ON THE GROUND STATE OF THE NONLINEAR SCHRÖDINGER EQUATION: ASYMPTOTIC BEHAVIOR AT THE ENDPOINT POWERS

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Abstract. We consider the ground states of the nonlinear Schrödinger equation, which stand for radially symmetric and exponentially decaying solutions on the full space. We investigate their behaviors at both endpoint powers of the nonlinearity, up to some rescaling to infer non-trivial limits. One case corresponds to the limit towards a Gaussian function called Gausson, which is the ground state of the stationary logarithmic Schrödinger equation. The other case, for dimension at least three, corresponds to the limit towards the Aubin-Talenti algebraic soliton. We prove strong convergence with explicit bounds for both cases, and provide detailed asymptotics. These theoretical results are illustrated with numerical approximations.

1. Introduction

We consider the ground states of the stationary nonlinear Schrödinger equation

$$-\Delta\phi + \phi = |\phi|^{2\sigma}\phi, \quad x \in \mathbb{R}^d, \tag{1.1}$$

with emphasis on the dependence of the solution upon the parameter $\sigma > 0$ in the nonlinearity. It has been known since the breakthrough works [3, 4] that ground states, defined as a minimizer of the action, exist in $H^1(\mathbb{R}^d)$ provided that the nonlinearity is H^1-subcritical: $0 < \sigma < \infty$ for $d = 1, 2$ and $0 < \sigma < \frac{2}{d-2}$ for $d \geq 3$. The uniqueness of such solutions, up to translation and sign change, was established in [4] for $d = 1$, and completely settled in [23] for $d \geq 2$, after a series of important steps, cited in [23]. The ground states are the (unique) positive, radially symmetric solutions to (1.1). We recall that $\phi \in \mathcal{C}^2(\mathbb{R}^d)$, and that $\phi, \nabla\phi$ decay exponentially (see e.g. [10, Theorem 8.1.1]).

In the present paper, we examine the behavior of the ground states when the parameter σ in the nonlinearity goes to the endpoint values, $\sigma = 0$ in any dimension, and $\sigma = \sigma_*(d) := \frac{2}{d-2}$ when $d \geq 3$. In what follows, we omit the dependence on d in σ_*.

For the limit $\sigma \to 0$, the Taylor expansion

$$|\phi|^{2\sigma} = \exp\left(\sigma \ln |\phi|^2\right) = 1 + \sigma \ln |\phi|^2 + \mathcal{O}(\sigma^2) \tag{1.2}$$

suggests, in order to get a nontrivial limit, to consider, instead of (1.1),

$$\Delta u + \frac{1}{\sigma}\left(|u|^{2\sigma} - 1\right) u = 0. \tag{1.3}$$

As we work on the whole space $\mathbb{R}^d$, this amounts to considering the rescaling

$$u(x) = \phi\left(\frac{x}{\sqrt{\sigma}}\right). \tag{1.4}$$

Formally, when σ goes to zero, the solution u to (1.3) is expected to converge in some sense to a solution of the stationary logarithmic Schrödinger equation,

$$\Delta u + u \ln\left(|u|^2\right) = 0. \tag{1.5}$$

Equation (1.5) is the stationary counterpart of the time dependent logarithmic Schrödinger equation,

$$i\partial_t \psi + \Delta\psi = \lambda\psi \ln\left(|\psi|^2\right), \tag{1.6}$$

with $\lambda \in \mathbb{R}$ ($\lambda = -1$ here) initially introduced in [5]. It was remarked there that (1.5) has explicit ground states for any $d \in \mathbb{N}$, called Gaussons (see also [6]),

$$u_0(x) = e^{\frac{d-|x|^2}{2}}. \tag{1.7}$$

The Cauchy problem for (1.6) in the case $\lambda < 0$ was studied initially in [11], and the orbital stability of the Gaussons was proven in [9] in the radial case, and in [1] for the general case. The fact that the Gausson (1.7) is the only (up to translation) positive, $\mathcal{C}^2$ solution of (1.5) vanishing at infinity, was proven in [14]. Uniqueness of positive, radially symmetric solutions of (1.5) vanishing at infinity as well as their derivative was established in [29], for $1 \le d \le 9$. Viewing ground states as solutions of a constrained minimization problem (minimization of the action on the Nehari manifold), uniqueness of ground states (up to translation and phase modification) was proven in [1].

The convergence of ground states of (1.1) to ground states for (1.5) was considered for the first time in [31]. The scaled equation (1.3) is also considered in [19], where the limit $\sigma \to 0$ is addressed, for x belonging to some bounded and convex domain. In the case $x \in \mathbb{R}^d$, it is proven in [31] that ground states to (1.3) converge to ground states of (1.5) in $H^1(\mathbb{R}^d) \cap \mathcal{C}^{2,\alpha}(\mathbb{R}^d)$ for any $\alpha \in (0,1)$. In the present paper, we revisit this convergence result by providing a rate of convergence in $\mathcal{O}(\sigma)$ as suggested by (1.2).

In [31], based on a result from [16], the authors infer that for any $\sigma \in (0, \sigma_*)$, there is no positive solution to (1.1) or, equivalently, to (1.3), in view of (1.4), such that

$$\|\phi\|_{L^\infty} = \|u\|_{L^\infty} \le e^{d/2}.$$

However, since in [16], the assumption $d \ge 3$ is made, one should be cautious with low dimensions. Indeed, when $d = 1$, ground states for (1.1) are given explicitly by

$$\phi(x) = (1+\sigma)^{\frac{1}{2\sigma}} \cosh(\sigma x)^{-\frac{1}{\sigma}}.$$

We note that for $\sigma > 0$,

$$\|\phi\|_{L^\infty} = \phi(0) = (1+\sigma)^{\frac{1}{2\sigma}} = e^{\frac{1}{2\sigma}\ln(1+\sigma)} < e^{\frac{1}{2}},$$

so Theorem 1.3 in [31] cannot be true for $d = 1$. We refer to Remark 3.1 for a more precise discussion.

In the H^1-critical case $\sigma = \sigma_*$ (for $d \ge 3$), the existence of ground states goes back to [2] and [28] independently. In view of Pohozaev identity (see e.g. [10]), nontrivial $\dot{H}^1 \cap L^{2\sigma_*+2}$ solutions satisfy the following equation, instead of (1.1),

$$\Delta\phi_* + |\phi_*|^{2\sigma_*}\phi_* = 0. \tag{1.8}$$

Positive radially symmetric solutions to (1.8) cease to be unique, due to a scaling invariance: if $\phi_*(x)$ is a solution to (1.8), then so is $\lambda^{1/\sigma_*}\phi_*(\lambda x)$ for any $\lambda > 0$. Up to this scaling invariance, the radially symmetric positive solutions are unique, given by

$$\phi_*(x) = \frac{1}{(1+a|x|^2)^{(d-2)/2}}, \quad a = \frac{\sigma_*^2}{4(1+\sigma_*)} = \frac{1}{d(d-2)}. \tag{1.9}$$

For any $d \geq 3$, ϕ_* belongs to the *homogeneous* Sobolev space $\dot{H}^1(\mathbb{R}^d)$ (that is, $\nabla\phi_* \in L^2(\mathbb{R}^d)$), but $\phi_* \in L^2(\mathbb{R}^d)$ only if $d \geq 5$. Unlike the limit $\sigma \to 0$, it seems that the limit $\sigma \to \sigma_*$ has not been considered so far in the literature. Similar to the case $\sigma \to 0$, where the rescaling (1.4) was introduced in order to get a nontrivial limit, the limit $\sigma \to \sigma_*$ requires a modification in order to make the limit regular, and establish a connection with the algebraic soliton (1.9). This is discussed more precisely in Section 2.3.

We conclude this introduction by illustrating in Figure 1 the dependence of the L^∞-norm of the ground states of (1.3) for $d = 1, \dots, 5$. The dependence is monotonically decreasing in dimensions $d = 1$ and $d = 2$, whereas it is monotonically increasing and diverges as $\sigma \to \sigma_*$ in dimensions $d = 4$ and $d = 5$, suggesting a renormalization in order to study the limit towards ϕ_* in (1.9), as evoked above. For $d = 3$, the dependence first decreases for small values of σ and then increases and diverges at $\sigma_* = 2$.

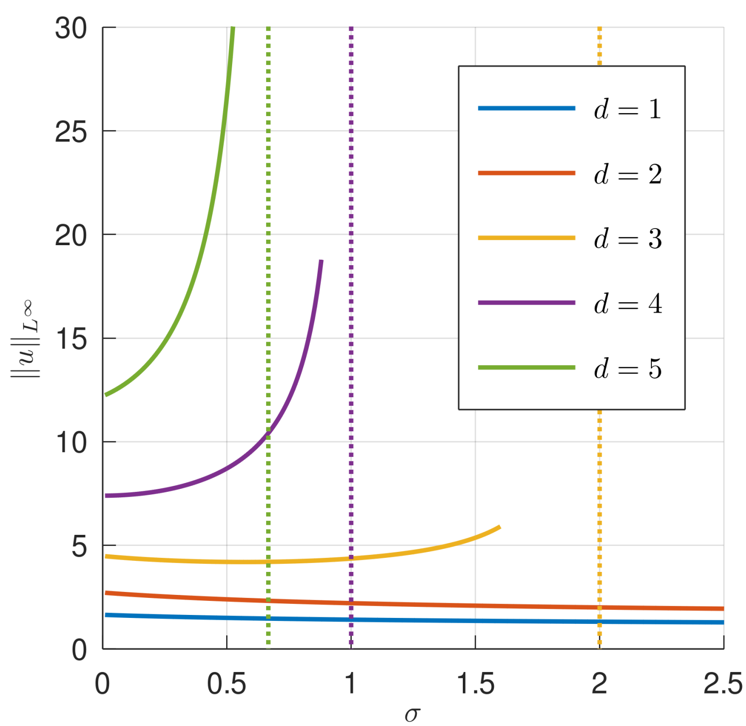

FIGURE 1. Maximum of the ground states $\|u\|_{L^\infty}$ versus σ for $d = 1, \dots, 5$ (see Remark 6.1 for comments on the limit $\sigma \to \sigma_*$).

Precise statements of the main results on the asymptotic behavior of the ground states at the endpoint powers are given in Section 2. Section 3 is dedicated to continuity properties with respect to the nonlinearity parameter $\sigma \in (0, \sigma_*)$. In Section 4, we consider the limit $\sigma \to 0$. The other endpoint $\sigma \to \sigma_*$ is studied in Section 5. Details about the numerical methods and the numerical approximations are presented in Section 6.

Notations. The differential element is denoted by d to avoid any confusion with the space dimension d. For radially symmetric functions and $1 \leq p < \infty$, we denote by $L^p_r = L^p_r(0,\infty)$ the set of functions $f = f(r)$ such that

$$\|f\|^p_{L^p_r} := \int_0^\infty r^{d-1}|f(r)|^p \mathrm{d}r < \infty,$$

and by $H^k_r = H^k_r(0,\infty)$ for $k \in \mathbb{N}^*$ the set of functions f such that

$$\|f\|^2_{H^k_r} := \int_0^\infty r^{d-1}\left(|f^{(k)}(r)|^2 + \ldots + |f'(r)|^2 + |f(r)|^2\right)\mathrm{d}r < \infty.$$

These definitions discard the measure of the unit sphere in $\mathbb{R}^d$ to lighten notations. This measure is only included in Section 6 for numerical computations. Finally we denote by

$$\langle f, g\rangle := \int_0^\infty r^{d-1} f(x)g(x)\mathrm{d}x$$

the scalar product on L^2_r.

2. Main results

In order to emphasize the dependence of the ground state profile upon the nonlinearity parameter σ, we denote the radially symmetric, positive, and monotonically decreasing solution to (1.3) by u_σ, and make the standard abuse of notation in $u_\sigma(x) = u_\sigma(r)$, $r = |x|$. In the radial coordinate r, studying (1.3) amounts to considering the family of solutions of the initial-value problem

$$\begin{cases} u''(r) + \dfrac{d-1}{r}u'(r) + \dfrac{1}{\sigma}(|u(r)|^{2\sigma} - 1)u(r) = 0, \quad r > 0, \\ u(0) = \alpha, \quad u'(0) = 0, \end{cases} \tag{2.1}$$

for $\alpha > 0$ and $\sigma \in (0, \sigma_*)$. This family of solutions is denoted by $u(r;\alpha,\sigma)$. It is known (see [16, Theorem 1.3]) that for each $\sigma \in (0,\sigma_*)$, there exists a unique value of $\alpha = \alpha(\sigma)$ such that $u_\sigma(r) := u(r;\alpha(\sigma),\sigma)$ is a positive, monotonically decreasing function in $\mathcal{C}^2(0,\infty) \cap L^\infty(0,\infty)$, with exponential decay as $r \to \infty$. This is the ground state profile of (1.3). Due to its uniqueness ([23]), it coincides up to a scalar multiplication with a minimizer of the variational problem

$$\inf_{u \in H^1_r}\left\{\|u'\|^2_{L^2_r} + \frac{1}{\sigma}\|u\|^2_{L^2_r} \;\middle|\; \|u\|_{L^{2\sigma+2}_r} = 1\right\}, \tag{2.2}$$

the existence of which was considered in [3, 4].

2.1. Continuity with respect to σ.

Related to the ground state profile $u_\sigma \in \mathcal{C}^2(0,\infty) \cap L^\infty(0,\infty)$, we also consider the linearized operator $\mathcal{L}_\sigma : H^2_r \to L^2_r$ given by

$$\mathcal{L}_\sigma = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{\sigma}\left(1 - u_\sigma^{2\sigma}\right) - 2u_\sigma^{2\sigma}. \tag{2.3}$$

In Section 3, we prove the following theorem, which allows us to control the dependence of $\alpha(\sigma)$ upon $\sigma \in (0,\sigma_*)$.

Theorem 2.1. *The mapping $\sigma \mapsto u_\sigma$ is $\mathcal{C}^1$ in $(0,\sigma_*)$ with values in H^1_r. Moreover, $\frac{\mathrm{d}u_\sigma}{\mathrm{d}\sigma} = \chi_\sigma$ where $\chi_\sigma \in H^2_r \cap \mathcal{C}^2$ is the unique solution in H^1_r to*

$$\mathcal{L}_\sigma \chi_\sigma = \frac{1}{\sigma^2}(1-|u_\sigma|^{2\sigma})u_\sigma + \frac{1}{\sigma}(\ln u_\sigma^2)|u_\sigma|^{2\sigma}u_\sigma. \tag{2.4}$$

The mapping $(0,\sigma_*) \ni \sigma \mapsto \alpha(\sigma) \in (0,\infty)$ is $\mathcal{C}^1$ and $\alpha'(\sigma) = \chi_\sigma(0)$.

2.2. **The limit $\sigma \to 0$.** We gather the main results of Section 4 in the following statement:

Theorem 2.2. *Let $d \geq 1$. As $\sigma \to 0$, the ground state profile u_σ of (2.1) converges to the Gausson u_0 given by (1.7) in $H^1_r \cap \mathcal{C}^{2,\alpha} \cap \mathcal{C}^\infty_{\mathrm{loc}}$ for any $0<\alpha<1$. We have the asymptotic expansion*

$$u_\sigma = u_0 + \sigma\mu_0 + \sigma e_\sigma,$$

where

$$\mu_0(r) = \frac{1}{12}\left[d(d-4) + 4(1-d)r^2 + r^4\right]u_0(r),$$

and for every $0 \leq s < 1$, e_σ goes to zero in $H^s_r \cap \mathcal{C}^\infty_{\mathrm{loc}}$ as $\sigma \to 0$. In particular, the mapping $\sigma \mapsto \alpha(\sigma)$ is continuously differentiable as $\sigma \to 0$, with

$$\alpha(0) = e^{d/2} \quad \text{and} \quad \alpha'(0) = \frac{d(d-4)}{12}e^{d/2}.$$

The computation of $\alpha'(0)$ is new, and illustrated numerically in Section 6. The computation of the correcting term μ_0 is new too, as well as the corresponding error estimate for e_σ. This result shows that some statements from [16] and [31] are flawed in the case $d \leq 3$. See Remark 3.1 for details.

2.3. **The limit $\sigma \to \sigma_*$.** The numerical data from Figure 1 suggests that $\alpha(\sigma) \to \infty$ as $\sigma \to \sigma_* = \frac{2}{d-2}$ for $d \geq 3$. In order to get the asymptotic dependence of $\alpha(\sigma) \to \infty$, we use the scaling transformation

$$u(r) = \alpha w(\rho), \quad \rho = \frac{\alpha^\sigma r}{\sqrt{\sigma}}, \quad \alpha > 0, \quad \sigma \in (0,\sigma_*). \tag{2.5}$$

If u satisfies (2.1), then w satisfies the initial-value problem:

$$\begin{cases} w''(\rho) + \frac{d-1}{\rho}w'(\rho) + |w(\rho)|^{2\sigma}w(\rho) = \epsilon w(\rho), \\ w(0) = 1, \quad w'(0) = 0, \end{cases} \tag{2.6}$$

where $\epsilon := \alpha^{-2\sigma}$. Every solution $u = u(r;\alpha,\sigma)$ of (2.1) is equivalent to the solution $w = w(\rho;\epsilon,\sigma)$ of (2.6). Again, if $u_\sigma(r) = u(r;\alpha(\sigma),\sigma)$ for some $\alpha = \alpha(\sigma)$ is the ground state (a positive, monotonically decreasing function in $\mathcal{C}^2(0,\infty) \cap L^\infty(0,\infty)$, with the fast (exponential) decay condition as $r \to \infty$), then $w_\sigma(\rho) = w(\rho;\epsilon(\sigma),\sigma)$ is the ground state for $\epsilon = \epsilon(\sigma)$, where

$$\epsilon(\sigma) := [\alpha(\sigma)]^{-2\sigma}. \tag{2.7}$$

The limit $\alpha(\sigma) \to \infty$ corresponds now to the limit $\epsilon(\sigma) \to 0$, where the limiting ground state is represented by the Aubin–Talenti algebraic soliton (1.9), rewritten as

$$w_*(\rho) = \frac{1}{(1+a\rho^2)^{\frac{1}{\sigma_*}}}, \quad a := \frac{\sigma_*^2}{4(1+\sigma_*)}, \quad \sigma_* = \frac{2}{d-2}. \tag{2.8}$$

We recall ([2, 28]) that the Aubin–Talenti algebraic soliton (2.8) coincides up to a scalar multiplication with the unique minimizer of the variational problem

$$\mathcal{S} := \inf_{w\in D^{1,2}_r(0,\infty)} \left\{ \|w'\|_{L^2_r} \;\middle|\; \|w\|_{L_r^{\frac{2d}{d-2}}} = 1 \right\}, \tag{2.9}$$

where $D^{1,2}_r(0,\infty)$ is the space of closure of $\mathcal{C}^\infty_{0,r}(0,\infty)$ under the norm $\|\nabla\cdot\|_{L^2_r}$. The minimizer of (2.9) gives the best constant of the Sobolev inequality

$$\|w\|_{L_r^{\frac{2d}{d-2}}} \le \mathcal{S}^{-\frac{1}{2}} \|w'\|_{L^2_r}. \tag{2.10}$$

Furthermore, it is only degenerate due to the one-parameter scaling transformation introduced before, $w_*(\rho) \mapsto \lambda^{1/\sigma_*} w_*(\lambda\rho)$ with $\lambda > 0$ (see [8] and the appendix in [7]). Changing u_σ satisfying (2.1) to w_σ satisfying (2.6) makes the limit $\sigma\to\sigma_*$ regular, since it corresponds to $\epsilon\to 0$ in (2.6). In particular, the parameter λ in the scaling invariance is naturally $\lambda = 1$, in view of the initial condition: $w_\sigma(0) = 1$. However, as the expression of ϵ is implicit, the convergence $w_\sigma \to w_*$ is quite delicate. In this direction, we prove in Section 5 the main result given by the following theorem.

Theorem 2.3. *Let $d \ge 3$. As $\sigma\to\sigma_*$, the ground state w_σ of (2.6) converges to the Aubin-Talenti algebraic soliton w_* given by (2.8) in $L^\infty_r \cap W^{1,\infty}_{\rm loc}$. Moreover, if $d \ge 5$, we have*

$$w_\sigma \to w_* \quad in\ H^1_r$$

and

$$\epsilon(\sigma) \underset{\sigma\to\sigma_*}{\sim} \frac{(1-\sigma_*)(\sigma_*-\sigma)}{2\sigma_*(1+\sigma_*)(2+\sigma_*)}.$$

For $\alpha(\sigma) = u_\sigma(0) = \|u_\sigma\|_{L^\infty(\mathbb{R}^d)}$ where u_σ is the ground state of (1.3), this yields the asymptotic behavior for $d \ge 5$:

$$\alpha(\sigma) \underset{\sigma\to\sigma_*}{\sim} C(d)(\sigma_*-\sigma)^{1/d-1/2},$$

for some explicit constant $C(d) > 0$.

3. Continuity properties in $\sigma \in (0, \sigma_*)$

3.1. **Some properties of the linearized operator.** The operator $\mathcal{L}_\sigma$, defined in (2.3), is a self-adjoint operator in L^2_r. Due to the exponential decay $u_\sigma(r) \to 0$ as $r\to\infty$, the essential spectrum of $\mathcal{L}_\sigma$ is located on $[\sigma^{-1},\infty)$ by Weyl's theorem.

Since u_σ is characterized variationally as a constrained minimizer of (2.2) with a single constraint, the Morse index of $\mathcal{L}_\sigma$ (the number of negative eigenvalues in L^2_r) is either 0 or 1, and as

$$\langle \mathcal{L}_\sigma u_\sigma, u_\sigma\rangle = -2\int_0^\infty r^{d-1}|u_\sigma(r)|^{2\sigma+2}\mathrm{d}r < 0,$$

the Morse index is exactly one. Moreover, due to non-degeneracy of constrained minimizers of (2.2) ([23, 33]), the kernel of $\mathcal{L}_\sigma$ is trivial and the rest of its spectrum in L^2_r is strictly positive and bounded away from 0. By Sturm's theorem, the uniquely defined solution $v \in \mathcal{C}^2(0,\infty)$ of the initial-value problem

$$\begin{cases} v''(r) + \dfrac{d-1}{r}v'(r) + \dfrac{1}{\sigma}(|u_\sigma(r)|^{2\sigma}-1)v(r) + 2|u_\sigma(r)|^{2\sigma}v(r) = 0, \\ v(0) = 1, \quad v'(0) = 0, \end{cases} \tag{3.1}$$

has a single node $r_0 \in (0,\infty)$ such that $v(r) > 0$ for $r \in [0, r_0)$ and $v(r) < 0$ for $r \in (r_0, \infty)$ with the divergence $v(r) \to -\infty$ as $r \to \infty$.

