JOURNAL OF MATHEMATICAL PHYSICS **46**, 053520 (2005)

Bifurcations from the endpoints of the essential spectrum in the linearized nonlinear Schrödinger problem

Scipio Cuccagna
DISMI, University of Modena and Reggio Emilia, 42100 Italy

Dmitry Pelinovsky
Department of Mathematics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

(Received 11 January 2005; accepted 9 March 2005; published online 10 May 2005)

We study bifurcations of eigenvalues from the endpoints of the essential spectrum in the linearized nonlinear Schrödinger problem in three dimensions. We show that a resonance and an eigenvalue of positive energy at the endpoint may bifurcate only to a real eigenvalue of positive energy, while an eigenvalue of negative energy at the endpoint may also bifurcate to complex eigenvalues. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1901345]

I. INTRODUCTION

We consider the nonlinear Schrödinger (NLS) equation in three dimensions,

$$i\psi_t = -\Delta\psi + U(x)\psi + F(|\psi|^2)\psi, \tag{1.1}$$

where $(x,t)\in\mathbb{R}^3\times\mathbb{R}$ and $\psi\in\mathbb{C}$. For suitable functions $U(x)$ and $F(|\psi|^2)$, the NLS equation (1.1) possesses special solutions,

$$\psi = \phi(x)e^{i\omega t}, \quad \omega > 0, \tag{1.2}$$

where $\phi(x)$ is an exponentially decreasing solution of the elliptic problem,

$$-\Delta\phi + \omega\phi + U(x)\phi + F(\phi^2)\phi = 0, \tag{1.3}$$

such that $\phi:\mathbb{R}^3\to\mathbb{R}$ and $\phi\in C^\infty$. Linearization of the nonlinear Schrödinger equation (1.1) with the ansatz,

$$\psi = (\phi(x) + \varphi(x)e^{izt} + \overline{\theta}(x)e^{-i\overline{z}t})e^{i\omega t}, \tag{1.4}$$

leads to the spectral problem,

$$\mathcal{L}\boldsymbol{\psi} = z\boldsymbol{\psi}, \tag{1.5}$$

where $\boldsymbol{\psi}=(\varphi,\theta)^T$ and the linear operator $\mathcal{L}$ on $L^2(\mathbb{R}^3\mapsto\mathbb{C}^2)$ takes the form $\mathcal{L}=\sigma_3\mathcal{H}$, where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} -\Delta+\omega+f(x) & g(x) \\ g(x) & -\Delta+\omega+f(x) \end{pmatrix}, \tag{1.6}$$

and

$$f(x) = U(x) + F(\phi^2) + F'(\phi^2)\phi^2, \quad g(x) = F'(\phi^2)\phi^2.$$

We assume that $U(x)\in C^\infty$ is exponentially decreasing and $F\in C^\infty, F(0)=0$, such that $f,g:\mathbb{R}^3\to\mathbb{R}$ are exponentially decaying C^∞-functions.

We denote the point spectrum of $\mathcal{L}$ as $\sigma_p(\mathcal{L})$ and the essential spectrum of $\mathcal{L}$ as $\sigma_e(\mathcal{L})$. We have shown in Cuccagna *et al.* (2005) that the spectrum of $\mathcal{L}$ is associated to the sign of the energy functional defined in $H^1(\mathbb{R}^3\mapsto\mathbb{C}^2)$,

$$h = \langle \boldsymbol{\psi}, \mathcal{H}\boldsymbol{\psi} \rangle. \tag{1.7}$$

In particular, an eigenvalue is of positive (respectively, negative) energy if $h > 0$ (respectively, $h < 0$). We showed in Cuccagna *et al.* (2005) that the nonsingular part of $\sigma_e(\mathcal{L})$ has always positive energy. We also proved that an embedded eigenvalue z of positive energy $h > 0$ disappears under a generic perturbation in the context of operator $\mathcal{L}$, while one of negative energy $h < 0$ bifurcates into isolated complex eigenvalues of $\sigma_p(\mathcal{L})$. The latter result generalizes an older work by Grillakis (1990), while the former one is new and consistent with the theory of embedded eigenvalues for standard Schrödinger operators.

In this paper we apply a generic perturbation to $\mathcal{L}$ in the case when the points $z = \pm\omega$, that are thresholds of $\sigma_e(\mathcal{L})$, are either eigenvalues or resonances of rank one. We show how the resonance or eigenvalue can either disappear in a different sheet of the Riemann surface associated to the resolvent of $\mathcal{L}$ or move away from the essential spectrum becoming an isolated real eigenvalue, or a pair of isolated complex eigenvalues. Furthermore we study the dependence of this singularity on the perturbation, obtaining an analogue of the similar work by Klaus and Simon (1980) on standard Schrödinger operators. We note that the resonance and eigenvalues at the endpoints are typically eliminated by hypothesis in the analysis of the NLS equation (1.1) and the linearized NLS problem (1.5) [Cuccagna (2001), Perelman (2004), Schlag (2004)]

One application of our result is the analysis of the NLS equation (1.1) in the case when operator $H_0 = -\Delta + U(x)$ supports $-\mu_1 < \cdots < -\mu_N$ negative eigenvalues and when the threshold 0 is either a resonance or an eigenvalue. It is well known [Tsai and Yau (2002)] that the NLS equation (1.1) admits then nonlinear standing wave solutions of form (1.2) with ω close to μ_n for any preassigned n and these standing wave solutions are small. Their stability properties depend crucially on the spectral properties of the related $\mathcal{L}$ which turns out to be a small perturbation of $\sigma_3(H_0 + \omega)$ by the smallness of the standing wave. In the case of $n = 1$, the discrete spectrum of $\mathcal{L}$ is close to that of $\sigma_3(H_0 + \omega)$, in particular has at least $2N$ elements with the point 0 of multiplicity 2. Our paper can be used to track the threshold singularity of operator $\mathcal{L}$ under perturbation.

Another possible application occurs when we add a small nonlinear perturbation $\epsilon\delta F(|\psi|^2)\psi$ to the main equation (1.1). Under appropriate conditions, the ground state can be shown to depend smoothly on ϵ. Now, if for $\epsilon = 0$ and a given value of ω operator $\mathcal{L}$ has resonances or eigenvalues at the thresholds, one can ask what happens to these singularities for nearby $\epsilon \neq 0$. The present paper gives a tool for analysis, avoiding details of specific applications.

For earlier work on "edge bifurcations," which is the name for bifurcations of resonances from the endpoints, see Kapitula and Sandstede [(2002), (2004)] where the main tool is the Evans function. Since the Evans function seems better suited to one-dimensional (1D) problems, our present work is based on theory by Jensen and Kato (1979) for scalar Schrödinger operators, applied here to the linearized NLS problem (1.5). Notice that our work is more general than Kapitula and Sandstede [(2002), (2004)] since it allows also eigenvalues at the endpoints and it does not depend on whether the solution $\phi(x)$ is a ground state. Furthermore we answer to a specific question [see Corollary 5.4 in Kapitula *et al.* (2004)] by showing that it is impossible for a resonant pole to become an unstable (complex) eigenvalue.

Our paper is structured as follows. The formalism of operator resolvent near the endpoints is exposed in Sec. II. Bifurcations of a simple resonance and a simple eigenvalue from the endpoint are described in Secs. III and IV, respectively. Section V gives the proof of Lemma 4.7.

II. OPERATOR RESOLVENT NEAR THE ENDPOINTS

Using standard Pauli matrices σ_2 and σ_3, we write $\mathcal{L}$ explicitly as

$$\mathcal{L} = (-\Delta + \omega + f(x))\sigma_3 + ig(x)\sigma_2, \tag{2.1}$$

such that $\sigma_3\mathcal{L}\sigma_3 = \mathcal{L}^*$. We also decompose the operator $\mathcal{L}$ into the unbounded differential part $\mathcal{L}_0$ and bounded potential part $V(x)$ as $\mathcal{L} = \mathcal{L}_0 + V(x)$, where $\mathcal{L}_0 = (-\Delta + \omega)\sigma_3$ and $V(x) = f(x)\sigma_3$

$+ig(x)\sigma_2$. We assume that $V(x)$ is continuous, exponentially decaying matrix-valued function, such that

$$|V_{i,j}(x)| \leq Ce^{-\alpha|x|}, \quad \forall\, x \in \mathbb{R}^3, \quad 1 \leq i,j \leq 2, \tag{2.2}$$

for some $\alpha > 0, C > 0$. In these notations, the spectral problem (1.5) is rewritten as

$$(\mathcal{L}_0 - z)\boldsymbol{\psi} = -V(x)\boldsymbol{\psi}. \tag{2.3}$$

We use the weighted H_s^τ and L_s^2 spaces defined as

$$H_s^\tau = \{f:(\omega - \Delta)^{r/2} f \in L_s^2\}, \tag{2.4}$$

$$L_s^2 = \{f:(1+|x|^2)^{s/2} f \in L^2\}. \tag{2.5}$$

We also use the standard Fourier transform in L^2,

$$f(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{ipx}\, \mathrm{d}x, \quad f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(p) e^{-ipx}\, \mathrm{d}p. \tag{2.6}$$

We denote the operator resolvent as $\mathcal{R}(z) = (\mathcal{L} - z)^{-1}$ and $\mathcal{R}_0(z) = (\mathcal{L}_0 - z)^{-1}$, such that

$$\mathcal{R}(z) = (I + \mathcal{R}_0(z)V)^{-1}\mathcal{R}_0(z). \tag{2.7}$$

The domain of the essential spectrum $\sigma_e(\mathcal{L})$ is located at $\mathcal{D}_e = (-\infty, -\omega] \cup [\omega, \infty)$, such that the points $z = \pm\omega$ are endpoints of $\sigma_e(\mathcal{L})$.

