Nonlinearity **19** (2006) 2695–2716 doi:10.1088/0951-7715/19/11/010

Translationally invariant nonlinear Schrödinger lattices

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Received 7 March 2006, in final form 6 September 2006
Published 13 October 2006
Online at stacks.iop.org/Non/19/2695

Recommended by J R Dorfman

Abstract

The persistence of stationary and travelling single-humped localized solutions in the spatial discretizations of the nonlinear Schrödinger (NLS) equation is addressed. The discrete NLS equation with the most general cubic polynomial function is considered. Constraints on the nonlinear function are found from the condition that the second-order difference equation for stationary solutions can be reduced to the first-order difference map. The discrete NLS equation with such an exceptional nonlinear function is shown to have a conserved momentum but admits no standard Hamiltonian structure. It is proved that the reduction to the first-order difference map gives a sufficient condition for existence of translationally invariant single-humped stationary solutions. Another constraint on the nonlinear function is found from the condition that the differential advance–delay equation for travelling solutions admits a reduction to an integrable normal form given by a third-order differential equation. This reduction gives a necessary condition for existence of single-humped travelling solutions. The nonlinear function which admits both reductions defines a four-parameter family of discrete NLS equations which generalizes the integrable Ablowitz–Ladik lattice. Particular travelling solutions of this family of discrete NLS equations are written explicitly.

PACS numbers: 05.45.Yv, 63.20.Pw

(Figures in this article are in colour only in the electronic version)

1. Introduction

We address spatial discretizations of the nonlinear Schrödinger (NLS) equation in one dimension,

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1.1}$$

which has a family of travelling wave solutions

$$u = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x - 2ct - s))\mathrm{e}^{\mathrm{i}c(x-ct)+\mathrm{i}\omega t+\mathrm{i}\theta}, \tag{1.2}$$

where $\omega \in \mathbb{R}_+$ and $(c, s, \theta) \in \mathbb{R}^3$ are free parameters of the solution family. The discrete counterpart of the NLS equation takes the form

$$\mathrm{i}\dot{u}_n + \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1}) = 0, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}, \tag{1.3}$$

where $h^2 > 0$ is parameter and $f : \mathbb{C}^3 \mapsto \mathbb{C}$ is a non-analytic function with the properties:

P1 (continuity) $f(u, u, u) = 2|u|^2 u$,
P2 (gauge covariance) $f(\mathrm{e}^{\mathrm{i}\alpha} v, \mathrm{e}^{\mathrm{i}\alpha} u, \mathrm{e}^{\mathrm{i}\alpha} w) = \mathrm{e}^{\mathrm{i}\alpha} f(v, u, w)\ \forall \alpha \in \mathbb{R}$,
P3 (symmetry) $f(v, u, w) = f(w, u, v)$,
P4 (reversibility) $\overline{f(v, u, w)} = f(\bar{v}, \bar{u}, \bar{w})$.

We refer to the model (1.3) as the *discrete NLS equation* or simply *the NLS lattice*. The discrete NLS equation (1.3) is a spatial semi-discretization of the continuous NLS equation (1.1), where the second partial derivative is replaced with the second-order central difference on the grid $x = nh$, $n \in \mathbb{Z}$ and the nonlinearity incorporates the effects of on-site and adjacent-site couplings. This model is derived in various branches of physics, most recently, in the context of Bose–Einstein condensates in periodic optical lattices and of arrays of coupled optical waveguides [KRB01]. A survey of results on stationary and travelling solutions of the discrete NLS equation (1.3) can be found in [EJ03].

The continuity property P1 is required if the discrete NLS equation (1.3) is to be matched to the continuous NLS equation (1.1) in the limit $h \to 0$. The gauge covariance P2, symmetry P3 and reversibility P4 properties originate from applications of the discrete NLS equation to the modelling of the envelope of modulated nonlinear dispersive waves in a non-dissipative isotropic system. We enforce two additional properties on the nonlinear function f:

P5 $f(v, u, w)$ is independent on h,
P6 $f(v, u, w)$ is a homogeneous cubic polynomial in (v, u, w).

Neither analysis nor applications require properties P5–P6. However, we use them to limit the search of all possible exceptional NLS lattices to a finite-dimensional parameter space. Indeed, we find immediately that all properties P1–P6 are satisfied if and only if the nonlinear function $f(u_{n-1}, u_n, u_{n-1})$ is represented by the ten-parameter family of cubic polynomials,

$$\begin{aligned} f &= \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) + \alpha_4 (|u_{n+1}|^2 + |u_{n-1}|^2) u_n \\ &\quad + \alpha_5 (\bar{u}_{n+1} u_{n-1} + u_{n+1} \bar{u}_{n-1}) u_n + \alpha_6 (u_{n+1}^2 + u_{n-1}^2) \bar{u}_n + \alpha_7 u_{n+1} u_{n-1} \bar{u}_n \\ &\quad + \alpha_8 (|u_{n+1}|^2 u_{n+1} + |u_{n-1}|^2 u_{n-1}) + \alpha_9 (u_{n+1}^2 \bar{u}_{n-1} + u_{n-1}^2 \bar{u}_{n+1}) \\ &\quad + \alpha_{10} (|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}), \end{aligned} \tag{1.4}$$

where the real-valued parameters $(\alpha_1, \ldots, \alpha_{10})$ satisfy the continuity constraint:

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8 + 2\alpha_9 + 2\alpha_{10} = 2. \tag{1.5}$$

This family generalizes the cubic on-site lattice when

$$(\text{dNLS}) \qquad f = 2|u_n|^2 u_n \tag{1.6}$$

and the integrable Ablowitz–Ladik (AL) lattice

$$(\text{AL}) \qquad f = |u_n|^2 (u_{n+1} + u_{n-1}). \tag{1.7}$$

These models, as well as their linear interpolation (referred to as the Salerno model), have been used as prototypical models for discrete NLS equations [EJ03].

We shall consider *persistence* of travelling wave solutions (1.2) in the discrete NLS equation (1.3) for sufficiently small h depending on the nonlinear function (1.4). Since both the translational and Gallileo invariances are broken, existence of a *stationary* solution for some $\omega \in I_1 \subset \mathbb{R}$,

$$u_n(t) = \phi(hn)\mathrm{e}^{\mathrm{i}\omega t}, \qquad \phi : \mathbb{Z} \mapsto \mathbb{C}, \tag{1.8}$$

does not guarantee existence of a *travelling* solution for some $(\omega, c) \in I_2 \subset \mathbb{R}^2$,

$$u_n(t) = \phi(hn - 2ct)\mathrm{e}^{\mathrm{i}\omega t}, \qquad \phi : \mathbb{R} \mapsto \mathbb{C}, \tag{1.9}$$

where ω is frequency and c is velocity of travelling solutions. In both cases (1.8) and (1.9), we are only interested in existence of *single-humped localized* solutions $\phi(z)$, which correspond to the sech-solutions (1.2) in the limit $h \to 0$. In what follows, *stationary* and *travelling* solutions stand for single-humped localized solutions. Direct substitutions of (1.8) and (1.9) to the discrete NLS equation (1.3) show that the sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ with $\phi_n = \phi(hn)$ satisfies the second-order difference equation,

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - \omega\phi_n + f(\phi_{n-1}, \phi_n, \phi_{n+1}) = 0, \qquad n \in \mathbb{Z}, \tag{1.10}$$

while the function $\phi(z)$ with $z = hn - 2ct$ satisfies the differential advance–delay equation,

$$2\mathrm{i}c\phi'(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} - \omega\phi(z) + f(\phi(z-h), \phi(z), \phi(z+h)), \qquad z \in \mathbb{R}. \tag{1.11}$$

The existence of stationary solutions of the second-order difference equation (1.10) has been proved for a large class of nonlinearities (1.4) by using the variational method [P06]. Two single-humped solutions exist in a general case: one solution is symmetric about a selected lattice node and the other solution is symmetric about a midpoint between two adjacent nodes.

There are two obstacles to the stationary solution (1.8) persisting as the travelling solution (1.9). First, the solution of the stationary problem (1.10) corresponds generally to a *piecewise continuous* solution $\phi(z)$ of the travelling problem (1.11) on $z \in \mathbb{R}$ [AEHV05]. Jump discontinuities of $\phi(z)$ may occur in between two adjacent nodes, e.g. between $z_n = hn$ and $z_{n+1} = (n+1)h$. Second, even if there exists a *continuous* solution $\phi(z)$ of the travelling problem (1.11) with $c = 0$, this solution may not persist as a *continuously differentiable* solution $\phi(z)$ of the problem (1.11) with $c \neq 0$. The stationary solution of the second-order difference equation (1.10) is said to be *translationally invariant* if the function $\phi_n = \phi(nh)$ on $n \in \mathbb{Z}$ can be extended to a one-parameter family of continuous solutions $\phi(z - s)$ on $z \in \mathbb{R}$ of the advance–delay equation (1.11) with $c = 0$, where $s \in \mathbb{R}$ is an arbitrary translation parameter.

The persistence problem for travelling waves in discrete lattices was recently considered in a number of publications. Kevrekidis [K03] suggested a method to define an exceptional nonlinear function f from the condition that the discrete NLS equation (1.3) preserves the *momentum* invariant,

$$M = \mathrm{i}\sum_{n\in\mathbb{Z}} (\bar{u}_{n+1}u_n - u_{n+1}\bar{u}_n). \tag{1.12}$$

No characterization of the nonlinear functions (1.4) that preserve the momentum conservation (1.12) was made in [K03] except for the integrable AL lattice (1.7). In addition, it was not shown that the preservation of momentum (1.12) guarantees the existence of a translationally invariant stationary solution.

Dmitriev *et al* [DKSYT06] generalized the approach of [K03] and derived some particular discretizations (1.4) from the condition that the discrete NLS equation (1.3) conserves the momentum (1.12) and the *power*

$$N = a \sum_{n \in \mathbb{Z}} |u_n|^2 + b \sum_{n \in \mathbb{Z}} (\bar{u}_{n+1} u_n + u_{n+1} \bar{u}_n), \tag{1.13}$$

where (a, b) are real-valued parameters. No general families of the cubic function f in (1.4) were identified in [DKSYT06].