3.2. **Proof of Theorem 2.1.** The existence and uniqueness of solution $\chi_\sigma = \mathcal{L}_\sigma^{-1} h \in H^2_r$ of (2.4) with

$$h(r) = \frac{1}{\sigma^2}(1 - |u_\sigma(r)|^{2\sigma}) u_\sigma(r) + \frac{1}{\sigma}(\ln u_\sigma(r)^2)|u_\sigma(r)|^{2\sigma} u_\sigma(r) \in L^2_r,$$

follows by the spectral theory since $\mathrm{Ker}(\mathcal{L}_\sigma) = \{0\}$. Moreover, bootstrapping yields $\chi_\sigma \in \mathcal{C}^2(0,\infty) \cap L^\infty(0,\infty)$ with the fast (exponential) decay $\chi_\sigma(r) \to 0$ as $r \to \infty$. The nonlinear operator function

$$F(u,\sigma) : H^2_r \times (0,\sigma_*) \to L^2_r, \quad F(u,\sigma) = -\Delta_r u + \frac{1}{\sigma}(1 - |u|^{2\sigma})u,$$

is $\mathcal{C}^1$ in (u,σ), and by definition $F(u_\sigma,\sigma) = 0$. As u_σ is positive and exponentially decreasing at infinity, the Jacobian $\mathcal{L}_\sigma = D_u F(u_\sigma,\sigma)$ maps H^2_r to L^2_r. In view of Section 3.1, this Jacobian is invertible. The implicit function theorem then implies that the mapping $(0,\sigma_*) \ni \sigma \mapsto u_\sigma \in H^2_r$ is $\mathcal{C}^1$. From Peano's Theorem (see e.g. [21, Chapter V]), the derivative $\frac{\mathrm{d}u_\sigma}{\mathrm{d}\sigma}$ also belongs to $\mathcal{C}^2$, and satisfies the same equation as χ_σ, that is (2.4). By uniqueness, we conclude $\frac{\mathrm{d}u_\sigma}{\mathrm{d}\sigma} = \chi_\sigma$, hence Theorem 2.1, since $\alpha(\sigma) = u_\sigma(0)$.

3.3. **Correspondence to earlier results.** Peano's Theorem also implies that the family of solutions of the initial-value problem (2.1) is $\mathcal{C}^1$ with respect to both α and σ with

$$v(r) := \partial_\alpha u(r;\alpha(\sigma),\sigma) \quad \text{and} \quad \phi(r) := \partial_\sigma u(r;\alpha(\sigma),\sigma),$$

where $v \in \mathcal{C}^2(0,\infty)$ solves (3.1) and $\phi \in \mathcal{C}^2(0,\infty)$ solves the linear inhomogeneous equation $\mathcal{L}_\sigma \phi = h$ with the initial condition $\phi(0) = \phi'(0) = 0$. Considering v goes back to [22], with a first application in [13] to prove uniqueness results, and considering ϕ goes back to [15]. In [16], both functions were used. By the linear superposition principle, we have

$$\phi(r) = \chi_\sigma(r) - \chi_\sigma(0) v(r), \tag{3.2}$$

where $\chi_\sigma = \mathcal{L}_\sigma^{-1} h \in \mathcal{C}^2(0,\infty) \cap L^\infty(0,\infty)$ was considered above.

The solution ϕ generally diverges as $r \to \infty$, if $\chi_\sigma(0) \neq 0$. We show that $\phi(r) < 0$ for small $r > 0$ in agreement with [16, Lemma 3.1]. Indeed, we have $\phi''(0) = -d^{-1} h(0)$ with

$$h(0) = \frac{1}{\sigma^2}(1 - \alpha^{2\sigma})\alpha + \frac{1}{\sigma}(\ln \alpha^2)\alpha^{2\sigma}\alpha \equiv \mathfrak{h}(\alpha,\sigma).$$

Since

$$\lim_{\sigma \to 0} \mathfrak{h}(\alpha,\sigma) = \frac{1}{2}(\ln \alpha^2)^2 \alpha > 0,$$

and

$$\frac{\partial}{\partial \sigma} \sigma^2 \mathfrak{h}(\alpha,\sigma) = \sigma (\ln \alpha^2)^2 \alpha > 0,$$

we have $\mathfrak{h}(\alpha,\sigma) > 0$ for every $\sigma \in (0,\sigma_*)$ and $\alpha > 0$. Therefore, $\phi''(0) < 0$ and $\phi(r) < 0$ for small $r > 0$.

Remark 3.1. We will show in Section 4.2 that

$$\alpha(0) = \alpha_0, \quad \alpha'(0) = \frac{d(d-4)}{12}\alpha_0,$$

where $\alpha_0 = u_0(0) = e^{d/2}$. These results imply the following.

- The results of Theorems 1.1 and 1.2 in [16] are incorrect for $d = 3$. Lemma 2.1 about $v(r)$ is correct and so are Lemmas 3.1–3.3 about $\phi(r)$. If $\alpha'(\sigma) = \phi_p(0) < 0$, as for $d = 3$ and small $\sigma > 0$, then $\phi(r)$ stays negative for all $r > 0$ and diverges $\phi(r) \to -\infty$ as $r \to \infty$. If $\alpha'(\sigma) = \phi_p(0) > 0$, as for $d \geq 5$ and small $\sigma > 0$, then $\phi(r)$ changes sign exactly once and diverges $\phi(r) \to +\infty$ as $r \to \infty$. The proofs of Theorems 1.1 and 1.2 in Sections 5-6 of [16] are supposed to handle both cases; however, the outcome shows that the first case is mishandled.
- The result of Theorem 1.3 in [31], based on the above mentioned result from [16], is incorrect for $1 \leq d \leq 3$: there are positive solutions to (1.1) such that $\|\phi\|_{L^\infty} \leq e^{d/2}$ when $d \leq 3$ and $\sigma \in (0, \sigma_*)$, given by $\phi_\sigma(x) = u_\sigma(x\sqrt{\sigma})$.

4. The limit $\sigma \to 0$: convergence to the Gausson

In this section, we use the fact, proved in [31, Theorem 1.1], that for any $d \geq 1$,

$$\|u_\sigma - u_0\|_{L^\infty_r} \underset{\sigma \to 0}{\longrightarrow} 0,$$

where the Gausson is given by

$$u_0(r) = e^{\frac{d-r^2}{2}}. \tag{4.1}$$

The main purpose of this section is to provide the proof of Theorem 2.2. We first recall the main steps from the proof of [31, Theorem 1.1], and explain why the convergence also holds in $\mathcal{C}^\infty_{\rm loc}(\mathbb{R}^d)$.

4.1. Leading order convergence.

To prove [31, Theorem 1.1], the authors establish a variational characterization of the ground states u_σ and u_0, from which they infer the convergence $u_\sigma \to u_0$ in $H^1(\mathbb{R}^d)$ and in $\mathcal{C}^{2,\alpha}(\mathbb{R}^d)$ for any $0 < \alpha < 1$ thanks to the following lemma:

Lemma 4.1 (Lemma 2.1 in [31]). *(i) For any $\eta > 0$, there exists $C_\eta > 0$ such that*

$$\frac{x^{2\sigma} - 1}{\sigma} \leq C_\eta x^{2\eta}$$

holds for all $\sigma \in (0, \eta)$ and $x \geq 0$.
(ii) Let $s > 0$, $\delta > 0$, then

$$\frac{x^s(x^\delta - 1)}{\delta} \underset{\delta \to 0}{\longrightarrow} x^s \ln x \quad \text{in} \quad \mathcal{C}^{m,\alpha}_{\rm loc}[0, \infty),$$

where m is the largest integer with $m < s$, and $\alpha \in (0, s - m)$.

We note that the second convergence actually holds in $\mathcal{C}^\infty_{\rm loc}(0, \infty)$: for any $0 < a < b < \infty$, the convergence holds uniformly on $[a, b]$ and the same is true for all derivatives, as can be checked directly. It follows from [31, Corollary 2.1] that the ground states u_σ are uniformly bounded in $L^\infty(\mathbb{R}^d)$,

$$\|u_\sigma\|_{L^\infty} = u_\sigma(0) \leq C, \quad \forall \sigma \in \left(0, \frac{2}{d}\right),$$

where the bound $\sigma < 2/d$ is here just to fix ideas. With these tools in hand, standard L^p estimates for elliptic equations (see e.g. [20, Theorem 9.11 & 9.19]) and a bootstrap argument imply the convergence $u_\sigma \to u_0$ as $\sigma \to 0$, in $W^{2k,p}_{\rm loc}$ for every integer $k \geq 0$ and every $p \in (1, \infty)$, hence in $\mathcal{C}^\infty_{\rm loc}$ by Sobolev embedding.

4.2. **Computations of $\alpha'(0)$ and μ_0.** The case $\sigma = 0$ may be viewed as a limiting case of Theorem 2.1. Recall that the Gausson u_0 is the unique positive, radially symmetric, solution in $\mathcal{C}^2(0,\infty) \cap L^\infty(0,\infty)$ of the limiting equation

$$u''(r) + \frac{d-1}{r}u'(r) + (\ln u(r)^2)u(r) = 0. \tag{4.2}$$

The associated linearized operator $\mathcal{L}_0 : \mathrm{Dom}(\mathcal{L}_0) \subset L^2_r \to L^2_r$ given by

$$\mathcal{L}_0 = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \ln u_0^2 - 2 = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + r^2 - d - 2 \tag{4.3}$$

is the (shifted) quantum harmonic Schrödinger operator with

$$\mathrm{Dom}(\mathcal{L}_0) = \Sigma^2 := \{f \in H^2(\mathbb{R}^d),\ x \mapsto |x|^2 f(x) \in L^2(\mathbb{R}^d)\}$$

in L^2_r. By taking the limit $\sigma \to 0^+$ in (2.4) and denoting $\mu_0 := \lim_{\sigma\to 0^+} \chi_\sigma$, we obtain the uniquely defined solution of the limiting problem $\mu_0 = \mathcal{L}_0^{-1} h_0$ with the limiting function

$$h_0 = \frac{1}{2}(\ln u_0^2)^2 u_0 \in L^2_r.$$

Note that both μ_0 and h_0 decays faster (super-exponentially) for $\sigma = 0$ compared to the case $\sigma > 0$. Since for any $R > 0$, $u_0(x) \geq e^{\frac{d-R^2}{2}} > 0$ on the ball $B(0,R)$ of radius R in $\mathbb{R}^d$, the uniform convergence of u_σ toward u_0 implies that u_σ is bounded away from 0 on $B(0,R)$ for $\sigma \leq \sigma(R)$ sufficiently small. The ODE theory implies that the mapping $(0,\sigma_*) \ni \sigma \mapsto u_\sigma \in H^2_r(0,R)$ is also $\mathcal{C}^1$ in the limit $\sigma \to 0^+$.

We can thus consider the dependence $\alpha(\sigma)$ and the solution $\chi_\sigma(r)$ in the limit $\sigma \to 0$. It follows from (4.1) that $\alpha_0 = u_0(0) = e^{d/2}$. Writing $\mathcal{L}_0\mu_0 = h_0$ explicitly, we obtain

$$-\mu_0''(r) - \frac{d-1}{r}\mu_0'(r) - (d-2)\mu_0(r) + r^2\mu_0(r) = \frac{1}{2}(d-r^2)^2 e^{\frac{d-r^2}{2}}. \tag{4.4}$$

Substitution $\mu_0(r) = \frac{1}{2}e^{\frac{d-r^2}{2}}\tilde{\mu}_0(r)$ converts (4.4) to the form

$$-\tilde{\mu}_0''(r) - \frac{d-1}{r}\tilde{\mu}_0'(r) + 2r\tilde{\mu}_0'(r) - 2\tilde{\mu}_0(r) = (d-r^2)^2,$$

polynomial solutions of which are available explicitly:

$$\tilde{\mu}_0(r) = \frac{1}{6}\left[d(d-4) + 4(1-d)r^2 + r^4\right].$$

This yields the expression

$$\mu_0(r) = \frac{1}{12}\left[d(d-4) + 4(1-d)r^2 + r^4\right] u_0(r), \tag{4.5}$$

which vanishes at the roots of the polynomial

$$d(d-4) + 4(1-d)r^2 + r^4 = 0 \quad \Leftrightarrow \quad r^2 = 2(d-1) \pm \sqrt{3d^2 - 4d + 4}. \tag{4.6}$$

There is only one positive root of r for $d \leq 4$ and two positive roots for $d \geq 5$.

Assuming for the moment that $\alpha'(0) = \mu_0(0)$, we get

$$\alpha'(0) = \frac{d(d-4)}{12}\alpha_0, \tag{4.7}$$

which is negative for $d \leq 3$, zero at $d = 4$, and positive for $d \geq 5$. The relation $\alpha'(0) = \mu_0(0)$ is a direct consequence of the property $e_\sigma \to 0$ in $\mathcal{C}^0_{\rm loc}$, which is a particular case of the error estimate from Theorem 2.2, proven below.

4.3. **Explicit computations for $d = 1$.** In the one-dimensional case, the ground state is known explicitly, and elementary computations can be carried out:

Proposition 4.2. *Let $d = 1$,*

$$u_\sigma(x) = (1+\sigma)^{1/(2\sigma)} \cosh\left(x\sqrt{\sigma}\right)^{-1/\sigma}$$

be the ground state associated to (1.3), and $u_0(x) = e^{(1-x^2)/2}$ be the one-dimensional Gausson. Consider the corrector

$$\mu_0(x) = e^{(1-x^2)/2}\left(-\frac{1}{4} + \frac{x^4}{12}\right) = \frac{1}{12}(x^4 - 3)u_0(x),$$

in agreement with (4.5) for $d = 1$. Then

$$\|u_\sigma - u_0 - \sigma\mu_0\|_{L^\infty(\mathbb{R})} + \|u_\sigma - u_0 - \sigma\mu_0\|_{L^1(\mathbb{R})} = \mathcal{O}(\sigma^2).$$

In particular, the relation $\alpha'(0) = \mu_0(0)$ follows for $d = 1$.

Remark 4.3. By interpolation, we also have, for any $p \in [1, \infty]$,

$$\|u_\sigma - u_0 - \sigma\mu_0\|_{L^p(\mathbb{R})} = \mathcal{O}(\sigma^2).$$

Similar estimates for momenta, $\| \langle x\rangle^k (u_\sigma - u_0 - \sigma\mu_0)\|_{L^p(\mathbb{R})}$, where $k > 0$, follow easily by the same argument as below. Controlling Sobolev norms of the error would require more work though; we leave out this aspect, which is somehow anecdotal.

Proof. We note that

$$\sigma \mapsto \|u_\sigma\|_{L^\infty} = u_\sigma(0) = (1+\sigma)^{1/(2\sigma)}$$

is (strictly) decreasing on $\mathbb{R}_+$ (as can be checked by elementary computations). We readily compute

$$\alpha(\sigma) = (1+\sigma)^{1/(2\sigma)} = e^{\frac{1}{2\sigma}\ln(1+\sigma)} = e^{1/2}\left(1 - \frac{1}{4}\sigma + \mathcal{O}(\sigma^2)\right), \tag{4.8}$$

in agreement with (4.7) for $d = 1$, and we focus on the remaining part defining u_σ.

For $x, \sigma \geq 0$, let

$$g_x(\sigma) := \ln\cosh\left(x\sqrt{\sigma}\right).$$

We have

$$g_x(0) = 0, \quad \text{and for } \sigma > 0, \quad g_x'(\sigma) = \frac{x}{2\sqrt{\sigma}}\tanh\left(x\sqrt{\sigma}\right).$$

Since we have the expansion

$$\tanh(y) = y - \frac{y^3}{3} + \mathcal{O}(y^5) \quad \text{for } 0 \leq y \leq \frac{\pi}{2},$$

we infer in particular

$$g_x(\sigma) = \sigma\frac{x^2}{2} - \sigma^2\frac{x^4}{12} + \mathcal{O}\left(\sigma^3 x^6\right), \quad 0 \leq x \leq \frac{1}{\sqrt{\sigma}}.$$

Therefore, we have

$$\tilde{u}_\sigma(x) := \cosh\left(x\sqrt{\sigma}\right)^{-1/\sigma} = \exp\left(\frac{-1}{\sigma}g_x(\sigma)\right) = e^{-x^2/2}e^{\sigma\frac{x^4}{12}+R(\sigma,x)}, \tag{4.9}$$

where there exists C such that for all $0 \le x \le 1/\sqrt{\sigma}$,

$$|R(\sigma,x)| \le C\sigma^2 x^6,$$

hence

$$\tilde{u}_\sigma(x) = e^{-x^2/2}\left(1+\sigma\frac{x^4}{12}+\mathcal{O}\left(\sigma^2(x^6+x^8)\right)\right), \quad 0 \le x \le 1/\sqrt{\sigma}.$$

On the other hand, for $x > 1/\sqrt{\sigma}$,

$$g_x(\sigma) \ge \ln\cosh(1), \quad \text{hence} \quad \tilde{u}_\sigma(x) \le e^{-\frac{\ln\cosh(1)}{\sigma}}.$$

Let $\tilde{u}_0(x) = e^{-x^2/2}$. We obviously have $\tilde{u}_0(x) = \mathcal{O}(e^{-1/(2\sigma)})$ for $x > 1/\sqrt{\sigma}$, so by symmetry, we infer

$$\tilde{u}_\sigma(x) = \tilde{u}_0(x)\left(1+\sigma\frac{x^4}{12}\right)+\mathcal{O}\left(\sigma^2\right) \quad \text{in } L^\infty(\mathbb{R}),$$

which, together with (4.8), yields the L^∞-estimate.

For the L^1-estimate, let $\tilde{v}_0(x) = \frac{x^4}{12}\tilde{u}_0(x)$, we consider

$$\|\tilde{u}_\sigma - \tilde{u}_0 - \sigma\tilde{v}_0\|_{L^1(\mathbb{R})} = 2\int_0^\infty e^{-x^2/2}\left|e^{\sigma\frac{x^4}{12}+R(\sigma,x)} - 1 - \sigma\frac{x^4}{12}\right|\mathrm{d}x.$$

Again, we distinguish the regions $0 < x \le 1/\sqrt{\sigma}$ and $x > 1/\sqrt{\sigma}$. From the above Taylor expansion, on the first region,

$$e^{\sigma\frac{x^4}{12}+R(\sigma,x)} - 1 = \sigma\frac{x^4}{12} + R_1(\sigma,x),$$

where there exists C_1 such that

$$|R_1(\sigma,x)| \le C_1\left(\sigma^2x^8 + \frac{1}{\sigma}\left(x\sqrt{\sigma}\right)^6\right) = C_1\sigma^2(x^8+x^6), \quad 0 \le x \le \frac{1}{\sqrt{\sigma}}.$$

This yields

$$\begin{aligned}\int_0^{1/\sqrt{\sigma}} e^{-x^2/2}\left|e^{\sigma\frac{x^4}{12}+R(\sigma,x)} - 1 - \sigma\frac{x^4}{12}\right|\mathrm{d}x &\lesssim \sigma^2\int_0^\infty e^{-x^2/2}\left(1+x^6+x^8\right)\mathrm{d}x \\ &\lesssim \sigma^2.\end{aligned}$$

We next show that the tail of the integral is actually much smaller. Changing variables,

$$\int_{1/\sqrt{\sigma}}^\infty \tilde{u}_\sigma(x)\mathrm{d}x = \int_{1/\sqrt{\sigma}}^\infty \frac{\mathrm{d}x}{(\cosh(x\sqrt{\sigma}))^{1/\sigma}} = \frac{1}{\sqrt{\sigma}}\int_1^\infty \frac{\mathrm{d}y}{(\cosh(y))^{1/\sigma}}.$$

Taylor formula for $f(y) = \ln\cosh y$ yields

$$f(y) = f(1) + (y-1)f'(1) + (y-1)^2\int_0^1 (1-\theta)f''(\theta y)\mathrm{d}\theta.$$

As

$$f'(y) = \tanh(y), \quad f''(y) = \frac{1}{\cosh^2 y} \ge 0,$$

we infer

$$f(y) \geq f(1) + (y-1)f'(1),$$

hence

$$\int_1^\infty \frac{\mathrm{d}y}{(\cosh(y))^{1/\sigma}} \leq \int_{y_\sigma}^\infty e^{-\frac{1}{\sigma}(f(1)+(y-1)f'(1))}\mathrm{d}y = \frac{\sigma}{f'(1)} e^{-\frac{1}{\sigma}f(1)},$$

which is $\mathcal{O}(\sigma^k)$ for all $k > 0$. Recalling the asymptotic formula

$$\int_M^\infty e^{-x^2/2}\mathrm{d}x \underset{M\to\infty}{\sim} \frac{1}{M} e^{-M^2/2},$$

we also have

$$\int_{1/\sqrt{\sigma}}^\infty \left(\tilde{u}_0(x) + \sigma \tilde{v}_0(x)\right)\mathrm{d}x = \mathcal{O}(\sigma^k) \quad \text{for all } k > 0,$$

hence the L^1-estimate of the proposition. □

4.4. **Asymptotic expansions for general $d \geq 1$.** To complete the proof of Theorem 2.2, we describe u_σ up to some $o(\sigma)$ in H^s_r for $0 \leq s < 1$. The convergence in $\mathcal{C}^\infty_{\mathrm{loc}}$ follows from rather classical arguments.