Let us consider bifurcations from the endpoint $z = \omega$, since bifurcations from the other endpoint $z = -\omega$ are obtained from the symmetry of the problem (2.3). When $z \notin \mathcal{D}_e$ but $|z - \omega|$ is small, we introduce the parametrization,

$$z = \omega - \zeta^2, \quad \operatorname{Re} \zeta > 0, \tag{2.8}$$

and consider the kernel of $R_0(\zeta) \equiv \mathcal{R}_0(\omega - \zeta^2), \operatorname{Re} \zeta > 0$ in the explicit form

$$R_0(\zeta) = \frac{\sigma_3}{4\pi|x-y|} \begin{bmatrix} e^{-\zeta|x-y|} & 0 \\ 0 & e^{-\sqrt{2\omega-\zeta^2}|x-y|} \end{bmatrix}. \tag{2.9}$$

When $\zeta \to 0$, the resolvent $R_0(\zeta)$ has the Taylor series expansion in $\mathcal{B}(H_s^{-1}, H_{-s}^1), s > \frac{3}{2}$,

$$R_0(\zeta) = R_0 - \zeta R_1 + \zeta^2 R_2 - \zeta^3 R_3 + \mathrm{O}(\zeta^4), \quad \operatorname{Re} \zeta > 0, \tag{2.10}$$

where

$$R_0 = \frac{\sigma_3}{4\pi|x-y|} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\sqrt{2\omega}|x-y|} \end{bmatrix}, \quad R_1 = \frac{1}{4\pi} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.11}$$

$$R_2 = \frac{\sigma_3}{8\pi} \begin{bmatrix} |x-y| & 0 \\ 0 & \dfrac{e^{-\sqrt{2\omega}|x-y|}}{\sqrt{2\omega}} \end{bmatrix}, \quad R_3 = \frac{1}{24\pi} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} |x-y|^2. \tag{2.12}$$

If the function $\boldsymbol{\psi}(x)$ solves the problem (2.3) for $z = \omega$, the components $\psi_1(x)$ and $\psi_2(x)$ satisfy the equations

$$\Delta\psi_1 = f\psi_1 + g\psi_2, \tag{2.13}$$

$$(\Delta - 2\omega)\psi_2 = g\psi_1 + f\psi_2. \tag{2.14}$$

Define

$$C_0 = \int_{\mathbb{R}^3} (f\psi_1 + g\psi_2)\mathrm{d}x. \tag{2.15}$$

The bounded linear operator $(I+R_0V)$ is defined in $L^2_{-s}, s>\frac{1}{2}$. When it has a kernel, then $|C_0| < \infty$ for the function $\boldsymbol{\psi}(x)$. The following two cases are different: (i) $C_0 \neq 0$ and (ii) $C_0=0$. The first case is referred to as the resonance and the second case is referred to as the eigenvalue of the linearized NLS problem (2.3).

Since $(f\psi_1+g\psi_2) \in H^2_s, s>0$ and $f(x), g(x)$ decay exponentially, it follows from (2.14) that $\psi_2(x)$ decays exponentially too, such that $\psi_2 \in H^2_s, s>0$. When $C_0 \neq 0, \psi_1(x)$ decays algebraically as $1/|x|$, such that $\psi_1 \in H_{-s}, s>\frac{1}{2}$. When $C_0=0, \psi_1(x)$ decays more rapidly as $1/|x|^2$, such that $\psi_1 \in H_{-s}, s>-\frac{1}{2}$, including the energy space $H^1_0 \subset L^2$. We summarize that

$$C_0 \neq 0, \quad \mathrm{Ker}(I+R_0V) \in H^1_{-s}, \quad s>\tfrac{1}{2}, \tag{2.16}$$

$$C_0 = 0, \quad \mathrm{Ker}(I+R_0V) \in H^1_{-s}, \quad s>\tfrac{1}{2}. \tag{2.17}$$

In either case, we study the kernel of the adjoint operator $\mathrm{Ker}(I+V^*R_0)$ and the generalized kernel $N_g(I+R_0V)=\cup_{n=1}^{\infty}\mathrm{Ker}(I+R_0V)^n$ in the following two lemmas.

Lemma 2.1: Let $\boldsymbol{\psi} \in \mathrm{Ker}(I+R_0V), \boldsymbol{\psi} \in H^1_{-s}, s>\frac{1}{2}$. *Then,* $\boldsymbol{\phi}=V^*\sigma_3\boldsymbol{\psi} \in \mathrm{Ker}(I+V^*R_0), \boldsymbol{\phi} \in H^{-1}_s, s>\frac{1}{2}$, *such that* $V^*\sigma_3$ *is an injection of* $\mathrm{Ker}(I+R_0V)$ *to* $\mathrm{Ker}(I+V^*R_0)$.

Proof: It follows from direct computations for $\boldsymbol{\psi} \neq \mathbf{0}$ that

$$(I+V^*R_0)V^*\sigma_3\boldsymbol{\psi} = V^*(I+R_0V^*)\sigma_3\boldsymbol{\psi} = V^*\sigma_3(I+R_0V)\boldsymbol{\psi} = \mathbf{0},$$

such that $\boldsymbol{\phi}=V^*\sigma_3\boldsymbol{\psi} \in \mathrm{Ker}(I+V^*R_0)$ and $\boldsymbol{\phi} \in H^{-1}_s, s>\frac{1}{2}$. We show that $\boldsymbol{\phi} \neq \mathbf{0}$. Since $V^*\sigma_3=\sigma_3V$, then $\boldsymbol{\phi}=\sigma_3V\boldsymbol{\psi}=\mathbf{0}$ implies that $V\boldsymbol{\psi}=\mathbf{0}$ and $(\mathcal{L}_0-\omega)\boldsymbol{\psi}=\mathbf{0}$, or equivalently, $\Delta\psi_1=0$ and $(\Delta-2\omega)\psi_2=0$. However, if $\boldsymbol{\psi} \in H^1_{-s}, s>\frac{1}{2}$, then the latter equations imply that $\boldsymbol{\psi}=\mathbf{0}$, which is impossible. ■

Lemma 2.2: The generalized kernel $N_g(I+R_0V)$ *in* $H^1_{-s}, s>\frac{1}{2}$, *coincides with* $\mathrm{Ker}(I+R_0V)$.

Proof: Let $\boldsymbol{\psi} \in \mathrm{Ker}(I+R_0V)$. The generalized kernel $N_g(I+R_0V)$ is bigger than the kernel $\mathrm{Ker}(I+R_0V)$ iff there exists a solution of the derivative equation,

$$(I+R_0V)\boldsymbol{\psi}_1 = \boldsymbol{\psi}, \quad \boldsymbol{\psi}_1 \in H^1_{-s}, \quad s>\tfrac{1}{2}. \tag{2.18}$$

Then,

$$\langle\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle = \langle(I+R_0V)\boldsymbol{\psi}_1, V^*\sigma_3\boldsymbol{\psi}\rangle = \langle\boldsymbol{\psi}_1, V^*(I+R_0V^*)\sigma_3\boldsymbol{\psi}\rangle = \langle\boldsymbol{\psi}_1, V^*\sigma_3(I+R_0V)\boldsymbol{\psi}\rangle = 0,$$

such that $\langle\boldsymbol{\psi}, \sigma_3V\boldsymbol{\psi}\rangle = -\langle\boldsymbol{\psi}, \sigma_3(\mathcal{L}_0-\omega)\boldsymbol{\psi}\rangle=0$. Since $\sigma_3(\mathcal{L}_0-\omega)=-\Delta+\omega-\omega\sigma_3$ and $\boldsymbol{\psi} \neq \mathbf{0}$, the quadratic form $\langle\boldsymbol{\psi}, \sigma_3V\boldsymbol{\psi}\rangle$ is nonzero for $\boldsymbol{\psi} \in H^1_{-s}, s>\frac{1}{2}$, such that no solution $\boldsymbol{\psi}_1(x)$ exists in the problem (2.18). ■

Since geometric and algebraic dimensions of the kernel of $(I+R_0V)$ coincide in $H^1_{-s}, s>\frac{1}{2}$, we introduce a natural splitting,

$$H^1_{-s} = \mathrm{Ker}(I+R_0V) \oplus [\mathrm{Ker}(I+V^*R_0)]^{\perp}, \tag{2.19}$$

$$H^{-1}_s = [\mathrm{Ker}(I+R_0V)]^{\perp} \oplus \mathrm{Ker}(I+V^*R_0), \tag{2.20}$$

where $\perp$ is defined in terms of the pairing of H^1_{-s} and H^{-1}_s. We denote $\mathcal{S}_0$ as the projection of H^1_{-s} to $\mathrm{Ker}(I+R_0V)$, associated to the splitting (2.19), and $\mathcal{S}_0^*$ as the dual projection in the dual space

H_s^{-1}, associated to the splitting (2.20). In what follows, we assume that the dimension of $\mathrm{Ker}(I+R_0V)$ is one. The two cases in (2.16) and (2.17) are considered separately in Secs. III and IV.