Oster *et al* [OJE03] addressed the most general *Hamiltonian* discrete NLS equation in the form,

$$\mathrm{i}\dot{u}_n = \frac{\partial H}{\partial \bar{u}_n}, \qquad H = \sum_{n \in \mathbb{Z}} \left(\frac{|u_{n+1} - u_n|^2}{h^2} - F(u_n, u_{n+1}) \right), \tag{1.14}$$

where $F : \mathbb{C}^2 \mapsto \mathbb{R}$ is a symmetric potential function of quartic polynomials which satisfies the gauge invariance property: $F(\mathrm{e}^{\mathrm{i}\alpha} u, \mathrm{e}^{\mathrm{i}\alpha} w) = F(u, w)\ \forall \alpha \in \mathbb{R}$. The most general function F is

$$\begin{aligned} F = {} & \delta_1 (|u_n|^2 + |u_{n+1}|^2)(u_n \bar{u}_{n+1} + \bar{u}_n u_{n+1}) + \delta_2 |u_n|^2 |u_{n+1}|^2 \\ & + \delta_3 (u_n^2 \bar{u}_{n+1}^2 + \bar{u}_n^2 u_{n+1}^2) + \delta_4 (|u_n|^4 + |u_{n+1}|^4), \end{aligned} \tag{1.15}$$

where $(\delta_1, \delta_2, \delta_3, \delta_4)$ are real-valued parameters. Only the case $\delta_2 = 4\delta_3$ was treated in [OJE03] based on relevant applications for modelling of arrays of optical waveguides. When $\delta_1 = \delta_2 = \delta_3 = 0$, the Hamiltonian discrete NLS equation includes the dNLS lattice (1.6). It was found in [OJE03] that there exists a configuration in parameters $(\delta_1, \delta_2, \delta_3, \delta_4)$ which provides a reduced Peierls–Nabarro potential and enhanced mobility of localized modes. Additional computations of the localized stationary solutions of the same model were reported in [OJ05].

Ablowitz and Musslimani [AM03] outlined problems in the numerical approximations for travelling wave solutions with $c \neq 0$ in the context of earlier contradictory publications (also see review in [PR05]). An asymptotic method was developed in [AM03] to show that if the solution exists for $c = 0$ it can be continued to non-zero values of c as perturbation series expansions in powers of c.

Berger *et al* [BMR04] derived the necessary condition for persistence of travelling solutions from the integrable AL lattice and applied this condition to a number of discretizations including the Salerno model. It was found that the necessary condition given by the Melnikov integral for homoclinic orbits fails to settle the persistence question for a class of Hamiltonian perturbations generalizing the Salerno model.

Pelinovsky and Rothos [PR05] computed the normal form for bifurcations of travelling solutions near the special values of parameters $\omega = (\pi - 2)/h^2$ and $c = 1/h$. The normal form is represented by the third-order ODE related to the third-order derivative NLS equation [PR05]. The existence of embedded solitons in the third-order derivative NLS equation has been considered in the past (see [YA03, PY05] and references therein) and the previous results suggested that the dNLS lattice (1.6) did not support bifurcation of *single-humped* travelling solutions in a sharp contrast to the case of the AL lattice (1.7). (*Double-humped* travelling solutions in the normal form reduction were predicted for the dNLS lattice (1.6) in [PR05].)

We shall review the results and conjectures of the previous works in the context of the cubic function (1.4). Our results are based on the recent works [BOP05, DKY05, DKY06], where similar ideas are developed for monotonic kinks of the discrete ϕ^4 theory. See also [IP06, OPB05, RDKS06] for other relevant results on monotonic kinks in the discrete Klein–Gordon equation.

Reductions of second-order systems that satisfy first-order equations are old in classical field theory and have been exploited for the discrete Klein–Gordon equation by Speight [S97, S99]. We note however that analysis of single-humped localized solutions of the discrete NLS equation is different from the previous publications. Due to non-monotonicity of the single-humped solutions, the first-order equations must be quadratic with respect to the difference operator $|u_{n+1} - u_n|/h$ and the families of translationally invariant cubic NLS lattices are different from the families of translationally invariant ϕ^4 lattices.

Our main result is a derivation of a general four-parameter family of translationally invariant cubic NLS lattices characterized by the nonlinear function

$$\begin{aligned} f &= (1-\chi-4\xi-2\eta)|u_n|^2(u_{n+1}+u_{n-1}) + \chi u_n^2(\bar{u}_{n+1}+\bar{u}_{n-1}) \\ &\quad + \xi[(2|u_n|^2+|u_{n+1}|^2+|u_{n-1}|^2)u_n + (\bar{u}_{n+1}u_{n-1}+u_{n+1}\bar{u}_{n-1})u_n + (u_{n+1}^2+u_{n-1}^2)\bar{u}_n] \\ &\quad + \eta(|u_{n+1}|^2+|u_{n-1}|^2)(u_{n+1}+u_{n-1}) \\ &\quad + \nu[u_{n+1}^2\bar{u}_{n-1}+u_{n-1}^2\bar{u}_{n+1}-|u_{n+1}|^2u_{n-1}-|u_{n-1}|^2u_{n+1}], \end{aligned} \tag{1.16}$$

where (χ, ξ, η, ν) are real-valued parameters. When $\chi = \xi = \eta = \nu = 0$, the function (1.16) recovers the AL lattice (1.7), which admits the four-parameter family of travelling solutions of the differential advance–delay equation (1.11):

$$\phi(z) = \frac{1}{h}\sinh(\kappa h)\ \mathrm{sech}\,(\kappa z - s)\ \mathrm{e}^{(\mathrm{i}\beta z/h)+\mathrm{i}\theta}, \tag{1.17}$$

where $(s, \theta) \in \mathbb{R}^2$ and new parameters $\kappa \in \mathbb{R}_+$ and $\beta \in [0, 2\pi]$ parametrize the domain $(\omega, c) \in I_2 \subset \mathbb{R}^2$:

$$\omega = \frac{2}{h}\beta c + \frac{2}{h^2}(\cos\beta\cosh(\kappa h) - 1), \qquad c = \frac{1}{h^2\kappa}\sin\beta\sinh(\kappa h). \tag{1.18}$$

The exact travelling solutions (1.17) exist between the two limits $c = 0$ and $c = 1/h$. Another one-parameter family of exact solutions exists for the nonlinearity (1.16) with $\chi < \frac{1}{2}$ and $\xi = \eta = \nu = 0$:

$$\phi(z) = \frac{1}{h\sqrt{1-2\chi}}\sinh(\kappa h)\,\mathrm{sech}\,(\kappa z)\,\mathrm{e}^{\mathrm{i}\pi z/2h}, \tag{1.19}$$

which corresponds to the values $\kappa \in \mathbb{R}_+$ and $\beta = \pi/2$ in the parametrization (1.18).

The content of the paper is structured as follows. In section 2, the class of *exceptional* nonlinear functions in the general family of cubic polynomials (1.4) is identified from the condition that the second-order difference equation (1.10) admits a reduction to the first-order difference equation. It is shown that the discrete NLS equation (1.3) with an exceptional nonlinearity preserves the momentum conservation (1.12) but admits no Hamiltonian in the form (1.14).

In section 3, it is proved that all localized solutions of the second-order difference equation (1.10) are real-valued for sufficiently small h. Existence of an additional reduction to the first-order difference equation which generalizes the density flux of the discrete NLS equation (1.3) is also discussed in connection with the existence of real-valued solutions.

In section 4, it is proved that the first-order difference equation admits a one-parameter family of real-valued localized solutions for any $\omega > 0$ and sufficiently small h, and these localized solutions are *translationally invariant*. Unlike the case of monotonic kinks in [BOP05], the proof of the existence of a translationally invariant single-humped localized solution is based on a construction of two sequences which depend continuously on the initial value. One sequence is monotonically decreasing to zero and the other sequence is monotonically increasing until the turning point and it decreases monotonically beyond the turning point. The main outcome of this analysis is a clear evidence that the reduction to the first-order difference equation gives *a sufficient condition* for the existence of translationally invariant stationary solutions.

In section 5, the exceptional nonlinear functions are tested against the reduction of the differential advance–delay equation (1.11) to the normal form near the special values of parameters $\omega = (\pi - 2)/h^2$ and $c = 1/h$. Reductions of the normal form to integrable Hirota [H73] and Sasa–Satsuma [SS91] equations, which possess exact localized solutions, give *a necessary condition* for existence of travelling solutions in the differential advance–delay equation (1.11) near the special values of parameters (ω, c).

In section 6, properties of the translationally invariant NLS lattice with the nonlinear function (1.16) are summarized and further questions on existence of travelling solutions are posed.

2. Reduction to the first-order difference equation

We shall consider the stationary solution (1.8) of the discrete NLS equation (1.3), which satisfies the second-order difference equation (1.10). We obtain the constraints on the function $f(\phi_{n-1}, \phi_n, \phi_{n+1})$ in the family of cubic polynomials (1.4) from the condition that the second-order difference equation (1.10) admits a conserved quantity

$$E_n = \frac{1}{h^2}|\phi_{n+1} - \phi_n|^2 - \frac{1}{2}\omega\left(\phi_n\bar{\phi}_{n+1} + \bar{\phi}_n\phi_{n+1}\right) + g(\phi_n, \phi_{n+1}) = E_0, \qquad \forall n \in \mathbb{Z}, \tag{2.1}$$

where $g : \mathbb{C}^2 \mapsto \mathbb{R}$ is a non-analytic function with the properties:

- S1 (continuity) $g(u, u) = |u|^4$ and $\frac{\partial g}{\partial \bar{u}}(u, u) = \frac{\partial g}{\partial \bar{w}}(u, u) = |u|^2 u$,
- S2 (gauge invariance) $g(e^{i\alpha}u, e^{i\alpha}w) = g(u, w)\ \forall \alpha \in \mathbb{R}$,
- S3 (symmetry) $g(u, w) = g(w, u)$,
- S4 (reversibility) $\overline{g(u, w)} = g(\bar{u}, \bar{w})$,
- S5 $g(u, w)$ is independent on h,
- S6 $g(u, w)$ is a homogeneous quartic polynomial in (u, w).

