J. Differential Equations 253 (2012) 2796–2824

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Normal form for the symmetry-breaking bifurcation in the nonlinear Schrödinger equation

D.E. Pelinovsky [a,*], T.V. Phan [b]

[a] Department of Mathematics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada
[b] Department of Mathematics, University of Tennessee, Knoxville, TN 37996, United States

ARTICLE INFO

Article history:
Received 7 February 2012
Available online 13 August 2012

ABSTRACT

We derive and justify a normal form reduction of the nonlinear Schrödinger equation for a general pitchfork bifurcation of the symmetric bound state that occurs in a double-well symmetric potential. We prove persistence of normal form dynamics for both supercritical and subcritical pitchfork bifurcations in the time-dependent solutions of the nonlinear Schrödinger equation over long but finite time intervals.

1. Introduction

We consider the nonlinear Schrödinger (NLS) equation with a focusing power nonlinearity and an external potential (also known as the Gross–Pitaevskii equation),

$$i\Psi_t = -\Psi_{xx} + V(x)\Psi - |\Psi|^{2p}\Psi, \tag{1.1}$$

where $\Psi(x,t): \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is the wave function, $p\in\mathbb{N}$ is the nonlinearity power, and $V(x):\mathbb{R}\to\mathbb{R}$ is the external, symmetric, double-well potential satisfying the following conditions:

(H1) $V(x)\in L^\infty(\mathbb{R})$ and $xV'(x)\in L^\infty(\mathbb{R})$;
(H2) $\lim_{|x|\to\infty} V(x) = 0$;
(H3) $V(-x) = V(x)$ for all $x\in\mathbb{R}$;
(H4) $L_0 = -\partial_x^2 + V(x)$ has the lowest eigenvalue $-E_0 < 0$;
(H5) $V(x)$ has a non-degenerate local maximum at $x = 0$ and two minima at $x = \pm x_0$ for some $x_0 > 0$.

* Corresponding author.
E-mail addresses: dmpeli@math.mcmaster.ca (D.E. Pelinovsky), phan@math.utk.edu (T.V. Phan).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.07.007

The easiest way to think about the double-well potential $V(x)$ is to consider the sum of two single-well potentials centered at two symmetric points,

$$V(x) = \frac{1}{2}\big[V_0(s-x) + V_0(s+x)\big], \quad s \geqslant 0, \tag{1.2}$$

where the single-well potential $V_0(x)$ satisfies (H1)–(H4) and has a global minimum at $x = 0$ and no other extremum points. For sufficiently large $s > s_*$, where s_* is the inflection point of V_0, that is, $V_0''(s_*) = 0$, the sum of two single-well potentials (1.2) becomes a double-well potential we would like to consider. We note, however, that not every double-well potential V satisfying (H1)–(H5) can be represented by the sum (1.2).

The symmetric double-well potentials are used in the atomic physics of Bose–Einstein condensation [1] through a combination of parabolic and periodic (optical lattice) potentials. Similar potentials were also examined in the context of nonlinear optics, e.g. in optically induced photo-refractive crystals [8] and in a structured annular core of an optical fiber [13]. Physical relevance and simplicity of the model make the topic fascinating for a mathematical research. We note that the defocussing NLS equation is also relevant both for Bose–Einstein condensation and nonlinear optics. We take the focusing NLS equation for simplicity to be precise in mathematical statements throughout our work.

Bifurcations of stationary states and their stability in the NLS equation (1.1) under the assumptions (H1)–(H5) on the potential $V(x)$ were recently considered by Kirr et al. [10].

Let $\Psi(x,t) = e^{iEt}\phi(x;E)$ be a stationary state and $\phi(x;E)$ be a solution of the stationary nonlinear Schrödinger equation

$$\big(-\partial_x^2 + V\big)\phi - \phi^{2p+1} + E\phi = 0. \tag{1.3}$$

Via standard regularity theory, if $V \in L^\infty(\mathbb{R})$, then any weak solution $\phi(\cdot;E) \in H^1(\mathbb{R})$ of the stationary equation (1.3) belongs to $H^2(\mathbb{R})$. Moreover, if $-E \notin \sigma(L_0)$, then the solution $\phi(\cdot;E) \in H^2(\mathbb{R})$ decays exponentially fast to zero as $|x| \to \infty$.

Existence of symmetric stationary states ϕ for any $E > E_0$ bifurcating from the lowest eigenvalue $-E_0$ of the operator $L_0 = -\partial_x^2 + V(x)$ was first considered by Jeanjean and Stuart [6]. Kirr et al. [10] continued this research theme and obtained the following bifurcation theorem.

Theorem 1. *(See [10].) Consider the stationary NLS equation* (1.3) *with* $p \geqslant \frac{1}{2}$ *and* $V(x)$ *satisfying* (H1)–(H5).

(i) *There exists a* C^1 *curve* $(E_0,\infty) \ni E \mapsto \phi(\cdot;E) \in H^2(\mathbb{R})$ *of positive symmetric states bifurcating from the zero solution at* $E = E_0$. *This curve undertakes the symmetry-breaking (pitchfork) bifurcation at a finite* $E_* \in (E_0,\infty)$, *for which the second eigenvalue* $\lambda(E)$ *of the operator*

$$L_+(E) = -\partial_x^2 + V(x) - (2p+1)\phi^{2p}(x;E) + E \tag{1.4}$$

passes from positive values for $E < E_*$ *to negative values for* $E > E_*$ *with* $\lambda(E_*) = 0$.

(ii) *Let* $\phi_*(x) = \phi(x;E_*)$ *be the positive symmetric state at the bifurcation point and* $\psi_* \in H^2(\mathbb{R})$ *be the anti-symmetric eigenvector of* $L_+(E_*)$ *corresponding to the second eigenvalue* $\lambda(E_*) = 0$. *Assume that* $\lambda'(E_*) \neq 0$, *hence* $\lambda'(E_*) < 0$. *The* C^1 *curve* $(E_0,\infty) \ni E \mapsto \phi(\cdot;E) \in H^2(\mathbb{R})$ *intersects transversely at* $E = E_*$ *with the* C^1 *curve of positive asymmetric states* $E \mapsto \varphi_\pm(\cdot;E) \in H^2(\mathbb{R})$ *that extends to* $E > E_*$ *if* $\mathcal{Q} < 0$ *and to* $E < E_*$ *if* $\mathcal{Q} > 0$, *where*

$$\begin{aligned}\mathcal{Q} = {} & 2p^2(2p+1)^2\big\langle \phi_*^{2p-1}\psi_*^2, L_+^{-1}(E_*)\phi_*^{2p-1}\psi_*^2\big\rangle_{L^2} \\ & + \frac{1}{3}p(2p+1)(2p-1)\big\langle \psi_*^2, \phi_*^{2p-2}\psi_*^2\big\rangle_{L^2}.\end{aligned} \tag{1.5}$$

The asymmetric states φ_+ *and* φ_- *are centered at the left and the right well of* V, *respectively.*

Orbital stability of the stationary state $\phi(x;E)$ in the NLS equation (1.1) depends on the number of negative eigenvalues of $L_+(E)$ and $L_-(E)$, where

$$L_-(E) = -\partial_x^2 + V(x) - \phi^{2p}(x;E) + E. \tag{1.6}$$

Since $L_-(E)\phi(E) = 0$ and $\phi(x;E) > 0$ for all $x \in \mathbb{R}$ and $E > E_0$, the spectrum of $L_-(E)$ is non-negative for any $E > E_0$. This fact simplifies the stability analysis of the stationary states [5,4].

Let us denote $N_s(E) = \|\phi(\cdot;E)\|_{L^2}^2$ and $N_a(E) = \|\varphi_+(\cdot;E)\|_{L^2}^2 = \|\varphi_-(\cdot;E)\|_{L^2}^2$. In what follows, we always assume that

$$N_s'(E_*) = 2\langle \partial_E\phi_*, \phi_*\rangle_{L^2} > 0, \quad \text{where } \partial_E\phi_*(x) = \partial_E\phi(x;E_*), \tag{1.7}$$

that is, $N_s(E)$ is increasing near the bifurcation point $E = E_*$. The following stability theorem was also proven by Kirr et al. [10].

Theorem 2. *(See [10].) Assume $N_s'(E_*) > 0$ in addition to conditions of Theorem 1. Then the symmetric state ϕ is orbitally stable for $E \leqslant E_*$ and unstable for $E > E_*$. If in addition, $\mathcal{Q} < 0$, then $N_a(E)$ is an increasing function of $E > E_*$ if $\mathcal{S} > 0$ and it is a decreasing function of $E > E_*$ if $\mathcal{S} < 0$, where*

$$\mathcal{S} = N_s'(E_*) + \mathcal{Q}^{-1}\left(\lambda'(E_*)\|\psi_*\|_{L^2}^2\right)^2. \tag{1.8}$$

Consequently, the asymmetric states $\varphi_\pm$ near $E = E_*$ are orbitally stable for $\mathcal{S} > 0$ and unstable for $\mathcal{S} < 0$.

For any potential $V(x)$ represented by (1.2) with a sufficiently large s, we show in Appendix A that $\lambda'(E_*) < 0$, $N_s'(E_*) > 0$, and $\mathcal{Q} < 0$ for any $p \geqslant \frac{1}{2}$, hence, the stable symmetric state ϕ for $E < E_*$ becomes unstable for $E > E_*$ and the asymmetric states $\varphi_\pm$ exist for $E > E_*$. In the limit $s \to \infty$, the boundary $\mathcal{S} = 0$ is equivalent to $p = p_*$, where

$$p_* = \frac{3+\sqrt{13}}{2} \approx 3.3028. \tag{1.9}$$

If $p < p_*$, the asymmetric states $\varphi_\pm$ are stable for $E > E_*$. If $p > p_*$, both symmetric and asymmetric states are unstable for $E > E_*$. Therefore, we can classify the symmetry-breaking bifurcation at $E = E_*$ as the supercritical (if $\mathcal{S} > 0$) or the subcritical (if $\mathcal{S} < 0$) pitchfork bifurcations with respect to the squared L^2-norm, which is a conserved quantity of the NLS equation (1.1) in time. The functions $N_s(E)$ and $N_a(E)$ in the two different cases are shown schematically on Fig. 1, where stable branches are depicted by solid line and the unstable branches are depicted by dotted lines.

The classification into the supercritical and subcritical pitchfork bifurcations is usually based on the analysis of the normal form equations obtained from the center manifold reductions and the near identity transformations. It is the goal of this paper to derive and to justify the normal form equations for time-dependent perturbations to stationary states. We shall look at the long but finite temporal dynamics of the normal form equations, avoiding the complexity of the time evolution at infinite time intervals. To enable near identity transformations up to any polynomial order, we shall only consider the integer values of p. Our main result is the following normal form equation, which is nothing but the classical Duffing oscillator:

$$N_s'(E_*)\ddot{A} + \lambda'(E_*)\|\psi_*\|_{L^2}^2\left(\mathcal{N}_0 - N_s(E_*)\right)A - \mathcal{Q}\mathcal{S}A^3 = 0, \tag{1.10}$$

where $A(t)$ is a real-valued amplitude, $\mathcal{N}_0 = \|\Psi_0\|_{L^2}^2$ is given by the initial condition of the NLS equation, and all other coefficients are the same as in Theorems 1 and 2.

Trajectories of the Duffing equation (1.10) with $\lambda'(E_*) < 0$ and $N_s'(E_*) > 0$ on the phase plane (A, B), where $B = \dot{A}$, are shown on Fig. 2 for four distinct cases of different values of

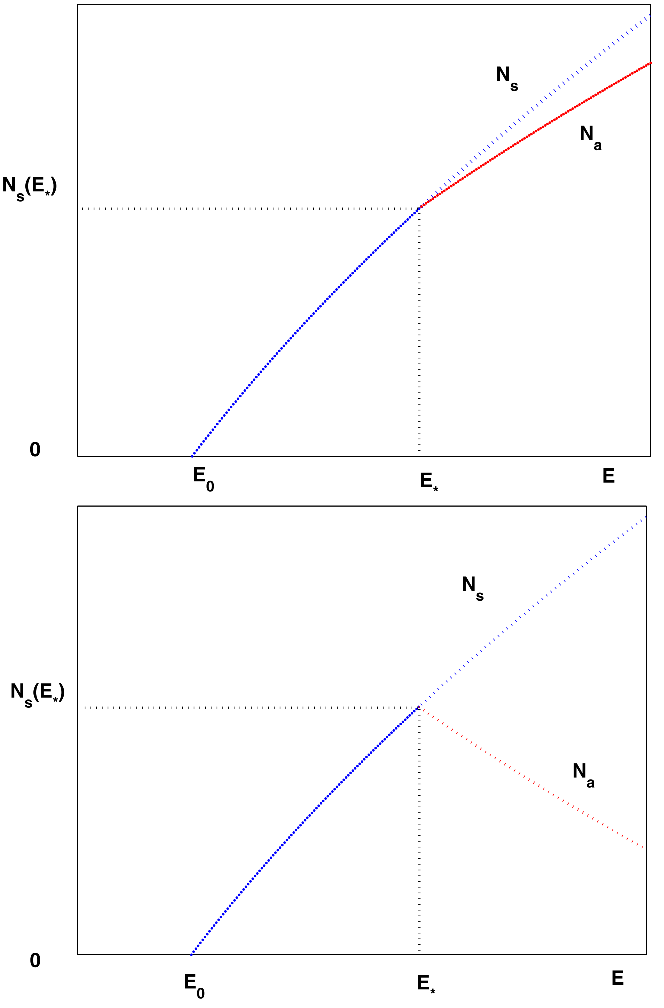

Fig. 1. Schematic representation of the supercritical (top) and subcritical (bottom) pitchfork bifurcations. Unstable stationary states are shown by dotted curves.

$\Delta N := \mathcal{N}_0 - N_s(E_*)$ and $\mathcal{QS}$. Two left panels show a typical supercritical pitchfork bifurcation, where a zero equilibrium state is stable for $\Delta N < 0$ (bottom) and unstable for $\Delta N > 0$ (top), whereas a pair of stable nonzero equilibrium states bifurcates for $\Delta N > 0$ (top). Two right panels show a typical subcritical pitchfork bifurcation, where a pair of unstable nonzero equilibrium states exists for $\Delta N < 0$ (bottom) and disappears for $\Delta N > 0$ (top) resulting in the change of stability of the zero equilibrium state.