III. BIFURCATION OF A SIMPLE RESONANCE

Here we assume that $\mathrm{Ker}(I+R_0V)\subsetneq H_0^1$, such that $z=\omega$ is a resonance of $\sigma_e(\mathcal{L})$ but not an eigenvalue. It is clear from (2.13) in the case of $C_0\neq 0$ that there is only one eigenvector $\boldsymbol{\psi}(x)$ which decays as $1/|x|$ and belongs to $H_{-s}^1, s>\frac{1}{2}$. Therefore, the resonance at $z=\omega$ is always simple, such that the dimension of $\mathrm{Ker}(I+R_0V)$ in $H_{-s}^1, s>\frac{1}{2}$ is one. Since $C_0\neq 0$, we normalize the eigenvector $\boldsymbol{\psi}\in\mathrm{Ker}(I+R_0V)$ by the condition

$$\int_{\mathbb{R}^3}(f\psi_1+g\psi_2)\mathrm{d}x=\sqrt{4\pi}, \tag{3.1}$$

such that

$$R_1V\boldsymbol{\psi}=\frac{1}{\sqrt{4\pi}}\mathbf{e}_1, \quad \mathbf{e}_1=\begin{bmatrix}1\\0\end{bmatrix}.$$

By Lemma 2.2, we have $\langle\boldsymbol{\psi},V^*\sigma_3\boldsymbol{\psi}\rangle\neq 0$, such that the spectral projection, associated with the splitting (2.19), is

$$\mathcal{S}_0=\boldsymbol{\psi}\frac{\langle\cdot,V^*\sigma_3\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\psi},V^*\sigma_3\boldsymbol{\psi}\rangle}. \tag{3.2}$$

Following the analysis of Jensen and Kato (1979), we study the Taylor series expansions of $(I+R_0(\zeta)V)$ near $z=\omega$, or equivalently $\zeta=0$.

Lemma 3.1: Let $\mathcal{S}=\mathcal{I}-\mathcal{S}_0$. The following statements are true:

(i) *Operator $\mathcal{S}(I+R_0V)\mathcal{S}$ is invertible in $\mathcal{B}(\mathcal{S}H_{-s}^1,\mathcal{S}H_{-s}^1)$ with the inverse denoted by $\mathcal{K}$.*

(ii) *Operator $\mathcal{S}_0^*V^*\sigma_3R_1V\mathcal{S}_0=\mathcal{S}_0^*V^*R_1V\mathcal{S}_0$ is invertible in $\mathcal{B}(\mathcal{S}_0H_{-s}^1,\mathcal{S}_0^*H_s^{-1})$ with the inverse $\boldsymbol{\psi}\langle\cdot,\boldsymbol{\psi}\rangle$.*

Proof: To prove (i), we notice that the splitting in (2.19) and (2.20) is invariant for $(I+R_0V)$, such that $\mathcal{S}(I+R_0V)\mathcal{S}$ is the restriction of $(I+R_0V)$ on $[\mathrm{Ker}(I+V^*R_0)]^{\perp}$. Since it has an empty kernel and R_0V is compact, the statement (i) follows from the Fredholm alternative theorem.

To prove (ii), we denote the dual of $\boldsymbol{\psi}$ as $\boldsymbol{\phi}\in\mathcal{S}_0^*H_s^{-1}\subseteq\mathrm{Ker}(I+V^*R_0)$, such that $\langle\boldsymbol{\psi},\boldsymbol{\phi}\rangle=1$. Then $\mathcal{S}_0^*=\boldsymbol{\phi}\langle\cdot,\boldsymbol{\psi}\rangle$ and

$$\mathcal{S}_0^*V^*R_1V\mathcal{S}_0\boldsymbol{\psi}=\frac{1}{\sqrt{4\pi}}\mathcal{S}_0^*V^*\mathbf{e}_1=\frac{1}{\sqrt{4\pi}}\boldsymbol{\phi}\langle\mathbf{e}_1,V\boldsymbol{\psi}\rangle=\boldsymbol{\phi},$$

where the last equality is due to normalization (3.1). Therefore, $\mathcal{S}_0^*V^*R_1V\mathcal{S}_0=\boldsymbol{\phi}\langle\cdot,\boldsymbol{\phi}\rangle$, which has the inverse $\boldsymbol{\psi}\langle\cdot,\boldsymbol{\psi}\rangle$. ■

Lemma 3.2: The following expansion is valid in $\mathcal{B}(H_{-s}^1,H_{-s}^1)$ for $s>\frac{5}{2}$ near $\zeta=0$:

$$(I+R_0(\zeta)V)^{-1}=-\zeta^{-1}\boldsymbol{\psi}\langle\cdot,V^*\sigma_3\boldsymbol{\psi}\rangle+\mathcal{K}+\mathrm{O}(\zeta), \quad \mathrm{Re}\,\zeta>0. \tag{3.3}$$

Proof: Let

$$\mathcal{X}=\mathcal{S}H_{-s}^1\oplus\mathcal{S}_0H_{-s}^1=H_{-s}^1, \quad \mathcal{Y}=\mathcal{S}H_{-s}^1\oplus\mathcal{S}_0^*H_s^{-1}$$

and

$$\mathcal{B} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \zeta^{-\frac{1}{2}}\mathcal{S}_0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \zeta^{-\frac{1}{2}}\mathcal{S}_0^* V^* \sigma_3 \end{bmatrix}.$$

If $\mathcal{S}\mathbf{u}=\mathbf{0}$, then $\mathbf{u} \in \mathcal{S}_0 H^1_{-s}$ and $\mathcal{S}_0^* V^* \sigma_3$ is injective by Lemma 2.1 and definition of $\mathcal{S}_0^*$. As a result, $\mathcal{B}:\mathcal{X} \mapsto H^1_{-s}$ is an isomorphism, while $\mathcal{C}:H^1_{-s} \mapsto \mathcal{Y}$ is injective. Let $\mathcal{A} \equiv \mathcal{C}(I+R_0(\zeta)V)\mathcal{B}$. Then,

$$\mathcal{A} = \mathcal{C}(I+R_0V)\mathcal{B} - \zeta\mathcal{C}R_1V\mathcal{B} + \mathrm{O}(\zeta^2) = \begin{bmatrix} \mathcal{S}(I+R_0V)\mathcal{S} & 0 \\ 0 & -\mathcal{S}_0^* V^* \sigma_3 R_1 V \mathcal{S}_0 \end{bmatrix} + \mathrm{O}(\zeta).$$

If $\mathcal{A}$ is invertible, $\mathcal{B}$ is surjective, and $\mathcal{C}$ is injective, then, by Lemma 3.12 of Jensen and Kato (1979), we have

$$(I+R_0(\zeta)V)^{-1} = \mathcal{B}\mathcal{A}^{-1}\mathcal{C},$$

such that the expansion (3.3) holds by the Neumann expansion argument. ■

Using (2.7), (2.10), and (3.3), we have the following result.

Corollary 3.3: The following expansion is valid in $\mathcal{B}(H_s^{-1},H^1_{-s})$ *for* $s>\frac{5}{2}$ *near* $\zeta=0$:

$$R(\zeta) = \zeta^{-1}\boldsymbol{\psi}\langle\cdot,\sigma_3\boldsymbol{\psi}\rangle + \mathrm{O}(1), \quad \mathrm{Re}\,\zeta>0. \tag{3.4}$$

In order to work in L^2 rather than in a weighted space $L^2_{-s}, s>\frac{1}{2}$, we use the Birman–Schwinger formulation of the spectral problem (2.3) for $V=B^*A$ [Cuccagna *et al.* (2005)];

$$(I+\mathcal{Q}_0(z))\boldsymbol{\Psi} = \mathbf{0}, \quad \mathcal{Q}_0(z) = A\mathcal{R}_0(z)B^*, \quad z \in \mathbb{C} \setminus \mathcal{D}_e, \tag{3.5}$$

where

$$\boldsymbol{\Psi} = -A\boldsymbol{\psi}, \quad \boldsymbol{\psi} = \mathcal{R}_0(z)B^*\boldsymbol{\Psi}. \tag{3.6}$$

It is clear from (2.9) that $Q_0(\zeta) \equiv \mathcal{Q}_0(\omega-\zeta^2)$, initially defined for $\mathrm{Re}\,\zeta>0$, admits an analytical extension in an open set around $\zeta=0$ with values in $\mathcal{B}(L^2,H^2)$, such that $Q_0=Q(0)$ is well defined. Moreover, for any positive integer n, the map $\boldsymbol{\psi} \mapsto -A\boldsymbol{\psi}$ is an isomorphism,

$$\mathrm{Ker}(I+R_0V)^n \subset L^2_{-s} \mapsto \mathrm{Ker}(I+Q_0)^n \subset L^2, \quad s>\tfrac{1}{2}, \tag{3.7}$$

such that the inverse map is $\boldsymbol{\Psi} \mapsto R_0B^*\boldsymbol{\Psi}$. By Lemma 2.2, there exists an $(I+Q_0)$-invariant splitting,

$$L^2 = \mathrm{Ker}(I+Q_0) \oplus [\mathrm{Ker}(I+Q_0^*)]^{\perp}. \tag{3.8}$$

We denote $\mathcal{P}_0$ by the projection of L^2 on $\mathrm{Ker}(I+Q_0)$ and $\mathcal{P}_0^*$ by the dual projection.

With the use of Corollary 3.3, we consider the family of operators $\mathcal{L}_1=\mathcal{L}+\epsilon V_1$, where the perturbation potential $V_1(x)$ satisfies the same assumption as the potential $V(x)$, while the unperturbed operator $\mathcal{L}$ has a simple resonance. Let $\mathcal{R}_1(z)=(\mathcal{L}_1-z)^{-1}$ and define $\mathcal{Q}(z)=A_1\mathcal{R}(z)B_1^*$ and $\mathcal{Q}_1(z)=A_1\mathcal{R}_1(z)B_1^*=(I+\epsilon\mathcal{Q}(z))^{-1}\mathcal{Q}(z)$, where $V_1=B_1^*A_1$. We can always factorize V_1 so that $A_1=A$. It follows from (2.7) with $A_1=A$ that

$$\mathcal{Q}(z) = (I+A\mathcal{R}_0(z)B_1^*)^{-1}A\mathcal{R}_0(z)B_1^* = (I+\mathcal{Q}_0(z))^{-1}A\mathcal{R}_0(z)B_1^*.$$

We again use parametrization (2.8) and denote $Q(\zeta) \equiv \mathcal{Q}(\omega-\zeta^2), \mathrm{Re}\,\zeta>0$. Since $(I+Q_0(\zeta))^{-1}$ can be extended meromorphically from $\mathrm{Re}\,\zeta>0$ to $\mathrm{Re}\,\zeta \leqslant 0$, then $Q(\zeta)$ is a meromorphic function of $\zeta \in \mathbb{C}$. Similarly, $Q_1(\zeta)$ is also a meromorphic function of $\zeta \in \mathbb{C}$.