The most general polynomial $g(\phi_n, \phi_{n+1})$ that satisfies properties S1–S6 takes the form

$$\begin{aligned} g = {} & \gamma_1(|\phi_n|^2 + |\phi_{n+1}|^2)(\bar{\phi}_{n+1}\phi_n + \phi_{n+1}\bar{\phi}_n) + \gamma_2|\phi_n|^2|\phi_{n+1}|^2 \\ & + \gamma_3(\phi_n^2\bar{\phi}_{n+1}^2 + \bar{\phi}_n^2\phi_{n+1}^2) + \gamma_4(|\phi_n|^4 + |\phi_{n+1}|^4), \end{aligned} \tag{2.2}$$

where the real-valued parameters $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ satisfy the continuity constraint:

$$4\gamma_1 + \gamma_2 + 2\gamma_3 + 2\gamma_4 = 1. \tag{2.3}$$

In the limit $h \to 0$, the second-order difference equation (1.10) reduces to the second-order ODE:

$$\phi'' - \omega\phi + 2|\phi|^2\phi = 0, \tag{2.4}$$

while the first-order difference equation (2.1) transforms to the first integral of (2.4)

$$E = |\phi'|^2 - \omega|\phi|^2 + |\phi|^4. \tag{2.5}$$

The first-order ODE (2.5) with $E = 0$ admits a unique localized solution $\phi(z) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(z-s))$ with $s \in \mathbb{R}$ and $\omega > 0$, which agrees with the exact solution (1.2) for $z = x$ and $c = 0$. Therefore, the first-order difference equation (2.1) is a spatial discretization of the first-order ODE (2.5).

The precise methodology of discretization of the first-order ODE (2.5) was developed in [DKY05] to construct *translationally invariant monotonic kinks* in the discrete Klein–Gordon equation. Application of this method to the discrete ϕ^4 equation was also described in [DKY05] but more general families of this equation with translationally invariant kinks were reported in [BOP05]. Recently, a similar method was extended to the discrete NLS equation in [DKSYT06] and some examples of this equation with *translationally invariant single-humped localized solutions* were derived. We shall list here all *such* discrete NLS equations with the cubic function (1.4). In particular, we derive four elementary results on the existence of the reduction (2.1) with the nonlinear function (2.2) and its relations to the conservation of M in (1.12), N in (1.13) and H in (1.14).

Lemma 2.1. *The reduction of the second-order difference equation (1.10) to the first-order difference equation (2.1) exists provided that the nonlinear function (1.4) satisfies the constraints:*

$$\alpha_4 = \alpha_1 - \alpha_6, \qquad \alpha_5 = \alpha_6, \qquad \alpha_7 = \alpha_1 - 2\alpha_6, \qquad \alpha_{10} = \alpha_8 - \alpha_9, \tag{2.6}$$

such that $(\alpha_1, \alpha_2, \alpha_3, \alpha_6, \alpha_8, \alpha_9) \in \mathbb{R}^6$ *are free parameters.*

Proof. We apply symbolic computations with Wolfram's Mathematica to the relation between the quartic polynomial g and the cubic polynomial f:

$$\begin{aligned} g(\phi_n, \phi_{n+1}) - g(\phi_{n-1}, \phi_n) &= \tfrac{1}{2}(\bar{\phi}_{n+1} - \bar{\phi}_{n-1}) f(\phi_{n-1}, \phi_n, \phi_{n+1}) \\ &\quad + \tfrac{1}{2}(\phi_{n+1} - \phi_{n-1}) \overline{f(\phi_{n-1}, \phi_n, \phi_{n+1})}. \end{aligned} \tag{2.7}$$

As a result, we find that $\alpha_1 = 2\gamma_1$, $\alpha_2 = \gamma_2$, $\alpha_3 = 2\gamma_3$ and $\alpha_8 = \gamma_4$ under the constraints (2.6). ■

Corollary 2.2. *When $\phi_n \in \mathbb{R}$, the first-order difference equation (2.1) is characterized by the symmetric quartic polynomial function*

$$g = \beta_1 \phi_n^2 \phi_{n+1}^2 + \beta_2(\phi_n^2 + \phi_{n+1}^2)\phi_n\phi_{n+1} + \beta_3(\phi_n^4 + \phi_{n+1}^4), \tag{2.8}$$

where $\beta_1 = \gamma_2 + 2\gamma_3 = \alpha_2 + \alpha_3$, $\beta_2 = 2\gamma_1 = \alpha_1$ and $\beta_3 = \gamma_4 = \alpha_8$ under the continuity constraint

$$\beta_1 + 2\beta_2 + 2\beta_3 = 1. \tag{2.9}$$

Remark 2.3. Corollary 2.2 recovers equation (18) of [BOP05], where real-valued nonlinear functions of the second-order difference equation (1.10) are considered in the context of the discrete ϕ^4 model. It also agrees with equation (19) of [DKSYT06] in the context of the discrete NLS equation.

Lemma 2.4. *The discrete NLS equation (1.3) conserves the momentum invariant (1.12) provided that the constraints (2.6) are met.*

Proof. Computing time-derivative of (1.12) and using the stroboscopical summation in n, we convert the result to the following irreducible remainder:

$$\dot{M} = (\alpha_1 - \alpha_4 - \alpha_6)\sum_{n\in\mathbb{Z}}(|u_{n+1}|^2 - |u_n|^2)(u_{n+1}\bar{u}_n + \bar{u}_{n+1}u_n))$$
$$+ (\alpha_4 - \alpha_5 - \alpha_7)\sum_{n\in\mathbb{Z}}(|u_{n+1}|^2(u_n\bar{u}_{n-1} + \bar{u}_n u_{n-1}) - |u_{n-1}|^2(u_n\bar{u}_{n+1} + \bar{u}_n u_{n+1}))$$
$$+ (\alpha_5 - \alpha_6)\sum_{n\in\mathbb{Z}}(u_{n+1}u_n\bar{u}_{n-1}^2 + \bar{u}_{n+1}\bar{u}_n u_{n-1}^2 - u_{n+1}^2\bar{u}_n\bar{u}_{n-1} - \bar{u}_{n+1}^2 u_n u_{n-1})$$
$$+ (\alpha_8 - \alpha_9 - \alpha_{10})\sum_{n\in\mathbb{Z}}(|u_{n+1}|^2 - |u_{n-1}|^2)(u_{n+1}\bar{u}_{n-1} + \bar{u}_{n+1}u_{n-1}).$$

The constraints (2.6) give the most general solution of the system of homogeneous linear equations, which follows from the momentum conservation $\dot{M} = 0$. ■

Corollary 2.5. *There exists a one-to-one correspondence between the set of nonlinear functions (1.4) that conserves the momentum (1.12) and the set of nonlinear functions that supports the reduction to the first-order difference equation (2.1).*

Remark 2.6. Corollary 2.5 proves the conjecture of [K03] for the case of a discrete NLS equation (1.3) with the cubic polynomial function (1.4). A similar result for a discrete Klein–Gordon equation was obtained in the most general case in [K03, appendix] by explicit computations.

Lemma 2.7. *The discrete NLS equation (1.3) conserves the power invariant (1.13) provided that the linear homogeneous system has a non-trivial solution for* (a, b)*:*

$$\begin{bmatrix} \alpha_2 - \alpha_3 - \alpha_8 & \alpha_4 - \alpha_1 - \alpha_6 \\ \alpha_7 & -2\alpha_3 \\ \alpha_{10} & \alpha_7 - \alpha_4 - \alpha_5 \\ \alpha_9 & \alpha_6 - \alpha_5 \\ 0 & \alpha_8 + \alpha_9 - \alpha_{10} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.10}$$

which specifies four quadratic constraints on parameters $(\alpha_1, \ldots, \alpha_{10}) \in \mathbb{R}^{10}$.

Proof. Computing time-derivative of (1.13) and using the stroboscopical summation in n, we convert the result to the following irreducible remainder:

$$-\mathrm{i}\dot{N} = [a(\alpha_2 - \alpha_3 - \alpha_8) + b(\alpha_4 - \alpha_1 - \alpha_6)]\sum_{n\in\mathbb{Z}}(|u_{n+1}|^2 - |u_n|^2)(\bar{u}_{n+1}u_n - \bar{u}_n u_{n+1})$$
$$+ [a\alpha_7 - 2b\alpha_3]\sum_{n\in\mathbb{Z}}(\bar{u}_n^2 u_{n-1}u_{n+1} - u_n^2\bar{u}_{n-1}\bar{u}_{n+1}) + [a\alpha_{10} + b(\alpha_7 - \alpha_4 - \alpha_5)]$$
$$\times \sum_{n\in\mathbb{Z}}(|u_{n-1}|^2(\bar{u}_n u_{n+1} - \bar{u}_{n+1}u_n) + |u_{n+1}|^2(\bar{u}_n u_{n-1} - \bar{u}_{n-1}u_n))$$
$$+ [a\alpha_9 + b(\alpha_6 - \alpha_5)]\sum_{n\in\mathbb{Z}}(u_{n+1}^2\bar{u}_n\bar{u}_{n-1} - \bar{u}_{n+1}^2 u_n u_{n-1} + u_{n-1}^2\bar{u}_n\bar{u}_{n+1} - \bar{u}_{n-1}^2 u_n u_{n+1})$$
$$+ b(\alpha_8 + \alpha_9 - \alpha_{10})\sum_{n\in\mathbb{Z}}(|u_{n+1}|^2 - |u_{n-1}|^2)(u_{n+1}\bar{u}_{n-1} - \bar{u}_{n+1}u_{n-1}).$$

The linear system (2.10) ensures that $\dot{N} = 0$. ■

Corollary 2.8. *Simultaneous conservation of momentum M and power N is possible if and only if parameters $(\alpha_1, \ldots, \alpha_{10}) \in \mathbb{R}^{10}$ satisfy the constraints (2.6) and additional constraints*

$$\alpha_9 = 0, \qquad \alpha_6(\alpha_2 - \alpha_3 - 2\alpha_8) = 0, \qquad \alpha_6(\alpha_1 - 2\alpha_6) - \alpha_3\alpha_8 = 0. \tag{2.11}$$

Lemma 2.9. *The discrete NLS equation (1.3) has the Hamiltonian structure (1.14) provided that*

$$\alpha_2 = 2\alpha_3 = 2\alpha_8, \qquad \alpha_5 = \alpha_7 = \alpha_9 = \alpha_{10} = 0, \tag{2.12}$$

such that $(\alpha_1, \alpha_2, \alpha_4, \alpha_6) \in \mathbb{R}^4$ are free parameters.