4.4.1. *Derivation.* For $z, \sigma > 0$, we denote the nonlinearity in (1.3) by

$$f(z,\sigma) = \left(z^{2\sigma} - 1\right) z,$$

where we note that $f(z, 0) = 0$. We write an asymptotic expansion for u_σ for small $\sigma > 0$ as

$$u_\sigma = u_0 + \sigma\mu_0 + \sigma e_\sigma = u_0 + \sigma v_\sigma, \tag{4.10}$$

where u_0 is the Gausson (4.1) satisfying (4.2), μ_0 is the first-order correction (4.5) satisfying (4.4), and the remainder term e_σ is expected to vanish as $\sigma \to 0$ to ensure that $v_\sigma \to \mu_0$. Plugging this expression into (1.3), and using (4.2), we obtain

$$\sigma\left(v_\sigma'' + \frac{d-1}{r} v_\sigma'\right) = -\frac{1}{\sigma} f\left(u_0 + \sigma v_\sigma, \sigma\right) + u_0 \ln u_0^2.$$

Consider the decomposition

$$f\left(u_0 + \sigma v_\sigma, \sigma\right) = f\left(u_0 + \sigma v_\sigma, \sigma\right) - f(u_0, \sigma) + f(u_0, \sigma).$$

Taylor formula yields, since $f(z, 0) = 0$,

$$f(u_0, \sigma) = \sigma \partial_\sigma f(u_0, 0) + \frac{\sigma^2}{2} \partial^2_{\sigma\sigma} f(u_0, 0) + \frac{\sigma^3}{2} \int_0^1 (1-\theta)^2 \partial^3_{\sigma\sigma\sigma} f(u_0, \theta\sigma)\mathrm{d}\theta.$$

We readily compute

$$\partial_\sigma f(z, 0) = z \ln z^2, \quad \partial^2_{\sigma\sigma} f(z, 0) = z\left(\ln z^2\right)^2, \quad \partial^3_{\sigma\sigma\sigma} f(z,\sigma) = z^{2\sigma+1}\left(\ln z^2\right)^3,$$

so

$$\frac{1}{\sigma^2}\left(f\left(u_0, \sigma\right) - \sigma u_0 \ln u_0^2\right)\Big|_{\sigma=0} = \frac{1}{2} u_0 \left(\ln u_0^2\right)^2 = h_0.$$

Next, we write

$$\begin{aligned} f\left(u_0+\sigma v_\sigma,\sigma\right)-f(u_0,\sigma) &= \sigma v_\sigma \int_0^1 \partial_z f\left(u_0+\theta\sigma v_\sigma,\sigma\right)\mathrm{d}\theta \\ &= 2\sigma^2 v_\sigma \int_0^1 \left(u_0+\theta\sigma v_\sigma\right)^{2\sigma\theta}\mathrm{d}\theta \\ &\quad + \sigma v_\sigma \int_0^1 \Big(\left(u_0+\theta\sigma v_\sigma\right)^{2\sigma\theta}-1\Big)\,\mathrm{d}\theta. \end{aligned}$$

Assuming $v_\sigma \to \mu_0$ as $\sigma \to 0$, we get

$$\frac{1}{\sigma^2}\left(f\left(u_0+\sigma v_\sigma,\sigma\right)-f(u_0,\sigma)\right) \underset{\sigma\to 0}{\longrightarrow} 2\mu_0 + \mu_0 \ln u_0^2 = (d+2-r^2)\mu_0.$$

Reordering terms, we expect μ_0 to solve $\mathcal{L}_0\mu_0 = h_0$, hence to be given explicitly by (4.5). The correction term v_σ solves

$$\begin{aligned} v_\sigma'' + \frac{d-1}{r}v_\sigma' = &-\frac{1}{2}u_0\left(\ln u_0^2\right)^2 - \frac{\sigma}{2}\left(\ln u_0^2\right)^3 u_0 \int_0^1 (1-\theta)^2 u_0^{2\theta\sigma}\mathrm{d}\theta \\ &+ 2v_\sigma \int_0^1 \left(u_0+\theta\sigma v_\sigma\right)^{2\sigma\theta}\mathrm{d}\theta \\ &+ v_\sigma \int_0^1 \frac{\left(u_0+\theta\sigma v_\sigma\right)^{2\sigma\theta}-1}{\sigma}\mathrm{d}\theta. \end{aligned}$$

Recalling that $\sigma v_\sigma = u_\sigma - u_0$, if we denote the potential

$$\begin{aligned} V_\sigma(r) := &-\int_0^1 \frac{\left((1-\theta)u_0(r)+\theta u_\sigma(r)\right)^{2\sigma}-1}{\sigma}\mathrm{d}\theta \\ &-2\int_0^1 \left((1-\theta)u_0(r)+\theta u_\sigma(r)\right)^{2\sigma}\mathrm{d}\theta, \end{aligned} \tag{4.11}$$

associated to the Schrödinger operator

$$\widetilde{\mathcal{L}}_\sigma := -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + V_\sigma,$$

then the equation on v_σ writes

$$\widetilde{\mathcal{L}}_\sigma v_\sigma = \frac{1}{2}u_0\left(\ln u_0^2\right)^2 + \frac{\sigma}{2}\left(\ln u_0^2\right)^3 u_0 \int_0^1 (1-\theta)^2 u_0^{2\theta\sigma}\mathrm{d}\theta =: h_\sigma. \tag{4.12}$$

As we want to show that e_σ vanishes as $\sigma \to 0$, we need to invert the Schrödinger operator $\widetilde{\mathcal{L}}_\sigma$, considering the right hand side of (4.12) as a source term. Unfortunately, such operator could have a zero eigenvalue. However, we prove that in the limit $\sigma \to 0$, $\widetilde{\mathcal{L}}_\sigma$ is close in some sense to the shifted harmonic oscillator $\mathcal{L}_0$, ruling out the aforementioned scenario.

On a formal level, not only the error term e_σ in (4.10) is expected to vanish as $\sigma \to 0$, but also it is likely to satisfy $e_\sigma = \mathcal{O}(\sigma)$. However, as can be observed in the case of μ_0, every time a new term is derived in the asymptotic expansion in σ of u_σ, it turns out to be u_0 multiplied by a polynomial whose degree increases at every step. This makes it delicate to prove a quantitative error bound, even to show that $e_\sigma = \mathcal{O}(\sigma)$ in $L^2(\mathbb{R}^d)$ for $d \geq 2$. Also, to prove $e_\sigma = \mathcal{O}(\sigma)$, we would have to expand V_σ in powers of σ, which would involve $u_0 \ln\left((1-\theta)u_0 + \theta u_\sigma\right)$. Controlling this term in L_r^2

essentially requires to know some uniform bound from below for u_σ, which we could not derive. Therefore, we rely on the study of invertibility of the Schrödinger operator $\widetilde{\mathcal{L}}_\sigma$.

4.4.2. *Spectrum of the radial shifted harmonic oscillator.* Recall that the harmonic oscillator $H = -\Delta + |x|^2$ on $\mathbb{R}^d$ has its eigenvalues $(\Omega_n)_{n\in\mathbb{N}}$ and eigenfunctions $(f_n)_{n\in\mathbb{N}}$ satisfying

$$\left|\begin{aligned}&\Omega_n = (\omega_{n_1} + \ldots + \omega_{n_d}),\\ &f_n = \psi_{n_1} \ldots \psi_{n_d},\\ &n_1 + \ldots + n_d = n,\end{aligned}\right.$$

where $\omega_k = 2k+1$ and ψ_k denotes the k-th Hermite function. Note that $(\omega_k)_{k\in\mathbb{N}}$ and $(\psi_k)_{k\in\mathbb{N}}$ are respectively eigenvalues and eigenfunctions of the one-dimensional harmonic oscillator. For k even (resp. odd), ψ_k is even (resp. odd).

Restricted on radial functions, the operator

$$H_{\text{rad}} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + r^2$$

then admits eigenfunctions $(g_n)_{n\in\mathbb{N}}$ such that

$$g_n = \psi_{n_1} \ldots \psi_{n_d}, \quad n_1 = n_2 = \ldots = n_d \text{ even},$$

with sorted eigenvalues $\Lambda_n = \omega_{n_1} + \ldots + \omega_{n_d}$. In particular for $n = 0$, g_0 denotes a radial Gaussian function associated to first eigenvalues $\Lambda_0 = d$, while the second eigenfunction $\Lambda_1 = 5d$ corresponds to $n_1 = \ldots = n_d = 2$.

Recalling that radial shifted harmonic oscillator writes $\mathcal{L}_0 = H_{\text{rad}} - d - 2$ from (4.3), and denoting by $(\lambda_0^{(k)})_{k\in\mathbb{N}}$ and $(\varphi_0^{(k)})_{k\in\mathbb{N}}$ its sorted eigenvalues and eigenfunctions, we thus infer that $\lambda_0^{(0)} = -2$ and that $\lambda_0^{(1)} = 4d - 2$, so that $\lambda_0^{(k)} \geq 2$ for all $k \geq 1$ and all $d \geq 1$.

4.4.3. *Properties of the Schrödinger operator $\widetilde{\mathcal{L}}_\sigma$.*

Lemma 4.4. *Let $\sigma > 0$. The potential V_σ, defined in* (4.11)*, is radially symmetric, non-decreasing, and*

$$\lim_{r\to\infty} V_\sigma(r) = \frac{1}{\sigma}.$$

Moreover, there exists $K > 0$ such that for all $\sigma \in (0, 2/d]$, $V_\sigma \geq -K$. As a consequence, $\widetilde{\mathcal{L}}_\sigma$ is a self-adjoint accretive operator such that $\sigma_c(\widetilde{\mathcal{L}}_\sigma) = [1/\sigma, \infty)$ and $\sigma_p(\widetilde{\mathcal{L}}_\sigma) \subset [-K, 1/\sigma]$.

In the above statement, the upper bound $\sigma \leq 2/d$ is arbitrary, to avoid to distinguish the case $d \leq 2$ (where σ has no upper bound otherwise) from the general case.

Proof. As recalled in the beginning of Section 4, we know that $u_\sigma \to u_0$ in $L^\infty(\mathbb{R}^d)$, so there exists C_∞ such that

$$\|u_\sigma\|_{L^\infty} \leq C_\infty, \quad \forall \sigma \in [0, 2/d].$$

We infer

$$((1-\theta)u_0 + \theta u_\sigma)^{2\sigma} \leq C_\infty^{2\sigma}, \quad \forall \theta \in [0,1],$$

so

$$V_\sigma \geq \frac{1 - C_\infty^{2\sigma}}{\sigma} - 2C_\infty^{2\sigma} \underset{\sigma\to 0}{\longrightarrow} -\ln C_\infty^2 - 2,$$

hence $V_\sigma \geq -K$ for some uniform $K > 0$. The rest of the lemma follows easily. ☐

Thus there exists a set of sorted eigenvalues $\lambda_\sigma^{(0)} \le \lambda_\sigma^{(1)} \le \dots$, $(\lambda_\sigma^{(j)})_{j\in J}$, and eigenvectors $(\varphi_\sigma^{(j)})_{j\in J}$, $J\subset\mathbb{N}$, such that

$$(\widetilde{\mathcal{L}}_\sigma + K)\varphi_\sigma^{(j)} = (\lambda_\sigma^{(j)}+K)\varphi_\sigma^{(j)}, \tag{4.13}$$

with $\lambda_\sigma^{(j)} \in [-K, 1/\sigma]$, and $\|\varphi_\sigma^{(j)}\|_{L^2_r} = 1$ for all $j\in J$.

Our goal now is to prove that for $\sigma>0$ sufficiently small, the point spectrum of $\widetilde{\mathcal{L}}_\sigma$ is uniformly away from zero. We shall argue by comparison with the limiting case of the shifted harmonic operator $\mathcal{L}_0$, which will be made possible thanks to compactness properties.

Lemma 4.5. *For all $\varepsilon>0$, there exist $\sigma_0>0$ and $R>0$ such that if $0<\sigma\le\sigma_0$ and $r\ge R$,*

$$V_\sigma(r)+K \ge \frac{1}{\varepsilon}.$$

Proof. Let $\delta>0$ to be determined later. For any $z\le\delta$, $\frac{1-z^{2\sigma}}{\sigma} \ge \frac{1-\delta^{2\sigma}}{\sigma}$.

On the other hand, for any $r\ge0$ and $\theta\in[0,1]$,

$$0<(1-\theta)u_0(r)+\theta u_\sigma(r) = u_0(r)+\theta\left(u_\sigma(r)-u_0(r)\right) \le u_0(r)+\|u_\sigma-u_0\|_{L^\infty}.$$

Let $R>0$ such that for all $r\ge R$, $u_0(r)<\delta/2$, and let $\sigma_0>0$ such that for all $\sigma\le\sigma_0$, $\|u_\sigma-u_0\|_{L^\infty}<\delta/2$. For $r\ge R$ and $\sigma\le\sigma_0$,

$$\int_0^1 \frac{1-\left((1-\theta)u_0(r)+\theta u_\sigma(r)\right)^{2\sigma}}{\sigma}\mathrm{d}\theta \ge \frac{1-\delta^{2\sigma}}{\sigma}.$$

To control the other term defining V_σ, note that for $\sigma\le\sigma_0$ and $r\ge0$,

$$\int_0^1 \left((1-\theta)u_0(r)+\theta u_\sigma(r)\right)^{2\sigma} \le \left(\|u_0\|_{L^\infty}+\frac{\delta}{2}\right)^{2\sigma},$$

so we come up with

$$V_\sigma(r)+K \ge \frac{1-\delta^{2\sigma}}{\sigma} - 2\left(\|u_0\|_{L^\infty}+\frac{\delta}{2}\right)^{2\sigma} + K, \quad \forall r\ge R,\ \forall\sigma\le\sigma_0.$$

The right hand side goes to $K-\ln\delta^2$ as σ goes to zero. Up to decreasing $\sigma_0>0$, we have

$$V_\sigma(r)+K \ge K-\frac{1}{2}\ln\delta^2, \quad \forall r\ge R,\ \forall\sigma\le\sigma_0,$$

and we conclude by picking $\delta>0$ such that $\varepsilon^{-1} = K-\frac{1}{2}\ln\delta^2$. □

We infer that weighted L^2_r estimates involving V_σ provide compactness in L^2_r of bounded family of H^1_r functions:

Lemma 4.6. *Let $(g_\sigma)_{\sigma>0}$ be a family in H^1_r such that there exist $\sigma_1>0$ and $C>0$ with*

$$\|g_\sigma\|^2_{H^1_r} + \int_0^\infty \left(V_\sigma(r)+K\right) g_\sigma(r)^2 r^{d-1}\mathrm{d}r \le C, \quad \forall\sigma\in(0,\sigma_1).$$

Then the family $(g_\sigma)_{\sigma>0}$ is relatively compact in L^2_r as $\sigma\to0$.

Proof. We show that the Fréchet-Kolmogorov Theorem for radially symmetric functions can be applied, by proving the equitightness property,

$$\lim_{R\to\infty}\limsup_{\sigma\to 0}\int_R^\infty g_\sigma(r)^2 r^{d-1}\mathrm{d}r = 0.$$

Let $\varepsilon > 0$, and consider R, σ_0 provided by Lemma 4.5: for $\sigma \le \sigma_0$,

$$\int_R^\infty g_\sigma(r)^2 r^{d-1}\mathrm{d}r = \int_R^\infty \frac{V_\sigma(r)+K}{V_\sigma(r)+K} g_\sigma(r)^2 r^{d-1}\mathrm{d}r \le C\varepsilon,$$

hence the lemma. □

Lemma 4.7. *Let $(\varphi_\sigma)_{\sigma>0}$ be a sequence of eigenfunctions of $\widetilde{\mathcal{L}}_\sigma$, normalized in L_r^2, such that the related eigenvalue λ_σ satisfies $-4 \le \lambda_\sigma \le 4$. Then there exists a subsequence $\sigma_n \to 0$ as $n \to \infty$ such that $\varphi_{\sigma_n} \to \varphi_0$ in L_r^2 and $\lambda_{\sigma_n} \to \lambda$ for some $\varphi_0 \in H_r^1$ and $\lambda \in [-4, 4]$. Moreover, φ_0 is an eigenfunction of $\mathcal{L}_0 = \Delta_r + r^2 - d - 2$, normalized in L_r^2, with related eigenvalue λ.*

Proof. We compute

$$\|\varphi_\sigma'\|_{L_r^2}^2 + \int_0^\infty V_\sigma(r)\varphi_\sigma(r)^2 r^{d-1}\mathrm{d}r = \left\langle \widetilde{\mathcal{L}}_\sigma\varphi_\sigma, \varphi_\sigma\right\rangle = \lambda_\sigma \left\langle \varphi_\sigma, \varphi_\sigma\right\rangle = \lambda_\sigma.$$

Thus,

$$\|\varphi_\sigma\|_{H_r^1}^2 + \int_0^\infty \left(V_\sigma(r)+K\right)\varphi_\sigma(r)^2 r^{d-1}\mathrm{d}r = 1 + \lambda_\sigma + K \le 5 + K,$$

and we can invoke Lemma 4.6. Up to a subsequence, $\varphi_{\sigma_n} \to \varphi_0$ in L_r^2 and $\lambda_{\sigma_n} \to \lambda$ for some $\varphi_0 \in H_r^1$ and $\lambda \in [-4, 4]$. It remains to show that $\mathcal{L}_0\varphi = \lambda\varphi$.

We readily check the pointwise convergence $V_\sigma(r) \to V_0(r) := -\ln u_0(r)^2 - 2 = r^2 - d - 2$ as $\sigma \to 0$. We thus have the convergences

$$\begin{aligned}
\Delta\varphi_\sigma &\underset{\sigma\to 0}{\longrightarrow} \Delta\varphi \quad \text{in } H^{-2},\\
V_\sigma\varphi_\sigma &\underset{\sigma\to 0}{\longrightarrow} V_0\varphi \quad \text{in } L^2_{\mathrm{loc}},\\
\lambda_\sigma\varphi_\sigma &\underset{\sigma\to 0}{\longrightarrow} \lambda\varphi \quad \text{in } L^2.
\end{aligned}$$

Passing to the limit in the equation $\widetilde{\mathcal{L}}_\sigma\varphi_\sigma \equiv -\Delta\varphi_\sigma + V_\sigma\varphi_\sigma = \lambda_\sigma\varphi_\sigma$, we obtain

$$-\Delta\varphi + V_0\varphi = \lambda\varphi \quad \text{in } H^{-1}_{\mathrm{loc}},$$

and $-\Delta + V_0 = \mathcal{L}_0$, hence the lemma. □

We infer the announced result:

Proposition 4.8. *There exists $\sigma_0 > 0$ such that for all $0 < \sigma \le \sigma_0$, $\sigma_p(\widetilde{\mathcal{L}}_\sigma) \cap [-1, 1] = \emptyset$.*

Proof. We may assume that along some sequence $\sigma_n \to 0$, $\lambda_{\sigma_n}^{(0)} \le 2$ (the lowest eigenvalue of $\widetilde{\mathcal{L}}_{\sigma_n}$), for otherwise the result is straightforward. Up to a subsequence, $\lambda_{\sigma_n}^{(0)} \to \lambda_0$, and φ_{σ_n} converges in L_r^2 to some normalized eigenfunction φ_0 of $\mathcal{L}_0$, associated to λ_0. Moreover, φ_0 is radially symmetric and nonincreasing (not an excited state), so necessarily $\varphi_0 = \varphi_0^{(0)}$ and $\lambda_0 = \lambda_0^{(0)} = -2$. As the limit is unique, no subsequence is needed.

Consider now the second eigenvalue $\lambda_\sigma^{(1)}$, and suppose

$$\nu_0 = \liminf_{\sigma\to 0}\lambda_\sigma^{(1)} = \lim_{n\to\infty}\lambda_{\sigma_n}^{(1)} \le 4,$$

for some sequence $\sigma_n \to 0$. Note that if $\nu_0 > 2$, the proposition is proven. Let $\varphi_{\sigma_n}^{(1)}$ be a normalized eigenfunction associated to $\lambda_{\sigma_n}^{(1)}$. Up to a subsequence, $\varphi_{\sigma_n}^{(1)}$ converges to an eigenfunction φ_0 of $\mathcal{L}_0$, with eigenvalue λ_0, from Lemma 4.7. In addition,

$$0 = \left\langle \varphi_{\sigma_n}^{(0)}, \varphi_{\sigma_n}^{(1)} \right\rangle \underset{n\to\infty}{\longrightarrow} \left\langle \varphi_0^{(0)}, \varphi_0 \right\rangle.$$

Therefore, $\lambda_0 > \lambda_0^{(0)}$, and $\lambda_0 \geq \lambda_0^{(1)} = 4d - 2 \geq 2$, hence the result. □

4.4.4. *Convergence.*

Lemma 4.9. *There exists $C > 0$ such that the following holds. Let σ_0 given by Proposition 4.8. For any $\sigma \leq \sigma_0$, for any $\psi \in H_r^1$ such that $\widetilde{\mathcal{L}}_\sigma \psi = g \in L_r^2$,*

$$\|\psi\|_{H_r^1}^2 + \int_0^\infty \left(V_\sigma(r) + K\right) \psi(r)^2 r^{d-1} \mathrm{d}r \leq C\|g\|_{L_r^2}^2, \quad \forall \sigma \leq \sigma_0.$$

Proof. Let U_σ be unitary operators on L_r^2 and h_σ such that $\widetilde{\mathcal{L}}_\sigma = U_\sigma^{-1} h_\sigma(\rho) U_\sigma$ by the spectral theorem. We know that $h_\sigma(\rho)$ belongs to the spectrum of $\widetilde{\mathcal{L}}_\sigma$ for almost all $\rho > 0$, and so $|h_\sigma(\rho)| \geq 1$ for almost all $\rho > 0$, as soon as $\sigma \leq \sigma_0$ from Proposition 4.8.