The normal form equations have been considered previously in a similar context with different mathematical techniques. In the limit of large separation of the two potential wells, Kirr et al. [9] derived a two-mode reduction of the NLS equation. Persistence of this reduction for periodic small-amplitude oscillations near stable stationary states was addressed by Marzuola and Weinstein [11]. These authors only considered small-amplitude periodic solutions of the normal form equations arising in the large separation limit. They used sophisticated analysis based on Strichartz estimates and wave operators for the linear Schrödinger equations.

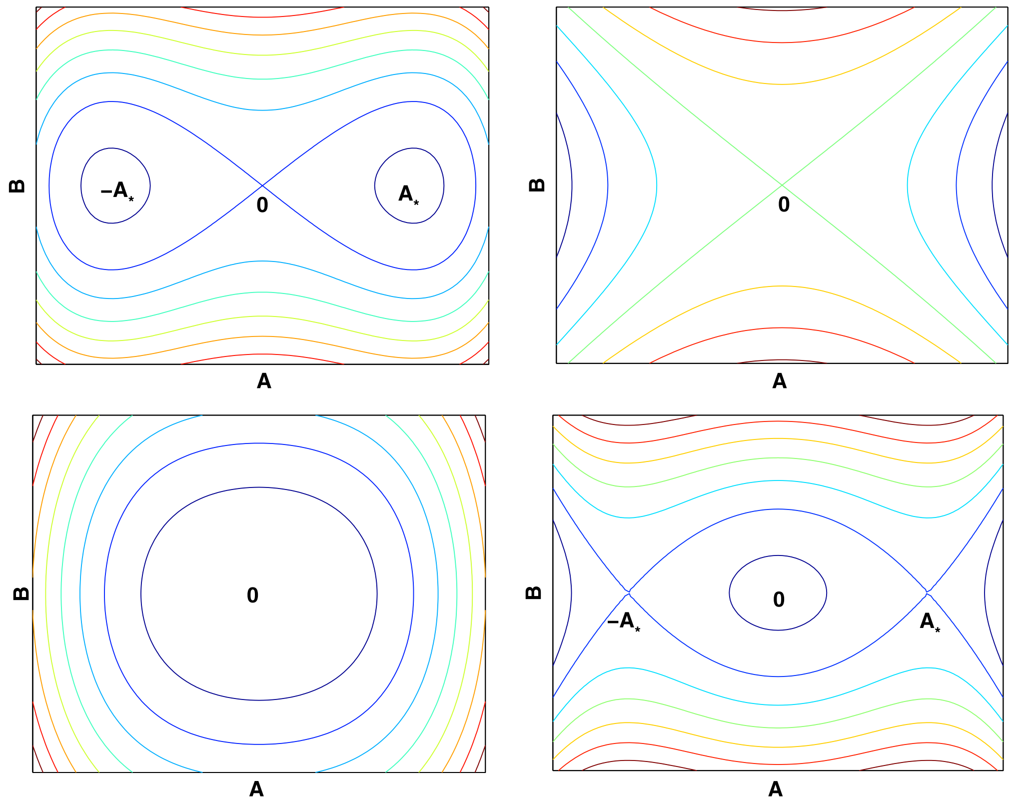

Fig. 2. Trajectories of the second-order system (1.11) on the phase plane (A, B) for $\Delta N > 0$ and $\mathcal{QS} < 0$ (top left); $\Delta N > 0$ and $\mathcal{QS} > 0$ (top right); $\Delta N < 0$ and $\mathcal{QS} < 0$ (bottom left); $\Delta N < 0$ and $\mathcal{QS} > 0$ (bottom right).

Similar but more formal reduction to the two-mode equations was developed by Sacchetti [14] using the semi-classical analysis. In comparison with [9,11], Sacchetti [14] considered the defocussing version of the NLS equation, where the anti-symmetric stationary state undertakes a similar symmetry-breaking bifurcation. Based on the two-mode reduction, Sacchetti [15] also reported the same threshold p_* as in (1.9) that separates the supercritical and subcritical pitchfork bifurcations. Recently, Fukuizumi and Sacchetti [3] justified the two-mode reduction rigorously in the semi-classical limit, up to an exponentially small error term.

Compared to these previous works, we shall deal with a *general* symmetry-breaking bifurcation of the symmetric states. We develop *simple* but *robust* analysis, which justifies a general normal form equation for the pitchfork bifurcation. Our analysis is based on the spectral decompositions and Gronwall inequalities. *Arbitrary* bounded solutions of the normal form equation are proved to shadow dynamics of time-dependent solutions of the NLS equation (1.1) near the stationary bound states for long but finite time intervals. Thus, we show how basic analytical methods can be used to treat time-dependent normal form equations for bifurcations in the nonlinear Schrödinger equations.

Our main result is formulated in the following theorem.

Theorem 3. *Let $E_* \in (E_0, \infty)$ be defined as in Theorem 1. Assume that $\lambda'(E_*) < 0$, $N_s'(E_*) > 0$, and $\mathcal{Q} < 0$. Fix $\mathcal{N}_0$ and define $\Delta N = \mathcal{N}_0 - N_s(E_*)$. There exists a sufficiently small, positive ε such that for all $\Psi_0 \in H^1$*

with $\mathcal{N}_0 = \|\Psi_0\|_{L^2}^2$ *and* $|\Delta N| < \varepsilon$, *there exist* $T > 0$ *and functions* $(\theta, E, A, B) \in C^1([0,T];\mathbb{R}^4)$ *such that the NLS equation* (1.1) *admits a solution* $\Psi \in C([0,T]; H^1(\mathbb{R}))$ *with* $\Psi(x,0) = \Psi_0(x)$ *in the form*

$$\Psi(x,t) = e^{i\theta(t)}\big[\phi\big(x; E(t)\big) + A(t)\psi\big(x; E(t)\big) + iB(t)\chi\big(x; E(t)\big)\big] + \tilde{\Psi}(x,t),$$

where $\psi(x;E)$ *and* $\chi(x;E)$ *satisfy*

$$L_+(E)\psi = -\Lambda^2(E)\chi, \qquad L_-(E)\chi = \psi,$$

subject to the normalization $\langle \chi, \psi\rangle_{L^2} = 1$, *and* $\Lambda^2(E)$ *admits the asymptotic expansion*

$$\Lambda^2(E) = -\lambda'(E_*)\|\psi_*\|_{L^2}^2(E - E_*) + \mathcal{O}(E - E_*)^2 \quad \text{as } E \to E_*.$$

Moreover, there are positive constants α_0, α_1, α_2, α_3, *and* α_4 *such that* $T \leqslant \alpha_0|\Delta N|^{-1/2}$,

$$\big|\dot{\theta}(t) - E_*\big| \leqslant \alpha_1|\Delta N|, \qquad \big|E(t) - E_*\big| \leqslant \alpha_2|\Delta N|, \qquad \big\|\tilde{\Psi}(\cdot,t)\big\|_{H^1} \leqslant \alpha_3|\Delta N|, \quad \text{for all } t \in [0,T],$$

and the trajectories of (A,B) *in the ellipsoidal domain,*

$$D = \big\{(A,B) \in \mathbb{R}^2\colon\ A^2 + |\Delta N|^{-1}B^2 \leqslant \alpha_4|\Delta N|\big\},$$

are homeomorphic to those of the truncated system,

$$\begin{cases} \dot{A} = B, \\ \dot{B} = \big(-\lambda'(E_*)\|\psi_*\|_{L^2}^2(\Delta N)A + \mathcal{Q}\mathcal{S}A^3\big)/N_s'(E_*). \end{cases} \tag{1.11}$$

Remark 1. The upper bound on T is generally sufficient to contain many oscillations of the Duffing equation (1.10) that follows from the system of two equations (1.11) because the characteristic time of this system is $\mathcal{O}(|\Delta N|^{-1/2})$ as $|\Delta N| \to 0$. As we show in Appendix A, all other coefficients of the Duffing equation (1.10) remain bounded as $s \to \infty$, therefore, the result of Theorem 3 is sufficient to contain many oscillations of the Duffing equation even in the case of large well separation of the potential V in (1.2).

The article is organized as follows. In Section 2, we derive modulation equations for dynamics of time-dependent solutions of the NLS equation near the stationary bound states at the onset of the symmetry-breaking (pitchfork) bifurcation. In Section 3, we justify the dynamics of the time-dependent modulation equations and give a proof of Theorem 3. Appendix A presents computations of quantities $\mathcal{Q}$ and $\mathcal{S}$ in the limit of large s in order to justify the main assumptions of Theorem 3. Appendix B shows how to recover results on the existence and stability of stationary states from the system of modulation equations used in the proof of Theorem 3.

2. Modulation equations for dynamics of bound states

We derive a set of modulation equations which describe temporal dynamics of solutions of the NLS equation (1.1) near the stationary bound states at the onset of the symmetry-breaking bifurcation. From the results in [10], we only use the statement of Theorem 1(i) on the existence of the symmetry-breaking bifurcation for the symmetric stationary state of the NLS equation (1.1) under assumptions (H1)–(H5). Compared to the result of Theorem 1(i), we restrict our work to integer values of p in order to deal with power series expansions without technical limitations.

Furthermore, we use the decomposition of the solution Ψ to the NLS equation (1.1) into a sum of the stationary state $e^{i\theta}\phi(\cdot;E)$ with slowly varying parameters (θ, E) and the remainder terms, which satisfies certain symplectic orthogonality conditions. Existence and uniqueness of this decomposition for small remainder terms follow from standard arguments based on the Implicit Function Theorem (see, e.g., [2,12]). Note that we are using these decompositions for large but finite time intervals of the NLS equation (1.1), therefore, we are not using asymptotic stability results for solitary waves of the NLS equations.

2.1. Primary decomposition near the symmetric stationary state

Let $\phi(x;E)$ be a solution of the stationary NLS equation (1.3) with properties

$$\phi(\cdot;E) \in H^2(\mathbb{R}): \quad \phi(-x;E) = \phi(x;E) > 0 \quad \text{for all } x \in \mathbb{R}.$$

As stated in Theorem 1(i), there exists a C^1 curve $E \mapsto \phi(\cdot;E) \in H^2(\mathbb{R})$ for all $E \in (E_0, \infty)$. If $p \in \mathbb{N}$, this curve is actually C^∞ by the bootstrapping arguments.

We decompose a solution of the NLS equation (1.1) as a sum of the symmetric stationary state with slowly varying parameters and the remainder terms,

$$\Psi(x,t) = e^{i\theta(t)}\big[\phi\big(x;E(t)\big) + u(x,t) + iw(x,t)\big], \tag{2.1}$$

where (E,θ) are coordinates of the stationary state and (u,w) are the remainder terms. Direct substitution of (2.1) into (1.1) shows that the real functions (u,w) satisfy the system of time evolution equations,

$$u_t = L_-(E)w + N_-(u,w) + (\dot{\theta} - E)w - \dot{E}\partial_E\phi, \tag{2.2}$$

$$-w_t = L_+(E)u + N_+(u,w) + (\dot{\theta} - E)(\phi + u), \tag{2.3}$$

where $L_+(E)$ and $L_-(E)$ are defined by (1.4) and (1.6) and the nonlinear terms are given explicitly by

$$N_+(u,w) = -(\phi+u)\big(\phi^2 + 2\phi u + u^2 + w^2\big)^p + \phi^{2p}\big(\phi + (2p+1)u\big),$$
$$N_-(u,w) = -w\big[\big(\phi^2 + 2\phi u + u^2 + w^2\big)^p - \phi^{2p}\big].$$

For any $p \in \mathbb{N}$, we can use the Taylor series expansions

$$N_+(u,w) = -p(2p+1)\phi^{2p-1}u^2 - p\phi^{2p-1}w^2 - \frac{1}{3}p(2p+1)(2p-1)\phi^{2p-2}u^3 - p(2p-1)\phi^{2p-2}uw^2 + \mathcal{O}\big(u^2+w^2\big)^2, \tag{2.4}$$

$$N_-(u,w) = -2p\phi^{2p-1}uw - p(2p-1)\phi^{2p-2}u^2w - p\phi^{2p-2}w^3 + \mathcal{O}\big(u^2+w^2\big)^2. \tag{2.5}$$

The linearized system associated with the time evolution equations (2.2) and (2.3) is determined by the spectrum of the linearized operator

$$\mathcal{L}(E) = \begin{bmatrix} 0 & L_-(E) \\ -L_+(E) & 0 \end{bmatrix}. \tag{2.6}$$

The generalized kernel of $\mathcal{L}(E)$ is at least two-dimensional, thanks to the exact eigenvectors

$$L_-(E)\phi = 0, \qquad L_+(E)\partial_E\phi = -\phi. \tag{2.7}$$

To determine (E,θ) uniquely in the neighborhood of the stationary state (for small u and w), we add the standard conditions of symplectic orthogonality [2,12],

$$\langle \phi, u \rangle_{L^2} = 0, \qquad \langle \partial_E \phi, w \rangle_{L^2} = 0. \tag{2.8}$$

These conditions ensure that the remainder terms are orthogonal to the generalized eigenvectors (2.7) with the account of the symplectic structure of the time evolution equations (2.2) and (2.3).

Under symplectic orthogonality conditions (2.8), the rate of changes of (E, θ) are uniquely determined from the projection equations

$$\begin{bmatrix} \langle \partial_E \phi, \phi - u \rangle_{L^2} & -\langle \phi, w \rangle_{L^2} \\ -\langle \partial_E^2 \phi, w \rangle_{L^2} & \langle \partial_E \phi, \phi + u \rangle_{L^2} \end{bmatrix} \begin{bmatrix} \dot{E} \\ \dot{\theta} - E \end{bmatrix} = \begin{bmatrix} \langle \phi, N_-(u, w) \rangle_{L^2} \\ -\langle \partial_E \phi, N_+(u, w) \rangle_{L^2} \end{bmatrix}. \tag{2.9}$$

We shall now study eigenvectors at the onset of the symmetry-breaking bifurcation at $E = E_*$ in order to build a frame for the secondary decomposition of the perturbations (u, w) near these eigenvectors.

2.2. Linear eigenvectors

As stated in Theorem 1(i), there exists a bifurcation value $E_* \in (E_0, \infty)$ such that the second eigenvalue $\lambda(E)$ of $L_+(E)$ satisfies $\lambda(E_*) = 0$. We shall denote $\phi_*(x) = \phi(x; E_*)$ at the bifurcation value $E = E_*$. In many cases, we will suppress the x-argument in the function $\phi(x; E)$ to emphasize the E-dependence of this function. In this setting, we have the following result.

