

Transverse Instability of Vector Solitons and Generation of Dipole Arrays

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We develop a theory of modulational instability of multiparameter solitary waves and analyze the transverse instability of composite (or vector) optical solitons in a saturable nonlinear medium. We demonstrate theoretically and experimentally that a soliton stripe breaks up into an array of (2+1)-dimensional dipole-mode vector solitons, thus confirming the robust nature of those solitons as fundamental composite structures of incoherently coupled fields.

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Nonlinearity-induced instabilities are observed in many different branches of physics, and they provide probably the most dramatic manifestation of strongly nonlinear effects that can occur in Nature. Transverse (or symmetry-breaking) instabilities of solitary waves were predicted theoretically almost 30 years ago [1] (see also the recent review paper [2]), but only recently both transverse and spatiotemporal instabilities were observed experimentally for different types of (bright and dark) spatial optical solitons [3].

When two (or more) components of an optical beam are coupled together, they can form the so-called vector (or composite) soliton [4]. Interaction between coupled waves brings many new features to the stability properties of composite solitary waves (see, e.g., Refs. [5,6]). In particular, as was first shown by Musslimani *et al.* [6], the incoherent coupling between the components of a composite dark-bright soliton can lead to a strong suppression of the transverse instabilities, the effect recently employed to generate vortices via a decay of dark solitons in a two-component mixture of the Bose-Einstein condensates [7].

Instabilities of solitary waves that develop under the action of higher-order perturbations or temporal effects are known to initiate a breakup of a stripe soliton, i.e., a quasi-one-dimensional soliton created in a bulk medium. Several different scenarios of the soliton breakup dynamics are known [2]. The most interesting scenario is associated with the generation of *new localized structures* stable in higher dimensions. In the scalar case, this corresponds to a breakup of a bright-soliton stripe into (2+1)-dimensional bright solitons in a self-focusing medium, or to the generation of an array of vortices due to the transverse instability of a dark soliton. In this Letter, we predict *analytically*, verify *numerically*, and demonstrate *experimentally* a novel type of transverse instability of vector solitons: a decay of a composite soliton stripe into an array of the dipole-mode vector solitons. We believe this effect provides, along with a direct investigation of the linear stability of those solitons [8], evidence of the robust nature of the dipole-

mode vector solitons which resemble “molecules of light” in their dynamics and interaction [9].

The interaction of two mutually incoherent optical beams propagating in a saturable bulk medium is described by the normalized equations for the slowly varying envelope amplitudes [8,10]:

$$\begin{aligned} i \frac{\partial E_1}{\partial z} + \Delta_{\perp} E_1 - F(I)E_1 &= 0, \\ i \frac{\partial E_2}{\partial z} + \Delta_{\perp} E_2 - F(I)E_2 &= 0, \end{aligned} \quad (1)$$

where Δ_{\perp} is the Laplacian defined for the transverse variables (x, y) , z is the propagation coordinate, and the function $F(I) = 1/(1 + I)$ characterizes a saturable nonlinearity of the medium, where $I = |E_1|^2 + |E_2|^2$ is the total intensity (see details and normalization in Refs. [6,8]). System (1) describes, in a greatly simplified isotropic approximation, the screening spatial solitons in photorefractive optical materials [10].

Stationary (1+1)-dimensional localized solutions of the system (1) have the form $E_{1,2}(x, z) = \Phi_{1,2}(x) \exp(i\beta_{1,2}z)$, where β_1 and β_2 are two independent propagation constants. A two-hump composite soliton is described by a single-hump nodeless mode in one component, say, $\Phi_1(x)$, and a double-hump mode in the other component, $\Phi_2(x)$. The field profiles for the two-component composite soliton are shown in Fig. 1.