Writing $g = U_\sigma^{-1} h_\sigma(\rho) U_\sigma \psi$, we have $\psi = U_\sigma^{-1} \frac{1}{h_\sigma(\rho)} U_\sigma g$, and thus

$$\|\psi\|_{L_r^2} = \left\| U_\sigma^{-1} \frac{1}{h_\sigma} U_\sigma g \right\|_{L_r^2} = \left\| \frac{1}{h_\sigma} U_\sigma g \right\|_{L_r^2} \leq \|U_\sigma g\|_{L_r^2} = \|g\|_{L_r^2}.$$

For the remaining part to estimate,

$$\begin{aligned}
\|\psi'\|_{L_r^2}^2 + \int_0^\infty \left(V_\sigma(r) + K\right) \psi(r)^2 r^{d-1} \mathrm{d}r &= \left\langle (\widetilde{\mathcal{L}}_\sigma + K)\psi, \psi \right\rangle \\
&= \left\langle U_\sigma^{-1} \left(h_\sigma + K\right) U_\sigma \psi, \psi \right\rangle \\
&= \left\langle U_\sigma^{-1} \left(h_\sigma + K\right) \frac{1}{h_\sigma} U_\sigma g, U_\sigma^{-1} \frac{1}{h_\sigma} U_\sigma g \right\rangle \\
&= \int_0^\infty \frac{h_\sigma(\rho) + K}{h_\sigma(\rho)^2} \left(U_\sigma g(\rho)\right)^2 \rho^{d-1} \mathrm{d}\rho.
\end{aligned}$$

Since the map $z \mapsto \frac{z+K}{z^2}$ is bounded in $\mathbb{R} \setminus [-1, 1]$, we infer that there exists $C > 0$ such that

$$\int_0^\infty \frac{h_\sigma(\rho) + K}{h_\sigma(\rho)^2} \left(U_\sigma g(\rho)\right)^2 \rho^{d-1} \mathrm{d}\rho \leq C \int_0^\infty \left(U_\sigma g(\rho)\right)^2 \rho^{d-1} \mathrm{d}\rho = C\|g\|_{L_r^2}^2,$$

hence the result. □

We can now prove the end of Theorem 2.2:

Corollary 4.10. *The family $(v_\sigma)_\sigma$ is bounded in H_r^1, and converges strongly in L_r^2 to μ_0 given by (4.5). By interpolation, the convergence holds in H_r^s for all $0 \leq s < 1$.*

Proof. Denote by h_σ the right hand side of (4.12). It is easy to check that

$$h_\sigma \underset{\sigma\to 0}{\longrightarrow} h_0 = \frac{1}{2} u_0 \left(\ln u_0^2\right)^2 \quad \text{in } L_r^2.$$

Lemma 4.9 implies that $(v_\sigma)_\sigma$ is bounded in H^1_r, and together with Lemma 4.6, we infer that up to a subsequence, v_σ converges strongly in L^2_r, to some $v \in H^1_r$. Passing to the limit in (4.12), which we rewrite as

$$-\Delta v_\sigma + V_\sigma v_\sigma = h_\sigma,$$

and arguing like in the proof of Lemma 4.7, we come up with

$$-\Delta v + V_0 v = h, \quad \text{that is} \quad \mathcal{L}_0 v = h_0.$$

We infer that $v = \mu_0$, and by uniqueness of the limit, the whole sequence $(v_\sigma)_\sigma$ is converging to μ_0 in L^2_r as $\sigma \to 0$, hence the result. □

Lemma 4.11. *The family $(v_\sigma)_\sigma$ converges to μ_0 in $\mathcal{C}^\infty_{\rm loc}$.*

Proof. We recall that v_σ satisfies (4.12). Moreover, from the convergence of u_σ to u_0 in $\mathcal{C}^\infty_{\rm loc}$ and using the fact that u_0 is strictly positive on every bounded set (and thus far from 0), we deduce that V_σ and h_σ are bounded in $\mathcal{C}^\infty(K)$ uniformly in σ for every compact set K. Thus, using regularity theory for elliptic equations (see for instance [20, Theorems 9.11 & 9.19]) and bootstraping (with the first step using that v_σ is unformly bounded in $H^1(\mathbb{R}^d)$), we deduce that v_σ is also uniformly bounded in every $W^{m,p}(K)$ for any compact set K. This leads to the conclusion by Sobolev embeddings. □

5. The limit $\sigma \to \sigma_*$: convergence to the algebraic soliton

5.1. **Some properties of the ground state.** As pointed out in Section 2, the parameter $\epsilon(\sigma)$ in (2.6) is implicit and is defined from the condition that $w_\sigma(\rho) = w(\rho; \epsilon(\sigma), \sigma)$ is positive and monotonically decreasing with the fast (exponential) decay condition as $\rho \to \infty$. Here, we derive some estimates involving ϵ and w_σ, thanks to Pohozaev identitites.

Multiplying (2.6) by $\rho^{d-1} w_\sigma(\rho)$ and integrating on $(0,\infty)$ by parts with

$$\rho^{d-1} w_\sigma(\rho) w_\sigma'(\rho)|_{\rho=0}^{\rho\to\infty} = 0,$$

we obtain

$$\|w_\sigma\|^{2\sigma+2}_{L^{2\sigma+2}_r} = \epsilon(\sigma)\|w_\sigma\|^2_{L^2_r} + \|w_\sigma'\|^2_{L^2_r}. \tag{5.1}$$

Multiplying (2.6) by $\rho^d w_\sigma'(\rho)$ and integrating by parts on $(0,\infty)$, with

$$\rho^d (w_\sigma'(\rho))^2|_{\rho=0}^{\rho\to\infty} = \rho^d w_\sigma^2(\rho)|_{\rho=0}^{\rho\to\infty} = 0,$$

we get

$$\frac{d}{1+\sigma}\|w_\sigma\|^{2\sigma+2}_{L^{2\sigma+2}_r} = d\epsilon(\sigma)\|w_\sigma\|^2_{L^2_r} + (d-2)\|w_\sigma'\|^2_{L^2_r}. \tag{5.2}$$

In what follows, we can express $d \geq 3$ by using σ_* due to $d = 2 + \frac{2}{\sigma_*}$.

Eliminating $\|w_\sigma\|^{2\sigma+2}_{L^{2\sigma+2}_r}$ from (5.1) and (5.2) yields

$$\epsilon(\sigma) = \frac{(\sigma_* - \sigma)\|w_\sigma'\|^2_{L^2_r}}{\sigma(1+\sigma_*)\|w_\sigma\|^2_{L^2_r}}. \tag{5.3}$$

On the other hand, eliminating $\|w_\sigma'\|^2_{L^2}$ from (5.1) and (5.2) yields

$$\|w_\sigma\|^{2\sigma+2}_{L^{2\sigma+2}_r} = \frac{\sigma_*(1+\sigma)\epsilon(\sigma)}{(\sigma_* - \sigma)}\|w_\sigma\|^2_{L^2_r}, \tag{5.4}$$

and since $\rho \mapsto w_\sigma(\rho)$ is nonincreasing on $(0,\infty)$, interpolation yields

$$\|w_\sigma\|_{L_r^{2\sigma+2}}^{2\sigma+2} \le \|w_\sigma\|_{L_r^\infty}^{2\sigma} \|w_\sigma\|_{L_r^2}^2 = \|w_\sigma\|_{L_r^2}^2.$$

We infer from (5.4) that

$$0 < \epsilon(\sigma) \le \frac{(\sigma_* - \sigma)}{\sigma_*(1+\sigma)}, \tag{5.5}$$

hence $\epsilon(\sigma) = \mathcal{O}(\sigma_* - \sigma)$. Comparison (5.3) with (5.5) implies that the ratio $\|w'_\sigma\|_{L_r^2}^2 / \|w_\sigma\|_{L_r^2}^2$ is uniformly bounded. Therefore, the quantity

$$-\frac{\epsilon(\sigma)}{\sigma_* - \sigma} = \frac{\epsilon(\sigma_*) - \epsilon(\sigma)}{\sigma_* - \sigma}$$

is bounded, and has converging subsequences. We shall prove that $\epsilon(\sigma) \sim c(d)(\sigma_* - \sigma)$ as $\sigma \to \sigma_*$, for some explicit $c(d) > 0$, and no subsequence is needed.

5.2. **Convergence in $L_r^\infty \cap W_{\rm loc}^{1,\infty}$.** The first convergence result announced in Theorem 2.3 is a direct consequence of the following property.

Proposition 5.1. *Let $d \ge 3$ and $\sigma \in (0, \sigma_*)$. For w_σ the solution to (2.6), with $\epsilon = \mathcal{O}(\sigma_* - \sigma)$, and w_* given by (2.8), there exists $C_0 > 0$ independent of $\sigma \in (0,\sigma_*)$ such that for every $R > 0$,*

$$\sup_{0\le\rho\le R} |w_\sigma(\rho) - w_*(\rho)| + \sup_{0\le\rho\le R} |w'_\sigma(\rho) - w'_*(\rho)| \le C_0(\sigma_* - \sigma) e^{C_0 R}. \tag{5.6}$$

In addition, there exists $C > 0$ independent of $\sigma \in (0,\sigma_*)$ such that

$$\|w - w_*\|_{L_r^\infty} = \sup_{\rho\ge 0} |w_\sigma(\rho) - w_*(\rho)| \le C\left(\ln \frac{1}{\sigma_* - \sigma}\right)^{-2/\sigma_*}. \tag{5.7}$$

Proof. Let $\varphi_\sigma(\rho) = w_*(\rho) - w_\sigma(\rho)$ denote the error, and consider $q_\sigma = \varphi_\sigma^2 + (\varphi'_\sigma)^2$. We compute

$$q'_\sigma(\rho) = 2\varphi_\sigma\varphi'_\sigma + 2\varphi'_\sigma\varphi''_\sigma,$$

so using (2.6) (and its limiting case $\sigma = \sigma_*$ for which $\epsilon(\sigma_*) = 0$),

$$q'_\sigma(\rho) = 2\varphi_\sigma\varphi'_\sigma + 2\varphi'_\sigma\left(-\frac{d-1}{\rho}\varphi'_\sigma - w_*^{2\sigma_*+1} + w_\sigma^{2\sigma+1} - \epsilon(\sigma) w_\sigma\right).$$

Young inequality implies $2\varphi_\sigma\varphi'_\sigma \le q_\sigma$. Recall that we have the uniform estimates

$$0 < w_\sigma(\rho), w_*(\rho) \le 1,$$

and $\epsilon = \mathcal{O}(\sigma_* - \sigma)$. We decompose

$$w_*^{2\sigma_*+1} - w_\sigma^{2\sigma+1} = \underbrace{w_*^{2\sigma_*+1} - w_*^{2\sigma+1}}_{=:S_\sigma} + \underbrace{w_*^{2\sigma+1} - w_\sigma^{2\sigma+1}}_{=:G_\sigma}.$$

Taylor formula yields

$$G_\sigma = (2\sigma+1)\varphi_\sigma \int_0^1 \left(w_\sigma + \theta(w_* - w_\sigma)\right)^{2\sigma} \mathrm{d}\theta,$$

and by the above uniform bounds,

$$|G_\sigma| \le (2\sigma+1)|\varphi_\sigma|.$$

Invoking Young inequality again, we infer, for some $C > 0$ independent of $\sigma \leq \sigma_*$ and $\rho \geq 0$,

$$q_\sigma'(\rho) \leq C q_\sigma(\rho) + S_\sigma(\rho)^2 + C(\sigma_* - \sigma)^2.$$

For the source term S_σ, Taylor formula for the map $z \mapsto w_*^z$ yields

$$S_\sigma = w_*^{2\sigma+1}\left(w_*^{2(\sigma_*-\sigma)} - 1\right) = 2(\sigma_* - \sigma) w_*^{2\sigma+1} \ln w_* \int_0^1 w_*^{2\theta(\sigma_*-\sigma)} \mathrm{d}\theta,$$

hence, since $0 < w_* \leq 1$,

$$|S_\sigma(\rho)| \leq 2(\sigma_* - \sigma) w_*(\rho)^{2\sigma+1} |\ln w_*(\rho)| = \mathcal{O}\left(\sigma_* - \sigma\right).$$

As $q_\sigma(0) = 0$, Grönwall lemma implies that there exists $C_0, C_1 > 0$ independent of $\sigma \in [\sigma_*/2, \sigma_*]$ such that for every $R > 0$,

$$\sup_{0 \leq \rho \leq R} q_\sigma(\rho) \leq C_1 (\sigma_* - \sigma)^2 e^{2C_0 R},$$

hence (5.6).

Pick $R_\sigma = \frac{1}{2C_0} \ln \frac{1}{\sigma_* - \sigma}$. We have

$$\begin{aligned} |w_\sigma(R_\sigma)| \leq |w_*(R_\sigma)| + |w_\sigma(R_\sigma) - w_*(R_\sigma)| &\lesssim R_\sigma^{-2/\sigma_*} + (\sigma_* - \sigma) e^{C_0 R_\sigma} \\ &\lesssim \left(\ln \frac{1}{\sigma_* - \sigma}\right)^{-2/\sigma_*}, \end{aligned}$$

where we have used the explicit decay for w_*. Since w_σ and w_* are positive decreasing, for $\rho \geq R_\sigma$, we have

$$|w_\sigma(\rho) - w_*(\rho)| \leq w_\sigma(\rho) + w_*(\rho) \leq w_\sigma(R_\sigma) + w_*(R_\sigma) \lesssim \left(\ln \frac{1}{\sigma_* - \sigma}\right)^{-2/\sigma_*}.$$

On the other hand, (5.6) yields

$$\sup_{0 \leq \rho \leq R_\sigma} |w_\sigma(\rho) - w_*(\rho)| \leq C_0 (\sigma_* - \sigma) e^{C_0 R_\sigma} \lesssim \sqrt{\sigma_* - \sigma}.$$

Combining the two bounds together yields (5.7). □

5.3. **Convergence in H_r^1.** We now turn to the proof of the second convergence result of Theorem 2.3. Let $d \geq 5$. We consider the minimizer v_σ of the problem (2.2), and we denote

$$\mathcal{K}_\sigma := \inf_{v \in H_r^1} \left\{ \|v'\|_{L_r^2}^2 + \frac{1}{\sigma} \|v\|_{L_r^2}^2 \;\middle|\; \|v\|_{L_r^{2\sigma+2}}^{2\sigma+2} = 1 \right\},$$

so that v_σ is solution of the equation

$$v_\sigma'' + \frac{d-1}{r} v_\sigma' + \mathcal{K}_\sigma v_\sigma^{1+2\sigma} = \frac{1}{\sigma} v_\sigma. \tag{5.8}$$

Lemma 5.2. *There exists C_1, $C_2 > 0$ and $\varepsilon > 0$ such that for all $\sigma \in [\sigma_* - \varepsilon/2, \sigma_*]$, we have $C_1 \leq \mathcal{K}_\sigma \leq C_2$.*

Remark 5.3. This lemma is reminiscent of continuity properties of the best constant in Gagliardo-Nirenberg inequalities,

$$S_{\sigma,d}^{1/2} \|f\|_{L^{2\sigma+2}(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}^{1-\theta} \|\nabla f\|_{L^2(\mathbb{R}^d)}^{\theta}, \quad \theta = \frac{d\sigma}{2\sigma+2} = \frac{\sigma(1+\sigma_*)}{\sigma_*(1+\sigma)}.$$

It follows from [17, Section 2] (see also [18, Lemma 2.50]) that $\sigma \mapsto S_{\sigma,d}$ is continuous on $(0,\sigma_*]$. Note however that as $\sigma \to \sigma_*$, $\theta \to 1$, and the property $w_* \in L^2_r$ requires $d \geq 5$. Therefore, the bound from above in Lemma 5.2 is not an immediate consequence of that continuity, this is why we prove it.

Proof. By interpolation, for $v \in H^1_r$,

$$\|v\|_{L_r^{2\sigma+2}} \leq \|v\|_{L_r^2}^{1-\theta} \|v\|_{L_r^{2\sigma_*+2}}^{\theta}, \quad \theta = \frac{\sigma(1+\sigma_*)}{\sigma_*(1+\sigma)} \in (0,1),$$

and (2.10) yields

$$\|v\|^2_{L_r^{2\sigma_*+2}} \leq \mathcal{S}^{-1}\|v'\|^2_{L^2_r} \leq \mathcal{S}^{-1}\left(\|v'\|^2_{L^2_r} + \frac{1}{\sigma}\|v\|^2_{L^2_r}\right),$$

hence the bound from below.

We now turn to the bound from above. Let $\lambda > 0$ to be fixed, and denote

$$w_{*,\lambda}(r) = \lambda^{\frac{1}{\sigma_*}} \frac{w_*(\lambda r)}{\|w_*\|_{L_r^{2\sigma_*+2}}}$$

where w_* denotes the Aubin-Talenti algebraic soliton (2.8). Then we compute

$$\|w_{*,\lambda}\|^{2\sigma+2}_{L_r^{2\sigma+2}} = \lambda^{\frac{2\sigma+2}{\sigma_*}-d}\left(\frac{\|w_*\|_{L_r^{2\sigma+2}}}{\|w_*\|_{L_r^{2\sigma_*+2}}}\right)^{2\sigma_*+2} = \lambda^{\frac{2(\sigma-\sigma_*)}{\sigma_*}}\left(\frac{\|w_*\|_{L_r^{2\sigma+2}}}{\|w_*\|_{L_r^{2\sigma_*+2}}}\right)^{2\sigma_*+2},$$

thus we take

$$\lambda = \left(\frac{\|w_*\|_{L_r^{2\sigma+2}}}{\|w_*\|_{L_r^{2\sigma_*+2}}}\right)^{\frac{(\sigma_*+1)\sigma_*}{\sigma_*-\sigma}},$$

so that $\|w_{*,\lambda}\|_{L_r^{2\sigma+2}} = 1$. Moreover, we write

$$\begin{aligned}
\|w_*\|^{2\sigma+2}_{L_r^{2\sigma+2}} - \|w_*\|^{2\sigma_*+2}_{L_r^{2\sigma_*+2}} &= \int_0^\infty \rho^{d-1}\left(w_*(\rho)^{2\sigma+2} - w_*(\rho)^{2\sigma_*+2}\right)\mathrm{d}\rho \\
&= \int_0^\infty \rho^{d-1} w_*(\rho)^{2\sigma+2}\left(1 - w_*(\rho)^{2(\sigma_*-\sigma)}\right)\mathrm{d}\rho,
\end{aligned}$$

and we remark from Taylor expansion that

$$w_*(\rho)^{2(\sigma_*-\sigma)} - 1 = 2(\sigma_*-\sigma)\underbrace{\log w_*(\rho)}_{\leq 0}\int_0^1 \underbrace{w_*(\rho)^{2\theta(\sigma_*-\sigma)}}_{\in[0,1]}\mathrm{d}\theta,$$

hence

$$0 \geq w_*(\rho)^{2(\sigma_*-\sigma)} - 1 \geq 2(\sigma_*-\sigma)\log w_*(\rho).$$

This enables to write, since $0 < w_*(\rho) \leq 1$,

$$\begin{aligned}
0 \leq \|w_*\|^{2\sigma+2}_{L_r^{2\sigma+2}} - \|w_*\|^{2\sigma_*+2}_{L_r^{2\sigma_*+2}} &\leq 2(\sigma_*-\sigma)\int_0^\infty \rho^{d-1}|\log w_*(\rho)| w_*(\rho)^{2\sigma+2}\mathrm{d}\rho \\
&\leq K_1(\sigma_*-\sigma)\|w_*\|^{2\sigma+2-\eta}_{L_r^{2\sigma+2-\eta}} \leq K_2(\sigma_*-\sigma),
\end{aligned}$$

for fixed $\eta > 0$, $K_1 = K_1(\eta)$, $K_2 > 0$ uniform in σ. Moreover we have

$$\|w_*\|^{2\sigma_*+2}_{L_r^{2\sigma+2}} - \|w_*\|^{2\sigma+2}_{L_r^{2\sigma+2}} = 2\|w_*\|^{2\sigma+2}_{L_r^{2\sigma+2}}(\sigma_*-\sigma)\log\|w_*\|_{L_r^{2\sigma+2}}\int_0^1 \|w_*\|^{2\theta(\sigma_*-\sigma)}_{L_r^{2\sigma+2}}\mathrm{d}\theta,$$

with $\|w_*\|_{L_r^{2\sigma+2}}$ both bounded from above and from below by a positive constant, so

$$\left| \|w_*\|_{L_r^{2\sigma+2}}^{2\sigma_*+2} - \|w_*\|_{L_r^{2\sigma+2}}^{2\sigma+2} \right| \le K_3(\sigma_* - \sigma),$$

for $K_3 > 0$ uniform in σ. Thus,

$$\underbrace{(1 - K_3(\sigma_* - \sigma))^{\frac{\sigma_*}{2(\sigma_*-\sigma)}}}_{\underset{\sigma\to\sigma_*}{\longrightarrow} \exp(-\sigma_* K_3/2)} \le \lambda \le \underbrace{(1 + (K_2 + K_3)(\sigma_* - \sigma))^{\frac{\sigma_*}{2(\sigma_*-\sigma)}}}_{\underset{\sigma\to\sigma_*}{\longrightarrow} \exp(\sigma_*(K_2+K_3)/2)},$$

so we get the estimate $0 < K_4 \le \lambda \le K_5$ for constants K_4, K_5 uniform in σ. We then go back to the definition of $w_{*,\lambda}$, and we write that

$$\|w_{*,\lambda}\|_{L_r^2}^2 = \lambda^{2/\sigma_*-d} \frac{\|w_*\|_{L_r^2}^2}{\|w_*\|_{L_r^{2\sigma_*+2}}^2} = \lambda^{-2} \frac{\|w_*\|_{L_r^2}^2}{\|w_*\|_{L_r^{2\sigma_*+2}}^2} \le K_6,$$

and

$$\|w'_{*,\lambda}\|_{L_r^2}^2 = \lambda^{2/\sigma_*-d+2} \frac{\|w'_*\|_{L_r^2}^2}{\|w_*\|_{L_r^{2\sigma_*+2}}^2} = \frac{\|w'_*\|_{L_r^2}^2}{\|w_*\|_{L_r^{2\sigma_*+2}}^2} \le K_7,$$

for uniform constants K_6, $K_7 > 0$. This allows to conclude that

$$\mathcal{K}_\sigma \le \|w'_{*,\lambda}\|_{L_r^2}^2 + \frac{1}{\sigma}\|w_{*,\lambda}\|_{L_r^2}^2 \le K_8$$

for $K_8 > 0$ uniform in σ. □

Note that $(\sigma_* - \sigma)/(\sigma_*(\sigma+1)) + 1/\sigma_* = d/(2\sigma+2)$. For any $\varphi \in H_r^1$ and $\lambda > 0$, we then define φ_λ by

$$\varphi_\lambda(r) := \lambda^{d/(2\sigma+2)}\varphi(\lambda r).$$

We readily compute

$$\begin{aligned}
\|\varphi_\lambda\|_{L_r^2} &= \lambda^{(\sigma_*-\sigma)/(\sigma_*(\sigma+1))-1}\|\varphi\|_{L_r^2},\\
\|\varphi'_\lambda\|_{L_r^2} &= \lambda^{(\sigma_*-\sigma)/(\sigma_*(\sigma+1))}\|\varphi'\|_{L_r^2},\\
\|\varphi_\lambda\|_{L_r^{2\sigma+2}} &= \|\varphi\|_{L_r^{2\sigma+2}}.
\end{aligned}$$

We then define the application

$$T_\sigma : \left\{ \begin{array}{ccc} \mathbb{R}_+^* \times \left\{ \psi \in H_r^1 \;\middle|\; \|\psi\|_{L_r^2} = \|\psi\|_{L_r^{2\sigma+2}} = 1 \right\} & \to & \left\{ \psi \in H_r^1 \;\middle|\; \|\psi\|_{L_r^{2\sigma+2}} = 1 \right\} \\ (\lambda, \varphi) & \mapsto & \varphi_\lambda. \end{array} \right.$$

It is straightforward to see that the application T_σ is bijective.