Lemma 1. *There exist odd functions $\psi_*, \chi_* \in H^2(\mathbb{R})$ such that*

$$L_+(E_*)\psi_* = 0, \qquad L_-(E_*)\chi_* = \psi_*, \qquad \langle \chi_*, \psi_* \rangle_{L^2} = 1. \tag{2.10}$$

Moreover,

$$\lambda'(E_*) = 1 - 2p(2p+1)\frac{\langle \partial_E \phi_*, \phi_*^{2p-1} \psi_*^2 \rangle_{L^2}}{\|\psi_*\|_{L^2}^2}. \tag{2.11}$$

Proof. Let $g(x; E)$ be an eigenfunction of $L_+(E)$ for the eigenvalue $\lambda(E)$. By Sturm's Theorem, $g(x; E)$ is odd in x because $\lambda(E)$ is the second eigenvalue of $L_+(E)$. Because the second eigenvalue of $L_+(E)$ is simple and $L_+(E)$ is C^1 in E near $E = E_*$, asymptotic perturbation theory for simple eigenvalues of closed operators [7, Section 8.2.3] guarantees that $\lambda(E)$ and $g(E)$ are C^1 near $E = E_*$.

Let $\psi_*(x) = g(x; E_*)$. It follows that ψ_* is an odd function in x and $L_+(E_*)\psi_* = \lambda(E_*)\psi_* = 0$. Since $\phi_*(x) = \phi(x; E_*)$ is even, positive and $L_-(E_*)\phi_* = 0$, we see that zero is the lowest eigenvalue of $L_-(E_*)$. Therefore, it is a simple eigenvalue. Because ψ_* is odd, there is an odd function $\chi_* \in H^2(\mathbb{R})$ such that $L_-(E_*)\chi_* = \psi_*$. On the other hand, we have

$$\langle \chi_*, \psi_* \rangle_{L^2} = \left\langle L_-^{-1}(E_*)\psi_*, \psi_* \right\rangle_{L^2} > 0.$$

By rescaling ψ_* and χ_*, we get $\langle \chi_*, \psi_* \rangle_{L^2} = 1$. This completes the proof of the first part of the lemma.

To prove (2.11), we compute explicitly

$$L_+'(E_*) = 1 - 2p(2p+1)\phi_*^{2p-1}\partial_E \phi_*, \qquad L_-'(E_*) = 1 - 2p\phi_*^{2p-1}\partial_E \phi_*. \tag{2.12}$$

By differentiating the relation $L_+(E)g(E) = \lambda(E)g(E)$ at $E = E_*$, we get

$$L_+'(E_*)\psi_* + L_+(E_*)\partial_E \psi_* = \lambda'(E_*)\psi_*.$$

Taking the inner product of this equation with ψ_*, we get

$$\lambda'(E_*) = \frac{\langle L'_+(E_*)\psi_*, \psi_*\rangle_{L^2}}{\|\psi_*\|^2_{L^2}} = 1 - 2p(2p+1)\frac{\langle \partial_E\phi_*, \phi_*^{2p-1}\psi_*^2\rangle_{L^2}}{\|\psi_*\|^2_{L^2}}.$$

This completes the proof of the lemma. □

We would like now to extend the functions (ψ_*, χ_*) as the eigenvectors of the linearized system associated with the linearized operator $\mathcal{L}(E)$ in (2.6) near $E = E_*$. Note that the eigenvectors of the linearized (non-self-adjoint) system are different from the eigenvector $g(E)$ of the (self-adjoint) operator $L_+(E)$ introduced in the proof of Lemma 1. The following lemma gives the extension of (ψ_*, χ_*) near $E = E_*$.

Lemma 2. *There exists a sufficiently small, positive ϵ such that for all $|E - E_*| < \epsilon$, the linearized system*

$$L_+(E)\psi(E) = -\Lambda^2(E)\chi(E), \qquad L_-(E)\chi(E) = \psi(E) \tag{2.13}$$

admits a small eigenvalue $\Lambda^2(E)$ with an eigenvector $(\psi(E), \chi(E))$, which are C^1 functions near $E = E_*$ such that

$$\psi(E) = \psi_* + \mathcal{O}_{H^2}(E - E_*), \qquad \chi(E) = \chi_* + \mathcal{O}_{H^2}(E - E_*), \qquad \big\langle\chi(E), \psi(E)\big\rangle_{L^2} = 1, \tag{2.14}$$

and

$$\frac{d}{dE}\Lambda^2(E_*) = -\lambda'(E_*)\|\psi_*\|^2_{L^2}. \tag{2.15}$$

Consequently, if $\lambda'(E_*) < 0$, the eigenvalue $\Lambda(E)$ is real for $E > E_*$ and purely imaginary for $E < E_*$.

Proof. Recall that $L_-(E_*)\phi_* = 0$, $L_+(E_*)\psi_* = 0$ and $L_-(E_*)\chi_* = \psi_*$, where both ψ_* and χ_* are odd in x and ϕ_* is even in x. Let P_o be an orthogonal projection to the space of odd functions. Because the spectrum of $P_oL_-(E)P_o$ is bounded away from zero, the self-adjoint operator $P_oL_-(E)P_o$ is invertible for any E. Let us consider the following generalized eigenvalue problem,

$$P_oL_+(E)P_o\psi = -\gamma\big(P_oL_-(E)P_o\big)^{-1}\psi,$$

where γ is a new spectral parameter, ψ is an eigenfunction, and $P_oL_\pm(E)P_o$ are C^1 functions near $E = E_*$. $\gamma = 0$ is a simple eigenvalue of the generalized eigenvalue problem at $E = E_*$. By the same asymptotic perturbation theory for simple eigenvalues of closed operators [7, Section 8.2.3], there exists a solution for $\gamma = \Lambda^2(E)$, $\psi = \psi(E)$, and $\chi(E) = (P_oL_-(E)P_o)^{-1}\psi(E)$ in the linearized system (2.13) such that $\Lambda^2(E)$, $\psi(E)$, and $\chi(E)$ are C^1 functions near $E = E_*$ and the eigenvectors $\psi(E)$, and $\chi(E)$ satisfy the expansion (2.14).

By differentiating the problem $L_+(E)\psi(E) = -\Lambda^2(E)\chi(E)$ at $E = E_*$, taking the inner product with ψ_*, and using (2.11), (2.12), and (2.14), we get (2.15). Thus, it follows that if $\lambda'(E_*) < 0$ the eigenvalue $\Lambda(E)$ is real for $E > E_*$ and purely imaginary for $E < E_*$.

On the other hand, for small $|E - E_*|$, we have

$$\big\langle\chi(E), \psi(E)\big\rangle_{L^2} = \langle\chi_*, \psi_*\rangle_{L^2} + \mathcal{O}(E - E_*) = 1 + \mathcal{O}(E - E_*) > 0.$$

We can hence normalize $\psi(E)$ and $\chi(E)$ such that $\langle\chi(E), \psi(E)\rangle_{L^2} = 1$. This completes the proof of the lemma. □

Remark 2. Under the normalization $\langle \chi(E), \psi(E)\rangle_{L^2} = 1$, the L^2-norms of ψ and χ are no longer normalized to unity, in comparison with the normalization used in [10].

Remark 3. In what follows, we assume that $\lambda'(E_*) \neq 0$. As stated in Theorem 1(i), this necessarily implies that $\lambda'(E_*) < 0$.

Remark 4. Since eigenvalues of $L_+(E)$ are simple by the Sturm–Liouville theory, the geometric kernel of $\mathcal{L}(E_*)$ is exactly two-dimensional. The generalized kernel of $\mathcal{L}(E_*)$ is at least four-dimensional and includes the subspace

$$\mathcal{U}(E_*) = \text{span}\left\{\begin{bmatrix} 0 \\ \phi_* \end{bmatrix}, \begin{bmatrix} \partial_E \phi(E_*) \\ 0 \end{bmatrix}, \begin{bmatrix} \psi_* \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \chi_* \end{bmatrix}\right\}. \tag{2.16}$$

2.3. Secondary decomposition near the linear eigenvectors

Let us now decompose the perturbation terms into

$$u(x,t) = A(t)\psi(x;E) + U(x,t), \qquad w(x,t) = B(t)\chi(x;E) + W(x,t), \tag{2.17}$$

where (A, B) are coordinates of the decomposition and (U, W) are the remainder terms. The linear eigenvectors (ψ, χ) are solutions of the linearized system (2.13) for E near E_*. The remainder terms (U, W) are required to satisfy the conditions of symplectic orthogonality

$$\langle \phi, U\rangle_{L^2} = 0, \qquad \langle \partial_E \phi, W\rangle_{L^2} = 0, \qquad \langle \chi, U\rangle_{L^2} = 0, \qquad \langle \psi, W\rangle_{L^2} = 0. \tag{2.18}$$

These conditions ensure that the remainder terms (U, W) are orthogonal to four generalized eigenvectors of operator $\mathcal{L}(E)$ with respect to the symplectic structure of the time evolution equations (2.2) and (2.3).

Substitution of (2.17) into (2.2)–(2.3) shows that (U, W) satisfy the time evolution equations

$$\begin{aligned} U_t = {} & L_-(E)W + N_-(A\psi + U, B\chi + W) \\ & + (\dot{\theta} - E)(B\chi + W) - \dot{E}(\partial_E\phi + A\partial_E\psi) - (\dot{A} - B)\psi, \end{aligned} \tag{2.19}$$

$$\begin{aligned} -W_t = {} & L_+(E)U + N_+(A\psi + U, B\chi + W) \\ & + (\dot{\theta} - E)(\phi + A\psi + U) + B\dot{E}\partial_E\chi + \big(\dot{B} - \Lambda^2 A\big)\chi. \end{aligned} \tag{2.20}$$

Under the orthogonality conditions (2.18), the rate of changes of (θ, E, A, B) are uniquely determined from the projection equations

$$\mathcal{M}\begin{bmatrix} \dot{\theta} - E \\ \dot{E} \end{bmatrix} = \begin{bmatrix} -\langle \partial_E\phi, N_+(A\psi + U, B\chi + W)\rangle_{L^2} \\ \langle \phi, N_-(A\psi + U, B\chi + W)\rangle_{L^2} \end{bmatrix}, \tag{2.21}$$

where

$$\mathcal{M} = \begin{bmatrix} \langle \partial_E\phi, \phi + U\rangle_{L^2} & -\langle \partial_E^2\phi, W\rangle_{L^2} \\ -\langle \phi, W\rangle_{L^2} & \langle \partial_E\phi, \phi - U\rangle_{L^2} \end{bmatrix}, \tag{2.22}$$

and

$$\dot{A} - B = \big\langle \chi, N_-(A\psi + U, B\chi + W)\big\rangle_{L^2} + \dot{E}\big(\langle \partial_E \chi, U\rangle_{L^2} - A\langle \partial_E \psi, \chi\rangle_{L^2}\big) + (\dot{\theta} - E)\big(B\|\chi\|_{L^2}^2 + \langle \chi, W\rangle_{L^2}\big), \tag{2.23}$$

$$\dot{B} - \Lambda^2 A = -\big\langle \psi, N_+(A\psi + U, B\chi + W)\big\rangle_{L^2} + \dot{E}\big(\langle \partial_E \psi, W\rangle_{L^2} - B\langle \partial_E \chi, \psi\rangle_{L^2}\big) - (\dot{\theta} - E)\big(A\|\psi\|_{L^2}^2 + \langle \psi, U\rangle_{L^2}\big). \tag{2.24}$$

To prove Theorem 3, we shall control the dynamics of small (U, W), $(E - E_*, \dot{\theta} - E_*)$, and (A, B) in the system (2.19)–(2.24) on long but finite time intervals. Appendix B shows how to use the system (2.19)–(2.24) for analysis of existence and stability of stationary states near the bifurcation point E_*.

2.4. Conserved quantities

The NLS equation (1.1) admits two conserved quantities given by

$$\mathcal{N}[\Psi] = \int_{\mathbb{R}} \big|\Psi(x,t)\big|^2 dx \tag{2.25}$$

and

$$\mathcal{H}[\Psi] = \int_{\mathbb{R}} \left[\big|\Psi_x(x,t)\big|^2 + V(x)\big|\Psi(x,t)\big|^2 - \frac{1}{p+1}\big|\Psi(x,t)\big|^{2p+2}\right] dx. \tag{2.26}$$

They are referred to as the energy N and the Hamiltonian H, respectively.

Let $N_s(E) = \mathcal{N}[\phi(\cdot; E)]$ and $H_s(E) = \mathcal{H}[\phi(\cdot; E)]$. If Ψ_0 is an initial condition for the solution Ψ of the NLS equation (1.1), we define

$$\mathcal{N}_0 = \mathcal{N}[\Psi_0] \quad \text{and} \quad \mathcal{H}_0 = \mathcal{H}[\Psi_0].$$

Substitution of (2.1) and (2.17) into (2.25) and (2.26) gives

$$\mathcal{N}_0 = N_s(E) + \int_{\mathbb{R}} \big[(A\psi + U)^2 + (B\chi + W)^2\big] dx$$

and

$$\mathcal{H}_0 = H_s(E) + \int_{\mathbb{R}} \big[(A\psi_x + U_x)^2 + (B\chi_x + W_x)^2 + V(A\psi + U)^2 + V(B\chi + W)^2\big] dx - \frac{1}{p+1}\int_{\mathbb{R}} \big[\big((\phi + A\psi + U)^2 + (B\chi + W)^2\big)^{p+1} - \phi^{2p+2} - 2(p+1)\phi^{2p+1}(A\psi + U)\big] dx,$$

where we have used the stationary equation (1.3) and the symplectic orthogonality (2.18).

By direct computation, we can verify that

$$H_s'(E) + E N_s'(E) = 0, \tag{2.27}$$

for any E, for which $\phi(\cdot; E) \in H^2(\mathbb{R})$ exists.

Remark 5. In what follows, we assume that $N_s'(E_*) \neq 0$ (and, more precisely, $N_s'(E_*) > 0$). Under this condition, the generalized kernel of operator $\mathcal{L}(E_*)$ is exactly the four-dimensional subspace $\mathcal{U}(E_*)$ in (2.16) because $\langle \phi_*, \partial_E \phi(E_*)\rangle_{L^2} \neq 0$ and $\langle \chi_*, \psi_*\rangle_{L^2} = 1 \neq 0$.

