. 7. First ten eigenvalues of the problem (4.1) for $x_0 = 0.9$ and $M = 100$

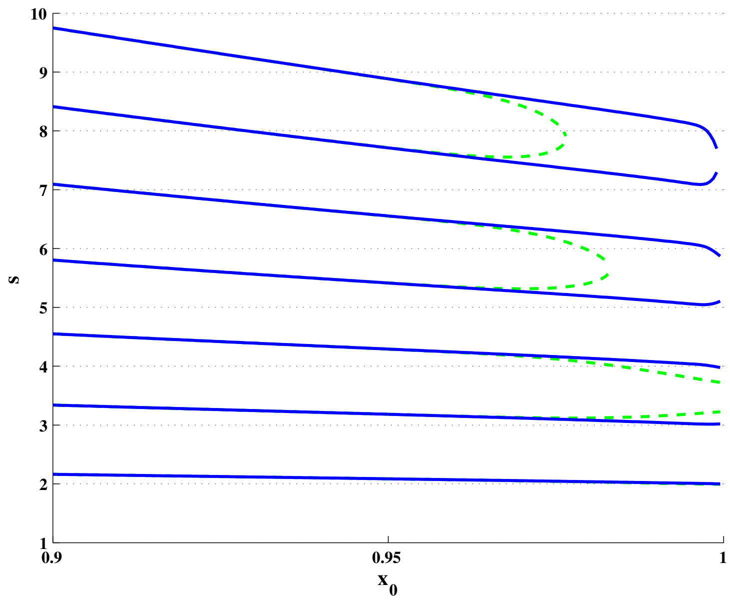

Fig. 8. Convergence of eigenvalues of the problem (4.1) for $\epsilon = 4$ and two values of $M = 100$ (dashed) and $M = 1000$ (solid)

7. Discussions

We have shown analytically that the stationary flow on the sphere is asymptotically stable whatever the Reynolds number may occur. This result is relevant for the flow of a viscous fluid (e.g. oil) over a sphere (e.g. a metal ball). We have also found that the linearized operator for symmetry-preserving perturbations has void spectrum in the energy space for sufficiently large Reynolds numbers. One can show by direct analysis that the full system (1.5)–(1.7) reduces to a scalar linear equation for symmetry-preserving (ϕ-independent) solutions:

$$\frac{\partial v_\phi}{\partial t} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\phi\right) = \nu\Delta_1 v_\phi, \tag{7.1}$$

where Δ_1 is given by (2.13) for $k = 1$. When $v_\phi(\theta, t) = -\Psi_0'(\theta)e^{\lambda t}$, the linear equation (7.1) reduces to the linear eigenvalue problem (2.17) which has no eigenvalues for $\nu \leq \frac{1}{2}$ ($\epsilon \geq 2$) in the space of functions such that $\int_0^\pi |\Psi_0'(\theta)|^2 \sin\theta d\theta < \infty$. Implications of this result to the well-posedness of the Cauchy problem for the linear time-dependent equation (7.1) with $\nu \leq \frac{1}{2}$ remain unclear.

We have also shown analytically and numerically that the stationary flow on the truncated spherical layer is asymptotically stable and all isolated eigenvalues are real for small Reynolds numbers and complex for large Reynolds numbers. The eigenvalues are always real for symmetry-preserving perturbations. The truncated spherical layer can be used to model the ice melting in Arctics due to global warming, when the near-stationary flow of ocean water moves from Arctics to Antarctica. However, our model does not include Coriolis and gravity forces, as well as location of continents to treat correctly this physical process.

The inclusion of the effects of rotation alter significantly the physical picture that emerges from the model considered here and makes the problem more complicated. In particular, the results of [28] are not applicable for the latter case. However, there are physical reasons [15] to believe that the two-dimensional NS equations modified by additional rotational terms will be relevant for modeling of the three-dimensional NS equations in spherical coordinates on a rotating earth [19]. In particular, we expect that the solution (1.9) will remain the same for components (v_θ, v_ϕ) and will be different in the component q, while the corresponding linearized problem (2.2)–(2.4) will be affected by additional rotational terms. Details of the eigenvalues of the modified linearized problem will be the subject of the forthcoming studies.

Acknowledgement. The authors thank Marina Chugunova and Bartosz Protas for useful discussions and remarks. The work was supported by the PREA and NSERC Discovery grants.

References

[1] G. K. Bachelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1967.

[2] P. B. Bailey, W. N. Everitt, J. Weidmann and A. Zettl, Regular approximations of singular Sturm–Liouville problems, *Results in Mathematics* **22** (1993), 3–22.