The main results of this section are formulated in the following two propositions.

Proposition 3.4: Let ϵ *be a small positive parameter. If* $\langle\psi,V_1^*\sigma_3\psi\rangle<0$, *then* $\sigma_p(\mathcal{L}_1)$ *includes a real eigenvalue* $z(\epsilon), z(\epsilon)<\omega$, *such that*

$$z(\epsilon) = \omega - \epsilon^2\langle\boldsymbol{\psi},V_1^*\sigma_3\boldsymbol{\psi}\rangle^2 + \mathrm{o}(\epsilon^2). \tag{3.9}$$

If $\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle > 0$, *then* $\sigma_p(\mathcal{L}_1)$ *does not include an eigenvalue in the neighborhood of* $z=\omega$. *In both cases, resonance at* $z=\omega$ *disappears at* $\epsilon \neq 0$.

Proof: It follows from Corollary 3.3 that

$$Q(\zeta) = \zeta^{-1} A \boldsymbol{\psi} \langle \cdot , B_1 \sigma_3 \boldsymbol{\psi} \rangle + Q_c(\zeta),$$

where $Q_c(\zeta)$ is bounded for small $|\zeta|$. Then for Re $\zeta > 0$, we have

$$Q_1(\zeta) = [I + \epsilon \zeta^{-1}(I + \epsilon Q_c(\zeta))^{-1} A \boldsymbol{\psi} \langle \cdot , B_1 \sigma_3 \boldsymbol{\psi} \rangle]^{-1}(I + \epsilon Q_c(\zeta))^{-1} Q(\zeta), \tag{3.10}$$

which can be extended meromorphically from Re $\zeta > 0$ to Re $\zeta \leqslant 0$. By Fredholm theorem, the first factor on the right-hand side of (3.10) has singularities at $\zeta = \zeta(\epsilon)$, where $\zeta(\epsilon)$ is the solution of the linear equation,

$$\zeta + \epsilon \langle \boldsymbol{\psi}, V_1^* \sigma_3 \psi \rangle - \epsilon^2 \langle Q_c(0) A \boldsymbol{\psi}, B_1 \sigma_3 \boldsymbol{\psi} \rangle + \mathrm{O}(\epsilon^3) = 0. \tag{3.11}$$

By implicit function theorem, there is a unique solution $\zeta = \zeta(\epsilon)$ for small ϵ, such that

$$\zeta(\epsilon) = - \epsilon \langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle + 0(\epsilon^2). \tag{3.12}$$

The map $\zeta = \sqrt{\omega - z}$ transforms the domain $\mathcal{D} = \{ z \in \mathbb{C} : z \notin [\omega, \infty) \}$ into the first sheet of the Riemann surface $\mathcal{D}_1 = \{ \zeta \in \mathbb{C} : \text{Re } \zeta > 0 \}$, which is connected with the second sheet $\mathcal{D}_2 = \{ \zeta \in \mathbb{C} : \text{Re } \zeta < 0 \}$. When the root of (3.11) belongs to $\mathcal{D}_1$, the corresponding point $z \in \mathcal{D}$ is the eigenvalue of $\mathcal{L}_1$, at least for small ϵ, since the singularities of $\mathcal{Q}_1(z) = A \mathcal{R}_1(z) B_1^*$ coincide with the singularities of $\mathcal{R}_1(z)$. When the root of (3.11) belongs to $\mathcal{D}_2$, the corresponding point z belongs to the compliment of the closure of $\mathcal{D}$ in the Riemann surface, which continues $\mathcal{D}$ across $z \in [\omega, \infty)$. As a result, it does not belong to the closure of $\mathcal{D}$, such that it is not an eigenvalue. ∎

Proposition 3.5: If $\epsilon > 0$ *and* $\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle < 0$, *the new eigenvalue* $z(\epsilon)$ *with the corresponding eigenvector* $\boldsymbol{\psi}_\epsilon(x)$ *has the positive energy norm* (1.7), *such that*

$$\langle \boldsymbol{\psi}_\epsilon, \mathcal{H} \boldsymbol{\psi}_\epsilon \rangle > 0, \quad \forall\, \epsilon > 0. \tag{3.13}$$

Proof: Using (3.5), we look for a solution of the problem:

$$(I + A R_0(\zeta(\epsilon))(B^* + \epsilon B_1^*))(\boldsymbol{\Psi} + \tilde{\boldsymbol{\Psi}}_\epsilon) = 0,$$

where $\boldsymbol{\Psi} = -A \boldsymbol{\psi}$ and $\tilde{\boldsymbol{\Psi}}_\epsilon \in [\text{Ker}(I + Q_0^*)]^\perp$. Projecting the equation on $[\text{Ker}(I + Q_0^*)]^\perp$ with operator $\mathcal{P}_0^*$, we have the problem,

$$F(\tilde{\boldsymbol{\Psi}}_\epsilon, \epsilon) = \mathcal{P}_0^*(I + A R_0(\zeta(\epsilon))(B^* + \epsilon B_1^*))\tilde{\boldsymbol{\Psi}}_\epsilon + \epsilon \mathcal{P}_0^* A R_0(\zeta(\epsilon)) B_1^* \boldsymbol{\Psi} + \mathcal{P}_0^* A [R_0(\zeta(\epsilon)) - R_0] B^* \boldsymbol{\Psi} = 0,$$

where $F(0,0) = 0$ and

$$\frac{\partial F}{\partial \tilde{\boldsymbol{\Psi}}_\epsilon}(0,0) = \mathcal{P}_0^*(I + Q_0^*).$$

Since $\mathcal{P}_0^*(I + Q_0^*)$ is an isomorphism in $[\text{Ker}(I + Q_0^*)]^\perp$, the function $\tilde{\boldsymbol{\Psi}}_\epsilon$ is a smooth function of ϵ, by implicit function theorem. Therefore, we define

$$\boldsymbol{\psi}_\epsilon = R_0(\zeta(\epsilon))(B^* + \epsilon B_1^*)(\boldsymbol{\Psi} + \tilde{\boldsymbol{\Psi}}_\epsilon).$$

Since $R_0(\zeta(\epsilon)) \in B(L_s^2, H_{-s}^2)$ and $(B^* + \epsilon B_1^*)(\boldsymbol{\Psi} + \tilde{\boldsymbol{\Psi}}_\epsilon) \in L_s^2, s > \frac{1}{2}$ are continuous in ϵ at $\epsilon = 0$, we conclude that

$$\lim_{\epsilon\to 0^+} \boldsymbol{\psi}_\epsilon(x) = \boldsymbol{\psi}(x), \quad \boldsymbol{\psi}_\epsilon \in L^2_{-s}, \quad s > \tfrac{1}{2}.$$

It follows from the system (2.13) and (2.14) in the case (2.16) that $\psi_1 \notin L^2(\mathbb{R})$ and $\psi_2 \in L^2(\mathbb{R})$. By Fatou lemma, we have the limit

$$\lim_{\epsilon\to 0} \langle \boldsymbol{\psi}_\epsilon, \mathcal{H}\boldsymbol{\psi}_\epsilon\rangle = \omega\|\psi_1\|^2_{L^2} - \omega\|\psi_2\|^2_{L^2} = +\infty.$$

By continuity, the inequality (3.13) holds for $\epsilon > 0$. ■

IV. BIFURCATION OF A SIMPLE EIGENVALUE

Here we assume that $\mathrm{Ker}(I+R_0V) \subseteq H^1_0$, such that $z=\omega$ is an eigenvalue of $\sigma_p(\mathcal{L})$. Let $\boldsymbol{\psi} \in \mathrm{Ker}(\mathcal{L}-\omega) \subset L^2$ and we assume that $\dim \mathrm{Ker}(\mathcal{L}-\omega)=1$. Let $\mathcal{P}_0$ be the spectral projection in L^2 onto $\mathrm{Ker}(\mathcal{L}-\omega)$, such that

$$\mathcal{P}_0 = \boldsymbol{\psi}\frac{\langle \cdot, \sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle}. \tag{4.1}$$

It is proved in Cuccagna *et al.* (2005), Sec. III that a simple eigenvalue has non zero energy (1.7) such that $\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle \neq 0$. Since $C_0=0$ in (2.15), it is clear that $R_1V\boldsymbol{\psi}=\mathbf{0}$. Furthermore, we have the following result.