3. Time-dependent normal form equations

Our goal is to prove the main result, Theorem 3, by using the decompositions developed in Section 2. We start by rewriting the main equations in the abstract form. In particular, we rewrite the modulation equations (2.21)–(2.24) for (θ, E, A, B) as

$$\begin{cases} \dot{\theta} - E = R_\theta(E, A, B, U, W), \\ \dot{E} = R_E(E, A, B, U, W), \\ \dot{A} - B = R_A(E, A, B, U, W), \\ \dot{B} - \Lambda^2(E) A = R_B(E, A, B, U, W), \end{cases} \tag{3.1}$$

and the system (2.19)–(2.20) for the remainder terms (U, W) as

$$\begin{cases} U_t = L_-(E) W + R_U(E, A, B, U, W), \\ -W_t = L_+(E) U + R_W(E, A, B, U, W), \end{cases} \tag{3.2}$$

where R_θ, R_E, R_A, R_B, R_U and R_W are some functionals on the solution. These functionals can be computed explicitly. Indeed, it follows from (2.21) that

$$\begin{bmatrix} R_\theta \\ R_E \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} -\langle \partial_E \phi, N_+(A\psi + U, B\chi + W)\rangle_{L^2} \\ \langle \phi, N_-(A\psi + U, B\chi + W)\rangle_{L^2} \end{bmatrix}, \tag{3.3}$$

where matrix $\mathcal{M}$ is given by (2.22). If $\|U\|_{L^2}, \|W\|_{L^2} \ll 1$ and $N_s'(E_*) \neq 0$, then $\mathcal{M}$ is invertible and

$$\mathcal{M}^{-1} = \langle \partial_E \phi, \phi\rangle_{L^2}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}\big(\|U\|_{L^2} + \|W\|_{L^2}\big). \tag{3.4}$$

On the other hand, Eqs. (2.23) and (2.24) yield

$$\begin{cases} R_A = \big\langle \chi, N_-(A\psi + U, B\chi + W)\big\rangle_{L^2} + R_E\big(\langle \partial_E \chi, U\rangle_{L^2} - A\langle \partial_E \psi, \chi\rangle_{L^2}\big) \\ \qquad + R_\theta\big(B\|\chi\|_{L^2}^2 + \langle \chi, W\rangle_{L^2}\big), \\ R_B = -\big\langle \psi, N_+(A\psi + U, B\chi + W)\big\rangle_{L^2} + R_E\big(\langle \partial_E \psi, W\rangle_{L^2} - B\langle \partial_E \chi, \psi\rangle_{L^2}\big) \\ \qquad - R_\theta\big(A\|\psi\|_{L^2}^2 + \langle \psi, U\rangle_{L^2}\big). \end{cases} \tag{3.5}$$

When the modulation equations (3.1) are substituted into the system (2.19)–(2.20), we obtain

$$\begin{cases} R_U = N_-(A\psi + U, B\chi + W) + R_\theta(B\chi + W) - R_E(\partial_E \phi + A\partial_E \psi) - R_A \psi, \\ R_W = N_+(A\psi + U, B\chi + W) + R_\theta(\phi + A\psi + U) + B R_E \partial_E \chi + R_B \chi. \end{cases} \tag{3.6}$$

Remark 6. If $A = B = 0$, then $U = W = 0$ is an invariant solution of the system (3.2), which gives zero values of R_θ, R_E, R_A, and R_B in the system (3.1) for any E.

The above remark inspires us to consider the power series expansions for solutions of the systems (3.1) and (3.2). Taking into account the spatial symmetry of eigenfunctions, we can see that R_θ and R_E are quadratic with respect to (A, B), whereas R_A and R_B are cubic with respect to (A, B). Moreover, we write

$$\begin{cases} \dot{\theta} - E = C_1(E)A^2 + C_2(E)B^2 + \tilde{R}_\theta(E, A, B, \tilde{U}, \tilde{W}), \\ \dot{E} = C_3(E)AB + \tilde{R}_E(E, A, B, \tilde{U}, \tilde{W}), \\ \dot{A} - B = C_4(E)A^2B + C_5(E)B^3 + \tilde{R}_A(E, A, B, \tilde{U}, \tilde{W}), \\ \dot{B} - \Lambda^2(E)A = C_6(E)A^3 + C_7(E)AB^2 + \tilde{R}_B(E, A, B, \tilde{U}, \tilde{W}), \end{cases} \tag{3.7}$$

and

$$\begin{cases} U = A^2\Theta(x; E) + B^2\Delta(x; E) + A^3U_1(x; E) + AB^2U_2(x; E) + \tilde{U}(x, t), \\ W = AB\Gamma(x; E) + A^2BW_1(x; E) + B^3W_2(x; E) + \tilde{W}(x, t), \end{cases} \tag{3.8}$$

where $\tilde{R}_\theta$, $\tilde{R}_E$, $\tilde{R}_A$, and $\tilde{R}_B$ are new error terms, whereas $\tilde{U}$ and $\tilde{W}$ are new remainder terms. Because all quadratic and cubic terms in (A, B) are taken into account in (3.7)–(3.8), the error and remainder terms are quartic with respect to (A, B).

Let us first explicitly compute the coefficients $C_1, C_2, \ldots, C_7$ and determine the functions $\Theta, \Delta, \ldots, W_2$. We shall then estimate the error and remainder terms in (3.7) and (3.8) as quartic with respect to (A, B). Working in a small neighborhood of $(0, 0)$ on the phase plane (A, B) and using $|\Delta N|$ as a small parameter, we consider an ellipsoidal region on the (A, B)-plane defined by

$$\exists\alpha > 0: \quad A^2 + |\Delta N|^{-1}B^2 \leqslant \alpha|\Delta N|. \tag{3.9}$$

Let $T > 0$ be the maximal time until which we consider solutions of the modulation equations (3.7) in the domain (3.9). We assume (and prove in Section 3.4) that there are positive constants α_0, α_1, and α_2 such that

$$T \leqslant \alpha_0|\Delta N|^{-1/2}, \tag{3.10}$$

and

$$|\dot{\theta} - E_*| \leqslant \alpha_1|\Delta N|, \qquad |E - E_*| \leqslant \alpha_2|\Delta N|. \tag{3.11}$$

The following theorem provides the control of the error terms of the system (3.7) and the remainder terms of the decomposition (3.8).

Lemma 3. *Assume (3.9)–(3.11). There exists a sufficiently small positive ε such that for any $|\Delta N| < \varepsilon$, there are positive constants c_1 and c_2 such that*

$$\sup_{t\in[0,T]} \left(\|\tilde{U}(\cdot, t)\|_{H^1} + \|\tilde{W}(\cdot, t)\|_{H^1}\right) \leqslant c_1(\Delta N)^2 \tag{3.12}$$

and

$$\sup_{t\in[0,T]} \left(|\tilde{R}_\theta| + |\tilde{R}_E| + |\tilde{R}_A| + |\tilde{R}_B|\right) \leqslant c_2(\Delta N)^2. \tag{3.13}$$

The proof of Lemma 3 is given in Sections 3.1 and 3.2.

3.1. Power series expansions

For explicit computations, we use the power series expansions (2.4)–(2.5) and the decompositions (2.17) and (3.8) to expand

$$\begin{aligned} N_+ &= -p(2p+1)\phi^{2p-1}\psi^2 A^2 - p\phi^{2p-1}\chi^2 B^2 - p(2p+1)\left(\frac{2p-1}{3}\phi^{2p-2}\psi^3 + 2\phi^{2p-1}\psi\Theta\right)A^3 \\ &\quad - p\big((2p-1)\phi^{2p-2}\psi\chi^2 + 2(2p+1)\phi^{2p-1}\psi\Delta + 2\phi^{2p-1}\chi\Gamma\big)AB^2 + \tilde{N}_+, \\ N_- &= -2p\phi^{2p-1}\psi\chi AB - p\big(\phi^{2p-2}\chi^3 + 2\phi^{2p-1}\chi\Delta\big)B^3 \\ &\quad - p\big((2p-1)\phi^{2p-2}\chi\psi^2 + 2\phi^{2p-1}\psi\Gamma + 2\phi^{2p-1}\chi\Delta\big)A^2 B + \tilde{N}_-, \end{aligned}$$

where $\tilde{N}_+$ and $\tilde{N}_-$ are of the form

$$\tilde{N}_+, \tilde{N}_- = \mathcal{O}\big((A^2+B^2)^2 + (A+B)(\tilde{U}+\tilde{W}) + \tilde{U}^2 + \tilde{W}^2\big). \tag{3.14}$$

From Eqs. (3.6)–(3.8) we have

$$\begin{cases} R_U = f_{1,1}(E)AB + f_{2,1}(E)A^2B + f_{0,3}(E)B^3 + \tilde{F}_U(E,A,B,\tilde{U},\tilde{W}), \\ R_W = g_{2,0}(E)A^2 + g_{0,2}(E)B^2 + g_{3,0}(E)A^3 + g_{1,2}(E)AB^2 + \tilde{F}_W(E,A,B,\tilde{U},\tilde{W}), \end{cases} \tag{3.15}$$

where

$$\begin{aligned} f_{1,1} &= -2p\phi^{2p-1}\psi\chi - C_3\partial_E\phi, \\ g_{2,0} &= -p(2p+1)\phi^{2p-1}\psi^2 + C_1\phi, \\ g_{0,2} &= -p\phi^{2p-1}\chi^2 + C_2\phi, \\ f_{2,1} &= C_1\chi - C_3\partial_E\psi - C_4\psi - p(2p-1)\phi^{2p-2}\chi\psi^2 - 2p\phi^{2p-1}\psi\Gamma - 2p\phi^{2p-1}\chi\Theta, \\ f_{0,3} &= C_2\chi - C_5\psi - p\phi^{2p-2}\chi^3 - 2p\phi^{2p-1}\chi\Delta, \\ g_{1,2} &= C_2\psi + C_3\partial_E\chi + C_7\chi - p\big((2p-1)\phi^{2p-2}\psi\chi^2 + 2(2p+1)\phi^{2p-1}\psi\Delta + 2\phi^{2p-1}\chi\Gamma\big), \\ g_{3,0} &= C_1\psi + C_6\chi - p(2p+1)\left(\frac{1}{3}(2p-1)\phi^{2p-2}\psi^3 + 2\phi^{2p-1}\psi\Theta\right), \end{aligned}$$

and

$$\begin{aligned} \tilde{F}_U &= \big(C_1A^2 + C_2B^2\big)W + (B\chi + W)R_\theta - (\partial_E\phi + A\partial_E\psi)R_E - R_A\psi \\ &\quad - 2p\phi^{2p-1}\big(A\psi\big(A^3U_1 + AB^2U_2 + \tilde{U}\big) + B\chi\big(A^2BW_1 + B^3W_2 + \tilde{W}\big)\big) \\ &\quad - 2p\phi^{2p-1}UW - p(2p-1)\phi^{2p-1}\big(\big(2AU\psi + U^2\big)(B\chi + W) + A^2\psi^2 W\big) \\ &\quad - p\phi^{2p-2}\big(3B^2\chi^2W + 3B\chi W^2 + W^3\big) + \tilde{N}_-, \\ \tilde{F}_W &= R_\theta(\phi + A\psi + U) + \big(C_1A^2 + C_2B^2\big)U + R_B\chi + R_E\partial_E\chi + \tilde{N}_+. \end{aligned}$$

Substituting (3.8) and (3.15) into the time evolution equations (3.2) and computing the time derivative of (E, A, B) using the modulation equations (3.7), we obtain

$$\begin{cases} L_+(E)\Theta + \Lambda^2(E)\Gamma + g_{2,0}(E) = 0, \\ L_+(E)\Delta + \Gamma + g_{0,2}(E) = 0, \\ L_-(E)\Gamma - 2\Theta - 2\Lambda^2(E)\Delta + f_{1,1}(E) = 0, \end{cases} \tag{3.16}$$

$$\begin{cases} L_+(E)U_1 + \Lambda^2(E)W_1 + g_{3,0}(E) = 0, \\ L_+(E)U_2 + 2W_1 + 3\Lambda^2(E)W_2 + g_{1,2}(E) = 0, \\ L_-(E)W_1 - 3U_1 - 2\Lambda^2(E)U_2 - f_{2,1}(E) = 0, \\ L_-(E)W_2 - U_2 - f_{0,3}(E) = 0, \end{cases} \tag{3.17}$$

and

$$\begin{cases} \tilde{U}_t = L_-(E)\tilde{W} + \tilde{R}_U(E, A, B, \tilde{U}, \tilde{W}), \\ -\tilde{W}_t = L_+(E)\tilde{U} + \tilde{R}_W(E, A, B, \tilde{U}, \tilde{W}), \end{cases} \tag{3.18}$$

where

$$\begin{aligned} \tilde{R}_U = \tilde{F}_U &- \big(A^2\partial_E\Theta + B^2\partial_E\Delta + A^3\partial_E U_1 + AB^2\partial_E U_2\big)(C_3AB + R_E) \\ &- \big(2A\Theta + 3AU_1 + B^2U_2\big)\big(C_4A^2B + C_5B^3 + \tilde{R}_A\big) \\ &- (2B\Delta + 2ABU_2)\big(C_6A^3 + C_7AB^2 + \tilde{R}_B\big), \\ \tilde{R}_W = \tilde{F}_W &+ \big(AB\partial_E\Gamma + A^2B\partial_E W_1 + B^3\partial_E W_2\big)(C_3AB + R_E) \\ &+ (B\Gamma + 2ABW_1)\big(C_4A^2B + C_5B^3 + \tilde{R}_A\big) \\ &+ \big(A\Gamma + A^2W_1 + 3B^2W_2\big)\big(C_6A^3 + C_7AB^2 + \tilde{R}_B\big). \end{aligned}$$

We shall now introduce two constrained L^2-spaces by

$$L^2_+(\mathbb{R}) = \big\{U \in L^2(\mathbb{R}):\ \langle\phi, U\rangle_{L^2} = \langle\chi, U\rangle_{L^2} = 0\big\}, \tag{3.19}$$

$$L^2_-(\mathbb{R}) = \big\{W \in L^2(\mathbb{R}):\ \langle\partial_E\phi, W\rangle_{L^2} = \langle\psi, W\rangle_{L^2} = 0\big\}. \tag{3.20}$$

Note that the orthogonal projections depend on E but we omit this dependence for the notational convenience. We can also define $H^s_\pm(\mathbb{R})$ as constrained H^s-spaces for any $s \geqslant 0$.