Asymptotic analysis.—We study analytically the linear stability of the (1+1)-dimensional composite solitons with respect to the transverse perturbations. We linearize Eqs. (1) as follows:

$$E_{1,2} = \{\Phi_{1,2}(x) + [u_{1,2}(x) + iw_{1,2}(x)]e^{ipy + \gamma z}\}e^{i\beta_{1,2}z},$$

where p is the transverse wave number and γ is the instability growth rate. The perturbation vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{w} = (w_1, w_2)$ satisfy the linear eigenvalue problem,

$$\begin{aligned} (\mathcal{L}_1 + p^2)\mathbf{u} &= -\gamma\mathbf{w}, \\ (\mathcal{L}_0 + p^2)\mathbf{w} &= \gamma\mathbf{u}, \end{aligned} \quad (2)$$

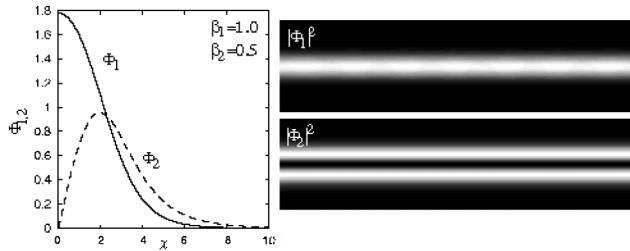


FIG. 1. Left: amplitudes (Φ_1 : solid line; Φ_2 : dashed line) of the (1+1)-dimensional composite soliton at $\beta_1 = 1$ and $\beta_2 = 0.5$. Right: top view of the intensity distribution of the beam components in a bulk geometry.

where the (2×2) matrix operators $\mathcal{L}_{0,1}$ are defined by the expressions explicitly written in Ref. [11].

The two-component multihump solitons are known to be stable with respect to the symmetry-preserving (longitudinal) perturbations [11,12]. Here we study the transverse instability of the multicomponent solitons. This problem can be solved analytically, in the limit of small perturbation wave numbers p^2 . Indeed, for small p^2 the discrete spectrum of the linearized problem for the multihump solitons at $\gamma = p^2 = 0$ is shifted due to the translational and phase-rotational symmetries [2]. For small p , such shifts of the zero eigenvalues can be calculated by means of the asymptotic expansion technique, assuming $p = p_1\epsilon + O(\epsilon^3)$, $\gamma = \gamma_1\epsilon + O(\epsilon^3)$, where ϵ is a small parameter.

We begin by considering a shift of the zero eigenvalue corresponding to the translation mode: $\mathbf{u} = \Phi'(x)$. This mode is responsible for the so-called “snake-type” instability. By expanding the solutions of Eqs. (2) into the asymptotic series

$$\begin{aligned}\mathbf{u} &= \Phi'(x) + \epsilon^2 \mathbf{u}^{(2)}(x) + O(\epsilon^4), \\ \mathbf{w} &= -\frac{1}{2} \gamma_1 \epsilon x \Phi(x) + O(\epsilon^3),\end{aligned}\quad (3)$$

we find (see Ref. [2] for details of the asymptotic technique in application to the soliton transverse instabilities) that the perturbation term $\mathbf{u}^{(2)}(x)$ does not diverge as $|x| \rightarrow \infty$ provided γ_1 is a solution of the equation:

$$\frac{1}{4} \gamma_1^2 N_{\text{tot}} + p_1^2 P_{\text{tot}} = O(\epsilon^2), \quad (4)$$

where the integral values

$$\begin{aligned}N_{\text{tot}} &= \int_{-\infty}^{\infty} (\Phi_1^2 + \Phi_2^2) dx, \\ P_{\text{tot}} &= \int_{-\infty}^{\infty} [(\Phi_1')^2 + (\Phi_2')^2] dx\end{aligned}$$

are both positively defined. Thus, it is clear that γ_1 is purely imaginary, and the corresponding linear eigenmode is *always stable*.

Now we analyze a shift of the double-zero eigenvalue corresponding to the phase-rotational modes: $\mathbf{w} = \Phi_1(x)\mathbf{e}_1$ and $\mathbf{w} = \Phi_2(x)\mathbf{e}_2$, where \mathbf{e}_1 and \mathbf{e}_2 are the unit vectors. We expand the solutions of Eqs. (2) into another form of the asymptotic series:

$$\begin{aligned}\mathbf{u} &= \gamma_1 \epsilon \left(c_1 \frac{\partial \Phi}{\partial \beta_1} + c_2 \frac{\partial \Phi}{\partial \beta_2} \right) + O(\epsilon^3), \\ \mathbf{w} &= c_1 \Phi_1(x) \mathbf{e}_1 + c_2 \Phi_2(x) \mathbf{e}_2 + \epsilon^2 \mathbf{w}^{(2)}(x) + O(\epsilon^4),\end{aligned}\quad (5)$$

where $c_{1,2}$ are arbitrary constants. The perturbation term $\mathbf{w}^{(2)}(x)$ does not diverge as $|x| \rightarrow \infty$ provided the coefficients $c_{1,2}$ satisfy the matrix eigenvalue problem:

$$\begin{aligned}\frac{\partial N_1}{\partial \beta_1} c_1 + \frac{\partial N_1}{\partial \beta_2} c_2 &= \mu N_1 c_1, \\ \frac{\partial N_2}{\partial \beta_1} c_1 + \frac{\partial N_2}{\partial \beta_2} c_2 &= \mu N_2 c_2,\end{aligned}\quad (6)$$

where $\mu = 2p_1^2/\gamma_1^2$, and $N_{1,2}(\beta_1, \beta_2) = \int_{-\infty}^{\infty} \Phi_{1,2}^2 dx$ are conserved powers of each mode calculated for a particular two-component multihump soliton. Eigenvalues of the Hessian matrix $\mathbf{U}_{ij} = \partial N_i / \partial \beta_j$ at the left of Eqs. (6) define the longitudinal stability of the two-component soliton, as shown in Ref. [12]. The two-hump composite soliton is stable when the Hessian matrix \mathbf{U} has one positive and one negative eigenvalue. The sign of two eigenvalues μ_{\pm} in Eq. (6) coincides with the sign of the eigenvalues of the Hessian matrix, i.e., $\mu = \mu_+ > 0$ and $\mu = \mu_- < 0$. The positive eigenvalue μ_+ corresponds to the *transverse instability* of the composite soliton characterized by the instability growth rate $\gamma_1 = p_1 \sqrt{2/\mu_+}$. This result is shown in Fig. 3 by a dashed line. The negative eigenvalue μ_- corresponds to a transversely stable linear eigenmode. Thus, there exists only one unstable eigenmode for the transverse instability of the two-hump composite soliton. This asymptotic analysis is valid only if the soliton is stable with respect to the longitudinal perturbations [13].

Numerical simulations.—In order to investigate the transverse modulational instability of the incoherently coupled beams, we solve Eqs. (1) numerically by the beam propagation method. As an input condition, we use the stationary solution perturbed as follows:

$$\begin{aligned}E_1(x, y; 0) &= [1 + \epsilon q(x, y)] \Phi_1(x), \\ E_2(x, y; 0) &= \Phi_2(x),\end{aligned}$$

where the function $q(x, y)$ represents a noise term with the amplitude $\epsilon = 10^{-5}$, and $q(x, y) \in [-1 : 1]$. By performing the numerical simulations with these initial conditions, we determine the wave number p which corresponds to the maximum growth rate, $p = p_{\text{max}} \approx 0.915$. However, this method does not provide a possibility to observe clearly the regular structures that are formed due to the instability development, because of the simultaneous growth of perturbations with different spatial frequencies and subsequent long-term transition dynamics of unstable modes. Knowing that the maximum growth rate corresponds to the wave number p_{max} , we perform additional numerical simulations by using a perturbation of the form $q = \cos(p_{\text{max}} y)$ with the amplitude $\epsilon = 0.01$. An example of the instability development for this case is shown in Figs. 2(a)–2(c), where the initial “neck-type” modulation [2(a)] and the subsequent formation of an array of the

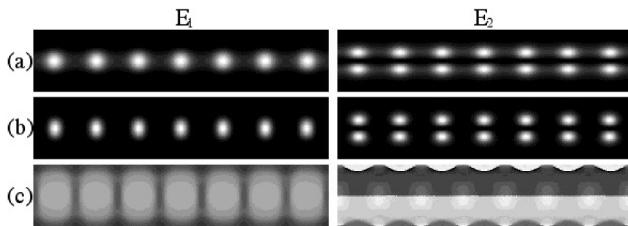


FIG. 2. Numerical results for the generation of a dipole array via the transverse modulational instability. (a) Intensity distribution at $z = 10$. (b),(c) Intensity and phase distribution at $z = 60$, respectively.

dipole-mode vector solitons [2(b) and 2(c)] are clearly observed.