Proof. Due to the symplectic structure (1.14), the potential function $F(u_n, u_{n+1})$ in (1.15) produces the nonlinear function $f(u_{n-1}, u_n, u_{n+1})$ in the form

$$f = \delta_1 \left[|u_{n+1}|^2 u_{n+1} + |u_{n-1}|^2 u_{n-1} + 2|u_n|^2(u_{n+1} + u_{n-1}) + u_n^2(\bar{u}_{n+1} + \bar{u}_{n-1}) \right]$$
$$+ \delta_2(|u_{n+1}|^2 + |u_{n-1}|^2)u_n + 2\delta_3(u_{n+1}^2 + u_{n-1}^2)\bar{u}_n + 4\delta_4|u_n|^2 u_n.$$

Therefore, $\alpha_2 = 2\delta_1$, $\alpha_4 = \delta_2$, $\alpha_6 = 2\delta_3$ and $\alpha_1 = 4\delta_4$ under the constraints (2.12). ■

Corollary 2.10. *None of the discrete NLS equations with the Hamiltonian structure (1.14) conserve momentum M.*

Remark 2.11. Corollary 2.10 agrees with the conclusion of [DKY06] obtained for the discrete ϕ^4 model.

Remark 2.12. Although it is claimed in [OJE03] from results of numerical simulations that travelling solutions may have enhanced mobility in the discrete NLS equation (1.3) with the Hamiltonian structure (1.14), no conservation of momentum is supported in the Hamiltonian model and the second-order difference equation (1.10) provides no reduction to the first-order difference equation (2.1).

3. Existence of real-valued stationary solutions

We shall study the correspondence between solutions of the difference equations (1.10) and (2.1). In particular, we prove that all solutions of the second-order difference equation (1.10) are real-valued for sufficiently small h. In addition, we consider existence of another reduction to the first-order difference equation which generalizes the density flux for the discrete NLS equation (1.3).

Lemma 3.1. *Let polynomials g and f be related by (2.7). If the sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ is any solution of the second-order difference equation (1.10), it satisfies the first-order difference equation (2.1). If the sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ is a non-constant real-valued solution of the first-order difference equation (2.1), it satisfies the second-order difference equation (1.10).*

Proof. By using the first-order difference equation (2.1), we obtain

$$E_n - E_{n-1} = \frac{1}{2}(\bar{\phi}_{n+1} - \bar{\phi}_{n-1}) \left(\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - \omega\phi_n \right)$$
$$+ \frac{1}{2}(\phi_{n+1} - \phi_{n-1}) \left(\frac{\bar{\phi}_{n+1} - 2\bar{\phi}_n + \bar{\phi}_{n-1}}{h^2} - \omega\bar{\phi}_n \right) + g(\phi_n, \phi_{n+1}) - g(\phi_{n-1}, \phi_n). \tag{3.1}$$

If g and f are related by (2.7) and $\{\phi_n\}_{n\in\mathbb{Z}}$ solves the second-order difference equation (1.10), then $E_n - E_{n-1} = 0$ and the first-order difference equation (2.1) holds with $E_n = E_0$ $\forall n \in \mathbb{Z}$. In the opposite direction, if $E_n - E_{n-1} = 0$, then the non-constant sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ satisfies the second-order equation,

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - \omega\phi_n + f(\phi_{n-1}, \phi_n, \phi_{n+1}) = \mathrm{i}(\phi_{n+1} - \phi_{n-1})\tilde{f}(\phi_{n-1}, \phi_n, \phi_{n+1}), \quad n \in \mathbb{Z},$$

where $\tilde{f} : \mathbb{C}^3 \mapsto \mathbb{R}$ is arbitrary function. If the sequence is also real-valued, then $\tilde{f} \equiv 0$. ■

Remark 3.2. The one-to-one correspondence between the second-order and first-order difference equations (1.10) and (2.1) exists only for non-constant real-valued solutions. Complex-valued solutions of the first-order equation (2.1) can give solutions of a more general second-order equation with an additional function $\tilde{f}(\phi_{n-1}, \phi_n, \phi_{n+1})$. This property can be used for a full time–space discretization of the NLS equation (1.1), when the derivative term $\phi'(z)$ in the differential advance–delay equation (1.11) is replaced by the difference term $(\phi_{n+1} - \phi_{n-1})/(2h)$ such that $\tilde{f} = c/h = \text{const}$. We will not consider full time–space discretizations of the NLS equation (1.1) in this paper.

Lemma 3.3. *All localized bounded solutions of the second-order difference equation (1.10) are real-valued for sufficiently small h.*

Proof. The nonlinear function (1.4) satisfies the relation

$$\bar{\phi}_n f(\phi_{n-1}, \phi_n, \phi_{n+1}) - \phi_n \overline{f(\phi_{n-1}, \phi_n, \phi_{n+1})} = (J_n - J_{n-1})Q_n + R_n,$$

where

$$J_n = \bar{\phi}_n\phi_{n+1} - \phi_n\bar{\phi}_{n+1}, \tag{3.2}$$

$$Q_n = (\alpha_2 - \alpha_3)|\phi_n|^2 + \alpha_8(|\phi_{n-1}|^2 + |\phi_{n+1}|^2) + \alpha_9(\bar{\phi}_{n-1}\phi_{n+1} + \phi_{n-1}\bar{\phi}_{n+1}) \\ + \alpha_6(\bar{\phi}_n\phi_{n+1} + \phi_n\bar{\phi}_{n+1} + \bar{\phi}_n\phi_{n-1} + \phi_n\bar{\phi}_{n-1})$$

and

$$R_n = (\alpha_7 - 2\alpha_6)(\bar{\phi}_n^2\phi_{n-1}\phi_{n+1} - \phi_n^2\bar{\phi}_{n-1}\bar{\phi}_{n+1}) + (\alpha_{10} - \alpha_8 - \alpha_9)[(\bar{\phi}_n\phi_{n-1} - \phi_n\bar{\phi}_{n-1})|\phi_{n+1}|^2 \\ + (\bar{\phi}_n\phi_{n+1} - \phi_n\bar{\phi}_{n+1})|\phi_{n-1}|^2].$$

Multiplying the second-order difference equation (1.10) by $\bar{\phi}_n$ and subtracting the complex conjugate equation, we obtain

$$(J_n - J_{n-1})\left(1 + h^2 Q_n\right) + h^2 R_n = 0. \tag{3.3}$$

It is obvious that

$$R_n = (\alpha_7 - 2\alpha_6)(\bar{\phi}_n\phi_{n-1}J_n - \phi_n\bar{\phi}_{n+1}J_{n-1}) + (\alpha_{10} - \alpha_8 - \alpha_9)(|\phi_{n-1}|^2 J_n - |\phi_{n+1}|^2 J_{n-1}),$$

such that

$$J_n = \frac{1 + h^2 Q_n^{(1)}}{1 + h^2 Q_n^{(2)}} J_{n-1}, \qquad n \in \mathbb{Z}, \tag{3.4}$$

where the sequences $\{Q_n^{(j)}\}_{n\in\mathbb{Z}}$, $j = 1, 2$, are bounded if the sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ is bounded. If h is sufficiently small, then $1 + h^2 Q_n^{(j)} > 0$, $j = 1, 2$, $n \in \mathbb{Z}$. If the sequence is also localized such that $\lim_{n\to\pm\infty} \phi_n = 0$, then $\lim_{n\to\infty} J_n = 0$ and $J_n = 0$ $\forall n \in \mathbb{Z}$. ■

Remark 3.4. By lemmas 3.1 and 3.3, all localized solutions of the second-order difference equation (1.10) are *real-valued* and *equivalent* to solutions of the first-order difference equation (2.1) for sufficiently small h.

Corollary 3.5. *The second-order difference equation (1.10) admits a conserved quantity* $J_n = J_0$ $\forall n \in \mathbb{Z}$ *for sufficiently small* h *if*

$$\alpha_7 = 2\alpha_6, \qquad \alpha_{10} = \alpha_8 + \alpha_9, \tag{3.5}$$

while all other parameters $(\alpha_1, ..., \alpha_{10}) \in \mathbb{R}^{10}$ *are arbitrary.*

Proof. Under the constraints (3.5), then $R_n \equiv 0, \forall\ n \in \mathbb{Z}$. If h is sufficiently small, then $1 + h^2 Q_n > 0$, $n \in \mathbb{Z}$ and it follows from the recurrence relation (3.3) that $J_n = J_0, \forall n \in \mathbb{Z}$. ∎

Corollary 3.6. *The second-order difference equation (1.10) admits a conserved quantity*

$$\tilde{J}_n = \left(\bar{\phi}_n\phi_{n+1} - \phi_n\bar{\phi}_{n+1}\right)\left[1 + \alpha_6 h^2(\bar{\phi}_n\phi_{n+1} + \phi_n\bar{\phi}_{n+1}) + \alpha_8 h^2(|\phi_n|^2 + |\phi_{n+1}|^2)\right] = \tilde{J}_0, \tag{3.6}$$

if

$$\alpha_2 = \alpha_3 + \alpha_8, \qquad \alpha_7 = \alpha_9 = \alpha_{10} = 0, \tag{3.7}$$

while all other parameters $(\alpha_1, \ldots, \alpha_{10}) \in \mathbb{R}^{10}$ *are arbitrary.*

Proof. By using the symbolic computations, we obtain that

$$\tilde{J}_n - \tilde{J}_{n-1} = (J_n - J_{n-1})(1 + h^2 Q_n) + h^2 R_n = 0,$$

under the constraints (3.7). ∎

Remark 3.7. The conserved quantity (3.6) was used in [OJ05] in the context of the Hamiltonian discrete NLS equation (1.14), since the constraints (3.7) are compatible with the constraints (2.12). We note that complex-valued stationary solutions were also reported in [OJ05] from the reduction $\tilde{J}_n = 0$. Such complex-valued solutions are only supported by the second-order difference equation (1.10) for large values of h. Analysis of this paper is valid for sufficiently small values of h.