Lemma 5.4. *Let*

$$\Lambda_\sigma := \inf_{\varphi\in H_r^1} \left\{ \|\varphi'\|_{L_r^2}^2 \;\middle|\; \|\varphi\|_{L_r^2} = \|\varphi\|_{L_r^{2\sigma+2}} = 1 \right\}.$$

We have

$$\mathcal{K}_\sigma = \left(\frac{1+\sigma_*}{\sigma_*-\sigma}\right)^{\frac{\sigma_*-\sigma}{\sigma_*(1+\sigma)}} \frac{\sigma_*(1+\sigma)}{\sigma(1+\sigma_*)}\Lambda_\sigma^{1+\frac{\sigma_*-\sigma}{\sigma_*(1+\sigma)}}.$$

Moreover, if v_σ is a minimizer of (2.2), *then*

$$\|v_\sigma\|_{L_r^2} = \sqrt{\frac{\sigma_*-\sigma}{1+\sigma_*}}\|v'_\sigma\|_{L_r^2}.$$

Proof. For any $v \in H^1_r$ such that $\|v\|_{L^{2\sigma+2}_r} = 1$, there exists a unique $\lambda > 0$ and $\varphi \in H^1_r$ such that $\|\varphi\|_{L^2_r} = \|\varphi\|_{L^{2\sigma+2}_r} = 1$ and $v = \varphi_\lambda$. Therefore,

$$\|v'\|^2_{L^2_r} + \frac{1}{\sigma}\|v\|^2_{L^2_r} = \lambda^{2(\sigma_*-\sigma)/(\sigma_*(1+\sigma))}\left(\|\varphi'\|^2_{L^2_r} + \frac{\lambda^{-2}}{\sigma}\underbrace{\|\varphi\|^2_{L^2_r}}_{=1}\right).$$

Minimizing the expression on the right hand side with respect to λ, leading to

$$\lambda_0 = \sqrt{\frac{1+\sigma_*}{\|\varphi'\|^2_{L^2_r}(\sigma_*-\sigma)}}, \tag{5.9}$$

we then get

$$\|v\|_{L^2_r} = \lambda_0^{\frac{\sigma_*-\sigma}{\sigma_*(1+\sigma)}-1} = \lambda_0^{\frac{\sigma_*-\sigma}{\sigma_*(1+\sigma)}}\|\varphi'\|_{L^2_r}\sqrt{\frac{\sigma_*-\sigma}{1+\sigma_*}} = \|v'\|_{L^2_r}\sqrt{\frac{\sigma_*-\sigma}{1+\sigma_*}}.$$

This is in particular the case when $v = v_\sigma$ which is a minimizer of (2.2) (and thus already a minimizer with respect to λ). Moreover, we then get that

$$\begin{aligned}\|v'\|^2_{L^2_r} + \frac{1}{\sigma}\|v\|^2_{L^2_r} &= \lambda_0^{(\sigma_*-\sigma)/(\sigma_*(1+\sigma))}\|\varphi'\|^2_{L^2_r}\left(1+\frac{\sigma_*-\sigma}{\sigma_*(1+\sigma)}\right)\\ &= \left(\frac{1+\sigma_*}{\sigma_*-\sigma}\right)^{(\sigma_*-\sigma)/(\sigma_*(1+\sigma))}\|\varphi'\|^{2(1-(\sigma_*-\sigma)/(\sigma_*(1+\sigma))}_{L^2_r}\frac{\sigma_*(1+\sigma)}{\sigma(1+\sigma_*)}.\end{aligned}$$

The relation between $\mathcal{K}_\sigma$ and Λ_σ follows from the minimization of the remaining expression on

$$\left\{\varphi \in H^1_r \;\middle|\; \|\varphi\|_{L^2_r} = \|\varphi\|_{L^{2\sigma+2}} = 1\right\}.$$

□

Lemma 5.5. *Let v_σ be a minimizer of* (2.2)*. We have the identities*

$$\|v'_\sigma\|^2_{L^2_r} = \mathcal{K}_\sigma\frac{\sigma(1+\sigma_*)}{\sigma_*(1+\sigma)} \quad \text{and} \quad \|v_\sigma\|^2_{L^2_r} = \mathcal{K}_\sigma\frac{\sigma(\sigma_*-\sigma)}{\sigma_*(1+\sigma)}.$$

Proof. These expressions are direct consequences of Lemma 5.4 and the fact that

$$\mathcal{K}_\sigma = \|v'_\sigma\|^2_{L^2} + \frac{1}{\sigma}\|v_\sigma\|^2_{L^2}.$$

□

Lemma 5.6. *Let*

$$\varphi_\sigma := \operatorname*{argmin}_{\varphi\in H^1_r}\left\{\|\varphi'\|^2_{L^2_r} \;\middle|\; \|\varphi\|_{L^2_r} = \|\varphi\|_{L^{2\sigma+2}_r} = 1\right\}. \tag{5.10}$$

Then the quantity

$$\Lambda_\sigma := \|\varphi'_\sigma\|^2_{L^2_r} = \left[\left(\frac{\sigma_*-\sigma}{1+\sigma_*}\right)^{\frac{\sigma_*-\sigma}{(\sigma+1)\sigma_*}}\frac{\sigma(1+\sigma_*)}{\sigma_*(1+\sigma)}\mathcal{K}_\sigma\right]^{\frac{1}{1+(\sigma_*-\sigma)/(\sigma_*(1+\sigma))}} \tag{5.11}$$

is both bounded and bounded away from 0 as σ varies. Moreover, denoting

$$a_\sigma := \frac{\sigma_*(1+\sigma)}{\sigma(1+\sigma_*)}\Lambda_\sigma^{1+\frac{2(\sigma_*-\sigma)}{\sigma_*(1+\sigma)}} \quad \text{and} \quad b_\sigma := \frac{\Lambda_\sigma}{1+\sigma_*},$$

φ_σ is the only radially symmetric, positive solution vanishing at infinity, to

$$\varphi_\sigma'' + \frac{d-1}{\rho}\varphi_\sigma' + a_\sigma \varphi_\sigma^{1+2\sigma} = \frac{\sigma_*-\sigma}{\sigma} b_\sigma \varphi_\sigma.$$

Proof. The first part follows directly from Lemma 5.2. Recalling that

$$v_\sigma(r) = \lambda_0^{(\sigma_*-\sigma)/(\sigma_*(1+\sigma))}\varphi_\sigma(\lambda_0 r),$$

with λ_0 given in (5.9), and that v_σ satisfies (5.8), we infer that φ_σ satisfies

$$\varphi_\sigma'' + \frac{d-1}{\rho}\varphi_\sigma' + \mathcal{K}_\sigma \lambda_0^{2\sigma((\sigma_*-\sigma)/(\sigma_*(1+\sigma))-2}\varphi_\sigma^{1+2\sigma} = \frac{\lambda_0^{-2}}{\sigma}\varphi_\sigma,$$

with

$$\lambda_0^{-2} = \frac{\sigma_*-\sigma}{1+\sigma_*}\|\varphi_\sigma'\|_{L_r^2}^2 = \frac{\sigma_*-\sigma}{1+\sigma_*}\Lambda_\sigma = (\sigma_*-\sigma)b_\sigma.$$

One can then directly compute that

$$\begin{aligned}\lambda_0^{2\sigma((\sigma_*-\sigma)/(\sigma_*(1+\sigma))-2} &= \lambda_0^{2(\sigma_*-\sigma)(\sigma/(\sigma_*(1+\sigma))-1/\sigma_*)} = \lambda_0^{-2\frac{\sigma_*-\sigma}{\sigma_*(1+\sigma)}} \\ &= \left(\frac{\sigma_*-\sigma}{1+\sigma_*}\Lambda_\sigma\right)^{\frac{\sigma_*-\sigma}{\sigma_*(1+\sigma)}},\end{aligned}$$

and

$$\mathcal{K}_\sigma = \left(\frac{1+\sigma_*}{\sigma_*-\sigma}\right)^{(\sigma_*-\sigma)/(\sigma_*(1+\sigma))}\frac{\sigma_*(1+\sigma)}{\sigma(1+\sigma_*)}\Lambda_\sigma^{1+(\sigma_*-\sigma)/(\sigma_*(1+\sigma))},$$

so we get the identity

$$\mathcal{K}_\sigma \lambda_0^{2\sigma((\sigma_*-\sigma)/(\sigma_*(1+\sigma))-2} = a_\sigma.$$

The fact that φ_σ is positive and thus unique from [23] follows from the same arguments as in the proof of [32, Theorem B]. ☐

Lemma 5.7. *Let φ_σ be defined by* (5.10). *We have $\|\varphi_\sigma\|_{L_r^\infty} \geq 1$.*

Proof. This follows from interpolation, as

$$1 = \|\varphi_\sigma\|_{L_r^{2\sigma+2}}^{2\sigma+2} \leq \|\varphi_\sigma\|_{L_r^\infty}^{2\sigma}\|\varphi_\sigma\|_{L_r^2}^2 = \|\varphi_\sigma\|_{L_r^\infty}^{2\sigma}.$$

☐

For our upcoming analysis, we will rely on the following result, which is a direct application of [26, Theorem 3] (as pointed out after the statement in [26], the continuity of C_p follows from the proof).

Lemma 5.8. *For any $p \in (1,\infty)$, there exists $C_p > 0$ such that for any $u \in L^p(\mathbb{R}^d)$ and for any $\varepsilon > 0$, $v = \Delta(-\Delta+\varepsilon)^{-1}u$ satisfies*

$$\|v\|_{L^p} \leq C_p\|u\|_{L^p}.$$

Moreover, C_p is locally bounded with respect to p.

Lemma 5.9. *Let $u \in H^1(\mathbb{R}^d)$ with $d \geq 3$ such that $\Delta u \in L^p(\mathbb{R}^d)$ for some $p \in [1, d/2)$. Then there exists $K_p > 0$ (locally bounded with respect to p) such that*

$$\|u\|_{L^q(\mathbb{R}^d)} \leq K_p\|\Delta u\|_{L^p(\mathbb{R}^d)} \quad \textit{where } \frac{1}{q} = \frac{1}{p} - \frac{2}{d} \leq 1.$$

Proof. Let $v = -\Delta u$, in particular one can write $u = \frac{c_d}{|x|^{d-2}} * v$, and the conclusion follows from Hardy-Littlewood-Sobolev inequality (see e.g. [26]). □

Lemma 5.10. *For any $\zeta \in (1, d/2)$ with $d \geq 3$, there exists $C_{\zeta,d} > 0$ (locally bounded with respect to ζ) such that for any $\sigma \in (\sigma_*/2, \sigma_*]$,*

$$\|\varphi_\sigma\|_{L^\gamma_r} \leq C_{\zeta,d} \|\varphi_\sigma\|^{1+2\sigma}_{L^{(1+2\sigma)\zeta}_r} \quad \text{for } \frac{1}{\gamma} = \frac{1}{\zeta} - \frac{2}{d}.$$

Proof. We prove the result on $\mathbb{R}^d$, which in particular implies the result for radial functions. From Lemma 5.6 we have

$$(-\Delta + (\sigma_* - \sigma) b_\sigma/\sigma)\varphi_\sigma = a_\sigma \varphi_\sigma^{1+2\sigma},$$

with $b_\sigma > 0$. Thus, we can write

$$\Delta\varphi_\sigma = \Delta(-\Delta + (\sigma_* - \sigma)b_\sigma/\sigma)^{-1} \left(a_\sigma \varphi_\sigma^{1+2\sigma}\right).$$

From Lemmas 5.8 and 5.9, along with the fact that a_σ is bounded uniformly with respect to σ, we infer

$$\|\varphi_\sigma\|_{L^\gamma_r} \leq K_\zeta \|\Delta\varphi_\sigma\|_{L^\zeta_r} \leq K_\zeta C_\zeta \|a_\sigma \varphi_\sigma^{1+2\sigma}\|_{L^\zeta_r} \leq C_{\zeta,d} \|\varphi_\sigma\|^{1+2\sigma}_{L^{(1+2\sigma)\zeta}_r}.$$

□

Lemma 5.11. *For $d \geq 5$, there exists $\gamma \in (1, 2)$ such that $\|\varphi\|_{L^\gamma_r}$ is uniformly bounded in σ, for σ close enough to σ_*.*

Proof. Take

$$\zeta = \begin{cases} 1 + \varepsilon & \text{if } d = 5, \\ \dfrac{2}{1 + 2(\sigma_* - \varepsilon)} & \text{if } d \geq 6, \end{cases}$$

for some $\varepsilon > 0$ small enough, then $\zeta \in (1, 2)$ as

$$\frac{2}{1 + 2\sigma_*} = \frac{2(d-2)}{d+2} = 2 - \frac{8}{d+2} \in [1, 2) \quad \text{for } d \geq 6.$$

Applying Lemma 5.10 we get for $d \geq 6$ that

$$\|\varphi_\sigma\|_{L^\gamma_r} \leq C \|\varphi_\sigma\|^{1+2\sigma}_{L^{(1+2\sigma)\zeta}_r} \quad \text{with } \frac{1}{\gamma} = \frac{1}{\zeta} - \frac{2}{d}.$$

First, since

$$\frac{1 + 2\sigma_*}{2} - \frac{2}{d} = \frac{1}{2}\frac{d+2}{d-2} - \frac{2}{d} = \frac{1}{2}\frac{d(d+2) - 4(d-2)}{d(d-2)} = \frac{1}{2}\frac{d^2 - 2d + 8}{d^2 - 2d} > \frac{1}{2},$$

we have that $\gamma < 2$ for ε small enough. On the other hand, we have $(1 + 2\sigma)\zeta \geq 2$ for $\sigma \geq \sigma_* - \varepsilon$, and

$$(1 + 2\sigma)\zeta \leq 2\frac{1 + 2\sigma_*}{1 + 2(\sigma_* - \varepsilon)} < 2 + 2\sigma_*$$

for ε small enough. Thus, by interpolation and as $\|\varphi_\sigma\|_{L^2_r} = \|\varphi_\sigma\|_{L^{2\sigma+2}_r} = 1$, we have $\|\varphi_\sigma\|_{L^{(1+2\sigma)\zeta}} \leq 1$, so that $\|\varphi_\sigma\|_{L^\gamma_r} \leq C$. The case $d = 5$ is performed similarly. □

Lemma 5.12. *Let $v_n \in H^1_r$ be a family of functions which are radially symmetric decreasing and uniformly bounded in H^1_r and in L^γ_r for some $\gamma \in [1, 2)$. Then there exists a subsequence v_{n_k} and $v \in H^1_r \cap L^\gamma_r$ a radially symmetric decreasing function such that $v_{n_k} \underset{k\to\infty}{\longrightarrow} v$ in L^2_r.*

Proof. Strauss lemma ([27], see also [10, Proposition 1.7.1]) implies that there exists a subsequence $(v_{n_k})_{k\geq 0}$ and $v \in H^1_r$ such that $v_{n_k} \to v$ in L^p_r for all $p \in (2, 2^*)$, where $2^* = \frac{2d}{d-2}$. Fatou's lemma yields $v \in L^\gamma_r$, and interpolation implies $v_{n_k} \to v$ in L^2_r. □

The next lemma is a straightforward consequence of [2] and [28]:

Lemma 5.13. *Let $u \in H^1_r$ be a radially symmetric decreasing function satisfying*

$$\Delta u + a u^{1+2\sigma_*} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

for some $a > 0$. Then there exists $\lambda \geq 0$ such that $u(x) = \lambda w_*(x\sqrt{a}\lambda^{\sigma_*})$.

We will now use the functional introduced in [32], for $0 < \sigma < \sigma_*$,

$$J_\sigma(f) = \frac{\|\nabla f\|^{d\sigma}_{L^2(\mathbb{R}^d)} \|f\|^{2-(d-2)\sigma}_{L^2(\mathbb{R}^d)}}{\|f\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^d)}},$$

and its limiting expression from [28],

$$J_{\sigma_*}(f) = \frac{\|\nabla f\|^{d\sigma_*}_{L^2(\mathbb{R}^d)}}{\|f\|^{2\sigma_*+2}_{L^{2\sigma_*+2}(\mathbb{R}^d)}}.$$

It follows from [28] (case $\sigma = \sigma_*$) and [32, Theorem B] (case $0 < \sigma < \sigma_*$) that the minimizers of I_σ, $0 < \sigma \leq \sigma_*$, are ground state solutions (hence positive) to

$$\frac{d\sigma}{2}\Delta\psi + \psi^{2\sigma+1} = \frac{(d-2)}{2}(\sigma_* - \sigma)\psi. \tag{5.12}$$

There is uniqueness (up to translation) in the case $\sigma < \sigma_*$, but no longer in the case $\sigma = \sigma_*$ due to the additional scaling invariance. For any $\lambda > 0$,

$$w_{*,\lambda} := \lambda^{1/\sigma_*} w_*(\lambda\rho)$$

is the unique positive, radially symmetric solution to

$$w''_{*,\lambda} + \frac{d-1}{\rho} w'_{*,\lambda} + w^{2\sigma_*+1}_{*,\lambda} = 0, \quad w_{*,\lambda}(0) = \lambda^{1/\sigma_*}, \; w'_{*,\lambda}(0) = 0.$$

Denote, for $\sigma \leq \sigma_*$,

$$\ell_\sigma := \inf_{f \in H^1(\mathbb{R}^d)} J_\sigma(f).$$

As recalled above, it follows from [17] (see also [18]) that $\sigma \mapsto \ell_\sigma$ is continuous on $(0, \sigma_*]$. In addition, the value in the endpoint case σ_* is classical (see [28] or [18, Theorem 2.49]),

$$\ell_{\sigma_*} = \left(\frac{d(d-2)}{4}\right)^{d/(d-2)} 2^{2/(d-2)} \pi^{(2d+2)/d-2} \Gamma\left(\frac{d+1}{2}\right)^{-2/(d-2)}.$$

On the other hand, from [32], for $\sigma < \sigma_*$, J_σ is attained by ψ^* which is the unique (from [23]) positive solution to

$$\frac{d\sigma}{2}\Delta\psi^* + \mathcal{K}_\sigma(\sigma+1)(\psi^*)^{2\sigma+1} = \frac{d-2}{2}(\sigma_* - \sigma)\psi^*.$$

Moreover, it satisfies

$$\|\psi^*\|_{L^2(\mathbb{R}^d)} = \|\nabla\psi^*\|_{L^2(\mathbb{R}^d)} = 1.$$

We have the following result.

Lemma 5.14. *The quantity* Λ_σ *from* (5.11) *is continuous on* $(0,\sigma_*]$. *In particular,*

$$\Lambda_\sigma \underset{\sigma\to\sigma_*}{\longrightarrow} \left[\left(\frac{d(d-2)}{4}\right)^{\frac{d}{d-2}} 2^{\frac{2}{d-2}}\pi^{\frac{2d+2}{d-2}}\Gamma\left(\frac{d+1}{2}\right)^{-\frac{2}{d-2}}\right]^{\frac{2}{d\sigma_*}} =: \Lambda_{\sigma_*}$$

Proof. We first show that $\ell_\sigma = \Lambda_\sigma^{d\sigma/2}$. As $\|\varphi_\sigma\|_{L^{2\sigma+2}(\mathbb{R}^d)} = \|\varphi_\sigma\|_{L^2(\mathbb{R}^d)} = 1$, we infer

$$J_\sigma(\varphi_\sigma) = \|\nabla\varphi_\sigma\|_{L^2(\mathbb{R}^d)}^{d\sigma/2} = \Lambda_\sigma^{d\sigma/2} \geq \ell_\sigma$$

by definition of ℓ_σ.

On the other hand, as mentioned above we know that the minimum of ℓ_σ is attained for a positive, radially symmetric function ψ^*. We resume some arguments from the proof of [32, Theorem B]. For any positive, radially symmetric function f, define $f_{a,b}(x) = af(bx)$. We have

$$J_\sigma(f_{a,b}) = \frac{a^{d\sigma}b^{(1-d/2)d\sigma}\|\nabla f\|_{L^2(\mathbb{R}^d)}^{d\sigma} a^{2-(d-2)\sigma}b^{-\frac{d}{2}(2-(d-2)\sigma)}\|f\|_{L^2}^{2-(d-2)\sigma}}{a^{2\sigma+2}b^{-d}\|f\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2}} = J_\sigma(f).$$

Then we can find $a,b>0$ such that $\|f_{a,b}\|_{L^2(\mathbb{R}^d)} = \|f_{a,b}\|_{L^{2\sigma+2}(\mathbb{R}^d)} = 1$, so that $\|\nabla f_{a,b}\|_{L^2(\mathbb{R}^d)}^2 \geq \Lambda_\sigma$, and

$$J_\sigma(f_{a,b}) = \|\nabla f_{a,b}\|_{L^2(\mathbb{R}^d)}^{d\sigma} \geq \Lambda_\sigma^{d\sigma/2}.$$

Thus, $J_\sigma(f) \geq \Lambda_\sigma^{d\sigma/2}$, and we get by minimizing over $f \in H^1(\mathbb{R}^d)$ that $\ell_\sigma \geq \Lambda_\sigma^{d\sigma/2}$, so we get the equality.