Lemma 4.1: Let $\mathbf{u},\mathbf{v}$ *be two functions in* $H^{-1}_s, s>\frac{5}{2}$, *such that* $R_1\mathbf{u}=R_1\mathbf{v}=0$ *and* $\langle \mathbf{e}_1,\mathbf{u}\rangle = \langle \mathbf{e}_1,\mathbf{v}\rangle = 0$, *where* $\mathbf{e}_1=(1,0)^T$. *Then,*

$$\langle R_2\mathbf{u},\mathbf{v}\rangle = -\langle R_0\mathbf{u}, R_0\mathbf{v}\rangle. \tag{4.2}$$

Proof: The proof is given with a direct computation

$$\langle R_2\mathbf{u},\mathbf{v}\rangle = \lim_{\zeta\to 0}\zeta^{-2}\langle [R_0(\zeta)-R_0]\mathbf{u},\mathbf{v}\rangle = \lim_{\zeta\to 0}\zeta^{-2}\left\langle \sigma_3\begin{bmatrix} \frac{1}{p^2+\zeta^2}-\frac{1}{p^2} & 0 \\ 0 & \frac{1}{p^2+2\omega-\zeta^2}-\frac{1}{p^2+2\omega}\end{bmatrix}\hat{\mathbf{u}},\hat{\mathbf{v}}\right\rangle$$

$$= -\left\langle \begin{bmatrix} \frac{1}{p^4} & 0 \\ 0 & \frac{1}{(p^2+2\omega)^2}\end{bmatrix}\hat{\mathbf{u}},\hat{\mathbf{v}}\right\rangle = -\langle R_0\mathbf{u}, R_0\mathbf{v}\rangle,$$

where $\hat{\mathbf{u}}(p)$ is the Fourier transform of $\mathbf{u}(x)$, defined by (2.6). ■

We apply the splitting of $H^1_{-s}, s>-\frac{1}{2}$, defined by (2.19), with projection $\mathcal{S}_0$ to $\mathrm{Ker}(I+R_0V)$, such that $\mathcal{S}=\mathcal{I}-\mathcal{S}_0$.

Lemma 4.2: The following statements are true:

(i) $\mathcal{P}_0^*V^*R_2\sigma_3V\mathcal{P}_0 = -\mathcal{P}_0^*\sigma_3\mathcal{P}_0$ *and* $\mathcal{S}_0^*V^*\sigma_3R_2V\mathcal{S}_0 = -\mathcal{S}_0^*\sigma_3\mathcal{S}_0$.
(ii) *Operator* $\mathcal{S}_0^*\sigma_3\mathcal{S}_0$ *is invertible in* $\mathcal{B}(\mathcal{S}_0H^1_{-s}, \mathcal{S}_0^*H^{-1}_s)$, *with the inverse* $\mathcal{P}_0\sigma_3$.

Proof: To prove (i), we note that $\sigma_3V\mathcal{P}_0\mathbf{u}$ and $V\mathcal{P}_0\mathbf{v}$ for any $\mathbf{u},\mathbf{v}\in H^1_{-s}, s>-\frac{1}{2}$ satisfy assumptions of Lemma 4.1 and, therefore,

$$\langle R_2\sigma_3 V\mathcal{P}_0\mathbf{u}, V\mathcal{P}_0\mathbf{v}\rangle = -\langle R_0\sigma_3 V\mathcal{P}_0\mathbf{u}, R_0 V\mathcal{P}_0\mathbf{v}\rangle = -\langle \sigma_3\mathcal{P}_0\mathbf{u}, \mathcal{P}_0\mathbf{v}\rangle = -\langle \mathcal{P}_0^*\sigma_3\mathcal{P}_0\mathbf{u}, \mathbf{v}\rangle.$$

The second part of (i) follows from the relations $\mathcal{P}_0\mathcal{S}_0=\mathcal{S}_0$ and $\mathcal{S}_0^*\mathcal{P}_0^*=\mathcal{S}_0^*$.

To prove (ii), let $\boldsymbol{\phi}\in\mathcal{S}_0^* H_s^{-1}\subseteq \mathrm{Ker}(I+V^*R_0)$ be the dual of $\boldsymbol{\psi}$, such that $\langle\boldsymbol{\psi},\boldsymbol{\phi}\rangle=1$. Therefore,

$$\mathcal{S}_0 = \boldsymbol{\psi}\langle\cdot,\boldsymbol{\phi}\rangle, \quad \mathcal{S}_0^* = \boldsymbol{\phi}\langle\cdot,\boldsymbol{\psi}\rangle, \quad \mathcal{S}_0^*\sigma_3\mathcal{S}_0 = \boldsymbol{\phi}\langle\cdot,\boldsymbol{\phi}\rangle\langle\boldsymbol{\psi},\sigma_3\boldsymbol{\psi}\rangle,$$

such that

$$(\mathcal{S}_0^*\sigma_3\mathcal{S}_0)^{-1} = \boldsymbol{\psi}\frac{\langle\cdot,\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\psi},\sigma_3\boldsymbol{\psi}\rangle} = \mathcal{P}_0\sigma_3.$$

■

Lemma 4.3: The following expansion is valid in $\mathcal{B}(H^1_{-s},H^1_{-s})$ for $s>\frac{5}{2}$ near $\zeta=0$:

$$(I+R_0(\zeta)V)^{-1} = -\zeta^{-2}\mathcal{P}_0 V + \zeta^{-1}\mathcal{P}_0 V R_3 V\mathcal{P}_0 V + \mathrm{O}(1), \quad \mathrm{Re}\,\zeta>0. \tag{4.3}$$

Proof: The proof is similar to that of Lemma 3.2. Let

$$\mathcal{X} = \mathcal{S}H^1_{-s}\oplus\mathcal{S}_0H^1_{-s} = H^1_{-s}, \quad \mathcal{Y} = \mathcal{S}H^1_{-s}\oplus\mathcal{S}_0^*H_s^{-1}$$

and

$$\mathcal{B} = \begin{bmatrix}\mathcal{S} & 0\\ 0 & \zeta^{-1}\mathcal{S}_0\end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix}\mathcal{S} & 0\\ 0 & \zeta^{-1}\mathcal{S}_0^*V^*\sigma_3\end{bmatrix}.$$

If $\mathcal{S}\mathbf{u}=\mathbf{0}$, then $\mathbf{u}\in\mathrm{Ker}(I+R_0V)$ and $V^*\sigma_3$ is injective in $\mathrm{Ker}(I+V^*R_0)$ by Lemma 2.1. As a result, $\mathcal{B}:\mathcal{X}\mapsto\mathcal{X}$ is surjective, while $\mathcal{C}:\mathcal{X}\mapsto\mathcal{Y}$ is injective. Let $\mathcal{A}\equiv\mathcal{C}(I+R_0(\zeta)V)\mathcal{B}$. Using the Taylor series expansion (2.10), we have

$$\mathcal{A} = \mathcal{A}_0 - \zeta\mathcal{A}_1 + \mathrm{O}(\zeta^2),$$

where

$$\mathcal{A}_0 = \begin{bmatrix}\mathcal{S}(I+R_0V)\mathcal{S} & 0\\ 0 & \mathcal{S}_0^*V^*\sigma_3R_2V\mathcal{S}_0\end{bmatrix}, \quad \mathcal{A}_1 = \begin{bmatrix}\mathcal{S}R_1V\mathcal{S} & 0\\ 0 & \mathcal{S}_0^*V^*\sigma_3R_3V\mathcal{S}_0\end{bmatrix}.$$

By Neumann expansions, we have $\mathcal{A}^{-1}=\mathcal{A}_0^{-1}+\zeta\mathcal{A}_0^{-1}\mathcal{A}_1\mathcal{A}_0^{-1}+\mathrm{O}(\zeta^2)$, such that

$$\mathcal{A}^{-1} = \begin{bmatrix}\mathcal{K}_0 & 0\\ 0 & -(\mathcal{S}_0^*\sigma_3\mathcal{S}_0)^{-1}\end{bmatrix} + \zeta\begin{bmatrix}\mathcal{K}_0\mathcal{S}R_1V\mathcal{S}\mathcal{K}_0 & 0\\ 0 & (\mathcal{S}_0^*\sigma_3\mathcal{S}_0)^{-1}\mathcal{S}_0^*V^*\sigma_3R_3V\mathcal{S}_0(\mathcal{S}_0^*\sigma_3\mathcal{S}_0)^{-1}\end{bmatrix} + \mathrm{O}(\zeta^2),$$

where $\mathcal{K}_0=(\mathcal{S}(I+R_0V)\mathcal{S})^{-1}$. Since $(I+R_0(\zeta)V)^{-1}=\mathcal{B}\mathcal{A}^{-1}\mathcal{C}$, the expansion (4.3) holds. ■

Using (2.7), (2.10), and (4.3), as well as $\mathcal{P}_0VR_0=-\mathcal{P}_0$ and $\mathcal{P}VR_1=0$, we have the following result.

Corollary 4.4: The following expansion is valid in $\mathcal{B}(H_s^{-1},H^1_{-s})$ for $s>\frac{5}{2}$ near $\zeta=0$:

$$R(\zeta) = \zeta^{-2}\mathcal{P}_0 - \zeta^{-1}\mathcal{P}_0VR_3V\mathcal{P}_0 + \mathrm{O}(1), \quad \mathrm{Re}\,\zeta>0. \tag{4.4}$$

Similar to Sec. III, we use Corollary 4.4 and consider the family of operators $\mathcal{L}_1=\mathcal{L}+\epsilon V_1(x)$, where the perturbation potential $V_1(x)$ satisfies the same assumption as the potential $V(x)$, while the unperturbed operator $\mathcal{L}$ has a simple eigenvalue. Let $\mathcal{R}_1(z)=(\mathcal{L}_1-z)^{-1}$ and define $\mathcal{Q}(z)=A\mathcal{R}(z)B_1^*$ and $\mathcal{Q}_1(z)=A\mathcal{R}_1(z)B_1^*=(I+\epsilon\mathcal{Q}(z))^{-1}\mathcal{Q}(z)$, where $V_1=B_1^*A$. As in Sec. III, functions $Q(\zeta)$ and $Q_1(\zeta)$ can be meromorphically extended from $\mathrm{Re}\,\zeta>0$ to $\mathrm{Re}\,\zeta\leqslant 0$. The main result of this section is formulated in the following proposition.