The following two lemmas describe solutions of the linear inhomogeneous systems (3.16) and (3.17).

Lemma 4. *There exists a sufficiently small, positive ϵ such that for all $|E - E_*| < \epsilon$, the linear inhomogeneous system (3.16) admits a unique solution $\Theta, \Delta \in H^2_+(\mathbb{R})$ and $\Gamma \in H^2_-(\mathbb{R})$. Moreover, these solutions are even in x, C^1 in E, and satisfy*

$$\exists C > 0:\quad \big\|\partial_E^\alpha\Theta\big\|_{H^2} + \big\|\partial_E^\alpha\Delta\big\|_{H^2} + \big\|\partial_E^\alpha\Gamma\big\|_{H^2} \leqslant C, \quad \alpha = 0, 1. \tag{3.21}$$

Proof. To solve the linear inhomogeneous system (3.16) near $E = E_*$, we recall that operators $L_+(E_*)$ and $L_-(E_*)$ are not invertible. Hence we set $g_{2,0}, g_{0,2} \in L^2_-(\mathbb{R})$ and $f_{1,1} \in L^2_+(\mathbb{R})$, according to (3.19) and (3.20). These constraints set up uniquely the coefficients C_1, C_2, and C_3,

$$C_1 = p(2p+1)\frac{\langle\partial_E\phi, \phi^{2p-1}\psi^2\rangle_{L^2}}{\langle\partial_E\phi, \phi\rangle_{L^2}},\ C_2 = p\frac{\langle\partial_E\phi, \phi^{2p-1}\chi^2\rangle_{L^2}}{\langle\partial_E\phi, \phi\rangle_{L^2}},\ C_3 = -2p\frac{\langle\psi, \phi^{2p}\chi\rangle_{L^2}}{\langle\partial_E\phi, \phi\rangle_{L^2}}. \tag{3.22}$$

For E close to E_*, the existence and uniqueness of the solution $\Theta, \Delta \in H^2_+(\mathbb{R})$ and $\Gamma \in H^2_-(\mathbb{R})$ of the system (3.16) follow from the gap between zero (or small) eigenvalues of $L_\pm(E)$ and the rest of the spectrum of $L_\pm(E)$ by using the following arguments.

For any fixed E such that $|E - E_*|$ is sufficiently small, for any $\Gamma \in L^2_-(\mathbb{R})$, there exists a unique solution $\Theta_\Gamma, \Delta_\Gamma \in H^2_+(\mathbb{R})$ of the system

$$L_+(E)\Theta_\Gamma + \Lambda^2(E)\Gamma = -g_{2,0}(E), \qquad L_+(E)\Delta_\Gamma + \Gamma = -g_{0,2}(E). \tag{3.23}$$

Then, for any two $\Gamma_1, \Gamma_2 \in L^2_-(\mathbb{R})$, we have

$$L_+(E)(\Theta_{\Gamma_1} - \Theta_{\Gamma_2}) = \Lambda^2(E)(\Gamma_2 - \Gamma_1), \qquad L_+(E)(\Delta_{\Gamma_1} - \Delta_{\Gamma_2}) = \Gamma_2 - \Gamma_1.$$

The standard regularity theorem for elliptic equations implies that there is a positive constant $C(E)$ such that

$$\|\Theta_{\Gamma_1} - \Theta_{\Gamma_2}\|_{H^2} \leqslant C(E)\big|\Lambda(E)\big|^2\|\Gamma_2 - \Gamma_1\|_{L^2}, \qquad \|\Delta_{\Gamma_1} - \Delta_{\Gamma_2}\|_{H^2} \leqslant C(E)\|\Gamma_2 - \Gamma_1\|_{L^2}.$$

We recall that $\Lambda^2(E) \to 0$ as $E \to E_*$. From these estimates, we apply the fixed-point arguments and obtain the existence of $\Gamma \in H^2_-(\mathbb{R})$ which solves the equation

$$L_-(E)\Gamma - 2\Theta_\Gamma - 2\Lambda^2(E)\Delta_\Gamma = -f_{1,1}(E). \tag{3.24}$$

To estimate the H^2-norm of the three solutions, we write Eqs. (3.23) as

$$L_+(E_*)\Theta_\Gamma = -g_{2,0}(E) - \big(L_+(E) - L_+(E_*)\big)\Theta_\Gamma - \Lambda^2(E)\Gamma$$

and

$$L_+(E_*)\Delta_\Gamma = -g_{0,2}(E) - \big(L_+(E) - L_+(E_*)\big)\Delta_\Gamma - \Gamma.$$

Because $\|L_+(E) - L_+(E_*)\|_{L^\infty} = \mathcal{O}(E - E_*)$ and $\Lambda^2(E) = \mathcal{O}(E - E_*)$ are sufficiently small for $|E - E_*| < \epsilon$, the standard regularity theorem for elliptic equations implies again that there exists an E-independent constant $C > 0$ such that

$$\|\Theta_\Gamma\|_{H^2} \leqslant C\big(1 + \epsilon\|\Gamma\|_{H^2}\big), \qquad \|\Delta_\Gamma\|_{H^2} \leqslant C\big(1 + \|\Gamma\|_{H^2}\big). \tag{3.25}$$

From these estimates and Eq. (3.24), we get for some constants $C, \tilde{C} > 0$,

$$\|\Gamma\|_{H^2} \leqslant C\big(1 + \|\Theta_\Gamma\|_{H^2} + \epsilon\|\Delta_\Gamma\|_{H^2}\big) \leqslant \tilde{C}\big(1 + \epsilon\|\Gamma\|_{H^2}\big).$$

Hence, if $|E - E_*| < \epsilon$ is small enough, such that $\tilde{C}\epsilon < 1$, there is an E-independent constant $C > 0$ such that $\|\Gamma\|_{H^2} \leqslant C$. From this estimate and estimate (3.25), we also obtain the H^2-estimates of Θ and Δ.

Because $L_\pm(E)$, $g_{2,0}(E)$, $g_{0,2}(E)$, and $f_{1,1}(E)$ are all C^1 in E, we see that Θ, Δ, and Γ are all C^1 in E. By differentiating the system (3.16) and using the same method as the one we just used, we also obtain the H^2 estimates for $\partial_E\Theta$, $\partial_E\Delta$, and $\partial_E\Gamma$. □

Lemma 5. *There exists a sufficiently small, positive ϵ such that for all $|E - E_*| < \epsilon$, the linear inhomogeneous system (3.17) admits a unique solution $U_1, U_2 \in H^2_+(\mathbb{R})$ and $W_1, W_2 \in H^2_-(\mathbb{R})$. Moreover, these solutions are odd in x, C^1 in E, and satisfy*

$$\exists C > 0: \quad \big\|\partial_E^\alpha U_j\big\|_{H^2} + \big\|\partial_E^\alpha W_j\big\|_{H^2} \leqslant C, \quad j = 1, 2,\ \alpha = 0, 1. \tag{3.26}$$

Proof. To solve the linear inhomogeneous system (3.17) near $E = E_*$, we set $g_{3,0}, g_{1,2} \in L^2_-(\mathbb{R})$ and $f_{2,1}, f_{0,3} \in L^2_+(\mathbb{R})$ according to (3.19) and (3.20). These constraints set up uniquely the coefficients C_4, C_5, C_6, and C_7,

$$\begin{aligned}
C_4 &= -C_3\langle\partial_E\psi,\chi\rangle_{L^2} + C_1\|\chi\|^2_{L^2} - 2p\big\langle\chi^2,\phi^{2p-1}\Theta\big\rangle_{L^2} - p(2p-1)\big\langle\chi^2,\phi^{2p-2}\psi^2\big\rangle_{L^2}\\
&\quad - 2p\big\langle\chi\psi,\phi^{2p-1}\Gamma\big\rangle_{L^2},\\
C_5 &= C_2\|\chi\|^2_{L^2} - 2p\big\langle\chi^2,\phi^{2p-1}\Delta\big\rangle_{L^2} - p\big\langle\chi^2,\phi^{2p-2}\chi^2\big\rangle_{L^2},\\
C_6 &= -C_1\|\psi\|^2_{L^2} + 2p(2p+1)\big\langle\psi^2,\phi^{2p-1}\Theta\big\rangle_{L^2} + \frac{1}{3}p(2p+1)(2p-1)\big\langle\psi^2,\phi^{2p-2}\psi^2\big\rangle_{L^2},\\
C_7 &= -C_3\langle\partial_E\chi,\psi\rangle_{L^2} - C_2\|\psi\|^2_{L^2} + 2p(2p+1)\big\langle\psi^2,\phi^{2p-1}\Delta\big\rangle_{L^2} + 2p\big\langle\psi\chi,\phi^{2p-1}\Gamma\big\rangle_{L^2}\\
&\quad + p(2p-1)\big\langle\psi^2,\phi^{2p-2}\chi^2\big\rangle_{L^2}.
\end{aligned} \tag{3.27}$$

The rest of the proof is similar to that of Lemma 4. □

Finally, we estimate the error terms $\tilde{R}_\theta$, $\tilde{R}_E$, $\tilde{R}_A$, and $\tilde{R}_B$ in the modulation equations (3.7).

Lemma 6. *There is $C > 0$ such that*

$$|\tilde{R}_{\theta,E,A,B}| \leqslant C\big(\big(A^2+B^2\big)^2 + (A+B)\big(\|\tilde{U}\|_{L^2} + \|\tilde{W}\|_{L^2}\big) + \|\tilde{U}\|^2_{L^2} + \|\tilde{W}\|^2_{L^2}\big). \tag{3.28}$$

Proof. We recall from (3.3) and (3.4) that

$$\begin{bmatrix} R_\theta \\ R_E \end{bmatrix} = \big(\langle\partial_E\phi,\phi\rangle^{-1}_{L^2} + \mathcal{O}\big(\|U\|_{L^2} + \|W\|_{L^2}\big)\big)\begin{bmatrix} -\langle\partial_E\phi, N_+(A\psi+U, B\chi+W)\rangle_{L^2} \\ \langle\phi, N_-(A\psi+U, B\chi+W)\rangle_{L^2} \end{bmatrix}. \tag{3.29}$$

By using the symmetry properties of ϕ, ψ, and χ, as well as Lemmas 4 and 5, we get

$$\begin{aligned}
\big\langle\phi, N_-(A\psi+U, B\chi+W)\big\rangle_{L^2} &= \langle\partial_E\phi,\phi\rangle_{L^2} C_3 AB + \langle\phi,\tilde{N}_-\rangle_{L^2},\\
-\big\langle\partial_E\phi, N_+(A\psi+U, B\chi+W)\big\rangle_{L^2} &= \langle\partial_E\phi,\phi\rangle_{L^2}\big[C_1A^2 + C_2B^2\big] - \langle\partial_E\phi,\tilde{N}_+\rangle,
\end{aligned} \tag{3.30}$$

where $C_1(E)$, $C_2(E)$, and $C_3(E)$ are defined in (3.22). It then follows from (3.7), (3.14), (3.29), and (3.30) that $\tilde{R}_E$ and $\tilde{R}_\theta$ satisfy (3.28).

The computation of the terms $\tilde{R}_A$ and $\tilde{R}_B$ can be done exactly the same way using (3.5) and the above estimates on $\tilde{R}_E$ and $\tilde{R}_\theta$. □

3.2. Analysis of the remainder terms

We consider the remainder terms $\tilde{U}$ and $\tilde{W}$ satisfying the time evolution equations (3.18). Let us denote $\tilde{\mathbf{Z}} = (\tilde{U}, \tilde{W})$, $\tilde{\mathbf{R}} = (\tilde{R}_U, -\tilde{R}_W)$, and rewrite this system in the matrix-vector notations as

$$\partial_t\tilde{\mathbf{Z}} = \mathcal{L}(E)\tilde{\mathbf{Z}} + \tilde{\mathbf{R}}(E, A, B, \tilde{\mathbf{Z}}), \tag{3.31}$$

where operator $\mathcal{L}(E)$ is defined by (2.6).

Let $P_c(E)$ be the projection operator associated to the complement of the four-dimensional subspace spanned by

$$\mathcal{W}(E) := \operatorname{span}\left\{\begin{bmatrix}\phi(E)\\0\end{bmatrix}, \begin{bmatrix}0\\\partial_E\phi(E)\end{bmatrix}, \begin{bmatrix}0\\\psi(E)\end{bmatrix}, \begin{bmatrix}\chi(E)\\0\end{bmatrix}\right\}. \tag{3.32}$$

Compared to $\mathcal{U}(E_*)$ in (2.16), the subspace $\mathcal{W}(E_*)$ in (3.32) is associated with the generalized kernel of the adjoint linearization operator $\mathcal{L}^*(E_*)$. The remainder terms (U, W) are required to satisfy the conditions of symplectic orthogonality (2.18), which imply that (U, W) belong to the orthogonal complement of $\mathcal{W}(E)$.

We shall single out quartic terms in (A, B) from the nonlinear term of the system (3.31). To do so, we expand

$$\tilde{\mathbf{R}} = \sum_{i+j=4} \mathbf{f}_{ij}(E) A^i B^j + \hat{\mathbf{F}}(E, A, B, \tilde{\mathbf{Z}}), \tag{3.33}$$

where the C^1 functions $\mathbf{f}_{ij}(E) = P_c^*(E)\mathbf{f}_{ij}(E) \in H^2(\mathbb{R})$ can be computed explicitly near $E = E_*$, whereas the function $\hat{\mathbf{F}}(E, A, B, \tilde{\mathbf{Z}})$ satisfies the bounds

$$\exists C_s > 0: \quad \|\hat{\mathbf{F}}\|_{H^s} \leqslant C_s\big(|A|^5 + |B|^5 + \big(|A| + |B|\big)\|\tilde{\mathbf{Z}}\|_{H^s} + \|\tilde{\mathbf{Z}}\|_{H^s}^2\big), \tag{3.34}$$

for any $s > \frac{1}{2}$, recalling that $H^s(\mathbb{R})$ is a Banach algebra for any $s > \frac{1}{2}$. Without loss of generality, we can work for $s = 1$.