We finally conclude by the continuity of $\ell_\sigma > 0$. ☐

Proposition 5.15. *We have* $\varphi_\sigma \underset{\sigma\to\sigma_*}{\longrightarrow} \varphi_*$ *in* H_r^1, *where* $\varphi_*(r) = \lambda_* w_*({\lambda_*}^{\sigma_*}\sqrt{a_{\sigma_*}}r)$ *with*

$$a_{\sigma_*} = \lim_{\sigma\to\sigma_*} a_\sigma \quad \textit{and} \quad \lambda_* = \left(\frac{\|w_*\|_{L_r^2}}{\Lambda_{\sigma_*}^{d/4}}\right)^{1/\sigma_*}.$$

Proof. By Lemma 5.6 and Lemma 5.10, φ_σ satisfies the assumption of Lemma 5.12. Thus, there exists $\varphi_* \in H_r^1$ radially symmetric decreasing function such that $\varphi_{\sigma_n} \underset{n\to\infty}{\longrightarrow} \varphi_*$ in L_r^2. Moreover, using the fact that $\Lambda_\sigma \underset{\sigma\to\sigma_*}{\longrightarrow} \Lambda_{\sigma_*}$ from Lemma 5.14, we infer:

$$\begin{aligned}
&\Delta\varphi_{\sigma_n} \underset{n\to\infty}{\longrightarrow} \Delta\varphi_* \quad \text{in } H^{-2}(\mathbb{R}^d),\\
&a_{\sigma_n}\varphi_{\sigma_n}^{1+2\sigma_n} \underset{n\to\infty}{\longrightarrow} \Lambda_{\sigma_*}\varphi_*^{1+2\sigma_*} \quad \text{in } \mathcal{D}'(\mathbb{R}^d),\\
&(\sigma_*-\sigma_n)b_{\sigma_n}\varphi_{\sigma_n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(\mathbb{R}^d).
\end{aligned}$$

For the second claimed convergence, write

$$\varphi_*^{1+2\sigma_*} - \varphi_{\sigma_n}^{1+2\sigma_n} = \varphi_*^{1+2\sigma_*} - \varphi_*^{1+2\sigma_n} + \varphi_*^{1+2\sigma_n} - \varphi_{\sigma_n}^{1+2\sigma_n},$$

In view of the pointwise estimate $|\varphi_*|^{1+2\sigma_n} \leq |\varphi_*|\max\left(1,|\varphi_*|^{2\sigma_*}\right)$, Lebesgue Dominated Convergence Theorem yields

$$\varphi_*^{1+2\sigma_*} - \varphi_*^{1+2\sigma_n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

For the remaining difference, write similarly

$$\begin{aligned}|\varphi_*^{1+2\sigma_n} - \varphi_{\sigma_n}^{1+2\sigma_n}| &\lesssim |\varphi_* - \varphi_{\sigma_n}| \left(|\varphi_*|^{2\sigma_n} + |\varphi_\sigma|^{2\sigma_n}\right)\\ &\lesssim |\varphi_* - \varphi_{\sigma_n}| \left(\max\left(1, |\varphi_*|^{2\sigma_*}\right) + \max\left(1, |\varphi_{\sigma_n}|^{2\sigma_*}\right)\right).\end{aligned}$$

Using the strong convergence $\varphi_{\sigma_n} \to \varphi_*$ in L^2_r, and the boundedness of $(\varphi_{\sigma_n})_n$ in $H^1_r \subset L^{2\sigma_*+2}_r$, Lebesgue Dominated Convergence Theorem yields again

$$\varphi_*^{1+2\sigma_*} - \varphi_{\sigma_n}^{1+2\sigma_n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Note that $a_{\sigma_*} = \Lambda_{\sigma_*}$ from the explicit expressions in Lemma 5.6. Again from Lemma 5.6, we know that φ_σ satisfies

$$\Delta\varphi_\sigma + a_\sigma\varphi_\sigma^{1+2\sigma} = \frac{\sigma_* - \sigma}{\sigma} b_\sigma\varphi,$$

so we infer that φ_* satisfies

$$\Delta\varphi_* + \Lambda_{\sigma_*}\varphi_*^{1+2\sigma_*} = 0.$$

Invoking Lemma 5.13, there exists $\lambda_* \geq 0$ such that $\varphi_*(r) = \lambda_* w_*({\lambda_*}^{\sigma_*}\sqrt{\Lambda_{\sigma_*}}r)$. Moreover, since $\|\varphi_{\sigma_n}\|_{L^2_r} = 1$, we also get $\|\varphi_*\|_{L^2_r} = 1$ by strong convergence in L^2_r. Thus,

$$1 = \|\varphi_*\|_{L^2_r} = \lambda_* \left\| w_*({\lambda_*}^{\sigma_*}\sqrt{\Lambda_{\sigma_*}}\cdot)\right\|_{L^2_r} = {\lambda_*}^{1-\frac{\sigma_* d}{2}}\Lambda_{\sigma_*}^{-\frac{d}{4}}\|w_*\|_{L^2_r},$$

with $1 - d\sigma_*/2 = -\sigma_*$. Hence we can deduce that

$$\lambda_* = \left(\frac{\|w_*\|_{L^2_r}}{a_{\sigma_*}^{d/4}}\right)^{1/\sigma_*} = \left(\frac{\|w_*\|_{L^2_r}}{\Lambda_{\sigma_*}^{d/4}}\right)^{1/\sigma_*}.$$

Since the limit φ_* is uniquely characterized, no subsequence is needed. In order to infer that $\varphi_\sigma \underset{\sigma\to\sigma_*}{\longrightarrow} \varphi_*$ in H^1_r, we only have to prove that $\|\varphi'_\sigma\|_{L^2_r} \underset{\sigma\to\sigma_*}{\longrightarrow} \|\varphi'_*\|_{L^2_r}$. On the one hand, we know that $\|\varphi'_\sigma\|^2_{L^2_r} \underset{\sigma\to\sigma_*}{\longrightarrow} \Lambda_{\sigma_*}$ from Lemma 5.6. On the other hand, we can explicitly compute

$$\begin{aligned}\|\varphi'_*\|_{L^2_r} &= {\lambda_*}^{1+\sigma_*}\Lambda_{\sigma_*}^{1/2}\left\|w'_*({\lambda_*}^{\sigma_*}\sqrt{\Lambda_{\sigma_*}}\cdot)\right\|_{L^2_r} = {\lambda_*}^{1+\sigma_*-d\sigma_*/2}\Lambda_{\sigma_*}^{1/2-d/4}\|w'_*\|_{L^2_r}\\ &= \Lambda_{\sigma_*}^{1/2-d/4}\|w'_*\|_{L^2_r} = \Lambda_{\sigma_*}^{-\frac{1}{2\sigma_*}}\|w'_*\|_{L^2_r}.\end{aligned}$$

Since w_* is the Aubin-Talenti soliton,

$$\Lambda_{\sigma_*} = \ell_{\sigma_*}^{2/(d\sigma_*)} = \frac{\|w'_*\|^2_{L^2_r}}{\|w_*\|^2_{L^{2\sigma_*+2}_r}}.$$

Proceeding like we did in Subsection 5.1, we check that w_* satisfies the identity

$$\|w'_*\|^2_{L^2_r} = \|w_*\|^{2\sigma_*+2}_{L^{2\sigma_*+2}_r},$$

and we infer $\Lambda_{\sigma_*}^{-\frac{1}{2\sigma_*}}\|w'_*\|_{L^2_r} = \Lambda_{\sigma_*}^{1/2}$, hence the result. □

There remains to prove that $\varphi_\sigma(0) \underset{\sigma\to\sigma_*}{\longrightarrow} \alpha$ to end the proof of the convergence towards the algebraic soliton.

Lemma 5.16. *There exists $C_5 > 0$ such that $\beta_\sigma := \|\varphi_\sigma\|_{L^\infty} \leq C_5$.*

Proof. Let ω_σ be defined by $\varphi_\sigma(r) = \beta_\sigma \omega_\sigma(\rho)$ where $\rho = \sqrt{a_\sigma}\beta_\sigma^\sigma r$. Then ω_σ is a solution to

$$\begin{cases} \omega_\sigma'' + \dfrac{d-1}{\rho}\omega_\sigma' + \omega_\sigma^{1+2\sigma} = \dfrac{\sigma_* - \sigma}{\sigma}\dfrac{b_\sigma}{a_\sigma \beta_\sigma^{2\sigma}}\omega_\sigma, \\ \omega_\sigma(0) = 1, \quad \omega_\sigma'(0) = 0. \end{cases}$$

It is important to observe that by uniqueness of radially symmetric, positive solutions going to zero at infinity to (2.6) (from [23], see also [16, Theorem 1.3]), we have, since $w_\sigma(0) = \omega_\sigma(0)$,

$$\omega_\sigma = w_\sigma, \quad \epsilon = \frac{\sigma_* - \sigma}{\sigma}\frac{b_\sigma}{a_\sigma \beta_\sigma^{2\sigma}}.$$

Proposition 5.1 implies for instance that

$$\|w_\sigma'\|_{L_r^2((0,1))} \underset{\sigma\to\sigma_*}{\longrightarrow} \|w_*'\|_{L_r^2((0,1))} > 0.$$

Thus, writing $I_\sigma = \big(0, 1/(\sqrt{a_\sigma}\beta_\sigma^\sigma)\big)$, we have

$$\|\varphi_\sigma'\|_{L_r^2(I_\sigma)} = a_\sigma^{\frac{1}{2}-\frac{d}{4}}\beta_\sigma^{1+\frac{\sigma}{2}-\frac{\sigma d}{2}}\|w_\sigma'\|_{L_r^2((0,1))} = a_\sigma^{-\frac{1}{2\sigma_*}}\beta_\sigma^{1-\frac{\sigma}{\sigma_*}}\|w_\sigma'\|_{L_r^2((0,1))}.$$

Therefore, we get that

$$\beta_\sigma = \left(\frac{a_\sigma^{1/(2\sigma_*)}\|\varphi_\sigma'\|_{L_r^2(I_\sigma)}}{\|w_\sigma'\|_{L_r^2((0,1))}}\right)^{\frac{1}{1-\sigma/\sigma_*}}. \tag{5.13}$$

By contradiction, if $\beta_{\sigma_n} \underset{\sigma\to\sigma_*}{\longrightarrow} \infty$ for some $\sigma_n \underset{n\to\infty}{\longrightarrow} \sigma_*$, one would get that

$$\|\varphi_{\sigma_n}'\|_{L_r^2(I_{\sigma_n})} \underset{n\to\infty}{\longrightarrow} 0$$

from the convergence of φ_σ in H_r^1, and since $I_\sigma \to \{0\}$ as $\sigma \to \sigma_*$. From (5.13), we would then have that $\beta_{\sigma_n} \underset{n\to\infty}{\longrightarrow} 0$, a contradiction. □

Proposition 5.17. *We have* $\varphi_\sigma(0) \underset{\sigma\to\sigma_*}{\longrightarrow} \lambda_*$.

Proof. We know that $\beta_\sigma = \varphi_\sigma(0)$ is bounded from Lemma 5.16, and bounded away from 0 by Lemma 5.7. Take any converging subsequence of β_σ denoted by β_{σ_n}, and denote by β the limit. With the same notations from the previous lemma, we have once again that ω_{σ_n} converges in $W^{1,\infty}_{r,\text{loc}}$ to w_*. On the other hand, from the convergence of φ_σ to φ_* and of β_{σ_n} to β, we also know that

$$w_{\sigma_n} \to \frac{1}{\beta}\varphi_*\left(\frac{\cdot}{\sqrt{\Lambda_{\sigma_*}}\beta^{\sigma_*}}\right) = \frac{\lambda_*}{\beta}w_*\left(\left(\frac{\lambda_*}{\beta}\right)^{\sigma_*}\cdot\right) \quad \text{in } H_r^1.$$

By comparison, we thus get that $\beta = \lambda_*$. Since the limit is unique, the conclusion holds for the whole sequence. □

5.4. **End of the proof of Theorem 2.3.** In view of (5.3) and (5.5),

$$\frac{\epsilon(\sigma)}{\sigma_* - \sigma} = \frac{\epsilon(\sigma) - \epsilon(\sigma_*)}{\sigma_* - \sigma} = \frac{\|w_\sigma'\|_{L_r^2}^2}{\sigma(1+\sigma_*)\|w_\sigma\|_{L_r^2}^2}.$$

The convergence $w_\sigma \to w_*$ in H^1_r implies

$$\lim_{\sigma\to\sigma_*} \frac{\epsilon(\sigma_*)-\epsilon(\sigma)}{\sigma_*-\sigma} = -\frac{\|w_*'\|^2_{L^2_r}}{\sigma_*(1+\sigma_*)\|w_*\|^2_{L^2_r}}, \tag{5.14}$$

and thus $\sigma \mapsto \epsilon(\sigma)$ is $\mathcal{C}^1$ on $(0,\sigma_*]$, with $\epsilon'(\sigma_*)$ given by the above quantity. To compute this ratio, we use Emden–Fowler transformation,

$$\rho = e^t, \quad W_*(t) = e^{t/\sigma_*} w_*(\rho). \tag{5.15}$$

This transformation, applied to the expression (2.8), yields

$$W_*(t) = \frac{e^{\frac{t}{\sigma_*}}}{(1+ae^{2t})^{\frac{1}{\sigma_*}}}, \tag{5.16}$$

and thus

$$\|w_*\|^2_{L^2_r} = \int_0^\infty \frac{\rho^{1+\frac{2}{\sigma_*}}\mathrm{d}\rho}{(1+a\rho^2)^{\frac{2}{\sigma_*}}} = \int_{-\infty}^\infty \frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}\mathrm{d}t.$$

On the other hand, integration by parts gives

$$\|w_*'\|^2_{L^2_r} = \frac{4a^2}{\sigma_*^2}\int_0^\infty \frac{\rho^{3+\frac{2}{\sigma_*}}\mathrm{d}\rho}{(1+a\rho^2)^{2+\frac{2}{\sigma_*}}} = \frac{4a(1+\sigma_*)}{\sigma_*^2(2+\sigma_*)}\int_0^\infty \frac{\rho^{1+\frac{2}{\sigma_*}}\mathrm{d}\rho}{(1+a\rho^2)^{1+\frac{2}{\sigma_*}}}.$$

Similarly, we get

$$\|w_*'\|^2_{L^2_r} = \frac{4a(1+\sigma_*)}{\sigma_*^2(2+\sigma_*)}\int_{-\infty}^\infty \frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t.$$

To proceed further, we use the identity

$$\int_{-\infty}^\infty e^{\frac{2t}{\sigma_*}}\frac{(1-ae^{2t})}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t = \frac{\sigma_*}{2}\int_{-\infty}^\infty \frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}\mathrm{d}t = 0, \tag{5.17}$$

to further obtain

$$\begin{aligned}
\int_{-\infty}^\infty \frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t &= a\int_{-\infty}^\infty \frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t \\
&= -\frac{\sigma_*}{4}\int_{-\infty}^\infty e^{\frac{2t}{\sigma_*}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}\mathrm{d}t \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}\mathrm{d}t. \qquad (5.18)
\end{aligned}$$

Therefore,

$$\|w_*'\|^2_{L^2_r} = \frac{4(1+\sigma_*)}{\sigma_*^2(2+\sigma_*)}\int_{-\infty}^\infty \frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t = \frac{2(1+\sigma_*)}{\sigma_*^2(2+\sigma_*)}\int_{-\infty}^\infty \frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}\mathrm{d}t.$$

On the other hand, for $\sigma_* < 1$ (or, equivalently, $d \geq 5$), we have

$$\int_{-\infty}^\infty e^{\frac{2t}{\sigma_*}}\frac{(1-a(1-\sigma_*)e^{2t})}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}\mathrm{d}t = \frac{\sigma_*}{2}\int_{-\infty}^\infty \frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}-1}}\mathrm{d}t = 0,$$

which yields

$$\|w_*\|_{L^2_r}^2 = \int_{-\infty}^{\infty} \frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}} \mathrm{d}t = \frac{1}{a(1-\sigma_*)} \int_{-\infty}^{\infty} \frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}} \mathrm{d}t.$$

Using the explicit expressions for $\|w_*'\|_{L^2_r}^2$ and $\|w_*\|_{L^2_r}^2$ in (5.14) yields

$$\epsilon'(\sigma_*) = -\frac{2a(1-\sigma_*)}{\sigma_*^3(2+\sigma_*)} = \frac{(\sigma_*-1)}{2\sigma_*(1+\sigma_*)(2+\sigma_*)} < 0, \tag{5.19}$$

where we have used $a = \frac{\sigma_*^2}{4(1+\sigma_*)}$. Since by definition,

$$\epsilon(\sigma) = (\alpha(\sigma))^{-2\sigma} \Longleftrightarrow \alpha(\sigma) = \epsilon(\sigma)^{-1/(2\sigma)}, \tag{5.20}$$

the asymptotics $\epsilon(\sigma) \sim (\sigma - \sigma_*)\epsilon'(\sigma_*)$ as $\sigma \to \sigma_*$ yields the final claim of Theorem 2.3.

5.5. **Further properties of the ground state near the algebraic soliton.** Related to the algebraic soliton w_*, we introduce the linearized operator $\mathcal{M}_0 : H^2_r \subset L^2_r \to L^2_r$ given by

$$\mathcal{M}_0 = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{d-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{1+2\sigma_*}{(1+a\rho^2)^2}. \tag{5.21}$$

It is a self-adjoint operator in L^2_r with the essential spectrum located on $[0,\infty)$ by Weyl's theorem.

Since w_* is characterized variationally as a constrained minimizer of (2.9) with a single constraint, the Morse index of $\mathcal{M}_0$ (the number of negative eigenvalues in L^2_r) is either 0 or 1. Since

$$\langle \mathcal{M}_0 w_*, w_* \rangle = -2 \int_0^{\infty} \rho^{d-1} |w_*(\rho)|^{2\sigma+2} \mathrm{d}\rho < 0,$$

the Morse index is exactly one. To characterize solutions of the homogeneous equation $\mathcal{M}_0 \mathfrak{v} = 0$, we note that $\rho = 0$ is a regular singular point with two linearly independent solution $1 + \mathcal{O}(\rho^2)$ and $\rho^{2-d}[1+\mathcal{O}(\rho)]$. Since the second solution is singular and does not belong to L^2_r, we define the unique solution $\mathfrak{v} \in \mathcal{C}^2(0,\infty)$ of the initial-value problem

$$\left\{ \begin{array}{l} \mathfrak{v}''(\rho) + \frac{d-1}{\rho}\mathfrak{v}'(\rho) + \frac{1+2\sigma_*}{(1+a\rho^2)^2}\mathfrak{v}(\rho) = 0, \\ \mathfrak{v}(0) = 1, \quad \mathfrak{v}'(0) = 0. \end{array} \right. \tag{5.22}$$

We invoke [12, Theorem 8.1, p. 92]:

$$X = \begin{pmatrix} \mathfrak{v} \\ \mathfrak{v}' \end{pmatrix}$$

solves $X' = (A + V(\rho) + R(\rho))X$ with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V(\rho) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{d-1}{\rho} \end{pmatrix}, \quad R(\rho) = \begin{pmatrix} 0 & 0 \\ -\frac{1+2\sigma_*}{(1+a\rho^2)^2} & 0 \end{pmatrix}.$$

Since V' and R are integrable on $(1,\infty)$, with $\mu_1 = \mu_2 = 0$ in the notations of [12, Theorem 8.1, p. 92], $\mathfrak{v}(\rho)$ does not diverge as $\rho \to \infty$: it satisfies

$$\mathfrak{v}(\rho) \to \mathfrak{v}_\infty \quad \text{as} \quad \rho \to \infty, \tag{5.23}$$

with uniquely defined $\mathfrak{v}_\infty \in \mathbb{R}$. Since the Morse index is exactly one, Sturm's theorem implies that $\mathfrak{v}(\rho)$ has a single node such that $\mathfrak{v}(\rho) > 0$ for $\rho \in [0,\rho_0)$ and $\mathfrak{v}(\rho) < 0$ for $\rho \in (\rho_0,\infty)$ so that $\mathfrak{v}_\infty \leq 0$. However, due to degeneracy of the minimizers of (2.9) by the scaling transformation, we prove in the following lemma that $\mathfrak{v}_\infty = 0$ so that $\mathfrak{v} \in H^2_r \subset L^2_r$ if $d \geq 5$. Since $\mathcal{M}_0 : H^2_r \subset L^2_r \to L^2_r$

is not Fredholm due to 0 being an embedded eigenvalue in the end point of the essential spectrum, we also characterize solutions of the inhomogeneous equation $\mathcal{M}_0\mathfrak{g} = f$ for a given $f \in L^2_r$.

Lemma 5.18. *The exact solution of* (5.22) *is given by*

$$\mathfrak{v}(\rho) = \frac{1 - a\rho^2}{(1 + a\rho^2)^{1+1/\sigma_*}}, \tag{5.24}$$

hence $\mathfrak{v} \in H^2_r$ *if* $d \geq 5$. *For every* $f \in L^2_r$ *and* $d \geq 5$, *there exists a unique solution* $\mathfrak{g} = \mathcal{M}_0^{-1} f$ *satisfying* $\mathfrak{g}(0) = 0$, $\mathfrak{g}'(0) = 0$, *and*

$$\mathfrak{g}_\infty := \lim_{\rho\to\infty} \mathfrak{g}(\rho) = 0$$

if and only if $\langle \mathfrak{v}, f\rangle = 0$.