Proposition 4.5: Let ϵ be a small positive parameter and let $\langle R_3V\boldsymbol{\psi},V^*\sigma_3\boldsymbol{\psi}\rangle\neq 0$. Then,

(i) *eigenvalue at $z=\omega$ disappears as $\epsilon\neq 0$.*

(ii) *Let* $\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle < 0$. *Then* $\sigma_p(\mathcal{L}_1)$ *near* $z=\omega$ *includes one real eigenvalue* $z(\epsilon)<\omega$ *if* $\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle > 0$ *and two complex eigenvalues* $z_{1,2}(\epsilon)$ *if* $\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle < 0$. *Asymptotic approximations of the eigenvalues* $z(\epsilon)$ *and* $z_{1,2}(\epsilon)$ *are given by*

$$z(\epsilon) = \omega + \epsilon \frac{\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle} + O(\epsilon^{3/2}) \tag{4.5}$$

and

$$\mathrm{Re}(z_{1,2}(\epsilon)) = \omega + \epsilon \frac{\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle} + O(\epsilon^2), \tag{4.6}$$

$$\mathrm{Im}(z_{1,2}(\epsilon)) = \pm \epsilon^{3/2} \sqrt{\frac{\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle}} \frac{\langle R_3 V \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi} \rangle \langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle^2} + O(\epsilon^2). \tag{4.7}$$

(iii) *Let* $\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle > 0$. *Then* $\sigma_p(\mathcal{L}_1)$ *near* $z=\omega$ *includes one real eigenvalue* $z(\epsilon)<\omega$, *given by* (4.5), *if* $\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle < 0$ *and no eigenvalues if* $\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle > 0$.

The proof of Proposition is based on the following elementary result.

Lemma 4.6: Consider a quadratic equation,

$$\zeta^2 - \epsilon \zeta F(\epsilon, \zeta) + \epsilon G(\epsilon, \zeta) = 0, \tag{4.8}$$

where $F(\epsilon,\zeta)$ *and* $G(\epsilon,\zeta)$ *are analytic in* ϵ *and* ζ *at the point* (0, 0), *such that* $G(0,0) \neq 0$ *and*

$$\left.\frac{\partial F(0,\zeta)}{\partial \zeta}\right|_{\zeta=0} = \left.\frac{\partial G(0,\zeta)}{\partial \zeta}\right|_{\zeta=0} = 0. \tag{4.9}$$

Then, for small ϵ, *the quadratic equation* (4.8) *has exactly two solutions* $\zeta_{1,2}(\epsilon)$, *such that* $|\zeta_j(\epsilon) - \zeta_{j0}(\epsilon)| = O(\epsilon^{3/2})$, *where* $\zeta_{j0}(\epsilon), j=1, 2$, *are solutions of the quadratic equation*

$$\zeta^2 - \epsilon \zeta F(0,0) + \epsilon G(0,0) = 0. \tag{4.10}$$

Proof: Let $\mu = \epsilon^{1/2}$ and substitute $\zeta = \mu \xi$. Introducing another parameter λ, we rewrite the quadratic equation (4.8) in the form,

$$\xi^2 - \mu \xi F(\lambda \mu^2, \lambda \mu \xi) + G(\lambda \mu^2, \lambda \mu \xi) = 0. \tag{4.11}$$

The case $\lambda=1$ gives (4.8), while the case $\lambda=0$ gives (4.10). Since $G(0,0)\neq 0$ by assumption, there exist two analytical solutions of (4.11), by the implicit function theorem, which are defined for small $\mu>0$ and $\lambda \in [0,1]$. Since

$$\xi(1,\mu) - \xi(0,\mu) = \int_0^1 \partial_\lambda \xi(\lambda,\mu) d\lambda,$$

we apply implicit differentiation of (4.11) and find that

$$[2\xi - \mu F(\lambda\mu^2,\lambda\mu\xi) - \lambda\mu^2 \xi \partial_2 F(\lambda\mu^2,\lambda\mu\xi) + \lambda\mu \partial_2 G(\lambda\mu^2,\lambda\mu\xi)]\partial_\lambda \xi - \mu^3 \xi \partial_1 F(\lambda\mu^2,\lambda\mu\xi)$$
$$- \mu^2 \xi^2 \partial_2 F(\lambda\mu^2,\lambda\mu\xi) + \mu^2 \partial_1 G(\lambda\mu^2,\lambda\mu\xi) + \mu\xi \partial_2 G(\lambda\mu^2,\lambda\mu\xi) = 0,$$

where ∂_1 and ∂_2 are derivatives in the first and second arguments. Under constraints (4.9), we have $\partial_\lambda \xi = O(\mu^2)$, such that $|\zeta(1,\mu) - \zeta(0,\mu)| = O(\mu^3)$. ■

Proof of Proposition 4.5: It follows from Corollary 4.4 that

$$Q(\zeta) = \zeta^{-2} A \mathcal{P}_0 B_1^* - \zeta^{-1} A \mathcal{P}_0 V R_3 V \mathcal{P}_0 B_1^* + Q_c(\zeta), \tag{4.12}$$

where $Q_c(\zeta)$ is bounded for small ζ. As a result, for Re $\zeta>0$, we have

$$Q_1(\zeta) = [I + \epsilon(I + \epsilon Q_c(\zeta))^{-1}[\zeta^{-2}A\mathcal{P}_0B_1^* - \zeta^{-1}A\mathcal{P}_0VR_3V\mathcal{P}_0B_1^*]]^{-1}(I + \epsilon Q_c(\zeta))^{-1}Q(\zeta),$$

which can be extended meromorphically for Re $\zeta \leqslant 0$. Singularities of $Q_1(\zeta)$ near $\zeta=0$ correspond to zeros of

$$\det\left[\zeta^2 + \epsilon(I + \epsilon Q_c(\zeta))^{-1}\left(A\boldsymbol{\psi}\frac{\langle B_1^* \cdot , \sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle} - \zeta A\boldsymbol{\psi}\frac{\langle R_3V\mathcal{P}_0B_1^* \cdot , V^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle}\right)\right].$$

The determinant equation can be written as the quadratic equation (4.8), where $F(\epsilon,\zeta)$ and $G(\epsilon,\zeta)$ are defined for Re $\zeta > 0$ as

$$F(\epsilon,\zeta) = \frac{\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle^2}\langle B_1^*(I + \epsilon Q_c(\zeta))^{-1}A\boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle, \tag{4.13}$$

$$G(\epsilon,\zeta) = \frac{1}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle}\langle B_1^*(I + \epsilon Q_c(\zeta))^{-1}A\boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle, \tag{4.14}$$

and they can be analytically continued to Re $\zeta \leqslant 0$. It is clear from (4.13) and (4.14) that

$$F(0,\zeta) = \frac{\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle^2}, \quad G(0,\zeta) = \frac{\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle},$$

such that the condition (4.9) is satisfied. By Lemma 4.6, there exist two solutions of (4.8) in the $O(\epsilon^{3/2})$-neighborhood of solutions of (4.10), when $\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle \neq 0$. Solutions of (4.10) are expanded as

$$\zeta_{\pm 0}(\epsilon) = \pm\,\epsilon^{1/2}\sqrt{-\frac{\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle}} + \frac{\epsilon}{2}\frac{\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle\langle \boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle^2} + O(\epsilon^{3/2}). \tag{4.15}$$

When $\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle / \langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle < 0$, there is a unique real eigenvalue of operator $\mathcal{L}_1$ in the neighborhood of $z=\omega$, such that $z=\omega-\zeta_{+0}^2+O(\epsilon^{3/2})$, which results in (4.5). The other solution $\zeta_{-0}(\epsilon)$ corresponds to Re $\zeta<0$ and, by arguments in the proof of Proposition 3.4, it does not correspond to an eigenvalue of operator $\mathcal{L}_1$.

When $\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle / \langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle > 0$, we have to consider the $O(\epsilon)$ term of the asymptotic expansion (4.15). Due to the constraint $C_0=0$ in (2.15), we have

$$\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle = -\frac{1}{12\pi}\sum_{j=1}^{3}|(x_j, f\psi_1 + g\psi_2)|^2 \leqslant 0. \tag{4.16}$$

Since $\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle \neq 0$, then $\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle < 0$. Therefore, it follows from (4.15) that

$$\mathrm{Im}\,\zeta_{\pm 0}(\epsilon) = \pm\,\epsilon^{1/2}\sqrt{\frac{\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle}} + O(\epsilon^{3/2}),$$

$$\mathrm{Re}\,\zeta_{\pm 0}(\epsilon) = \frac{\epsilon}{2}\frac{\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle^2} + O(\epsilon^{3/2}).$$

In the case $\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle > 0$ and $\langle \boldsymbol{\psi}, \mathbf{V}_1^*\sigma_3\boldsymbol{\psi}\rangle > 0$, we have Re $\zeta_{\pm 0} < 0$, such that no eigenvalues of $\mathcal{L}_1$ exist in the neighborhood of $z=\omega$. In the case $\langle \boldsymbol{\psi}, \sigma_3\boldsymbol{\psi}\rangle < 0$ and $\langle \boldsymbol{\psi}, V_1^*\sigma_3\boldsymbol{\psi}\rangle < 0$, we have Re $\zeta_{\pm 0} > 0$, such that two complex eigenvalues of $\mathcal{L}_1$ exist in the neighborhood of $z=\omega$, with the asymptotic approximations (4.6) and (4.7). ■