[3] G. Ben-Yu, Spectral method for vorticity equations on spherical surface, *Math. Comput.* **64** (1995), 1067–1079.

[4] E. N. Blinova, A hydrodynamical theory of pressure and temperature waves and of centres of atmosphere action, *C. R. (Doklady) Acad. Sci URSS* **39** (1943), 257–260.

[5] E. N. Blinova, A method of solution of the nonlinear problem of atmospheric motions on a planetary scale, *Dokl. Acad. Nauk SSSR* **110** (1956), 975–977.

[6] S. Boatto and H. E. Cabral, Nonlinear stability of a latitudinal ring of point-vortices on a nonrotating sphere, *SIAM J. Appl. Math.* **64** (2003), 216–230.

[7] C. Cherfils and O. Lafitte, Analytic solutions of the Rayleigh equation for linear density profiles, *Phys. Rev. E* **62** (2000), 2967–2970.

[8] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw–Hill, New York, 1955.

[9] D. Crowdy, Point vortex motion on the surface of a sphere with impenetrable boundaries, *Physics of Fluids* **18** (2006), 036602.

[10] P. G. Drazin and W. H. Reed, *The Navier–Stokes Equations: A Classificaltion of Flows and Exact Solutions*, London Mathematical Society Lecture Note Series **334**, Cambridge University Press, Cambridge, 2006.

[11] P. G. Drazin and N. Riley, *Hydrodynamics Stability*, Cambridge University Press, Cambridge, 1981.

[12] N. Dunford and J. T. Schwartz, *Linear Operators. Part II: Spectral Theory*, John Wiley & Sons, New York, 1963.

[13] A. Furnier, H. Bunge, R. Hollerbach and I. Vilotte, Application of the spectral-element method to the axisymetric Navier–Stokes equations, *Geophys. J. Int.* **156** (2004), 682–700.
[14] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, 6th edition, Academic Press, 2005.
[15] R. Ibragimov, Shallow water theory and soltions of the free boundary problem on the atmospheric motion around the earth, *Physica Scripta* **61** (2000), 391–395.
[16] D. Iftimie and G. Raugel, Some results on the Navier–Stokes equations in thin 3D domains, *J. Diff. Eqs.* **169** (2001), 281–331.
[17] M. Jarund and A. Baede, The use of spectral technique in numerical weather prediction, *Lecture Notes in Applied Mathematics* **22** (1985), 1–41.
[18] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1995.
[19] N. E. Kochin, I. A. Kibel and N. W. Roze, *Theoretical Hydrodynamics VI* (in Russian), OGIZ, Gostexizdat, 1948.
[20] E. S. G. Leandro, J. A. Miranda and F. Moraes, Symmetric flows and Darcy's law in curved spaces, *J. Phys. A: Math. Gen.* **39** (2006), 1619–1632.
[21] J. L. Lions, R. Temam and S. Wang, New formulations of the primitive equations of atmosphere and applications, *Nonlinearity* **5** (1992), 237–288.
[22] J. L. Lions, R. Temam and S. Wang, On the equations of the large-scale ocean, *Nonlinearity* **5** (1992), 1007–1053.
[23] K. Mohseni and T. Colonius, Numerical treatment of polar singularities, *J. Comput. Phys.* **157** (2000), 787–795.
[24] R. K. Nagle, E. B. Saff and A. D. Snider, *Fundamentals of Differential Equations and Boundary Value Problems*, Pearson Education, New York, 2004.
[25] F. Parision, F. Moraes, J. A. Miranda and M. Widom, Saffman–Taylor problem on a sphere, *Phys. Rev. E* **63** (2001), 036307-10.
[26] J. Shen, On pressure stabilization method and projection method for unsteady Navier–Stokes equations, in: *Advances in Computer Methods for Partial Differential Equations*, 658–662, IMACS, New Brunswick, NJ, 1992.
[27] E. Simonnet, A variational crime on the tangent bundle of the sphere, *Numerische Mathematik* **85** (2000), 409–431.
[28] R. Temam and M. Ziane, Navier–Stokes equations in thin spherical domains, *Contemp. Math.* **209** (1997), 281–314.
[29] D. Williamson, A standard test for numerical approximation to the shallow water equations in spherical geometry, *J. Comput. Phys.* **102** (1992), 211–224.

Ranis N. Ibragimov and Dmitry E. Pelinovsky
Department of Mathematics
McMaster University
Hamilton
Ontario L8S 4K1
Canada

(accepted: October 2, 2006; published Online First: July 2, 2007)