Now, we mirror the decomposition (3.33) and expand $\tilde{\mathbf{Z}}$ as

$$\tilde{\mathbf{Z}} = \sum_{i+j=4} \mathbf{z}_{ij}(E) A^i B^j + \hat{\mathbf{Z}}(E, A, B), \tag{3.35}$$

where the C^1 functions $\mathbf{z}_{ij}(E) = P_c(E)\mathbf{z}_{ij}(E) \in H^2(\mathbb{R})$ can be computed explicitly near $E = E_*$. The new variable satisfies

$$\partial_t \hat{\mathbf{Z}} = \mathcal{L}(E)\hat{\mathbf{Z}} + \hat{\mathbf{R}}(E, A, B, \hat{\mathbf{Z}}), \tag{3.36}$$

where the residual term $\hat{\mathbf{R}}$ is computed from $\hat{\mathbf{F}}$ similar to how $\tilde{\mathbf{R}}$ is computed from $\tilde{\mathbf{F}}$. Therefore, the residual term satisfies the bound

$$\exists C > 0: \quad \|\hat{\mathbf{R}}\|_{H^1} \leqslant C\big(|A|^5 + |B|^5 + \big(|A| + |B|\big)\|\hat{\mathbf{Z}}\|_{H^1} + \|\hat{\mathbf{Z}}\|_{H^1}^2\big). \tag{3.37}$$

Because E depends on t, the spectral projections associated to the linearized operator $\mathcal{L}(E)$ are time-dependent. Since E is close to E_*, we can fix the value E_* before writing the time evolution problem (3.36) in the Duhamel form. In other words, we first rewrite (3.36) as

$$\partial_t \hat{\mathbf{Z}} = P_c^*(E_*)\mathcal{L}(E_*)P_c(E_*)\hat{\mathbf{Z}} + H_R(E)\hat{\mathbf{Z}} + \hat{\mathbf{R}}(E, A, B, \hat{\mathbf{Z}}), \tag{3.38}$$

where $H_R(E) = P_c^*(E)\mathcal{L}(E)P_c(E) - P_c^*(E_*)\mathcal{L}(E_*)P_c(E_*)$ is a 2×2 matrix-valued function. Thanks to C^1 smoothness of this function in E and x, it enjoys the bound

$$\exists C > 0: \quad \big\|H_R(E)\big\|_{C^1} \leqslant C|E - E_*|. \tag{3.39}$$

Then we use the Duhamel principle and write

$$\begin{aligned} \hat{\mathbf{Z}}(t) &= P_c^*(E_*)e^{t\mathcal{L}(E_*)}P_c(E_*)\hat{\mathbf{Z}}(0) \\ &\quad + P_c^*(E_*)\int_0^t e^{(t-\tau)\mathcal{L}(E_*)}P_c(E_*)\big[H_R\big(E(\tau)\big)\hat{\mathbf{Z}}(\tau) + \hat{\mathbf{R}}\big(E(\tau), A(\tau), B(\tau), \hat{\mathbf{Z}}(\tau)\big)\big]\,d\tau. \end{aligned} \tag{3.40}$$

Under assumptions $\lambda'(E_*) \neq 0$ and $N_s'(E_*) \neq 0$, the linearized operator $\mathcal{L}(E_*)$ has zero eigenvalue of algebraic multiplicity *four* and the rest of the spectrum is purely imaginary and bounded away from zero. Therefore, operator $P_c^*(E_*)e^{t\mathcal{L}(E_*)}P_c(E_*)$ forms a semi-group from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ for any $s \geqslant 0$ and there is $C_s > 0$ such that

$$\left\| P_c^*(E_*)e^{t\mathcal{L}(E_*)}P_c(E_*)\right\|_{H^s \to H^s} \leqslant C_s. \tag{3.41}$$

Local existence and uniqueness of solutions $\hat{\mathbf{Z}}(t)$ of the integral equation (3.40) follows for any $t \in [0, t_0]$, where $t_0 > 0$ is sufficiently small for fixed-point arguments. The solution can be continued over the maximal existence interval using standard continuation methods. We shall now use Gronwall's inequality to control $\|\hat{\mathbf{Z}}(t)\|_{H^1}$ over $t \in [0, T]$, where T is bounded by (3.10) and (A, B) belong to the domain (3.9).

Using (3.37), (3.39), (3.40), and (3.41), we obtain that for any $t \in [0, T]$, there is a positive constant C such that

$$\begin{aligned}\left\|\hat{\mathbf{Z}}(t)\right\|_{H^1} \leqslant{}& C\left\|\hat{\mathbf{Z}}(0)\right\|_{H^1} + C\int_0^t \left|E(\tau) - E_*\right|\left\|\hat{\mathbf{Z}}(\tau)\right\|_{H^1} d\tau + C\int_0^t \left|A(\tau)\right|\left\|\hat{\mathbf{Z}}(\tau)\right\|_{H^1} d\tau \\ &+ CT|\Delta N|^{5/2} + C\int_0^t \left\|\hat{\mathbf{Z}}(\tau)\right\|_{H^1}^2 d\tau.\end{aligned}$$

By Gronwall's inequality, we have

$$\sup_{t\in[0,T]}\left\|\hat{\mathbf{Z}}(t)\right\|_{H^1} \leqslant C\left(\left\|\hat{\mathbf{Z}}(0)\right\|_{H^1} + T|\Delta N|^{5/2} + T\sup_{t\in[0,T]}\left\|\hat{\mathbf{Z}}(t)\right\|_{H^1}^2\right)e^{C\int_0^T(|E(\tau)-E_*|+|A(\tau)|)\,d\tau}.$$

If A, T, and E are estimated by (3.9), (3.10), and (3.11) respectively, the last exponential term is bounded as $|\Delta N| \to 0$. Elementary continuation arguments give that if $\|\hat{\mathbf{Z}}(0)\|_{H^1} \leqslant C_0(\Delta N)^2$, then there is $C > 0$ such that

$$\sup_{t\in[0,T]}\left\|\hat{\mathbf{Z}}(t)\right\|_{H^1} \leqslant C(\Delta N)^2, \quad t \in [0, T],$$

or, by virtue of Eqs. (3.9) and (3.35), there is $C > 0$ such that

$$\sup_{t\in[0,T]}\left\|\tilde{\mathbf{Z}}(t)\right\|_{H^1} \leqslant C(\Delta N)^2, \quad t \in [0, T]. \tag{3.42}$$

Bound (3.42) provides the proof of the estimate (3.12). The estimate (3.13) follows from (3.28) and (3.42). All together, the proof of Lemma 3 is complete.

3.3. Conserved quantities

To complete the proof of Theorem 3, we need to show that the trajectories of the system (3.7) remain in the domain (3.9) for $t \in [0, T]$ and satisfy the estimates (3.10) and (3.11).

Estimates (3.11) follow from the first two equations of the system (3.7) in the domain (3.9) under the estimate (3.13) on the error terms and the estimate (3.10) on the maximal time T. Therefore, we shall only prove the estimates (3.9) and (3.10). To do so, we work with the last two equations of the system (3.7) and employ the conserved quantities (2.25) and (2.26).

Expanded at the quadratic terms in (A, B), the conserved quantity for $\mathcal{N}_0$ becomes

$$\mathcal{N}_0 = N_s(E) + A^2\|\psi\|_{L^2}^2 + B^2\|\chi\|_{L^2}^2 + \mathcal{O}\big(\big(A^2+B^2\big)^2 + (A+B)\big(\|\tilde U\|_{L^2} + \|\tilde W\|_{L^2}\big) + \|\tilde U\|_{L^2}^2 + \|\tilde W\|_{L^2}^2\big),$$

where (A, B) are defined in the domain (3.9) and the terms involving $(\tilde U, \tilde W)$ are controlled by the bound (3.12) to be of the higher order than the terms involving (A, B). To simplify our notations, we shall then rewrite $\mathcal{N}_0$ simply as

$$\mathcal{N}_0 = N_s(E) + A^2\|\psi\|_{L^2}^2 + B^2\|\chi\|_{L^2}^2 + \mathcal{O}\big(A^2+B^2\big)^2. \tag{3.43}$$

Computing the derivative of (3.43) in time and using system (3.7) up to the quadratic order, we obtain

$$N_s'(E)C_3(E) + 2\big(\|\psi(E)\|_{L^2}^2 + \Lambda^2(E)\|\chi(E)\|_{L^2}^2\big) = 0,$$

which is identically satisfied thanks to the identity

$$\|\psi\|_{L^2}^2 + \Lambda^2\|\chi\|_{L^2}^2 = \langle\psi, L_-\chi\rangle_{L^2} - \langle\chi, L_+\psi\rangle_{L^2} = \big\langle\psi, (L_- - L_+)\chi\big\rangle_{L^2}$$
$$= 2p\big\langle\psi, \phi^{2p}\chi\big\rangle_{L^2} = -\frac{1}{2}N_s'(E)C_3(E). \tag{3.44}$$

Expanded at the quadratic terms in (A, B), the conserved quantity for $\mathcal{H}_0$ becomes

$$\mathcal{H}_0 = H_s(E) + A^2\int_{\mathbb{R}}\big(\psi_x^2 + V\psi^2 - (2p+1)\phi^{2p}\psi^2\big)\,dx$$
$$+ B^2\int_{\mathbb{R}}\big(\chi_x^2 + V\chi^2 - \phi^{2p}\chi^2\big)\,dx + \mathcal{O}\big(A^2+B^2\big)^2. \tag{3.45}$$

Using the system (2.13) and the normalization $\langle\psi, \chi\rangle_{L^2} = 1$, we obtain

$$\int_{\mathbb{R}}\big(\psi_x^2 + V\psi^2 - (2p+1)\phi^{2p}\psi^2\big)\,dx = -\Lambda^2(E) - E\|\psi\|_{L^2}^2,$$
$$\int_{\mathbb{R}}\big(\chi_x^2 + V\chi^2 - \phi^{2p}\chi^2\big)\,dx = 1 - E\|\chi\|_{L^2}^2.$$

Using (3.43), we can further simplify (3.45) to the form

$$\mathcal{H}_0 = H_s(E) + E\big(N_s(E) - \mathcal{N}_0\big) - \Lambda^2(E)A^2 + B^2 + \mathcal{O}\big(A^2+B^2\big)^2. \tag{3.46}$$

We can now extend the conserved quantity $\mathcal{H}_0$ up to the quartic terms and write it abstractly as

$$\mathcal{H}_0 = H_s(E) + E\big(N_s(E) - \mathcal{N}_0\big) - \Lambda^2(E)A^2 + B^2$$
$$+ \frac{1}{2}D_1(E)A^4 + D_2(E)A^2B^2 + \frac{1}{2}D_3(E)B^4 + \mathcal{O}\big(A^2+B^2\big)^3, \tag{3.47}$$

where D_1, D_2, and D_3 are some coefficients, which can be computed explicitly. To avoid lengthy computations, we shall compute these coefficients from the derivative of (3.47) in time and using the identity (2.27) and the system (3.7) up to the quartic order. This procedure yields two relations between three coefficients D_1, D_2, and D_3,

$$D_1 + \Lambda^2 D_2 = \frac{1}{2} C_3 \big(\|\psi\|_{L^2}^2 + \partial_E \Lambda^2 \big) + C_4 \Lambda^2 - C_6,$$

$$D_2 + \Lambda^2 D_3 = \frac{1}{2} C_3 \|\chi\|_{L^2}^2 + C_5 \Lambda^2 - C_7.$$

Since $\Lambda^2(E_*) = 0$, coefficients $D_1(E_*)$ and $D_2(E_*)$ are determined uniquely from this system. In particular, using (2.11), (2.15), (3.22), and (3.44), we compute

$$\begin{aligned} D_1(E_*) &= -2p^2(2p+1) \frac{\langle \partial_E \phi_*, \phi_*^{2p-1} \psi_*^2 \rangle_{L^2} \langle \psi_*, \phi_*^{2p} \chi_* \rangle_{L^2}}{\langle \partial_E \phi_*, \phi_* \rangle_{L^2}} - C_6(E_*) \\ &= -p(2p+1) \frac{\langle \partial_E \phi_*, \phi_*^{2p-1} \psi_*^2 \rangle_{L^2} \|\psi_*\|_{L^2}^2}{\langle \partial_E \phi_*, \phi_* \rangle_{L^2}} - C_6(E_*). \end{aligned} \tag{3.48}$$

We have now all the ingredients to complete the proof of Theorem 3.

3.4. Analysis of dynamics as $E \to E_*$

Hamiltonian system (3.7) equipped with conserved quantities (3.43) and (3.47) is integrable in the sense of the Liouville up to the error terms controlled by Lemma 3. Using the conserved quantity (3.43) and the assumption that $N_s'(E_*) \neq 0$, we can exclude the variable E near $E = E_*$. Then, using the conserved quantity (3.47), we can plot the trajectories of the system (3.7) on the phase plane (A, B) and show the topological equivalence of the phase portraits of the system (3.7) to those of the second-order system (1.11).

We now proceed with the phase plane analysis for the system (3.7). We denote $\Delta N = \mathcal{N}_0 - N_s(E_*)$ and $\Delta H = \mathcal{H}_0 - H_s(E_*)$. Assuming that ΔN is small, we work in the domain (3.9) and use the expansions (3.43) and (3.47) to obtain

$$\Delta N = N_s'(E_*)(E - E_*) + A^2 \|\psi_*\|_{L^2}^2 + B^2 \|\chi_*\|_{L^2}^2 + \mathcal{O}\big((E - E_*)^2, (E - E_*)\big(A^2 + B^2\big), \big(A^2 + B^2\big)^2 \big)$$

and

$$\begin{aligned} \Delta H &= H_s(E) - H_s(E_*) + E\big(N_s(E) - N_s(E_*) - \Delta N \big) \\ &\quad - \Lambda^2(E) A^2 + B^2 + \frac{1}{2} D_1(E) A^4 + D_2(E) A^2 B^2 + \frac{1}{2} D_3(E) B^4 + \mathcal{O}\big(\big(A^2 + B^2\big)^6 \big) \\ &= -E \Delta N + \frac{1}{2} N_s'(E_*)(E - E_*)^2 + \lambda'(E_*) \|\psi_*\|_{L^2}^2 (E - E_*) A^2 + B^2 + \frac{1}{2} D_1(E_*) A^4 + D_2(E_*) A^2 B^2 \\ &\quad + \frac{1}{2} D_3(E_*) B^4 + \mathcal{O}\big((E - E_*)^3, (E - E_*)^2 \big(A^2 + B^2\big), (E - E_*)\big(A^2 + B^2\big)^2, \big(A^2 + B^2\big)^3 \big), \end{aligned}$$

where we have used (2.15) and (2.27). Note that for clarity of writing, we require in the two expansions above that all functions be C^2 near $E = E_*$. This property holds by the bootstrapping arguments for analytic nonlinearities with $p \in \mathbb{N}$.