4. Existence of translationally invariant stationary solutions

We shall consider existence of translationally invariant solutions $\{\phi_n\}_{n\in\mathbb{Z}}$ of the second-order difference equation (1.10). We show that such solutions exist if the second-order difference equation (1.10) is reduced to the first-order difference equation (2.1). Thus, the existence of the reduction to the first-order difference equation (2.1) gives the *sufficient* condition for existence of translationally invariant stationary solutions. By lemma 3.3, the analysis is only developed for *real-valued* localized solutions of the first-order difference equation (2.1). In this case, the initial-value problem for the first-order difference equation (2.1) with $E_0 = 0$ takes the implicit form,

$$\begin{cases} (\phi_{n+1} - \phi_n)^2 = h^2\omega\phi_n\phi_{n+1} - h^2 g(\phi_n, \phi_{n+1}), & n \in \mathbb{Z}, \\ \phi_0 = \varphi, \end{cases} \tag{4.1}$$

where $\varphi \in \mathbb{R}$ is the initial data, iterations in both positive and negative directions of $n \in \mathbb{Z}$ are considered and $g : \mathbb{R}^2 \mapsto \mathbb{R}$ is given by (2.8). Let $x = \phi_n$ and $y = \phi_{n+1}$ and rewrite $g(\phi_n, \phi_{n+1})$ as

$$g(x, y) = \beta_1 x^2 y^2 + \beta_2 xy(x^2 + y^2) + \beta_3(x^4 + y^4).$$

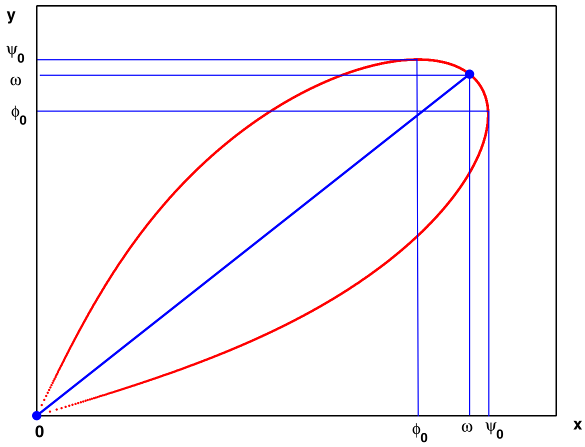

Figure 1. Solutions of the quartic equation (4.2) with $\beta_1 = \beta_2 = 0$ and $\beta_3 = 0.5$.

Due to the continuity constraint (2.9), we have $g(x, x) = x^4$ and $\partial_x g(x, x) = \partial_y g(x, x) = 2x^3$. We shall analyse the initial-value problem (4.1) for $\omega > 0$ and sufficiently small h and prove the existence of a particular single-humped localized sequence $\{\phi_n\}_{n\in\mathbb{Z}}$, which corresponds to the translationally invariant stationary solution of the second-order difference equation (1.10).

Lemma 4.1. *Let $Q(x, y) = \omega xy - g(x, y)$ and $\omega > 0$ is parameter. There exists $h_0 > 0$ and $L > \sqrt{\omega}$ such that the algebraic (quartic) equation*

$$(y - x)^2 = h^2 Q(x, y) \tag{4.2}$$

defines a convex, simply connected, closed curve inside the domain $0 \leqslant x \leqslant L$ and $0 \leqslant y \leqslant L$ for $0 < h < h_0$. The curve is symmetric about the diagonal $y = x$ and has two intersections with the diagonal at $y = x = 0$ and $y = x = \sqrt{\omega}$.

Proof. The statement of lemma is illustrated in figure 1. Symmetry of the curve about the diagonal $y = x$ follows from the fact that $g(x, y) = g(y, x)$. Intersections with the diagonal $y = x$ follow from the continuity constraint that gives $Q(x, x) = x^2(\omega - x^2)$. To prove convexity, we consider the behaviour of the curve for sufficiently small h in two domains: in the interval $0 \leqslant x \leqslant x_0$ and in a small neighbourhood of the point $y = x = \sqrt{\omega}$ with the radius $r_0 h^2$, where $x_0 = \sqrt{\omega} - r_0 h^2$ and $r_0 > \sqrt{\omega^3}/8$ is h-independent. Let $y = x + z$ and consider two branches of the algebraic equation (4.2) in the implicit form

$$F_\pm(x, z, h) = z \mp h\sqrt{Q(x, x+z)} = 0, \tag{4.3}$$

such that for $x \geqslant 0$

$$F_\pm(x, 0, h) = \mp hx\sqrt{\omega - x^2}, \qquad \partial_z F_\pm(x, 0, h) = 1 \mp \frac{h(\omega - 2x^2)}{2\sqrt{\omega - x^2}}.$$

It is clear that $F_\pm(x, 0, 0) = 0$ and $\partial_z F_\pm(x, 0, 0) = 1$, while $F_\pm(x, 0, h)$ is continuously differentiable in x and h and $\partial_z F_\pm(x, 0, h)$ is uniformly bounded in x and h on any compact subset of $x \in [0, x_0]$, where $x_0 = \sqrt{\omega} - r_0 h^2$ and $r_0 > \sqrt{\omega^3}/8$ is h-independent. By the implicit function theorem, there exists $h_0 > 0$, such that the implicit equations (4.3) define unique roots $z = \pm h S_\pm(x, h)$ in the domain $0 \leqslant x \leqslant x_0$ and $0 < h < h_0$, where $h S_\pm(x, h)$ are positive, continuously differentiable functions in x and h. Therefore, the algebraic equation (4.2) defines two branches of the curve located above and below the diagonal $y = x$. When h_0

is sufficiently small, the two branches are strictly increasing in the interval $0 \leqslant x \leqslant x_0$. In the limit $h_0 \to 0$, the two branches converge to the diagonal $y = x$ on $0 \leqslant x \leqslant \sqrt{\omega}$. Derivatives of the algebraic equation (4.2) in x are defined for any branch of the curve by

$$y'[2(y-x) - h^2(\omega x - \partial_y g(x, y))] = 2(y-x) + h^2(\omega y - \partial_x g(x, y)), \quad (4.4)$$

$$y''[2(y-x) - h^2(\omega x - \partial_y g(x, y))] = -2(y'-1)^2 + h^2(2\omega y' - \tilde{g}), \quad (4.5)$$

where $\tilde{g} = \partial_{xx}^2 g(x, y) - 2y'\partial_{xy}^2 g(x, y) - (y')^2 \partial_{yy}^2 g(x, y)$. It follows from the first derivative (4.4) that $y' = -1$ at the point $y = x = \sqrt{\omega}$ for any $h > 0$, such that there exists a small neighbourhood of the point $y = x = \sqrt{\omega}$ where the upper branch of the curve is strictly decreasing. Let B_h^+ be the upper semi-disc centred at $y = x = \sqrt{\omega}$ with a radius $r_0 h^2$, where $r_0 > \sqrt{\omega^3}/8$ is h-independent. Then, there exists a h-independent constant $C > 0$ such that

$$2(y-x) - h^2(\omega x - \partial_y g(x, y)) \geqslant Ch^2, \qquad (x, y) \in B_h^+.$$

It follows from the second derivative (4.5) for sufficiently small h that $y'' \leqslant -C_1/h^2$ in $(x, y) \in B_h^+$ with $C_1 > 0$. Therefore, the curve is convex in B_h^+. By using the rescaled variables in B_h^+:

$$x = \sqrt{\omega} - Xh^2, \qquad y = \sqrt{\omega} + Yh^2,$$

we find that the curvature of the curve in new variables is bounded by $Y''(X) \leqslant -C_1$ and, therefore, the first derivative $Y'(X)$ and so $y'(x)$ may only change by a finite number in $(x, y) \in B_h$. Therefore, there exists $0 < C_2 < \infty$ such that $y'(x) = C_2$ at a point x, where $x \leqslant x_0 = \sqrt{\omega} - r_0 h^2$. By the first part of the proof, the upper branch of the curve is monotonically increasing for $0 \leqslant x \leqslant x_0$. By the second part of the proof, it has a single maximum for $x_0 \leqslant x \leqslant \sqrt{\omega}$. Thus, the curve defined by the quartic equation (4.2) is convex for $y \geqslant x \geqslant 0$ (and for $0 \leqslant y \leqslant x$ by symmetry). ■

Corollary 4.2. *Let ψ_0 be a maximal value of y and φ_0 be the corresponding value of x on the upper branch of the curve defined by the quartic equation (4.2). Then, $\varphi_0 < \sqrt{\omega} < \psi_0$ and*

$$\varphi_0 = \sqrt{\omega} - \tfrac{3}{8}\sqrt{\omega^3}h^2 + \mathrm{O}(h^4), \qquad \psi_0 = \sqrt{\omega} + \tfrac{1}{8}\sqrt{\omega^3}h^2 + \mathrm{O}(h^4). \quad (4.6)$$

Lemma 4.3. *The initial-value problem (4.1) with $\omega > 0$ and $0 < h < h_0$ admits a unique monotonically decreasing sequence $\{\phi_n\}_{n=0}^{\infty}$ for any $0 < \varphi < \sqrt{\omega}$ that converges to zero from above as $n \to \infty$. The sequence $\{\phi_n\}_{n=0}^{\infty}$ is continuous with respect to h and φ.*