Proof. Differentiating $\alpha^{1/\sigma*} w_*(\alpha\rho)$ with respect to α at $\alpha = 1$ yields

$$\partial_\alpha \alpha^{1/\sigma*} w_*(\alpha\rho)|_{\alpha=1} = \frac{1 - a\rho^2}{\sigma_*(1 + a\rho^2)^{1+1/\sigma_*}}.$$

Multiplying it by σ_* yields (5.24) which satisfies the initial conditions $\mathfrak{v}(0) = 1$ and $\mathfrak{v}'(0) = 0$. Due to the decay $\mathfrak{v}(\rho) \sim \rho^{-2/\sigma_*}$ as $\rho \to \infty$, we have $\mathfrak{v}_\infty = 0$ in (5.23). Furthermore, $\mathfrak{v} \in L^2_r$ if $d \geq 5$, and due to smoothness, we have $\mathfrak{v} \in H^2_r$ if $d \geq 5$.

The second, linearly independent solution $\mathfrak{w} \in \mathcal{C}^2(0, \infty)$ of $\mathcal{M}_0\mathfrak{w} = 0$ is given by the Wronskian relation

$$\mathfrak{v}(\rho)\mathfrak{w}'(\rho) - \mathfrak{v}'(\rho)\mathfrak{w}(\rho) = \rho^{-(d-1)}, \quad \rho \in (0, \infty), \tag{5.25}$$

where the norming factor is uniquely chosen. It is clear from (5.25) that $\mathfrak{w}(\rho) \sim \rho^{-(d-2)}$ as $\rho \to 0$ with the singularity prescribed at the regular singular point $\rho = 0$. It is also clear from (5.25) that $\mathfrak{w}(\rho) \to \mathfrak{w}_\infty$ as $\rho \to \infty$ with $\mathfrak{w}_\infty \neq 0$. Solving $\mathcal{M}_0\mathfrak{g} = f$ by the variation of constant formula, we get

$$\mathfrak{g}(\rho) = \mathfrak{v}(\rho) \int_0^\rho \varrho^{d-1}\mathfrak{w}(\varrho)f(\varrho)\mathrm{d}\varrho - \mathfrak{w}(\rho) \int_0^\rho \varrho^{d-1}\mathfrak{v}(\varrho)f(\varrho)\mathrm{d}\varrho. \tag{5.26}$$

The lower limit of integration in (5.26) is chosen at 0 to satisfy the initial conditions $\mathfrak{g}(0) = \mathfrak{g}'(0) = 0$, e.g. if f is bounded at $\rho = 0$, then $\mathfrak{g}(\rho) \sim \rho^2$ as $\rho \to 0$. On the other hand, we use the Cauchy–Schwarz inequality and obtain for $\rho_0 \gg 1$,

$$\left|\int_{\rho_0}^\rho \varrho^{d-1}\mathfrak{w}(\varrho)f(\varrho)d\varrho\right| \leq C|\mathfrak{w}_\infty|\|f\|_{L^2_r}\|1\|_{L^2_r(\rho_0,\rho)} \leq C|\mathfrak{w}_\infty|\|f\|_{L^2_r}\rho^{\frac{d}{2}}.$$

Since $\mathfrak{v}(\rho) \sim \rho^{-(d-2)}$ as $\rho \to \infty$ and $d \geq 5$, the first term in (5.26) has the zero limit as $\rho \to \infty$. Then, we compute from the second term in (5.26) that

$$\mathfrak{g}_\infty := \lim_{\rho\to\infty} \mathfrak{g}(\rho) = -\mathfrak{w}_\infty \int_0^\infty \varrho^{d-1}\mathfrak{v}(\varrho)f(\varrho)\mathrm{d}\varrho,$$

where the last term is equivalent to $\langle \mathfrak{v}, f\rangle$, which is well-defined since $f, \mathfrak{v} \in L^2_r$ for $d \geq 5$. Thus $\mathfrak{g}_\infty = 0$ if and only if $\langle \mathfrak{v}, f\rangle = 0$. □

Remark 5.19. Resuming the Emden–Fowler transformation in the inhomogeneous case,

$$\rho = e^t, \quad W_*(t) = e^{t/\sigma_*} w_*(\rho), \quad \mathfrak{G}(t) = e^{t/\sigma_*}\mathfrak{g}(\rho), \quad F(t) = e^{t/\sigma_*} f(\rho),$$

the relation $\mathcal{M}_0\mathfrak{g} = f$ is transformed to the equivalent form

$$-\mathfrak{G}''(t) + \frac{1}{\sigma_*^2}\mathfrak{G}(t) - (1+2\sigma_*)|W_*(t)|^{2\sigma_*}\mathfrak{G}(t) = e^{2t}F(t), \tag{5.27}$$

where W_*, given by (5.16), is now exponentially decaying with the rate $e^{-|t|/\sigma_*}$ as $|t| \to \infty$. The homogeneous solution $\mathfrak{v}$ in (5.24) is related to the translational mode $W_*'(t)$ after the Emden–Fowler transformation, whereas the constraint $\langle \mathfrak{v}, f\rangle = 0$ is equivalent to the Fredholm condition $\int_{-\infty}^{\infty} e^{2t}W_*'(t)F(t)\mathrm{d}t = 0$ required to solve the linear inhomogeneous equation (5.27) to avoid the exponential growth of solutions at ∞.

The following proposition provides an alternative approach in the study of the asymptotic behavior of $\epsilon(\sigma) \to 0$ as $\sigma \to \sigma_*$ for $d \geq 5$ and recovers exactly the same expression for $\epsilon'(\sigma_*)$ given by (5.19).

Proposition 5.20. *For every $d \geq 5$, there exist unique solutions $\mathfrak{z}_*, \mathfrak{w}_* \in \mathcal{C}^2(0,\infty)$ of the linear inhomogeneous equations*

$$\mathcal{M}_0\mathfrak{z}_* = -w_* \tag{5.28}$$

and

$$\mathcal{M}_0\mathfrak{w}_* = (\ln w_*^2)w_*^{1+2\sigma_*}, \tag{5.29}$$

satisfying $\mathfrak{z}_*(0) = \mathfrak{z}_*'(0) = 0$ and $\mathfrak{w}_*(0) = \mathfrak{w}_*'(0) = 0$. If the mapping $(0,\sigma_*) \ni \sigma \mapsto \epsilon(\sigma) \in (0,\infty)$ is $\mathcal{C}^1$ at $\sigma = \sigma_*$, then $\epsilon(\sigma_*) = 0$ and $\epsilon'(\sigma_*)$ is given by (5.19)*.*

Proof. For $d \geq 5$, we have

$$w_* \in L_r^2 \quad \text{and} \quad (\ln w_*^2)w_*^{1+2\sigma_*} \in L_r^2.$$

Hence, solutions $\mathfrak{z}_* \in \mathcal{C}^2(0,\infty)$ and $\mathfrak{w}_* \in \mathcal{C}^2(0,\infty)$ of (5.28) and (5.29) are well defined by Lemma 5.18. However, we show that both $\mathfrak{z}_*(\rho)$ and $\mathfrak{w}_*(\rho)$ do not decay to 0 as $\rho \to \infty$.

For $\mathfrak{z}_*$, we check the Fredholm condition

$$\begin{aligned}\langle \mathfrak{v}, w_*\rangle &= \int_0^\infty \rho^{1+\frac{2}{\sigma_*}} \frac{(1-a\rho^2)}{(1+a\rho^2)^{1+\frac{2}{\sigma_*}}}\mathrm{d}\rho = \int_{-\infty}^\infty e^{2t+\frac{2t}{\sigma_*}} \frac{(1-ae^{2t})}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t \\ &= \sigma_* \int_{-\infty}^\infty e^{2t}W_*'(t)W_*(t)\mathrm{d}t = -\sigma_*\int_{-\infty}^\infty e^{2t}W_*^2(t)\mathrm{d}t < 0,\end{aligned}$$

where we have used the Emden–Fowler transformation (5.15) with $W_*(t)$ given by (5.16), and integrated by parts with the sufficient decay of $W_*^2(t) \sim e^{-2|t|/\sigma_*}$ at $\pm\infty$ since $\sigma_* < 1$ if $d \geq 5$. Since $\langle \mathfrak{v}, w_*\rangle \neq 0$, the unique solution $\mathfrak{z}_* \in \mathcal{C}^2(0,\infty)$ of (5.28) satisfying $\mathfrak{z}_*(0) = \mathfrak{z}_*'(0) = 0$ does not decay to 0 as $\rho \to \infty$.

For $\mathfrak{w}_*$, we use the Emden–Fowler transformation (5.15) and check the Fredholm condition

$$\begin{aligned}\langle \mathfrak{v}, (\ln w_*^2)w_*^{1+2\sigma_*}\rangle &= -\frac{2}{\sigma_*}\int_0^\infty \rho^{1+\frac{2}{\sigma_*}} \frac{(1-a\rho^2)}{(1+a\rho^2)^{3+\frac{2}{\sigma_*}}}\ln(1+a\rho^2)\mathrm{d}\rho \\ &= -\frac{2}{\sigma_*}\int_{-\infty}^\infty e^{2t+\frac{2t}{\sigma_*}} \frac{(1-ae^{2t})}{(1+ae^{2t})^{3+\frac{2}{\sigma_*}}}\ln(1+ae^{2t})\mathrm{d}t.\end{aligned}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{2+\frac{2}{\sigma_*}}} = \frac{2(1+\sigma_*)}{\sigma_*}e^{2t+\frac{2t}{\sigma_*}}\frac{(1-ae^{2t})}{(1+ae^{2t})^{3+\frac{2}{\sigma_*}}},$$

integration by parts removes the logarithmic term and yields

$$\begin{aligned}\langle \mathfrak{v}, (\ln w_*^2)w_*^{1+2\sigma_*}\rangle &= -\frac{1}{(1+\sigma_*)}\int_{-\infty}^{\infty}\ln(1+ae^{2t})\frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{2+\frac{2}{\sigma_*}}}\mathrm{d}t \\ &= \frac{2a}{(1+\sigma_*)}\int_{-\infty}^{\infty}\frac{e^{4t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{3+\frac{2}{\sigma_*}}}\mathrm{d}t > 0.\end{aligned}$$

Since $\langle \mathfrak{v}, (\ln w_*^2)w_*^{1+2\sigma_*}\rangle \neq 0$, the unique solution $\mathfrak{w}_* \in \mathcal{C}^2(0,\infty)$ of (5.29) satisfying $\mathfrak{w}_*(0) = \mathfrak{w}_*'(0) = 0$ does not decay to 0 as $\rho \to \infty$.

End of the proof. We have proved the existence and uniqueness of solutions $\mathfrak{z}_* \in \mathcal{C}^2(0,\infty)$ and $\mathfrak{w}_* \in \mathcal{C}^2(0,\infty)$ of the linear inhomogeneous equations (5.28) and (5.29). Let $w_\sigma(\rho) = w(\rho;\epsilon(\sigma),\sigma) \in \mathcal{C}^2(0,\infty)\cap L^\infty(0,\infty)$ be defined from the family of solutions of (2.6). Suppose that ϵ is $\mathcal{C}^1$ up to $\sigma = \sigma_*$. Differentiating (2.6) with respect to σ yields

$$\frac{\mathrm{d}w_\sigma}{\mathrm{d}\sigma}(\rho)|_{\sigma=\sigma_*} = \epsilon'(\sigma_*)\mathfrak{z}_*(\rho) + \mathfrak{w}_*(\rho). \tag{5.30}$$

Since $w_\sigma(\rho) \to 0$ as $\rho \to \infty$ for every $\sigma \in (0,\sigma_*)$, we require $\frac{\mathrm{d}w_\sigma}{\mathrm{d}\sigma}(\rho) \to 0$ as $\rho \to \infty$ for every $\sigma \in (0,\sigma_*)$ including the limit $\sigma \to \sigma_*^-$. By Lemma 5.18, this is possible if and only if $\epsilon'(\sigma_*)$ is chosen such that

$$-\epsilon'(\sigma_*)\langle \mathfrak{v}, w_*\rangle + \langle \mathfrak{v}, (\ln w_*^2)w_*^{1+2\sigma_*}\rangle = 0. \tag{5.31}$$

In order to derive the explicit expression (5.19), we integrate by parts with the use of the Emden–Fowler transformation (5.15) and (5.16):

$$\begin{aligned}\langle \mathfrak{v}, (\ln w_*^2)w_*^{1+2\sigma_*}\rangle &= -\frac{\sigma_*}{2(1+\sigma_*)^2}\int_{-\infty}^{\infty}e^{2t+\frac{2t}{\sigma_*}}\frac{d}{dt}\frac{1}{(1+ae^{2t})^{2+\frac{2}{\sigma_*}}}\mathrm{d}t \\ &= \frac{1}{(1+\sigma_*)}\int_{-\infty}^{\infty}\frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{2+\frac{2}{\sigma_*}}}\mathrm{d}t \\ &= -\frac{\sigma_*}{2a(1+\sigma_*)(2+\sigma_*)}\int_{-\infty}^{\infty}e^{\frac{2t}{\sigma_*}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t \\ &= \frac{1}{a(1+\sigma_*)(2+\sigma_*)}\int_{-\infty}^{\infty}\frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{1+\frac{2}{\sigma_*}}}\mathrm{d}t,\end{aligned}$$

where all integration by parts are justified due to the fast exponential decay at $\pm\infty$. Recalling (5.17) and (5.18), we have

$$\langle \mathfrak{v}, w_*\rangle = -\sigma_*\int_{-\infty}^{\infty}\frac{e^{2t+\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}\mathrm{d}t,$$

and

$$\langle \mathfrak{v}, (\ln w_*^2) w_*^{1+2\sigma_*} \rangle = \frac{1}{2a(1+\sigma_*)(2+\sigma_*)} \int_{-\infty}^{\infty} \frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}} \mathrm{d}t.$$

In order to show that one expression is proportional to the other one, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{e^{\frac{2t}{\sigma_*}-2t}}{(1+ae^{2t})^{\frac{2}{\sigma_*}-1}} = \frac{2(1-\sigma_*)}{\sigma_*} \frac{e^{\frac{2t}{\sigma_*}-2t}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}} - \frac{2a}{\sigma_*} \frac{e^{\frac{2t}{\sigma_*}}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}}.$$

Since $\sigma_* < 1$, integration by parts yields due to the exponential decay at $\pm\infty$ that

$$\langle \mathfrak{v}, (\ln w_*^2) w_*^{1+2\sigma_*} \rangle = \frac{(1-\sigma_*)}{2a^2(1+\sigma_*)(2+\sigma_*)} \int_{-\infty}^{\infty} \frac{e^{\frac{2t}{\sigma_*}-2t}}{(1+ae^{2t})^{\frac{2}{\sigma_*}}} \mathrm{d}t.$$

Replacing $t = -\tilde{t} - \frac{1}{2}\ln a^2$ yields finally

$$\begin{aligned} \langle \mathfrak{v}, (\ln w_*^2) w_*^{1+2\sigma_*} \rangle &= \frac{(1-\sigma_*)}{2(1+\sigma_*)(2+\sigma_*)} \int_{-\infty}^{\infty} \frac{e^{-\frac{2\tilde{t}}{\sigma_*}+2\tilde{t}} a^{-\frac{2}{\sigma_*}}}{(1+a^{-1}e^{-2\tilde{t}})^{\frac{2}{\sigma_*}}} \mathrm{d}\tilde{t} \\ &= -\frac{(1-\sigma_*)}{2\sigma_*(1+\sigma_*)(2+\sigma_*)} \langle \mathfrak{v}, w_* \rangle. \end{aligned}$$

Substituting this relation into (5.31) yields (5.19). □

Remark 5.21. We show that equation (5.29) for $\mathfrak{w}_*$ can be reduced to equation (5.28) for $\mathfrak{z}_*$ by using an elementary transformation. To do so, we rewrite (5.29) explicitly:

$$\mathfrak{w}_*''(\rho) + \frac{(2+\sigma_*)}{\sigma_*\rho} \mathfrak{w}_*'(\rho) + \frac{1+2\sigma_*}{(1+a\rho^2)^2} \mathfrak{w}_*(\rho) = \frac{2}{\sigma_*} \frac{\ln(1+a\rho^2)}{(1+a\rho^2)^2} \frac{1}{(1+a\rho^2)^{\frac{1}{\sigma_*}}}.$$

Substitution

$$\mathfrak{w}_*(\rho) = \frac{f(\rho)}{(1+a\rho^2)^{\frac{1}{\sigma_*}}}$$

brings this equation to the form

$$f''(\rho) + \frac{(2+\sigma_*)}{\sigma_*\rho} f'(\rho) - \frac{4a\rho}{\sigma_*(1+a\rho^2)} f'(\rho) + \frac{2\sigma_*}{(1+a\rho^2)^2} f(\rho) = \frac{2}{\sigma_*} \frac{\ln(1+a\rho^2)}{(1+a\rho^2)^2}.$$

Transformation

$$f(\rho) = \frac{1}{\sigma_*^2} \ln(1+a\rho^2) + g(\rho)$$

brings the right-hand-side to a rational function

$$g''(\rho) + \frac{(2+\sigma_*)}{\sigma_*\rho} g'(\rho) - \frac{4a\rho}{\sigma_*(1+a\rho^2)} g'(\rho) + \frac{2\sigma_*}{(1+a\rho^2)^2} g(\rho) = -\frac{1}{\sigma_*(1+\sigma_*)} \frac{(1+\sigma_*) - a\rho^2}{(1+a\rho^2)^2}.$$

By using the substitution

$$g(\rho) = b + c\rho^2 + h(\rho),$$

we obtain coefficients (b,c) to reduce the right-hand side to the constant function. Elementary computations give

$$b = -\frac{1}{2\sigma_*^2}\left[1 + \frac{1}{(1+\sigma_*)(2+\sigma_*)}\right], \quad c = \frac{1}{8(1+\sigma_*)(2+\sigma_*)},$$

and

$$h''(\rho) + \frac{d-1}{\rho} h'(\rho) - \frac{4a\rho}{\sigma_*(1+a\rho^2)} h'(\rho) + \frac{2\sigma_*}{(1+a\rho^2)^2} h(\rho) = \frac{4c(1-\sigma_*)}{\sigma_*}.$$

To summarize, the transformation

$$\mathfrak{w}_*(\rho) = \frac{\ln(1+a\rho^2) + \sigma_*^2 b + \sigma_*^2 c\rho^2}{\sigma_*^2 (1+a\rho^2)^{\frac{1}{\sigma_*}}} + \tilde{\mathfrak{w}}_*(\rho)$$

with the uniquely defined (b, c) reduces (5.29) to

$$\mathcal{M}_0 \tilde{\mathfrak{w}}_* = -\frac{(1-\sigma_*)}{2\sigma_*(1+\sigma_*)(2+\sigma_*)} w_*,$$

which coincides with (5.28) up to the scalar multiplication. The first term in $\mathfrak{w}_*$ is decaying as $\rho \to \infty$ if $d \geq 5$ but does not satisfy the initial condition $\mathfrak{w}_*(0) = 0$. To correct the solution, we use the homogeneous solution $\mathfrak{v}$ given by (5.24), which is also decaying as $\rho \to \infty$, and redefine $\mathfrak{w}_*$ in the equivalent form:

$$\mathfrak{w}_*(\rho) = \frac{\ln(1+a\rho^2) + \sigma_*^2 b + \sigma_*^2 c\rho^2}{\sigma_*^2 (1+a\rho^2)^{\frac{1}{\sigma_*}}} - b\mathfrak{v}(\rho) + \frac{(1-\sigma_*)}{2\sigma_*(1+\sigma_*)(2+\sigma_*)} \mathfrak{z}_*(\rho), \tag{5.32}$$

so that $\mathfrak{w}_*(0) = \mathfrak{w}_*'(0) = 0$ is satisfied. By using (5.32), we can rewrite (5.30) explicitly as

$$\begin{aligned} \frac{\mathrm{d}w_\sigma}{\mathrm{d}\sigma}(\rho)|_{\sigma=\sigma_*} &= \epsilon'(\sigma_*)\mathfrak{z}_*(\rho) + \mathfrak{w}_*(\rho), \\ &= \left[\epsilon'(\sigma_*) + \frac{(1-\sigma_*)}{2\sigma_*(1+\sigma_*)(2+\sigma_*)}\right] \mathfrak{z}_*(\rho) \\ &\quad + \frac{\ln(1+a\rho^2) + \sigma_*^2 b + \sigma_*^2 c\rho^2}{\sigma_*^2 (1+a\rho^2)^{\frac{1}{\sigma_*}}} - b\mathfrak{v}(\rho), \end{aligned}$$

Since $\mathfrak{z}_*$ does not decay to 0 as $\rho \to \infty$, we have $\frac{\mathrm{d}w_\sigma}{\mathrm{d}\sigma}(\rho)|_{\sigma=\sigma_*} \to 0$ as $\rho \to \infty$ if and only if $\epsilon'(\sigma_*)$ satisfies (5.19). Thus, both the explicit solution for (5.30) and the Fredholm condition (5.31) result in the same expression (5.19), which was found from the quotient (5.14).

Remark 5.22. In view of (5.19) and (5.20), we obtain the leading-order asymptotic divergence of $\alpha(\sigma)$ as $\sigma \to \sigma_*^-$ as

$$\alpha(\sigma) \sim \left(|\epsilon'(\sigma_*)|(\sigma_* - \sigma)\right)^{-\frac{1}{2\sigma_*}} \quad \text{as } \sigma \to \sigma_*^-.$$

Furthermore, we have

$$w_\sigma(\rho) \sim w_*(\rho) + (\sigma - \sigma_*)\frac{\mathrm{d}w_\sigma}{\mathrm{d}\sigma}(\rho)|_{\sigma=\sigma_*},$$

where the correction term

$$(\sigma - \sigma_*)\frac{\mathrm{d}w_\sigma}{\mathrm{d}\sigma}(\rho)|_{\sigma=\sigma_*} = (\sigma - \sigma_*)\left[\frac{\ln(1+a\rho^2) + \sigma_*^2 b + \sigma_*^2 c\rho^2}{\sigma_*^2 (1+a\rho^2)^{\frac{1}{\sigma_*}}} - b\mathfrak{v}(\rho)\right] \tag{5.33}$$

is positive for $\rho \in (0, \rho_0)$ and negative for $\rho \in (\rho_0, \infty)$ for some $\rho_0 > 0$.