The first conserved quantity for ΔN is useful to eliminate E in the domain (3.9) by

$$N_s'(E_*)(E - E_*) = \Delta N - A^2 \|\psi_*\|_{L^2}^2 - B^2 \|\chi_*\|_{L^2}^2 + \mathcal{O}(\Delta N)^2. \tag{3.49}$$

The second conserved quantity for ΔH can now be written in the form

$$G = \frac{(\Delta N)}{N_s'(E_*)}\lambda'(E_*)\|\psi_*\|_{L^2}^2 A^2 + B^2 + \frac{1}{2}F_1 A^4 + F_2 A^2 B^2 + \frac{1}{2}F_3 B^4 + \mathcal{O}(\Delta N)^3, \tag{3.50}$$

where

$$G = \Delta H + E_* \Delta N + \frac{(\Delta N)^2}{2N_s'(E_*)},$$
$$F_1 = \frac{\|\psi_*\|_{L^2}^4 (1 - 2\lambda'(E_*))}{N_s'(E_*)} + D_1(E_*),$$
$$F_2 = \frac{\|\psi_*\|_{L^2}^2 \|\chi_*\|_{L^2}^2 (1 - \lambda'(E_*))}{N_s'(E_*)} + D_2(E_*),$$
$$F_3 = \frac{\|\chi_*\|_{L^2}^4}{N_s'(E_*)} + D_3(E_*).$$

Using (1.5), (1.8), (2.15), and (3.48), we obtain

$$F_1 = -\frac{\mathcal{QS}}{N_s'(E_*)}.$$

In the domain (3.9), where $A^2 = \mathcal{O}(|\Delta N|)$ and $B^2 = \mathcal{O}(|\Delta N|^2)$, G can be rewritten in the form

$$G = \frac{(\Delta N)}{N_s'(E_*)}\lambda'(E_*)\|\psi_*\|_{L^2}^2 A^2 + B^2 - \frac{\mathcal{QS}}{2N_s'(E_*)} A^4 + \mathcal{O}\big(|\Delta N|^3\big). \tag{3.51}$$

This quantity is time-preserving for any $t \in [0, T]$. Dropping the error term $\mathcal{O}(|\Delta N|^3)$ in (3.51) produces the truncated system (1.11). We can now look at four different cases in the dynamics of the truncated system (1.11) for any $t \in [0, T]$. Recall that $\lambda'(E_*) < 0$, $N_s'(E_*) > 0$, and $\mathcal{Q} < 0$ by the assumptions of Theorem 3 (see also Appendix A).

3.4.1. Case $\Delta N > 0$ and $\mathcal{S} > 0$

It follows from (3.51) that the critical point $(A, B) = (0, 0)$ is a saddle point of G if $\Delta N > 0$ (recall that $\lambda'(E_*) < 0$). The level set $G = 0$ gives a curve on the phase plane (A, B) given by

$$B^2 = \frac{(\Delta N)}{N_s'(E_*)}\big|\lambda'(E_*)\big|\|\psi_*\|_{L^2}^2 A^2 + \frac{\mathcal{QS}}{N_s'(E_*)} A^4 + \mathcal{O}\big(|\Delta N|^3\big). \tag{3.52}$$

This curve contains the point $(0, 0)$ up to the terms of $\mathcal{O}(|\Delta N|^3)$.

If $\mathcal{Q} < 0$ and $\mathcal{S} > 0$, the curve $G = 0$ consists of two symmetric loops on the plane (A, B) that enclose the points $(\pm A_*, 0)$, where G is minimal. An elementary computation shows that

$$A_*^2 = \frac{\lambda'(E_*)\|\psi_*\|_{L^2}^2}{\mathcal{QS}}(\Delta N) + \mathcal{O}\big(|\Delta N|^2\big). \tag{3.53}$$

The points $(\pm A_*, 0)$ correspond to the stable asymmetric states $\varphi_\pm$, whereas the point $(0, 0)$ corresponds to the unstable symmetric state ϕ. Appendix B reviews the results of the stationary normal form equation that recovers the critical points $(0, 0)$ and $(\pm A_*, 0)$.

The critical points $(\pm A_*, 0)$ are minima of G (center points). Therefore, they are surrounded by continuous families of periodic orbits on the phase plane (A, B). Periodic orbits fill the domain enclosed by the two loops of the level $G = 0$. There are also periodic orbits for $G > 0$ that surround all three critical points $(\pm A_*, 0)$ and $(0, 0)$. Thus, the phase portrait of the full system (3.7) is topologically equivalent to the one on Fig. 2 (top left) for the truncated system (1.11).

3.4.2. Case $\Delta N > 0$ and $\mathcal{S} < 0$

The critical point $(A, B) = (0, 0)$ is again a saddle point of G if $\Delta N > 0$. It corresponds to the unstable symmetric state ϕ. No other critical points of G for small A and $B = 0$ exist if $\mathcal{S} < 0$. Therefore, trajectories on the phase plane (A, B) near $(0, 0)$ are all hyperbolic and they leave the neighborhood of $(0, 0)$ in a finite time, except for the stable manifolds. The phase portrait of the full system (3.7) is topologically equivalent to the one on Fig. 2 (top right) for the truncated system (1.11).

3.4.3. Case $\Delta N < 0$ and $\mathcal{S} > 0$

It follows from (3.51) that the critical point $(A, B) = (0, 0)$ is a minimum of G if $\Delta N < 0$. It corresponds to the stable symmetric state ϕ. No other critical points of G for small A and $B = 0$ exist if $\mathcal{S} > 0$. Therefore, $(0, 0)$ is a center point, which is surrounded by a continuous family of periodic orbits on the phase plane (A, B). The phase portrait of the full system (3.7) is topologically equivalent to the one on Fig. 2 (bottom left) for the truncated system (1.11).

3.4.4. Case $\Delta N < 0$ and $\mathcal{S} < 0$

The critical point $(A, B) = (0, 0)$ is again a minimum of G if $\Delta N < 0$. If $\mathcal{S} < 0$, there exist two symmetric maxima of G at the points $(\pm A_*, 0)$, where A_*^2 is given by (3.53). The points $(\pm A_*, 0)$ correspond to the unstable asymmetric states $\varphi_\pm$, whereas the point $(0, 0)$ corresponds to the stable symmetric state ϕ. Because the points $(\pm A_*, 0)$ are saddle points, the level curve of G at $A = A_*$ and $B = 0$ prescribes a pair of heteroclinic orbits connecting $(\pm A_*, 0)$. The pair of heteroclinic orbits encloses a continuous family of closed curves surrounding the center point $(0, 0)$ on the phase plane and corresponding to periodic orbits. The phase portrait of the full system (3.7) is topologically equivalent to the one on Fig. 2 (bottom right) for the truncated system (1.11).

3.5. The end of the proof of Theorem 3

All nontrivial solutions of the system of modulation equations (3.7) in the domain (3.9) are topologically equivalent to the ones given by the truncated system (1.11). Bound (3.10) on the maximal time $T > 0$, during which the solutions remain in the domain (3.9), follows directly from the integration of the system (1.11) over the time $t \in [0, T]$. Other bounds of Theorem 3 follow from Lemma 3. The proof of Theorem 3 is now complete.

Appendix A. Large separation of potential wells

The case of large separation of potential wells, when parameter s in the double-well potential (1.2) is large, gives a good example of explicit computations of numerical coefficients $\mathcal{Q}$ and $\mathcal{S}$. Knowing these numerical coefficients enables the explicit classification of the stationary states in Theorem 1 and 2 and verifies the conditions of Theorem 3. The following lemma gives the asymptotic result when $s \to \infty$.

Lemma 7. *Let V be given by (1.2). There exists a sufficiently large, positive s_0 such that for all $s > s_0$, we have $N_s'(E_*) > 0$, $\lambda'(E_*) < 0$, $\mathcal{Q} < 0$, and*

$$\mathcal{S} > 0 \quad \text{if } p < p_*, \quad \text{and} \quad \mathcal{S} < 0 \quad \text{if } p > p_*,$$

where p_* is the positive root of the equation $1 + 3p - p^2 = 0$.

Proof. We recall from [10] that

$$E_* \to E_0 \quad \text{and} \quad \psi_*^2 \to C_* \phi_*^2 \quad \text{in } L^\infty(\mathbb{R}) \text{ as } s \to \infty,$$

where $C_* > 0$ is a normalization constant. Using the exact identity

$$L_+^{-1}(E)\phi^{2p+1} = -\frac{1}{2p}\phi,$$

we can hence simplify the expression (1.5) to the form

$$\mathcal{Q} \to -\frac{4}{3}p(p+1)(2p+1)C_*^2 \|\phi_*\|_{L^{2p+2}}^{2p+2} \quad \text{as } s \to \infty. \tag{A.1}$$

Therefore, $\mathcal{Q} < 0$ for any $p \in \mathbb{N}$ if s is sufficiently large.

Because $E_* \to E_0$, we can approximate ϕ_* and $\partial_E \phi_*$ using the small-amplitude expansion for the symmetric states of the stationary equation (1.3),

$$\phi(x; E) = a\phi_0(x) + \mathcal{O}\big(a^{1+2p}\big), \qquad E = E_0 + a^{2p}\frac{\|\phi_0\|_{L^{2p+2}}^{2p+2}}{\|\phi_0\|_{L^2}^2} + \mathcal{O}\big(a^{4p}\big),$$

where $\phi_0 \in H^2(\mathbb{R})$ is the eigenfunction of the operator $L_0 = -\partial_x^2 + V(x)$ for the lowest eigenvalue $-E_0$ and $a \in \mathbb{R}$ is a small parameter of the expansion. As a result, we obtain

$$\big\langle \partial_E \phi_*, \phi_*^{2p+1} \big\rangle_{L^2} \to \frac{1}{2p}\|\phi_*\|_{L^2}^2, \quad \text{as } s \to \infty, \tag{A.2}$$

and

$$N_s'(E_*) = 2\langle \partial_E \phi_*, \phi_* \rangle_{L^2} \to \frac{1}{p}\frac{\|\phi_*\|_{L^2}^4}{\|\phi_*\|_{L^{2p+2}}^{2p+2}}, \quad \text{as } s \to \infty. \tag{A.3}$$

In addition, it follows from (2.11) and (A.2) that

$$\lambda'(E_*) \to -2p \quad \text{as } s \to \infty. \tag{A.4}$$

Therefore, $N_s'(E_*) > 0$ and $\lambda'(E_*) < 0$ as $s \to \infty$.

Substituting (A.1)–(A.4) into the expression (1.8), we obtain

$$\mathcal{S} \to \frac{(1 + 3p - p^2)\|\phi_*\|_{L^2}^4}{p(1+p)(1+2p)\|\phi_*\|_{L^{2p+2}}^{2p+2}} \quad \text{as } s \to \infty. \tag{A.5}$$

Therefore, $\mathcal{S} > 0$ for $p < p_*$ and $\mathcal{S} < 0$ for $p > p_*$, where p_* is given by the positive root of $1 + 3p - p^2 = 0$, that is, by (1.9). □

Appendix B. Stationary normal form equation

Recall that in Section 2, we have only used the statement of Theorem 1(i) from the results of Kirr et al. [10]. We show here how to recover the results of Theorems 1(ii) and 2 on the existence and stability of stationary states from the system of time evolution equations (2.19)–(2.24).

Theorems 1 and 2 were originally proved in [10] with the Lyapunov–Schmidt decomposition method that relies on an orthogonal decomposition with respect to the self-adjoint operator $L_+(E)$. On the other hand, the decomposition used in the derivation of the system of time evolution equations (2.19)–(2.24) relies on the symplectic orthogonality conditions (2.18). Therefore, it is important to establish that the system of modulation equations considered in our paper provides the same conclusions as Theorems 1(ii) and 2 do.

We start with the simplification of the system (2.19)–(2.24) for stationary solutions of the NLS equation (1.1). Because of the symplectic orthogonality conditions (2.18), we define the constrained H^2-space

$$H^2_E = \{U \in H^2(\mathbb{R}): \langle \phi(E), U\rangle_{L^2} = \langle \chi(E), U\rangle_{L^2} = 0\}, \tag{B.1}$$

where the subscript indicates that the orthogonal projections are E-dependent. The stationary solutions of the system (2.19)–(2.24) near $E = E_*$ are described by the following lemma.