A more special result occurs in the case when $\langle R_3V\boldsymbol{\psi}, V^*\sigma_3\boldsymbol{\psi}\rangle = 0$, which includes spherically symmetric potential $V(x)$ with spherically symmetric eigenvector $\boldsymbol{\psi}(x)$, see (4.16). In order to study this special case, we need to extend the theory of wave operators from Kato (1966) and Cuccagna *et al.* (2005). Following Cuccagna *et al.* (2005), we consider a decomposition of L^2 into the $\mathcal{L}$-invariant Jordan blocks:

$$L^2 = \sum_{z \in \sigma_p(\mathcal{L})} N_g(\mathcal{L} - z) \oplus X_c(\mathcal{L}), \quad X_c(\mathcal{L}) = \left[\sum_{z \in \sigma_p(\mathcal{L})} N_g(\mathcal{L}^* - z) \right]^{\perp}, \tag{4.17}$$

and, equivalently,

$$L^2 = \sum_{z \in \sigma_p(\mathcal{L})} N_g(\mathcal{L}^* - z) \oplus X_c(\mathcal{L}^*), \quad X_c(\mathcal{L}^*) = \left[\sum_{z \in \sigma_p(\mathcal{L})} N_g(\mathcal{L} - z) \right]^{\perp}, \tag{4.18}$$

where $\sigma_p(\mathcal{L}) = \sigma_p(\mathcal{L}^*)$ and $N_g(\mathcal{L} - z) = \cup_{n=1}^{+\infty} \mathrm{Ker}(\mathcal{L} - z)^n$. The invariant splittings (4.17) and (4.18) hold in the assumption that $\sigma_p(\mathcal{L}) \cap \sigma_e(\mathcal{L})$ is a union of simple eigenvalues, such that $N_g(\mathcal{L} - z) = \mathrm{Ker}(\mathcal{L} - z)$ for $z \in \mathcal{D}_e$. The action of $\mathcal{L}$ in $X_c(\mathcal{L})$ is given by the scattering theory of wave operators Kato (1966), which is based on the following existence result.

Lemma 4.7: Let $A(x)$ and $B(x)$ be exponentially decaying potentials and $\sigma_p(\mathcal{L}) \cap \sigma_e(\mathcal{L})$ be a union of simple eigenvalues, which includes the endpoints $z = \pm \omega$ without resonance. Let $\langle R_3 V \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi} \rangle = 0$. There exist isomorphisms $W: L^2 \mapsto X_c(\mathcal{L})$ and $Z: X_c(\mathcal{L}) \mapsto L^2$, which are inverse of each other, defined as follows:

$$\forall u \in L^2, \forall v \in X_c(\mathcal{L}^*), \langle Wu, v \rangle = \langle u, v \rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle A(\mathcal{L}_0 - \lambda - i\epsilon)^{-1} u, B(\mathcal{L}^* - \lambda + i\epsilon)^{-1} v \rangle \mathrm{d}\lambda, \tag{4.19}$$

and

$$\forall u \in X_c(\mathcal{L}), \forall v \in L^2, \langle Zu, v \rangle = \langle u, v \rangle + \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle A(\mathcal{L} - \lambda - i\epsilon)^{-1} u, B(\mathcal{L}_0 - \lambda + i\epsilon)^{-1} v \rangle \mathrm{d}\lambda. \tag{4.20}$$

We prove this result in Sec. V. Using Lemma 4.7, we consider bifurcation of the simple eigenvalue in the special case when $\langle R_3 V \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi} \rangle = 0$.

Proposition 4.8: Let ϵ be a small positive parameter and $\langle R_3 V \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi} \rangle = 0$. Then, Proposition 4.5 holds, but the asymptotic expansion (4.7) *is modified as follows*:

$$\mathrm{Im}(z_{1,2}(\epsilon)) = \pm \epsilon^{5/2} \frac{2\pi^2 |\hat{\psi}_1(0)|^2}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle} \sqrt{\frac{\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi} \rangle}} + \mathrm{O}(\epsilon^3), \tag{4.21}$$

where $\tilde{\boldsymbol{\psi}} = Z \mathcal{P}_c V_1 \boldsymbol{\psi}$ and $\hat{\boldsymbol{\psi}}(p)$ is the Fourier transform of $\tilde{\boldsymbol{\psi}}(x)$.

Proof: We use the splittings (4.17) and (4.18) and define operator $\mathcal{P}_c$ as the projection of L^2 on $X_c(\mathcal{L})$. It is clear from (4.12) that

$$Q_c(\zeta) = A(\mathcal{I} - \mathcal{P}_0) R(\zeta) B_1^* = \sum_{z \in \sigma_p(\mathcal{L}) \setminus \{\omega\}} A \mathcal{P}_z R(\zeta) B_1^* + A P_c \mathcal{R}(\zeta) B_1^*.$$

In the special case $\langle R_3 V \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi} \rangle = 0$, the quadratic equation (4.8) has $F(\epsilon, \zeta) = 0$ and

$$G(\epsilon, \zeta) = G(0, \zeta) + \epsilon \partial_1 G(0, \zeta) + \mathrm{O}(\epsilon^2),$$

where

$$G(0,\zeta) = \frac{\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi}\rangle},$$

and

$$\partial_1 G(0,\zeta) = -\frac{\langle B_1^* Q_c(\zeta) A \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi}\rangle} = -\frac{\langle (\mathcal{I} - \mathcal{P}_0) R(\zeta) V_1 \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi}\rangle}.$$

The term $\partial_1 G(0,\zeta)$ consists of the contribution from all eigenvalues of $\sigma_p(\mathcal{L})$ different from $z=\omega$ and from the contribution from $X_c(\mathcal{L})$. The first contribution can be estimated as follows:

$$\mathrm{Im} \sum_{z \in \sigma_p(\mathcal{L})\setminus\{\omega\}} \langle \mathcal{P}_z R(\zeta) V_1 \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi}\rangle = \mathrm{O}(\zeta^2).$$

This estimate is based on the expansion for real-valued $V_1(x)$ and $\boldsymbol{\psi}(x)$ [see Cuccagna *et al.* (2005)],

$$\sum_{z \in \sigma_p(\mathcal{L})\setminus\{\omega\}} \langle \mathcal{P}_z \mathcal{R}(z) V_1 \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi}\rangle = \sum_{z_j \in \mathbb{R}} (z - z_j)^{-1} \langle P_{z_j} V_1 \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi}\rangle + \sum_{z_j \in \mathbb{C}} [(z - z_j)^{-1} \langle P_{z_j} V_1 \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi}\rangle$$
$$+ (z - \overline{z_j})^{-1} \langle P_{\overline{z_j}} V_1 \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi}\rangle]. \tag{4.22}$$

Since $\mathcal{P}_{z_j}\sigma_3$ is self-adjoint for $z_j \in \mathbb{R}$, the factor $\langle P_{z_j} V_1 \boldsymbol{\psi}, V^* \sigma_3 \boldsymbol{\psi}\rangle$ is real. Then, the first term in (4.22) has the imaginary part of order $\mathrm{O}(\mathrm{Im}\, z)$ or $\mathrm{O}(\zeta^2)$ for $z=\omega-\zeta^2$. Similarly, the operator $(\mathrm{Re}\, z - z_j)^{-1}\mathcal{P}_{z_j}\sigma_3 + (\mathrm{Re}\, z - \overline{z_j})^{-1}\mathcal{P}_{\overline{z_j}}\sigma_3$ is self-adjoint for $z_j \in \mathbb{C}$, such that the second term in (4.22) has the imaginary part of order $\mathrm{O}(\mathrm{Im}\, z)$ or $\mathrm{O}(\zeta^2)$. The second contribution in $\partial_1 G(0,\zeta)$ can be estimated by using wave operators, which satisfy the following identities [Cuccagna *et al.* (2005)]:

$$P_c^* \sigma_3 = \sigma_3 P_c, \quad W^* \sigma_3 = \sigma_3 Z, \quad Z^* \sigma_3 = \sigma_3 W, \quad Z\mathcal{L} = \mathcal{L}_0 Z. \tag{4.23}$$

Since $\mathcal{P}_c V_1 \boldsymbol{\psi} \in X_c(\mathcal{L})$, there exists $\tilde{\boldsymbol{\psi}} \in L^2$, such that $\mathcal{P}_c V_1 \boldsymbol{\psi} = W\tilde{\boldsymbol{\psi}}$. As a result, we have

$$\langle \mathcal{P}_c R(\zeta) V_1 \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi}\rangle = \langle R(\zeta) V_1 \boldsymbol{\psi}, \sigma_3 W \tilde{\boldsymbol{\psi}}\rangle = \langle Z R(\zeta) V_1 \boldsymbol{\psi}, \sigma_3 \tilde{\boldsymbol{\psi}}\rangle = \langle R_0(\zeta) \tilde{\boldsymbol{\psi}}, \sigma_3 \tilde{\boldsymbol{\psi}}\rangle.$$

Since $\langle R_0 \tilde{\boldsymbol{\psi}}, \sigma_3 \tilde{\boldsymbol{\psi}}\rangle$ and $\langle R_1 \tilde{\boldsymbol{\psi}}, \sigma_3 \tilde{\boldsymbol{\psi}}\rangle$ are real valued, we finally have

$$\mathrm{Im}\, \partial_1 G(0,\zeta) = \frac{\langle R_1 \tilde{\boldsymbol{\psi}}, \sigma_3 \tilde{\boldsymbol{\psi}}\rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi}\rangle} \mathrm{Im}\, \zeta + \mathrm{O}(\zeta^2). \tag{4.24}$$

The quadratic equation (4.8) is now read as follows

$$\zeta^2 + \epsilon G(\epsilon, \zeta) = 0. \tag{4.25}$$

In the case $G(0,0) > 0$, we have from Lemma 4.6 that

$$\mathrm{Im}\, \zeta_{1,2}(\epsilon) = \pm \epsilon^{1/2} \sqrt{\frac{\langle \boldsymbol{\psi}, V_1^* \sigma_3 \boldsymbol{\psi}\rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi}\rangle}} + \mathrm{O}(\epsilon^{3/2}),$$

in addition, by expansion (4.24), we have from (4.25) that $2\, \mathrm{Re}\, \zeta\, \mathrm{Im}\, \zeta = -\epsilon^2\, \mathrm{Im}\, \zeta \partial_1 \partial_2 G(0,0) + \mathrm{O}(\epsilon^3, \epsilon^2 \zeta^2)$, such that

$$\mathrm{Re}\, \zeta_{1,2}(\epsilon) = -\frac{\epsilon^2}{2} \frac{\langle R_1 \tilde{\boldsymbol{\psi}}, \sigma_3 \tilde{\boldsymbol{\psi}}\rangle}{\langle \boldsymbol{\psi}, \sigma_3 \boldsymbol{\psi}\rangle} + \mathrm{O}(\epsilon^{5/2}).$$