Proof. By lemma 4.1 (see figure 1), there exists a unique lower branch of the curve in (4.2) below the diagonal $y = x$ for $0 < x < \sqrt{\omega}$ and the monotonically decreasing sequence $\{\phi_n\}_{n=0}^{\infty}$ with $0 < \varphi < \sqrt{\omega}$ satisfies the initial-value problem:

$$\begin{cases} \phi_{n+1} = \phi_n - hS_-(\phi_n, h), & n \in \mathbb{N}, \\ \phi_0 = \varphi, \end{cases} \quad (4.7)$$

where $hS_-(\phi, h) > 0$ is continuously differentiable with respect to ϕ and h. We shall prove that the monotonically decreasing sequence $\{\phi_n\}_{n=0}^{\infty}$ converges to zero from above. Since $Q(x, y)$ is a quartic polynomial, there exists a constant $C > 0$ that depends on ω and is independent of h, such that

$$(\phi_{n+1} - \phi_n)^2 \leqslant Ch^2\phi_n^2 \quad (4.8)$$

for all $\phi_{n+1} < \phi_n$. If h is sufficiently small, such that $Ch^2 < 1$, then $0 < \phi_{n+1} < \phi_n$, and the sequence $\{\phi_n\}_{n=0}^{\infty}$ is bounded from below by $\phi = 0$. By the Weierstrass theorem, the monotonically decreasing and bounded from below sequence $\{\phi_n\}_{n=0}^{\infty}$ converges as $n \to \infty$ to the fixed point $\phi = 0$. Continuity of the sequence $\{\phi_n\}_{n=0}^{\infty}$ in h and φ follows from smoothness of $hS_-(\phi, h)$ in h and φ. ■

Lemma 4.4. *There exists $N \geqslant 1$, such that the initial-value problem (4.1) with $\omega > 0$ and $0 < h < h_0$ admits a unique monotonically increasing sequence $\{\phi_n\}_{n=-\infty}^{N}$ for any $0 < \varphi < \sqrt{\omega}$ that converges to zero from above as $n \to -\infty$. The sequence $\{\phi_n\}_{n=-\infty}^{N}$ is continuous with respect to h and φ.*

Proof. By lemma 4.1 (see figure 1), there exists a unique upper branch of the curve in (4.2) above the diagonal $y = x$ for $0 < x < \sqrt{\omega}$ and the monotonically increasing sequence $\{\phi_n\}_{n=-\infty}^{0}$ with $0 < \varphi < \sqrt{\omega}$ satisfies the initial-value problem:

$$\begin{cases} \phi_{n+1} = \phi_n + hS_+(\phi_n, h), & (-n) \in \mathbb{N}, \\ \phi_0 = \varphi, \end{cases} \tag{4.9}$$

where $hS_+(\phi, h) > 0$ is continuously differentiable with respect to ϕ and h. Existence of $N \geqslant 1$ follows from the same equation (4.9) for $0 \leqslant n \leqslant N - 1$. The proof that the monotonically increasing sequence $\{\phi_n\}_{n=-\infty}^{N}$ converges to zero from above as $n \to -\infty$ is similar to the proof of lemma 4.3. Continuity of the sequence $\{\phi_n\}_{n=-\infty}^{N}$ in h and φ follows from smoothness of $hS_+(\phi, h)$ in h and φ. ■

Lemma 4.5. *The initial-value problem (4.1) with $\omega > 0$ and $0 < h < h_0$ admits a unique 2-periodic orbit $\{\phi_n\}_{n\in\mathbb{Z}}$ with $\phi_{n+1} \neq \phi_n$ and $\phi_{n+2} = \phi_n$ for any $\varphi_0 < \varphi < \psi_0$.*

Proof. By lemma 4.1 (see figure 1), the curve in (4.2) is symmetric about $y = x$ and has two branches in x below $y = x$ for $\sqrt{\omega} < x < \psi_0$. Therefore, any initial data on the branch between (φ_0, ψ_0) and (ψ_0, φ_0) leads to a unique 2-periodic orbit. ■

Corollary 4.6. *The initial-value problem (4.1) with $\omega > 0$ and $0 < h < h_0$ admits the following particular sequences.*

- *For any given $0 < \varphi \leqslant \psi_0$, there exists a symmetric single-humped sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ with maximum at $n = 0$, such that $\phi_n = \phi_{-n}$ (see figure 2(a)). The single-humped sequence is unique for $\varphi = \psi_0$ (see figure 2(b)). Let S_{on} denote the countable infinite set of values of $\{\phi_n\}_{n\in\mathbb{Z}}$ for the single-humped solution with $\varphi = \psi_0$.*
- *For any given $\sqrt{\omega} < \varphi < \psi_0$, there exists a symmetric double-humped sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ with local minimum at $n = 1$ and maxima at $n = 0$ and $n = 2$, such that $\phi_n = \phi_{-n+2}$ and $\varphi_0 < \phi_1 < \sqrt{\omega}$ (see figure 3(a)). The double-humped sequence becomes a unique 2-site top single-humped sequence for $\varphi = \sqrt{\omega}$ (see figure 3(b)). Let S_{off} denote the countable infinite set of values of $\{\phi_n\}_{n\in\mathbb{Z}}$ for the 2-site top single-humped solution with $\varphi = \sqrt{\omega}$.*
- *For any $\varphi \in (0, \psi_0)\backslash\{S_{\text{on}}, S_{\text{off}}\}$, there exists a unique non-symmetric single-humped sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ with $\phi_k \neq \phi_m$ for all $k \neq m$.*

Proposition 4.7. *The second-order difference equation (1.10) admits a translationally invariant single-humped sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ for any $0 < \phi_0 \leqslant \psi_0$ with $\phi_n = \phi(nh - s)$, where $n \in \mathbb{Z}$, $s \in \mathbb{R}$ and $\phi(z)$ is a continuous function. In the limit $h \to 0$, the function $\phi(z)$ converges pointwise to the function $\phi_s(z) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}z)$, i.e. there exists $C > 0$ and $s \in \mathbb{R}$ such that*

$$\max_{n\in\mathbb{Z}} |\phi_n - \phi_s(nh - s)| \leqslant Ch^2. \tag{4.10}$$

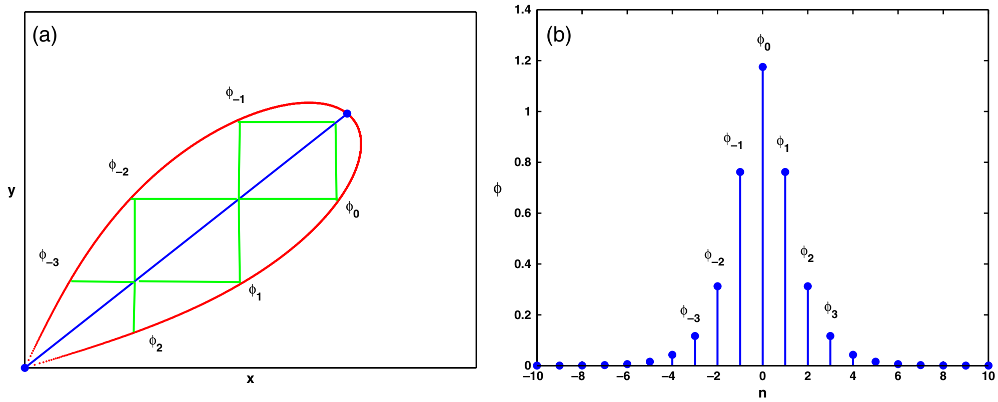

Figure 2. (*a*) A construction of the symmetric single-humped sequence from the solution of the quartic equation (4.2). (*b*) An example of the symmetric single-humped solution of the difference equation (4.1) for $\varphi = \psi_0$.

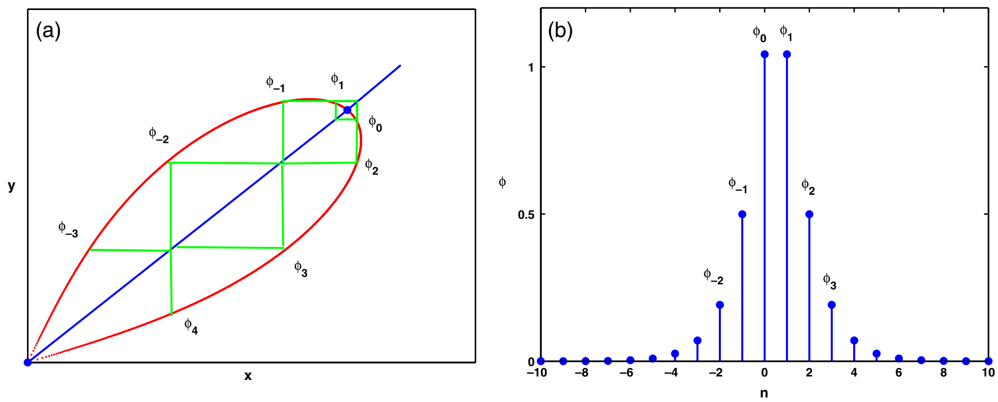

Figure 3. (*a*) A construction of the symmetric double-humped sequence from the solution of the quartic equation (4.2). (*b*) An example of the 2-site top single-humped solution of the difference equation (4.1) for $\varphi = \sqrt{\omega}$.

Proof. By the geometric construction of corollary 4.6, there exists a single-humped sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ for any $0 < \phi_0 \leqslant \psi_0$, which consists of the unique symmetric sequence for $\phi_0 \in S_{\text{on}}$, the unique 2-site top sequence for $\phi_0 \in S_{\text{off}}$ and the unique non-symmetric sequence for $\phi_0 \in (0, \psi_0)\backslash\{S_{\text{on}}, S_{\text{off}}\}$. By lemmas 4.3 and 4.4, the single-humped sequence is continuous in h and ϕ_0 such that it is translationally invariant. It remains to show the pointwise convergence of the sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ to the sequence $\{\phi_s(nh - s)\}_{n\in\mathbb{Z}}$ as $h \to 0$, where $\phi_s(z) = \sqrt{\omega}\,\text{sech}(\sqrt{\omega}z)$. By the translational invariance of the sequence, we can place the maximum at $n = 0$, such that $\phi_0 = \psi_0$, which is equivalent to the choice $s = 0$. Then, the bound (4.10) with $s = 0$ coincides with the bound for the convergence of the symmetric single-humped solution with $\{\phi_n\}_{n\in\mathbb{Z}} \subset S_{\text{on}}$ of the second-order difference equation (1.10) to the solution $\phi_s(hn) = \sqrt{\omega}\,\text{sech}(\sqrt{\omega}hn)$ of the second-order ODE (2.4). The standard proof of the error bound (4.10) uses property P3 of the nonlinear function f and relies on the formal power series in h^2 for exponentially decaying solutions (see appendix B in [BGKM91]). The formal series is different from the rigorous asymptotic solution of the difference map (1.10) along the stable and unstable manifolds by the exponentially small in h terms (see [T00]). A similar result (but with a different technique based on analysis of Fourier transforms) was proved in [FP99] for the discrete Fermi–Pasta–Ulam problem. ■