We determine the growth rate of the symmetric unstable mode numerically, and summarize the results in Fig. 3, where we also show the analytical result of the asymptotic analysis for small p (dashed line). The solid line shows an approximate fitting to the curve $\gamma = \alpha p(1 - p/p_c)^{1/2}$, where $\alpha \approx 0.89$ and $p_c \approx 1.36$ are found numerically.

As follows from our asymptotic analysis, the snake-type perturbation mode should always be stable. To verify this result in a general case and also to demonstrate the physical mechanism of the instability development from a perturbation mode of a different symmetry, we perturb the soliton stripe by applying a zigzag perturbation of the form

$$E_1 = \Phi_1(x + \varepsilon \sin(py)), \quad E_2 = \Phi_2(x + \varepsilon \sin(py)), \quad (7)$$

as shown in Fig. 4 (at $p = p_{\max}$). Several periodic oscillations of the stripe are clearly observed, and for $z > 25$ the periodic oscillations couple to an unstable symmetric mode, initiating a stripe breakup into a dipole array. The dipole components interact coherently, and therefore the neighboring lobes attract each other, breaking a regular periodic structure and creating a complex pattern, as shown in Fig. 4 at $z = 30$. Usually, such a regular pattern breaks up, displaying spatiotemporal chaotic dynamics at long propa-

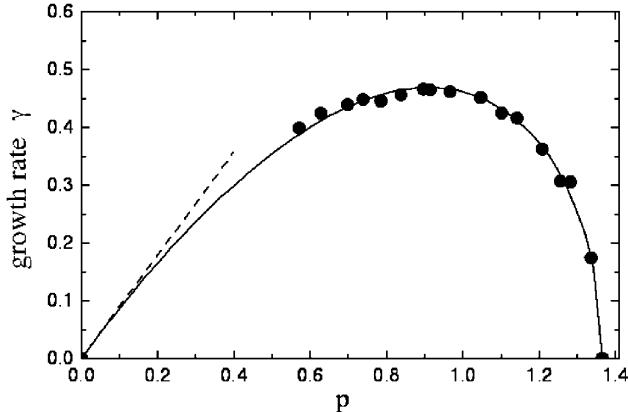


FIG. 3. Growth rate $\gamma(p)$ of the transverse modulational instability found numerically (filled circles) and analytically (dashed line). The solid line shows an approximate fitting to the curve $\gamma = \alpha p(1 - p/p_c)^{1/2}$.

gation distances. However, the study of stability of a periodic array of the dipoles is beyond the scope of our paper.

Experimental results.—To provide experimental evidence for our theoretical predictions, herewith we study the soliton transverse instability in a photorefractive strontium barium niobate (SBN) crystal. The experimental setup is similar to that described earlier in Ref. [14]. A beam from the solid-state laser (at 532 nm) is split into two parts. One beam is transmitted by a glass slide which transforms it into two parts with π relative phase difference in the middle. The other beam is transferred by a system of mirrors and combined with the second one to propagate in parallel. Such a composite beam is subsequently focused by a cylindrical lens onto an input face of a 15-mm-long SBN crystal, biased along its optical axis with a 3.6 kV/cm dc electric field which produced self-focusing saturable nonlinearity. Saturation is controlled by external illumination with a wide beam derived from a white light source. The cylindrical lens used transforms the circular symmetric beams into stripes with typical dimensions 3 mm wide and 20 μm thick. That provides the required input conditions for generating the coupled state of two quasi-one-dimensional bright stripes. The input and output facets of the crystal are imaged with a charge-coupled device camera. To ensure that both beams are *mutually temporally incoherent*, the fundamental component is reflected from a mirror oscillating with 1 kHz. The resulting frequency difference makes the beams mutually incoherent inside the crystal as its slow response is not able to follow the fast changes of their relative phase.