Remark 5.23. Due to the term $c\rho^2$ in (5.33) with $c \neq 0$, the first term in (5.33) is not in L_r^2 for $5 \leq d \leq 8$. This shows that the ground state near the algebraic soliton cannot be generally expanded as powers of $(\sigma_* - \sigma)$ in L_r^2.

Remark 5.24. For $d = 4$, we have $\sigma_* = 1$ so that the exact solution $\mathfrak{w}_*$ given by (5.32) is independent of $\mathfrak{z}_*$. It is clear that the solution (5.32) is non-decaying as $\rho \to \infty$ due to the $c\rho^2$ term. Nevertheless, the balance with the term $\mathfrak{z}_*$ is impossible since $w_* \notin L^2_r$ for $d = 4$ and $\mathfrak{z}_*$ is logarithmically growing as $\rho \to \infty$. This shows that the asymptotic behavior of $\epsilon(\sigma)$ as $\sigma \to \sigma^*$ is more complicated than the power expansion. Similarly, we do not have a balance between $\mathfrak{z}_*$ growing as $\mathcal{O}(\rho)$ and bounded $\mathfrak{w}_*$ for $d = 3$ ($\sigma_* = 2$). Modifications of the asymptotic behavior of the ground state near the algebraic soliton for $d = 4$ and $d = 3$ are discussed within the Gross–Pitaevskii equation with a harmonic potential in [24, 25].

6. Numerical approximations of the ground state

6.1. **Radial finite differences.** We first recall the definition of the radial Lebesgue spaces $L^p_r(\mathbb{R}^d)$ associated with the norms

$$\|u\|_{L^p_r} = \left(C(d) \int_0^\infty |u(r)|^p r^{d-1} \mathrm{d}r \right)^{1/p},$$

with the constants $C(d)$ given in Table 1.

d	1	2	3	4	5
$C(d)$	2	2π	4π	$2\pi^2$	$\frac{8}{3}\pi^2$

Table 1. Surface area of the unit sphere in $\mathbb{R}^d$ for $d = 1, \dots, 5$.

The d-dimensional radial Laplace operator

$$\Delta_r u = \frac{1}{r^{d-1}} \partial_r \left(r^{d-1} \partial_r u \right)$$

on the finite interval $[0, R]$, with Neumann boundary condition at $r = 0$ and Dirichlet boundary condition at $R > 0$, is then discretized as follows. We fix a mesh size $h = \frac{2R}{2M+1}$ with $M > 0$ an integer, and define both *regular* and *staggered* grid points as

$$r_j = jh \quad \text{and} \quad r_{j+\frac{1}{2}} = \left(j + \frac{1}{2} \right) h, \quad 0 \le j \le M,$$

so that we define the approximation of the radial Laplace operator Δ^h_r on the staggered grid as

$$\Delta^h_r u_{j+\frac{1}{2}} = \frac{1}{h^2} \frac{1}{r^{d-1}_{j+\frac{1}{2}}} \left(r^{d-1}_{j+1} u_{j+\frac{3}{2}} - (r^{d-1}_{j+1} + r^{d-1}_j) u_{j+\frac{1}{2}} + r^{d-1}_j u_{j-\frac{1}{2}} \right),$$

for $0 \le j \le M - 1$, with Neumann and Dirichlet boundary conditions imposed by

$$u_{-\frac{1}{2}} = u_{\frac{1}{2}} \quad \text{and} \quad u_{M+\frac{1}{2}} = 0.$$

Note that we have also introduced a ghost point $r_{-\frac{1}{2}}$ to approximate the Neumann boundary condition $u'(0) = 0$ with the second-order accuracy.

6.2. **Gradient flow with $L^{2\sigma+2}$ normalization.** The unique ground state with the profile u_σ satisfying (1.3) for $0<\sigma<\sigma_*$ can be numerically approximated as follows. We first recall the definition of the *Nehari manifold* for the variational problem (2.2):

$$\mathcal{N} = \left\{ \phi \in H^1(\mathbb{R}^d) \;\middle|\; I(\phi) = \frac{1}{\sigma}\int_{\mathbb{R}^d} |\phi|^{2\sigma+2}, \quad \phi \neq 0 \right\},$$

associated to the quadratic functional

$$I(\phi) = \frac{1}{2}\|\nabla\phi\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{\sigma}\|\phi\|^2_{L^2(\mathbb{R}^d)}.$$

We then perform, inspired by the method of [30], a normalized gradient flow scheme as follows. Starting from an initial radial state $\phi^0_\sigma(r) = e^{-r^2}$ for $r \in \mathbb{R}_+$, we realize a linearly implicit normalized gradient flow that writes for $n \in \mathbb{N}^*$ as

$$\begin{cases} \dfrac{\phi^{*,n+1}_\sigma - \phi^n_\sigma}{\tau} = \Delta_r \phi^{*,n+1}_\sigma + \dfrac{1}{\sigma}(|\phi^n_\sigma|^{2\sigma} - 1)\phi^{*,n+1}_\sigma, \\ \phi^{n+1}_\sigma = \lambda_{n+1}\phi^{*,n+1}_\sigma, \quad \lambda_{n+1} = \left(\dfrac{\sigma I(\phi^{*,n+1}_\sigma)}{\|\phi^{*,n+1}_\sigma\|^{2\sigma+2}_{L^{2\sigma+2}_r}} \right)^{\frac{1}{2\sigma}}. \end{cases} \tag{6.1}$$

We stop the algorithm when

$$\frac{\|\phi^{n+1}_\sigma - \phi^n_\sigma\|_{L^2_r}}{\tau} \leq \varepsilon \tag{6.2}$$

for a given threshold $\varepsilon > 0$. A fixed point $(\phi_\sigma, \phi^*_\sigma)$ of the iterative method (6.1) is then solution to $\phi_\sigma = \lambda\phi^*_\sigma$ with $\lambda^{2\sigma} = \sigma I(\phi^*_\sigma)/\|\phi^*_\sigma\|^{2\sigma+2}_{L^{2\sigma+2}_r}$ and

$$\frac{1-\lambda}{\tau}\phi_\sigma = \Delta_r\phi_\sigma + \frac{1}{\sigma}(|\phi_\sigma|^{2\sigma} - 1)\phi_\sigma.$$

Denoting $\gamma = 1 + \sigma(1-\lambda)/\tau$ and performing the rescaling

$$\mathrm{u}_\sigma(r) = \gamma^{-\frac{1}{1-2\sigma}}\phi_\sigma\left(\sqrt{\gamma}r\right),$$

we get the numerical approximation u_σ of the profile u_σ of the ground state. In the following interpretation of numerical results, we identify u_σ for $\sigma \in (0, \sigma_*)$.

Remark 6.1. The number of iterations needed in order to achieve the stopping criterion (6.2) greatly increases as $\sigma \to 0$ or as $\sigma \to \sigma_*$ for $d \geq 3$. This suggests that our numerical scheme is stiff with respect to both endpoint limits. In particular, we hardly go beyond $\sigma = 1.6$ for $d = 3$ and $\sigma = 0.9$ for $d = 4$ as $\sigma \to \sigma_*$.

In Figure 2 we plot the (approximated) ground state profile u_σ for σ varying between 0.1 and 8 in 2D, as well as the Gausson u_0 explicitly given by (1.7) and the expected root $r_0 = \sqrt{2+2\sqrt{2}}$ computed through equation (4.6), in both linear scale and logarithmic scale. We see that the successive ground states u_σ do converge towards the Gausson u_0 as $\sigma \to 0$. The crossing point $r_\sigma > 0$ between curves u_σ and u_0 also tends towards the expected root r_0 of equation (4.6) as $\sigma \to 0$. On the other hand, we observe in the limit $\sigma \to \infty$ that the ground state profile becomes steeper and steeper at the origin.

Recall that $\alpha(\sigma) = \|u_\sigma\|_{L^\infty} = u_\sigma(0)$ for $0 < \sigma < \sigma_*$, and $\alpha(0) = \|u_0\|_{L^\infty} = u_0(0) = e^{d/2}$. We plot in Figure 3 the dependence of $\alpha(\sigma) = u_\sigma(0)$ normalized by the value of $\alpha(0)$ versus $\sigma \in (0, 0.5)$

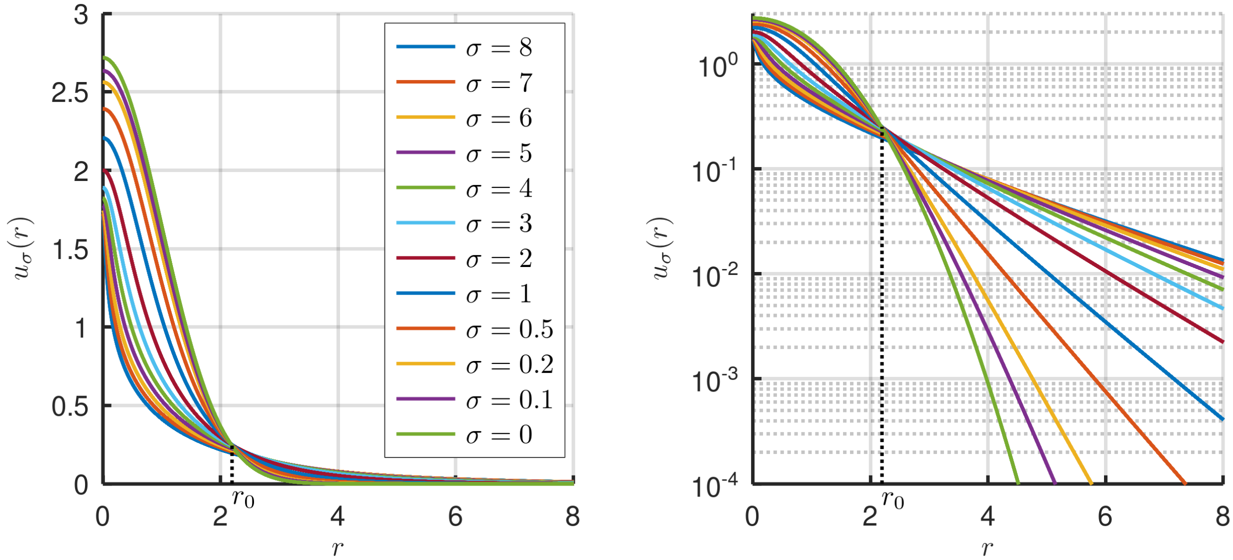

FIGURE 2. Ground state profile u_σ for $d = 2$ in linear scale *(left)* and logarithmic scale *(right)*, for different values of σ.

for $d = 3, 4, 5$, as well as the expected slopes at the origin explicitly given by (4.7). Each curve matches its respective slope at $\sigma = 0$.

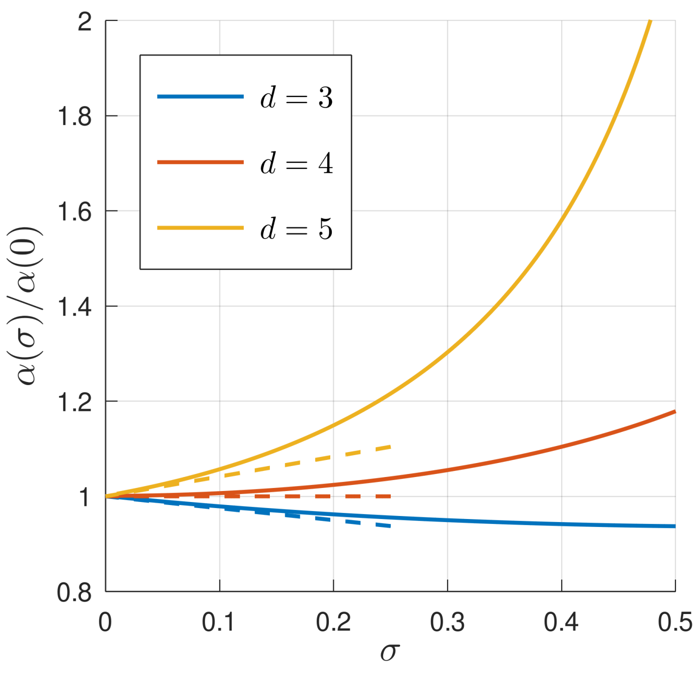

FIGURE 3. Dependence of $\alpha(\sigma)/\alpha(0)$ versus σ and the expected slopes (4.7) for $d = 3, 4, 5$.

6.3. **Gradient flow with L^∞ normalization.** We now illustrate the limit $\sigma \to \sigma_*$ for $d \geq 3$, which corresponds to the convergence to the algebraic soliton. One should first mention that this limit is very stiff for the $L^{2\sigma+2}$ normalization algorithm as the L^∞-norm of the ground states is unbounded when $\sigma \to \sigma_*$. This motivates the use of a new gradient flow approach, based on the formulation (2.6). We now perform, starting from the explicit initial radial state $w_\sigma^0(\rho) = w_*(\rho)$ for $\rho \in \mathbb{R}_+$, a linearly implicit normalized gradient flow that writes for $n \in \mathbb{N}^*$ as

$$\begin{cases} \dfrac{w_\sigma^* - w_\sigma^n}{\tau} = \Delta_\rho w_\sigma^* + |w_\sigma^n|^{2\sigma} w_\sigma^* - \epsilon_0(\sigma) w_\sigma^*, \\ w_\sigma^{n+1} = \dfrac{w_\sigma^*}{\|w_\sigma^*\|_{L^\infty}}, \end{cases} \tag{6.3}$$

where we use the approximation

$$\epsilon_0(\sigma) := \frac{(\sigma_* - 1)(\sigma_* - \sigma)}{2\sigma_*(1+\sigma_*)(2+\sigma_*)}$$

from equation (5.19) (instead of the implicit constant $\epsilon(\sigma)$). We stop the algorithm when

$$\frac{\|w_\sigma^{n+1} - w^n\|_{L_r^2}}{\tau} \leq \eta$$

for a fixed threshold $\eta > 0$. A fixed point $(\mathrm{w}_\sigma, \mathrm{w}_\sigma^*)$ of the iterative method (6.3) is then solution to $\mathrm{w}_\sigma^* = \mu \mathrm{w}_\sigma$ with $\mu = \|\mathrm{w}_\sigma^*\|_{L^\infty}$ and

$$\frac{\mu - 1}{\mu\tau} \mathrm{w}_\sigma = \Delta_\rho \mathrm{w}_\sigma + \mathrm{w}_\sigma^{2\sigma+1} - \epsilon_0(\sigma)\mathrm{w}_\sigma.$$

Therefore, the numerical value of $\epsilon(\sigma)$ in the formulation (2.6) is adjusted as

$$\underline{\epsilon}(\sigma) := \epsilon_0(\sigma) + \frac{\mu - 1}{\mu\tau},$$

where w_σ provides the numerical approximation of the profile w_σ.

In order to illustrate the convergence of the ground state profile w_σ to the algebraic soliton w_* as $\sigma \to \sigma_* = \frac{2}{3}$ for $d = 5$, we perform both normalized gradient flow methods on the range $\sigma \in [0.54, 0.66]$. More precisely:

- For $\sigma = 0.54$ and 0.58, we perform the gradient flow with $L_r^{2\sigma+2}$ normalization based on the numerical method (6.1), after which we rescale the solution to the profile w_σ by using the scaling transformation (2.5).
- For $\sigma = 0.62$ and 0.66, we perform the gradient flow with L^∞ normalization based on the numerical method (6.3).

In Figure 4 we plot the corresponding approximated ground states w_σ for different values of σ, as well as the algebraic soliton w_* for σ_* given by (2.8), in both linear scale and logarithmic scale. Once again, this illustrates the convergence of the ground state w_σ towards the algebraic soliton w_*.

In Figure 5, we plot the dependence of $\underline{\epsilon}(\sigma)$ versus $\sigma \in [0.5, 0.66]$, along with the predicted evolution slope $\epsilon_0(\sigma)$. The dependence $\underline{\epsilon}(\sigma)$ matches $\epsilon_0(\sigma)$ as $\sigma \to \sigma_*$.

Finally, we plot the difference $w_\sigma - w_*$ for several σ in Figure 6, as well as the expected limit crossing point

$$\rho_0 = \lim_{\sigma \to \sigma_*} \operatorname{argmin} \{\rho > 0 \mid w_\sigma(\rho) - w_*(\rho)\} > 0,$$

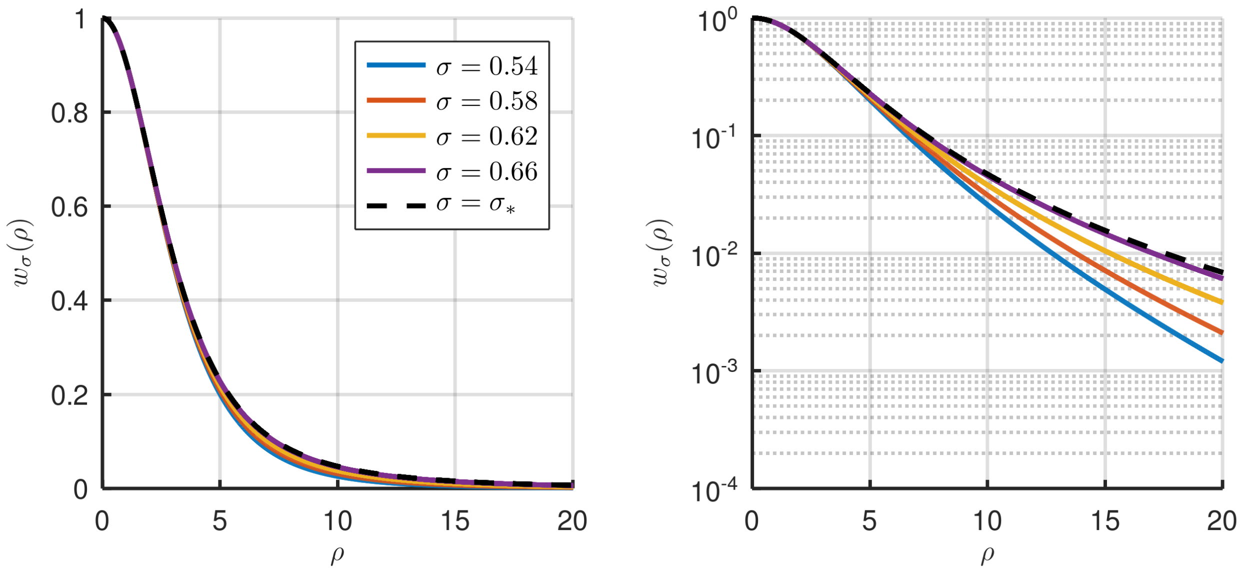

FIGURE 4. Ground state profile w_σ for $d = 5$ in the linear scale *(left)* and the logarithmic scale *(right)*.

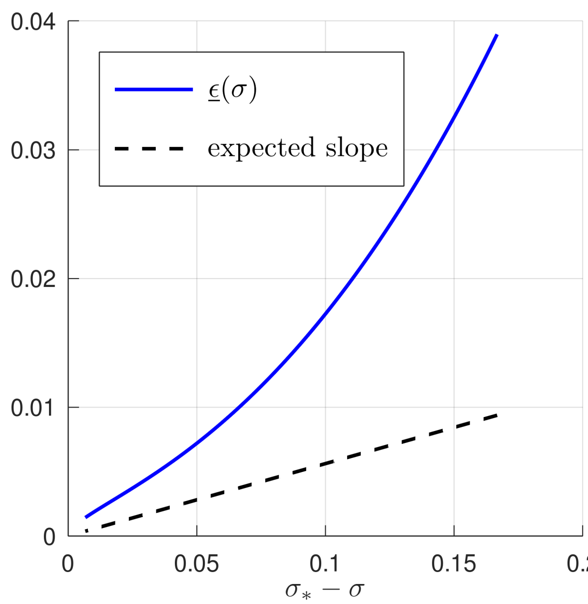

FIGURE 5. Dependence of $\underline{\epsilon}(\sigma)$ (solid curve) and the asymptotic approximation $\epsilon_0(\sigma)$ (dashed line) versus $\sigma \in [0.5, 0.66]$ for $d = 5$.

obtained from the asymptotic approximation (5.33). The value of ρ_0 was computed numerically as the unique positive root of the function given by (5.33).

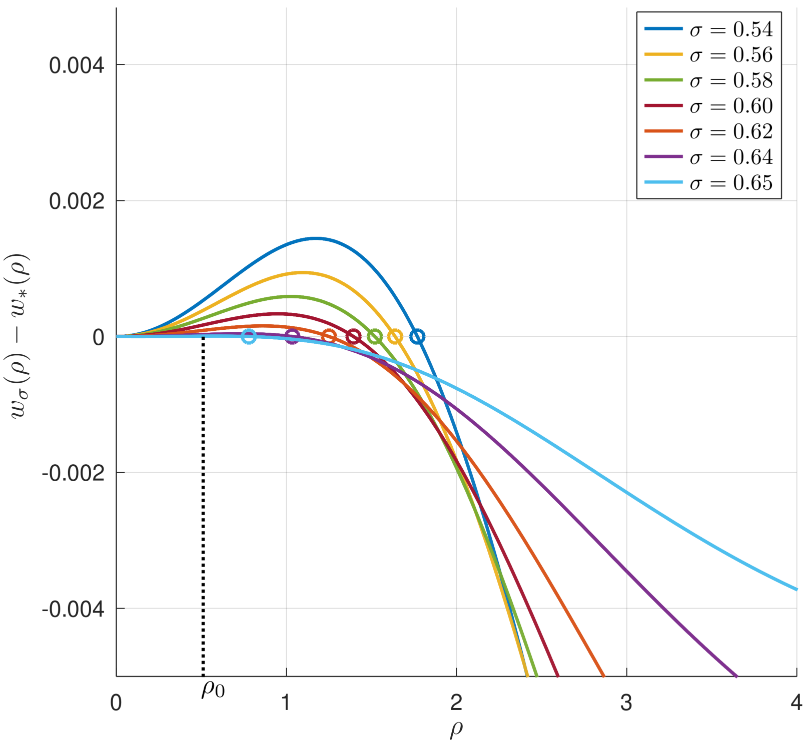

FIGURE 6. Difference $w_\sigma - w_*$ and expected crossing point ρ_0.

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