Example 4.8. For the AL lattice (1.7), when $\beta_1 = 1$ and $\beta_2 = \beta_3 = 0$, the second-order difference equation (1.10) admits the exact solution for translationally invariant single-humped solutions:

$$\phi_n = \psi_0 \operatorname{sech}(\kappa h n - s), \tag{4.11}$$

where $(s, \kappa) \in \mathbb{R}^2$ are arbitrary and (ψ_0, ω) are defined by

$$\psi_0 = \frac{\sinh(\kappa h)}{h}, \qquad \omega = \frac{4}{h^2} \sinh^2\left(\frac{\kappa h}{2}\right). \tag{4.12}$$

The exact stationary solution (4.11) follows from the exact travelling solution (1.17) for $\beta = 0$. The set S_{on} for the symmetric single-humped solution is defined by values of $\{\phi_n\}_{n\in\mathbb{Z}}$ for $s = 0$ and the set S_{off} for the 2-site top single-humped solution is defined for $s = \kappa h/2$. We find that

$$\psi_0 = \sqrt{\omega} \cosh\left(\frac{\kappa h}{2}\right) = \sqrt{\omega + \frac{\omega^2 h^2}{4}}, \tag{4.13}$$

in agreement with the asymptotic formula (4.6). The algebraic equation (4.2) can be reduced to the explicit roots for the one-step maps $hS_{\pm}(\phi, h)$ in the form

$$S_{\pm}(\phi, h) = \frac{\phi(\sqrt{4(\omega - \phi^2) + h^2\omega^2} \pm h(\omega - 2\phi^2))}{2(1 + h^2\phi^2)},$$

which satisfy all properties derived in lemma 4.1. In the limit $h \to 0$, the expression (4.11) with (4.12) converges to the solution (1.2) with $\omega = \kappa^2$ and $c = 0$.

Remark 4.9. For real-valued non-constant solutions, the second-order difference equation (1.10) can be reduced to another first-order difference equation:

$$\frac{1}{h}(\phi_{n+1} - \phi_n) = \tilde{g}(\phi_n, \phi_{n+1}), \tag{4.14}$$

where $\tilde{g}(\phi_n, \phi_{n+1})$ is a symmetric quadratic polynomial. The form (4.14) was used in [BOP05] (following pioneer papers [S97, S99]) in a search of another family of exceptional discretizations which admits *translationally invariant monotonic kinks* in the discrete ϕ^4 equation (see equation (21) in [BOP05]). By the implicit function theorem, the first-order difference equation (4.14) has only one branch of solutions on the plane (ϕ_n, ϕ_{n+1}). As a result, one cannot construct two (decreasing and increasing) sequences $\{\phi_n\}_{n=0}^{\infty}$ for *translationally invariant single-humped solutions*, such that other exceptional discretizations of [BOP05] are irrelevant for localized solutions of the discrete NLS equation (1.3). We also note that the squared equation (4.14) does not recover the first-order difference equation (2.1) due to parameter $\omega \neq 0$.

5. Existence of travelling solutions near $\omega = (\pi - 2)/h^2$ and $c = 1/h$

We shall consider the existence of continuously differentiable solutions of the differential advance–delay equation (1.11) with $c \neq 0$. A convenient parametrization of the solution is represented by the transformation of variables

$$\phi(z) = \frac{1}{h}\Phi(Z)\mathrm{e}^{\mathrm{i}\beta Z}, \qquad Z = \frac{z}{h}. \tag{5.1}$$

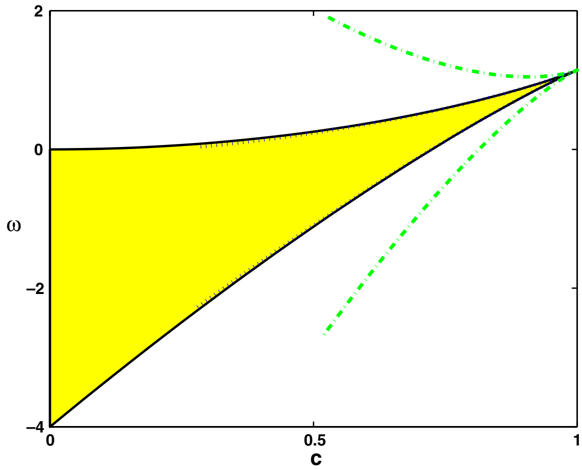

Figure 4. The boundary (5.2) of the domain of existence for travelling solutions of the differential advance–delay equation (1.11) and asymptotic approximations for the threshold on existence (5.6) (dotted curve) and the threshold on two-humped solutions (5.7) (dashed–dotted curve).

If parameters (ω, c) are expressed by (1.18), then $\Phi(Z)$ satisfies the differential advance–delay equation:

$$2\mathrm{i}\sin\beta\frac{\sinh(\kappa h)}{\kappa h}\Phi'(Z) + 2\cos\beta\cosh(\kappa h)\Phi(Z) = \Phi(Z+1)\mathrm{e}^{\mathrm{i}\beta} + \Phi(Z-1)\mathrm{e}^{-\mathrm{i}\beta}$$
$$+ f(\Phi(Z-1)\mathrm{e}^{-\mathrm{i}\beta}, \Phi(Z), \Phi(Z+1)\mathrm{e}^{\mathrm{i}\beta}), \qquad Z \in \mathbb{R},$$

where properties P2 and P6 of the function $f(v, u, w)$ have been used. If the single-humped localized solutions to the differential advance–delay equation exist, then the function $\Phi(Z)$ decays in Z with the real-valued rate κ, i.e. $\Phi(Z) \sim \mathrm{e}^{-\kappa h|Z|}$. (The function $\Phi(Z)$ may also contain decaying exponential terms with oscillatory tails due to the presence of complex eigenvalues.) The boundary on possible existence of travelling solutions corresponds to the line $\kappa = 0$, such that the travelling solutions may only exist in the exterior domain to the curve

$$\omega = \frac{2}{h}\beta c - \frac{4}{h^2}\sin^2\frac{\beta}{2}, \qquad c = \frac{1}{h}\sin\beta, \quad \beta \in [0, 2\pi]. \tag{5.2}$$

The boundary on existence of travelling solutions is shown in figure 4 for $c \geqslant 0$ by solid line. The shaded area on this figure corresponds to the non-existence domain with $\kappa \in \mathrm{i}\mathbb{R}$.

We apply the technique of the normal form reduction from [PR05] when parameters (ω, c) are close to the special values $\omega = (\pi - 2)/h^2$ and $c = 1/h$, which correspond to $\kappa = 0$ and $\beta = \pi/2$ on the boundary curve (5.2). By using a modified transformation of variables and parameters

$$\phi(z) = \frac{\epsilon}{h}\Phi(\zeta)\mathrm{e}^{\frac{\mathrm{i}\pi z}{2h}}, \qquad \zeta = \frac{\epsilon z}{h}, \qquad c = \frac{1+\epsilon^2 V}{h}, \qquad \omega = \frac{\pi - 2 + \epsilon^2\pi V + \epsilon^3\Omega}{h^2},$$

we rewrite the differential advance–delay equation (1.11) in the equivalent form

$$\mathrm{i}(\Phi(\zeta+\epsilon) - \Phi(\zeta-\epsilon) - 2\epsilon\Phi'(\zeta))$$
$$= \epsilon^3(2\mathrm{i}V\Phi'(\zeta) + \Omega\Phi(\zeta)) - \epsilon^2 f(-\mathrm{i}\Phi(\zeta-\epsilon), \Phi(\zeta), \mathrm{i}\Phi(\zeta+\epsilon)),$$

where $(\Omega, V) \in \mathbb{R}^2$ are rescaled parameters $(\omega, c) \in I_2 \subset \mathbb{R}^2$ near the point $\omega = (\pi - 2)/h^2$ and $c = 1/h$. By using the formal Taylor series expansions of the shift operators $\Phi(\zeta \pm \epsilon)$ in powers of ϵ and truncating the expansions at the order of $O(\epsilon^3)$, the differential advance–delay equation for $\Phi(\zeta)$ is converted to a third-order ODE which is related to the third-order derivative NLS equation [PR05]. The formal reduction can be proved with the rigorous technique of the centre manifold and normal forms (see analysis in [IP06] for kinks in the discrete ϕ^4 equation). Within this technique, the linear and nonlinear parts of the differential advance–delay equation for $\Phi(\zeta)$ are expanded as follows:

$$\Phi(\zeta+\epsilon) - \Phi(\zeta-\epsilon) - 2\epsilon\Phi'(\zeta) = \frac{\epsilon^3}{3}\Phi'''(\zeta) + O(\epsilon^5)$$

and

$$f(-i\Phi(\zeta-\epsilon), \Phi(\zeta), i\Phi(\zeta+\epsilon)) = (\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7)|\Phi|^2\Phi \\ + 2i\epsilon t(\alpha_2 + 2\alpha_8 - 2\alpha_9)|\Phi|^2\Phi'(\zeta) - 2i\epsilon(\alpha_3 - \alpha_8 - \alpha_9 + \alpha_{10})\Phi^2\bar{\Phi}'(\zeta) + O(\epsilon^2).$$