A typical result of our experiments is presented in Fig. 5. The images display the fundamental (E_1 , top) and the dipole (E_2 , bottom) components, after propagation through a 15-mm-long nonlinear SBN crystal. As a result of the stripe instability, an array of the robust dipole-mode vector solitons is indeed formed. We should note that the dipole-mode solitons are generated vertically, along the direction of the applied dc field, which appears as a nonpreferable direction in a photorefractive crystal, and it should be described by a theoretical model that takes into account the

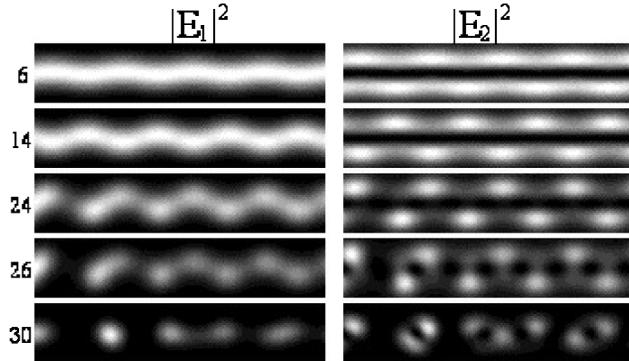


FIG. 4. Numerically found evolution of the snakelike perturbation applied to the soliton stripe. Stable oscillations of the snakelike mode ($z = 6, 14, 24$) couple to the unstable mode ($z = 26$) initiating the stripe breakup.

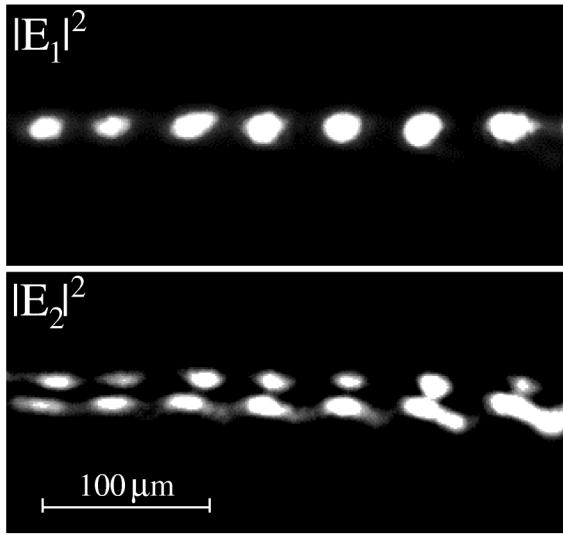


FIG. 5. Experimental observation of the transverse modulational instability of a soliton stripe and its decay into an array of the dipole-mode solitons. Input powers for both components are 20.0 and 8.0 μW , respectively.

effect of anisotropy [15]. As a result, the dipoles rotate, trying to align in the horizontal direction perpendicular to the applied electric field, as was recently discussed in Ref. [15]. Nevertheless, the initial decay of the stripe and the formation of the dipole array seem to be only weakly affected by the anisotropy [16], and the main qualitative features predicted in the framework of the isotropic model are well reproduced in the experiment.

The concept of the soliton transverse instabilities is rather general, and it is well known in many branches of physics (see, e.g., some examples in Ref. [2]). Therefore, we believe that our results can be generalized and be useful for other fields. For example, the model similar to our Eqs. (1) is employed for the study of the light beam interaction in a plasma, where the interesting phenomenon of the braided light was recently predicted in Ref. [17]. Additionally, our model finds its applications in the theory of the Bose-Einstein condensates (BECs), where the mixture states of two BEC species are analogous to optical vector solitons. Moreover, the concept of the transverse instability of the dark-soliton stripe has recently been employed in BEC for the first experimental observation of the stripe decay into vortex rings [7], the effect analogous to the instability of a vector soliton in the presence of higher dimensions.

In conclusion, we have developed an analytical asymptotic theory for analyzing the transverse modulational instability of multiparameter composite solitary waves. We have predicted theoretically and observed experimentally a novel type of the soliton transverse instability, a breakup of a vector-soliton stripe into an array of the dipole-mode vector solitons which are generated as robust objects even in the presence of anisotropy.

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