Lemma 8. *Assume that $N_s'(E_*) \neq 0$. There exists a sufficiently small, positive ϵ such that for all $|E - E_*| < \epsilon$, the nonlinear Schrödinger equation (1.1) admits a stationary solution*

$$\Psi = e^{it\mathcal{E}}\big[\phi(x;E) + A\psi(x;E) + U\big], \tag{B.2}$$

where

$$\mathcal{E} = E - \frac{\langle \partial_E \phi, N_+(A\psi + U, 0)\rangle_{L^2}}{\langle \partial_E\phi, \phi + U\rangle_{L^2}}, \tag{B.3}$$

while $U \in H^2_E$ and $A \in \mathbb{R}$ are uniquely defined from the implicit equations

$$L_+(E)U = G(A,E,U), \qquad \langle \psi(E), G(A,E,U)\rangle_{L^2} = 0 \tag{B.4}$$

with

$$G(A,E,U) = \Lambda^2 A\chi + \frac{\langle \partial_E \phi, N_+(A\psi + U, 0)\rangle_{L^2}}{\langle \partial_E\phi, \phi + U\rangle_{L^2}}(\phi + A\psi + U) - N_+(A\psi + U, 0). \tag{B.5}$$

Moreover, U and A^2 are C^1 functions near $E = E_*$ and there exist positive constants C_1 and C_2 such that

$$\|U\|_{H^2} \leqslant C_1|E - E_*|, \qquad A^2 \leqslant C_2|E - E_*|. \tag{B.6}$$

Proof. For real-valued stationary solutions, we can set $B = 0$ and $W = 0$ in the system (2.19)–(2.24), which give $w = 0$ and $N_-(A\psi + U, 0) = 0$. The modulation equations (2.21)–(2.24) become degenerate and can be rewritten in the form

$$\begin{gathered}
\langle \partial_E\phi, \phi - U\rangle_{L^2}\dot{E} = 0,\\
\langle \partial_E\phi, \phi + U\rangle_{L^2}(\dot{\theta} - E) = -\langle \partial_E\phi, N_+(A\psi + U, 0)\rangle_{L^2},\\
\dot{A} + \dot{E}\big(-\langle \partial_E\chi, U\rangle_{L^2} + A\langle \partial_E\psi, \chi\rangle_{L^2}\big) = 0,\\
-\Lambda^2 A + (\dot{\theta} - E)\big(A\|\psi\|^2_{L^2} + \langle \psi, U\rangle_{L^2}\big) = -\langle \psi, N_+(A\psi + U, 0)\rangle_{L^2}.
\end{gathered}$$

Assuming $N_s'(E_*) \neq 0$ and the smallness of $\|U\|_{L^2}$ for small $|E - E_*|$, we get $\langle \partial_E \phi, \phi - U \rangle_{L^2} \neq 0$ for small $|E - E_*|$. Hence, from the first equation, it follows that $\dot{E} = 0$. Then, from the third equation, we infer that $\dot{A} = 0$. The second equation gives (B.3) with the correspondence $\dot{\theta} = \mathcal{E}$. The fourth equation gives $\langle \psi(E), G(A, E, U) \rangle_{L^2} = 0$, where G is defined by (B.5) and the normalization $\langle \psi(E), \chi(E) \rangle_{L^2} = 1$ is used.

The time evolution system (2.19)–(2.20) implies that U becomes time-independent and satisfies the stationary equation $L_+(E)U = G(A, E, U)$. Thus, the system (B.4) is verified. It remains to prove the existence and uniqueness of small C^1 solutions U and A^2 of this system satisfying the estimates (B.6) for small $|E - E_*|$.

Since $L_+(E_*)\psi_* = 0$ and $\psi(E) \to \psi_*$ in $L^2(\mathbb{R})$ as $E \to E_*$, the Implicit Function Theorem gives the existence of a unique local map

$$\mathbb{R}^2 \ni (A, E) \mapsto U \in H^2(\mathbb{R}) \quad \text{near } (A, E) = (0, E_*), \tag{B.7}$$

such that U satisfies equation $L_+(E)U = G(A, E, U)$ subject to the constraint $\langle \chi(E), U \rangle_{L^2} = 0$, provided that $\langle \psi(E), G(A, E, U) \rangle_{L^2} = 0$. Let $U_{A,E}$ denote this map for small A and $|E - E_*|$. The map is C^∞ if $p \in \mathbb{N}$.

In addition, it follows from equation

$$L_+(E)\partial_E \phi(E) = -\phi(E) \tag{B.8}$$

that $U_{A,E}$ satisfies $\langle \phi(E), U_{A,E} \rangle_{L^2} = 0$ under the constraint $\langle \partial_E \phi(E), G(A, E, U) \rangle_{L^2} = 0$, which is identically satisfied. Therefore, $U_{A,E} \in H_E^2$, according to the definition (B.1).

We note that $G(A, E, U)$ is quadratic in A as $A \to 0$. Therefore, we proceed with a near identity transformation for the map (B.7),

$$U_{A,E} = A^2 \Theta(x; E) + \mathcal{O}_{H^2}\big(A^3\big), \tag{B.9}$$

where $\Theta \in H_E^2$ is a unique solution of the inhomogeneous equation

$$L_+(E)\Theta = p(2p+1)\phi^{2p-1}\psi^2 - p(2p+1)\frac{\langle \partial_E \phi, \phi^{2p-1}\psi^2 \rangle_{L^2}}{\langle \partial_E \phi, \phi \rangle_{L^2}}\phi. \tag{B.10}$$

It remains to control the value of A in terms of the small value of $|E - E_*|$. Substituting (2.15) and (B.9) to equation $\langle \psi(E), G(A, E, U) \rangle_{L^2} = 0$, we obtain

$$\lambda'(E_*)\|\psi_*\|_{L^2}^2 (E - E_*)A - Q A^3 + \mathcal{O}\big(A^4, A^2(E - E_*), (E - E_*)^2\big) = 0, \tag{B.11}$$

where

$$Q = \frac{1}{3}p(2p+1)(2p-1)\big\langle \psi_*^2, \phi_*^{2p-2}\psi_*^2 \big\rangle_{L^2} + 2p(2p+1)\big\langle \psi_*^2, \phi_*^{2p-1}\Theta_* \big\rangle_{L^2}$$
$$- p(2p+1)\frac{\langle \partial_E \phi_*, \phi_*^{2p-1}\psi_*^2 \rangle_{L^2}}{\langle \partial_E \phi_*, \phi_* \rangle_{L^2}}\|\psi_*\|_{L^2}^2.$$

Therefore, either $A = 0$ (and $U_{A,E} \equiv 0$) or A is a nonzero root of Eq. (B.11) such that A^2 is C^1 near $E = E_*$ satisfying the second estimate (B.6). Thanks to the expansion (B.9), the first estimate (B.6) is also satisfied. This concludes the proof of the lemma. □

We will now show that the results following from the normal form equation (B.11) are equivalent to the results of Theorems 1(ii) and 2. Using (B.8), we let

$$\Theta = \theta + p(2p+1)\frac{\langle \partial_E \phi, \phi^{2p-1}\psi^2\rangle_{L^2}}{\langle \partial_E \phi, \phi\rangle_{L^2}} \partial_E \phi, \qquad \theta = p(2p+1)L_+^{-1}(E)\phi^{2p-1}\psi^2.$$

On the other hand, expanding (B.3) gives

$$\mathcal{E} = E + p(2p+1)A^2 \frac{\langle \partial_E \phi, \phi^{2p-1}\psi^2\rangle_{L^2}}{\langle \partial_E \phi, \phi\rangle_{L^2}} + \mathcal{O}\big(A^3\big). \tag{B.12}$$

Using (2.11), we conclude that the normal form equation (B.11) is equivalent to equation

$$\lambda'(E_*)\|\psi_*\|_{L^2}^2(\mathcal{E} - E_*)A - \mathcal{Q}A^3 + \mathcal{O}\big(A^4, A^2(\mathcal{E} - E_*), (\mathcal{E} - E_*)^2\big) = 0, \tag{B.13}$$

where $\mathcal{Q}$ is given by (1.5) and $\mathcal{E}$ is a renormalized parameter of the stationary state ϕ.

We can see from (B.13) that besides zero solution $A = 0$ that corresponds to the symmetric state $\phi(x; E)$, there are two nonzero solutions that correspond to the asymmetric states,

$$\varphi_\pm(x; \mathcal{E}) = \phi(x; E) \pm A\psi(x; E) + A^2\Theta(x; E) + \mathcal{O}_{H^2}\big(A^3\big), \tag{B.14}$$

where $\mathcal{E}$ is related to E by the expansion (B.12) and A is a positive root of the stationary normal form equation (B.13) provided that $\operatorname{sign}((\mathcal{E} - E_*)\mathcal{Q}) = -1$ if $\lambda'(E_*) < 0$. If $\mathcal{Q} < 0$, the asymmetric states exist for $\mathcal{E} > E_*$. If $\mathcal{Q} > 0$, the asymmetric states exist for $\mathcal{E} < E_*$. We have recovered the statement of Theorem 1(ii).

To consider the statement of Theorem 2, we need the following result.

Lemma 9. *Let $\varphi_\pm$ be the asymmetric states (B.14) that exist for* $\operatorname{sign}((\mathcal{E} - E_*)\mathcal{Q}) = -1$ *near $\mathcal{E} = E_*$ and define*

$$L_+(A) = -\partial_x^2 + V(x) - (2p+1)\varphi_+^{2p} + \mathcal{E}(A),$$

where $\mathcal{E}(A)$ is given by the expansion (B.13). Then, the second eigenvalue of $L_+(A)$ is positive for $\mathcal{Q} < 0$ and negative for $\mathcal{Q} > 0$.

Proof. Using (B.11) and (B.14), we expand $\varphi_\pm$ for small A by

$$\varphi_\pm = \phi_* \pm A\psi_* + A^2\left(\Theta_* + \frac{Q}{\lambda'(E_*)\|\psi_*\|_{L^2}^2}\partial_E \phi_*\right) + \mathcal{O}_{H^2}\big(A^3\big). \tag{B.15}$$

Let $h(A)$ be the eigenfunction of $L_+(A)$ for the second eigenvalue $\mu(A)$. We have $h(0) = \psi_*$ and $\mu(0) = 0$ is a simple eigenvalue, so that the asymptotic perturbation theory for simple eigenvalues of closed operators [7, Section 8.2.3] applies. Using (B.13) and (B.15), we compute

$$L_+'(0) = -2p(2p+1)\phi_*^{2p-1}\psi_*$$

and

$$L_+''(0) = -2p(2p+1)(2p-1)\phi_*^{2p-2}\psi_*^2 - 4p(2p+1)\phi_*^{2p-1}\psi_*\left(\Theta_* + \frac{Q}{\lambda'(E_*)\|\psi_*\|_{L^2}^2}\partial_E\phi_*\right) + \frac{2\mathcal{Q}}{\lambda'(E_*)\|\psi_*\|_{L^2}^2}.$$

Algorithmic computations yield $h'(0) = 2\theta_*$, $\mu'(0) = 0$, and, after tedious computations,

$$\mu''(0) = \frac{\langle L_+''(0)\psi_* + 4L_+'(0)\theta_*, \psi_*\rangle_{L^2}}{\|\psi_*\|_{L^2}^2} = -\frac{4\mathcal{Q}}{\|\psi_*\|_{L^2}^2}.$$

Therefore, $\mu(A) > 0$ for small A if $\mathcal{Q} < 0$ and $\mu(A) < 0$ for small A if $\mathcal{Q} > 0$. □

Let $N_s(E) = \|\phi(\cdot; E)\|_{L^2}^2$ and $N_a(\mathcal{E}) = \|\varphi_\pm(\cdot; \mathcal{E})\|_{L^2}^2$. Using Eqs. (B.12) and (B.13), we expand $N_a(\mathcal{E})$ in the power series,

$$\begin{aligned} N_a(\mathcal{E}) &= N_s(E) + A^2\|\psi\|_{L^2}^2 + \mathcal{O}(A^4) \\ &= N_s(E_*) + N_s'(E_*)(E - E_*) + A^2\|\psi_*\|_{L^2}^2 + \mathcal{O}\big((E-E_*)^2, (E-E_*)A^2, A^4\big) \\ &= N_s(E_*) + \mathcal{S}(\mathcal{E} - E_*) + \mathcal{O}(\mathcal{E} - E_*)^2, \end{aligned}$$

where $\mathcal{S}$ is given by (1.8). Assuming that $\mathcal{Q} < 0$, the asymmetric states exist for $\mathcal{E} > E_*$. If $\mathcal{S} > 0$, then $N_a(\mathcal{E})$ increases with $\mathcal{E}$, whereas if $\mathcal{S} < 0$, then $N_a(\mathcal{E})$ decreases with $\mathcal{E}$.

Orbital stability and instability of asymmetric stationary states follow from the classical theorem of Grillakis, Shatah, and Strauss [5] because Lemma 9 shows that the operator $L_+(A)$ linearized at $\varphi_\pm(\mathcal{E})$ has one negative eigenvalue if $\mathcal{Q} < 0$ and $\mathcal{E} > E_*$. On the other hand, the symmetric state $\phi(E)$ is unstable for $E > E_*$ by a theorem of Grillakis [4] because Theorem 1(i) shows that $L_+(E)$ linearized at $\phi(E)$ has two negative eigenvalues for $E > E_*$. We have recovered the statement of Theorem 2.

References

[1] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, M.K. Oberthaler, Direct observation of tunneling and nonlinear self-trapping in a single bosonic Josephson junction, Phys. Rev. Lett. 95 (2005) 010402.
[2] S. Cuccagna, On asymptotic stability in energy space of ground states of NLS in 1D, J. Differential Equations 245 (2008) 653–691.
[3] R. Fukuizumi, A. Sacchetti, Bifurcation and stability for nonlinear Schrödinger equations with double well potential in the semiclassical limit, arXiv:1104.1511, 2011.
[4] M. Grillakis, Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system, Comm. Pure Appl. Math. 43 (1990) 299–333.
[5] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, J. Funct. Anal. 74 (1987) 160–197.
[6] H. Jeanjean, C. Stuart, Nonlinear eigenvalue problems having an unbounded branch of symmetric bound states, Adv. Differential Equations 4 (1999) 639–670.
[7] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1976.
[8] P.G. Kevrekidis, Z. Chen, B.A. Malomed, D.J. Frantzeskakis, M.I. Weinstein, Spontaneous symmetry breaking in photonic lattices: Theory and experiment, Phys. Lett. A 340 (2005) 275–280.
[9] E.W. Kirr, P.G. Kevrekidis, E. Shlizerman, M.I. Weinstein, Symmetry-breaking bifurcation in nonlinear Schrödinger/Gross–Pitaevskii equations, SIAM J. Math. Anal. 40 (2008) 566–604.
[10] E.W. Kirr, P.G. Kevrekidis, D.E. Pelinovsky, Symmetry-breaking bifurcation in the nonlinear Schrödinger equation with symmetric potentials, Comm. Math. Phys. 308 (2011) 795–844.
[11] J.L. Marzuola, M.I. Weinstein, Long time dynamics near the symmetry breaking bifurcation for nonlinear Schrödinger/Gross–Pitaevskii equations, Discrete Contin. Dyn. Syst. Ser. A 28 (2010) 1505–1554.

[12] T. Mizumachi, Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential, J. Math. Kyoto Univ. 48 (2008) 471–497.
[13] M. Ornigotti, G. Della Valle, D. Gatti, S. Longhi, Topological suppression of optical tunneling in a twisted annular fiber, Phys. Rev. A 76 (2007) 023833.
[14] A. Sacchetti, Nonlinear double well Schrödinger equations in the semiclassical limit, J. Stat. Phys. 119 (2005) 1347–1382.
[15] A. Sacchetti, Universal critical power for nonlinear Schrödinger equations with a symmetric double well potential, Phys. Rev. Lett. 103 (2009) 194101, 4 pp.