By rescaling the amplitude of $\Phi(\zeta)$ one can bring the term $|\Phi|^2\Phi$ to the order of $O(\epsilon^3)$, which matches the order of the term $\Phi'''(\zeta)$. However, the third-order ODE,

$$\frac{i}{3}\Phi''' - 2iV\Phi' - \Omega\Phi = |\Phi|^2\Phi,$$

has no single-humped localized solutions for any $(\Omega, V) \in \mathbb{R}^2$ (see [YA03,PR05]). Therefore, the necessary condition for existence of single-humped travelling solutions is the constraint on parameters of the cubic polynomial function (1.4)

$$\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7 = 0. \tag{5.3}$$

By rescaling the amplitude of $\Phi(\zeta)$ under the constraints (1.5), (2.6) and (5.3), the third-order ODE can be converted to the normalized form

$$\frac{i}{3}\Phi''' - 2iV\Phi' - \Omega\Phi + 2i|\Phi|^2\Phi' + i\gamma\Phi(|\Phi|^2)' = 0, \tag{5.4}$$

where

$$\gamma = -\frac{2(\alpha_3 - 2\alpha_9)}{(1 - 4\alpha_6 - 4\alpha_9)}.$$

Existence of single-humped localized solutions in the third-order ODE (5.4) is related to existence of embedded solitons in the third-order derivative NLS equation (see recent survey in [PY05]). We shall represent the basic facts about existence of single-humped localized solutions of the third-order ODE (5.4). By using the transformation of variables and parameters

$$\Phi = \lambda\Psi(Z)e^{-ikz}, \quad Z = \lambda\zeta, \quad \Omega = \tfrac{2}{3}k(\lambda^2 + k^2), \quad V = \tfrac{1}{6}(\lambda^2 - 3k^2),$$

we rewrite the ODE (5.4) in the form

$$\frac{i\lambda}{3}[\Psi''' - \Psi' + 6|\Psi|^2\Psi' + 3\gamma\Psi(|\Psi|^2)'] + k[\Psi'' - \Psi + 2|\Psi|^2\Psi] = 0, \tag{5.5}$$

where the real-valued parameters (λ, k) are arbitrary. The value $\lambda = 0$ defines the bifurcation line in the family of localized solutions of the ODE (5.5), which is equivalent to the curve $V = V_{\text{thr}}(\Omega)$ in the parameter plane (Ω, V), where

$$V_{\text{thr}}(\Omega) = -\frac{(3\Omega)^{2/3}}{2^{5/3}}. \tag{5.6}$$

Localized solutions may only exist above the threshold $V > V_{\text{thr}}(\Omega)$ for any $\Omega \in \mathbb{R}$. The asymptotic approximation of the threshold (5.6) is shown in figure 4 by dotted line. The threshold (5.6) is an asymptotic approximation of the exact curve (5.2) and the difference between the dotted and solid lines is almost invisible in figure 4.

- When $\gamma = 0$, the third-order ODE (5.5) is related to the integrable Hirota equation [H73], which admits the exact single-humped localized solution $\Psi = \operatorname{sech} Z$. The single-humped solution exists everywhere on the two-parameter plane (Ω, V) above the threshold (5.6), i.e. $V > V_{\text{thr}}(\Omega)$ and $\Omega \in \mathbb{R}$. This solution corresponds to the exact travelling solution (1.17) of the AL lattice.
- When $\gamma = 1$, the third-order ODE (5.5) is related to the integrable Sasa–Satsuma equation [SS91], which also admits the exact localized solutions everywhere on the two-parameter plane (Ω, V) above the threshold (5.6). (The exact solution and its properties are given in sections 4 and 5 of [SS91].) The localized solution has a single-humped profile for $V_{\text{thr}}(\Omega) < V < V_{\text{humps}}(\Omega)$ and a double-humped profile for $V > V_{\text{humps}}(\Omega)$ and $\Omega \neq 0$, where

$$V_{\text{humps}}(\Omega) = -\frac{(3\Omega)^{2/3}}{32^{4/3}}. \tag{5.7}$$

  The threshold (5.7) corresponds to the condition $\lambda^2 = k^2$ in the ODE (5.5). It is shown in figure 4 by a dashed–dotted curve. When $\Omega \to 0$ and $V > 0$, the distance between two humps diverges and the ODE (5.5) has again the single-humped solution for $\Omega = 0$ and $V > 0$ with $\Psi = (1/\sqrt{2})\operatorname{sech}(Z)$. This solution corresponds to $k = 0$ in the ODE (5.5).
- When $\gamma > -1$, the third-order ODE (5.4) has an exact single-humped localized solution for $\Omega = 0$ and $V > 0$ with $\Psi = (1/\sqrt{1+\gamma})\operatorname{sech} Z$. This solution matches the particular solution of the two-parameter family of exact solutions in the two integrable cases $\gamma = 0$ and $\gamma = 1$. It is shown in [PY05] by numerical analysis of the kernel of the linearization operator (see sections 3 and 4 in [PY05]) that the family of single-humped localized solutions for $\gamma \neq \{0, 1\}$ is isolated from other families of localized solutions, i.e. the one-parameter family with $\Omega = 0$ and $V > 0$ cannot be continued in the two-parameter plane (Ω, V) as a single-humped localized solution. The exact solution corresponds to the one-parameter travelling solution (1.19) for the NLS lattice (1.16) with $\chi < \frac{1}{2}$ and $\xi = \eta = \nu = 0$.

Results of the third-order ODE (5.4) give only the necessary condition on existence of travelling single-humped solution in the differential advance–delay equation (1.11) near the special values $\omega = (\pi - 2)/h^2$ and $c = 1/h$, i.e. if the smooth solution $\phi(z)$ to the differential advance–delay equation (1.11) exists, then it matches the analytical solutions of the third-order ODE (5.4). Persistence proof of true localized solutions is a delicate problem of rigorous analysis, which is left open even for travelling kinks of the discrete ϕ^4 equation [IP06]. (For comparison, persistence of solutions with exponentially small non-localized oscillatory tails was rigorously proved near the normal form equation in [IP06].)

6. Conclusion

We have considered stationary and travelling solutions of the discrete NLS equation (1.3) with the ten-parameter cubic nonlinearity (1.4) under the normalization constraint (1.5).

We have proved in lemma 2.1 that four constraints (2.6) are sufficient for reduction of the second-order difference equation (1.10) to the first-order difference equation (2.1). The main result (proposition 4.7) says that this reduction gives a sufficient condition for existence of translationally invariant stationary solutions for sufficiently small h that converge to the stationary solutions (1.2) with $c = 0$ in the continuum limit $h \to 0$. Furthermore, one more constraint (5.3) gives the necessary condition for existence of travelling solutions in the differential advance–delay equation (1.11) near the special values $\omega = (\pi - 2)/h^2$ and $c = 1/h$.

Combining all the constraints (1.5), (2.6) and (5.3), we have obtained the translationally invariant cubic NLS lattice with the nonlinear function (1.16), where $\alpha_3 = \chi$, $\alpha_6 = \xi$, $\alpha_8 = \eta$ and $\alpha_9 = \nu$. By lemmas 2.4, 2.7 and 2.9, this model conserves the momentum invariant (1.12), has no Hamiltonian structure (1.14) and may possess the power invariant (1.13). By corollary 2.8, the cubic NLS lattice with the nonlinearity (1.16) conserves N if $\nu = 0$ and

$$\text{(a)}\ \ \chi = \xi = 0, \qquad \text{(b)}\ \ \chi = 0,\ \xi = \tfrac{1}{4} - \eta, \qquad \text{(c)}\ \ \xi = \eta = 0, \qquad \text{(d)}\ \ \chi = \tfrac{1}{2} - 2\xi,\ \eta = 0.$$

Each of the four models has one free parameter. Models (a) and (d) were reported earlier in [DKSYT06] (see their equations (40) and (41)].

By lemmas 3.1 and 3.3, the second-order difference equation (1.10) has the reduction to the first-order difference equation (2.1) and may have an additional reduction to the first-order difference equation. By corollary 3.5, the cubic NLS lattice with the nonlinearity (1.16) possesses the first-order difference invariant $J_n = J_0$ in (3.2) if $\xi = \nu = 0$. The relevant model has two free parameters (χ, η). By corollary 3.6, the cubic NLS lattice with the nonlinearity (1.16) possesses the first-order difference invariant $\tilde{J}_n = J_0$ in (3.6) if $\eta = \nu = 0$ and $\chi = \frac{1}{2} - 2\xi$, which coincides with model (d).

Models (a) and (b) are related to the integrable Hirota equation (5.4) with $\gamma = 0$. Model (c) for $\chi < \frac{1}{2}$ admits one-parameter family of exact travelling solutions (1.19). For the particular value $\chi = -\frac{1}{2}$, this model is related to the integrable Sasa–Satsuma equation (5.4) with $\gamma = 1$. Model (d) is related to the third-order ODE (5.4) with $\gamma = -1$, which has no exact localized solutions.

We note that the linear interpolation between the dNLS and AL lattices (the Salerno model) fails to provide a model with translationally invariant stationary solutions. The rigorous proof of termination of travelling solutions in the Salerno model is an open problem as the previously known necessary condition for non-persistence fails in this case [BMR04].

Although the discrete NLS equation (1.3) with the nonlinear function (1.16) has translationally invariant stationary solutions (1.8), the existence of travelling solutions (1.9) for any $c \neq 0$ is an open problem. Persistence of travelling solutions for small values of c cannot be proved since infinitely many resonances with infinitely many Stokes constants appear in the limit $c \to 0$ [OPB05]. (It was shown in [OPB05] with numerical computations of the leading Stokes constant that none of the three particular discrete ϕ^4 lattices exhibits families of travelling solutions bifurcating from the family of translationally invariant stationary solutions.) Therefore, the naive method of asymptotic expansions in powers of c used in [AM03] is expected to fail beyond all orders of the asymptotic expansion.

Similarly, persistence of travelling solutions near the integrable normal form (5.4) cannot be proved since oscillatory tails are generic near the values $\omega = (\pi - 2)/h^2$ and $c = 1/h$ [IP06]. While the AL lattice has exact travelling solutions (1.17) between the two limiting cases $c = 0$

and $c = 1/h$, it is not clear if the translationally invariant NLS lattice with nonlinearity (1.16) exhibits any other families of travelling solutions on the plane (ω, c) besides the one-parameter family (1.19) for $\chi > -1$ and $\eta = 0$. Further results on existence of travelling solutions in the NLS lattice with the nonlinear function (1.16) will be considered in the forthcoming paper by means of numerical approximations (see [AEHV05, MCKC06]).

Acknowledgments

This work was initiated by discussions with I Barashenkov, M Johansson, P Kevrekidis and A Tovbis, to whom the author is grateful for useful remarks. The work was partly supported by the France–Canada SSHN Fellowship and by the EPSRC Visiting Research Fellowship.

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