Direct computations from (2.11) show that

$$\langle R_1\tilde{\boldsymbol{\psi}},\sigma_3\tilde{\boldsymbol{\psi}}\rangle = \frac{1}{4\pi}\left(\int_{\mathbb{R}^3}\tilde{\psi}_1\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^3}\tilde{\psi}_1^*\,\mathrm{d}x\right) = 2\pi^2|\hat{\psi}_1(0)|^2 \geqslant 0,$$

where $\hat{\psi}_1(p)$ is the Fourier transform of $\psi_1(x)$, defined by (2.6). Again, we have $\mathrm{Re}\,\zeta_{1,2}(\epsilon)>0$ in the case $\langle\boldsymbol{\psi},\sigma_3\boldsymbol{\psi}\rangle<0$ and $\hat{\psi}_1(0)\neq 0$, such that two complex eigenvalues of $\mathcal{L}_1$ exist in the neighborhood of $z=\omega$, with the asymptotic approximations (4.6) and (4.21). ■

V. PROOF OF LEMMA 4.7

According to Kato (1966), Lemma 4.7 is valid if we can prove that there exists $c>0$ such that $\forall\epsilon\neq 0$, the following bounds are true:

$$\int_{-\infty}^{\infty}\|A(\mathcal{L}_0 - i\epsilon - \lambda)^{-1}\mathbf{u}\|^2\,\mathrm{d}\lambda \leqslant c\|\mathbf{u}\|^2,\quad \mathbf{u}\in L^2, \tag{5.1}$$

$$\int_{-\infty}^{\infty}\|B(\mathcal{L}_0 - i\epsilon - \lambda)^{-1}\mathbf{u}\|^2\,\mathrm{d}\lambda \leqslant c\|\mathbf{u}\|^2,\quad \mathbf{u}\in L^2, \tag{5.2}$$

$$\int_{-\infty}^{\infty}\|B(\mathcal{L}^* - i\epsilon - \lambda)^{-1}\mathbf{u}\|^2\,\mathrm{d}\lambda \leqslant c\|\mathbf{u}\|^2,\quad \forall\,\mathbf{u}\in X_c(\mathcal{L}^*), \tag{5.3}$$

$$\int_{-\infty}^{\infty}\|A(\mathcal{L} - i\epsilon - \lambda)^{-1}\mathbf{u}\|^2\,\mathrm{d}\lambda \leqslant c\|\mathbf{u}\|^2,\quad \forall\,\mathbf{u}\in X_c(\mathcal{L}). \tag{5.4}$$

The bounds (5.1) and (5.2) are proved in Corollary to Theorem XIII.25 in Reed and Simon (1978). We prove the bound (5.4), while the bound (5.3) can be proved similarly. Following Cuccagna *et al.* (2005), we write

$$A(\mathcal{L}-z)^{-1}\mathbf{v} = (I+\mathcal{Q}_0^+(z))^{-1}A(\mathcal{L}_0-z)^{-1}\mathbf{v},\quad \mathbf{v}\in X_c(\mathcal{L}), \tag{5.5}$$

where $\mathcal{Q}_0^+(z)$ is continuation of $Q_0(z)$ from $\mathrm{Im}\,z>0$ to $\mathrm{Im}\,z\geqslant 0$. The operator $(I+\mathcal{Q}_0^+(z))^{-1}$ is uniformly bounded in z away from the eigenvalues of $\sigma_p(\mathcal{L})$. It has pole singularities at the eigenvalues of $\sigma_p(\mathcal{L})$, which were considered in Cuccagna *et al.* (2005), Lemma 4.3. The endpoint eigenvalues $z=\pm\omega$ were excluded from Cuccagna *et al.* (2005). Here we shall consider the eigenvalue $z=\omega$. We need to show that $A(\mathcal{L}-z)^{-1}\mathbf{v}$ has L^2-norm which is uniformly bounded in $\epsilon>0$, for $\mathrm{Im}\,z=\epsilon$ and $\mathrm{Re}\,z\approx\omega$. Near $z=\omega$, we have the following expansion in the space of operators $L^2\to L^2$:

$$(I+\mathcal{Q}_0^+(z))^{-1} = \frac{1}{\omega-z}A\mathcal{P}_0B^* + \mathrm{O}(1).$$

Due to the bounds (5.1) and (5.2), we only need to study $(\omega-z)^{-1}A\mathcal{P}_0V\mathcal{R}_0(z)\mathbf{v}$, for $\mathbf{v}\in X_c(\mathcal{L})$ near $z=\omega$. We use the relation

$$\langle VR_0\mathbf{v},\sigma_3\boldsymbol{\psi}\rangle = -\langle\mathbf{v},\sigma_3\boldsymbol{\psi}\rangle = 0,\quad \forall\,\mathbf{v}\in X_c(\mathcal{L}).$$

As a result,

$$\begin{aligned}\frac{1}{\omega-z}\mathcal{P}_0V\mathcal{R}_0(z)\mathbf{v} &= \frac{\boldsymbol{\psi}}{\omega-z}\frac{\langle V\mathcal{R}_0(z)\mathbf{v},\sigma_3\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\psi},\sigma_3\boldsymbol{\psi}\rangle} = \frac{\boldsymbol{\psi}}{\omega-z}\frac{\langle V[\mathcal{R}_0(z)-R_0]\mathbf{v},\sigma_3\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\psi},\sigma_3\boldsymbol{\psi}\rangle} = -\boldsymbol{\psi}\frac{\langle VR_0\mathcal{R}_0(z)\mathbf{v},\sigma_3\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\psi},\sigma_3\boldsymbol{\psi}\rangle}\\ &= \boldsymbol{\psi}\frac{\langle\mathcal{R}_0(z)\mathbf{v},\sigma_3\boldsymbol{\psi}\rangle}{\langle\boldsymbol{\psi},\sigma_3\boldsymbol{\psi}\rangle}.\end{aligned}$$

We need to show that $\langle \mathcal{R}_0(z)\mathbf{v}, \sigma_3 \boldsymbol{\psi}\rangle$ is in Hardy space H^2 for $\operatorname{Im} z > 0$, which is true if $\boldsymbol{\psi}(x)$ belong to the space of Rollnick potentials,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\boldsymbol{\psi}(x)||\boldsymbol{\psi}(x)|}{|x-y|^2} \mathrm{d}x\mathrm{d}y < \infty. \tag{5.6}$$

It is clear from (2.14) that $\psi_2(x)$ decays exponentially as $|x| \to \infty$. Since $C_0 = 0$ in (2.15) and $(x_j, f\psi_1 + g\psi_2) = 0, j = 1$, 2, 3 in (4.16), it follows from (2.13) that $\psi_1(x)$ decays algebraically as $|x|^{-3}$. As a result, the eigenvector $\boldsymbol{\psi}(x)$ satisfies the condition (5.6).

ACKNOWLEDGMENTS

The work was made possible due to the research program on "Resonance in linear and nonlinear Schrödinger equations," sponsored by the Fields Institute in August, 2003. One of the authors (D.P.) acknowledges partial support from NSERC Grant No. 5-36694.The other author (S.C) acknowledges full support from a special grant of the Italian Ministry of Education, University and Research. One of the authors (S.C.) thanks C. Zeng for discussion of Lemma 4.6.

[1] Cuccagna, S., "Stabilization of solutions to nonlinear Schrödinger equations," Commun. Pure Appl. Math. **54**, 1110–1145 (2001).
[2] Cuccagna, S., Pelinovsky, D., and Vougalter, V., "Spectra of positive and negative energies in the linearized NLS problem," Commun. Pure Appl. Math. **58**, 1–29 (2005).
[3] Grillakis, M., "Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system," Commun. Pure Appl. Math. **43**, 299–333 (1990).
[4] Jensen, A. and Kato, T., "Spectral properties of Schrödinger operators and time decay of the wave functions," Duke Math. J. **46**, 583–611 (1979).
[5] Kapitula, T. and Sandstede, B., "Edge bifurcations for near integrable systems via Evans function techniques," SIAM J. Math. Anal. **33**, 1117–1143 (2002).
[6] Kapitula, T. and Sandstede, B., "Eigenvalues and resonances using the Evans function," Discrete Contin. Dyn. Syst. **10**, 857–569 (2004).
[7] Kapitula, T., Kevrekidis, P., and Sandstede, B., "Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems," Physica D **195**, 263–282 (2004).
[8] Kato, T., "Wave operators and similarity for some non-selfadjoint operators," Math. Ann. **162**, 258–269 (1966).
[9] Klaus, M. and Simon, B., "Coupling constant thresholds in nonrelativistic quantum mechanics. I. Short-range two-body case," Ann. Phys. (N.Y.) **130**, 251–281 (1980).
[10] Perelman, G., "Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equation", Commun. Partial Differ. Equ. **29**, 1051–1095 (2004)
[11] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic, New York, 1978).
[12] Schlag, W., Stable manifolds for an orbitally unstable NLS, 2004 (unpublished).
[13] Tsai, T. P. and Yau, H. T. "Stable directions for excited states of nonlinear Schrödinger equations," Commun. Partial Differ. Equ. **27**, 2363–2402 (2002).