

Matrix stability theory for incoherent optical solitons

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Abstract

Revisited analysis of spectral stability of solitary waves is developed for a Hamiltonian system of N incoherently coupled nonlinear Schrödinger equations. A linearized problem for a non-self-adjoint matrix operator is studied by means of a pair of uncoupled constrained spectral problems for symmetric matrix Schrödinger operators. Counting lemmas on negative eigenvalues of the constrained problems recover and extend the known stability-instability results for solitary waves in Hamiltonian systems. Sharp bounds on the number and type of unstable eigenvalues in the linearized problem are found from diagonalization of two symmetric matrix Schrödinger operators. New computational algorithm is proposed to locate all unstable eigenvalues in the system of coupled nonlinear Schrödinger equations. The algorithm is applied to problems of oscillatory and transverse instabilities of multiple pulses in multi-channel optical systems.

Keywords: stability–instability theorems, multiple pulse optical solitons, incoherently coupled nonlinear Schrödinger equations, matrix Schrödinger operators, eigenvalues, eigenfunctions, and spectral decompositions.

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1 Introduction

Solitons occur as stationary localized solutions in a system of partial differential equations. When existence of stationary solutions is proved, the linearized stability problem explains typically the role of solitons in an initial-value problem with localized initial data. Since solitons model optical pulses in applied systems, stability of the stationary solutions motivates the use of solitons for all-optical signal processing, pulse transmission and data storage, while instability results generally in noise generation, information losses, and data switching.

This paper addresses stability of incoherent multicomponent solitons that occur in an N -channel optical fiber or in a N -component nonlinear optical waveguide. The problem is modeled by a system of N coupled nonlinear Schrödinger (NLS) equations normalized to the form:

$$i \frac{\partial \psi_n}{\partial z} + d_n \frac{\partial^2 \psi_n}{\partial x^2} + f_n(|\psi_1|^2, \dots, |\psi_N|^2) \psi_n = 0, \quad n = 1, \dots, N, \quad (1.1)$$

where $d_n \in \mathbb{R}$, $z \in \mathbb{R}_+$, $x \in \mathbb{R}$, $f_n : \mathbb{R}^N \rightarrow \mathbb{R}$, and $\psi_n(z, t) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$. We assume here that the nonlinear functions $f_n = f_n(|\psi_1|^2, \dots, |\psi_N|^2)$ satisfy two properties:

$$f_n(0, \dots, 0) = 0, \quad (1.2)$$

$$\frac{\partial f_n}{\partial |\psi_m|^2} = \frac{\partial f_m}{\partial |\psi_n|^2}. \quad (1.3)$$

We also assume that parameters d_n are all positive.

The system (1.1) occurs in various physical problems, e.g. for multichannel wavelength-division multiplexed optical beams [YB98], for birefringent optical fibers with two orthogonal polarizations [M87], and for self-trapped planar beams in biased photorefractive crystals [DN02]. In the optical context, the function $\psi_n(z, x)$ describes an envelope amplitude of the n^{th} channel (beam), z is propagation distance along the waveguide (fiber), and x is a retarded time or a transverse coordinate. The coupling between optical beams is nonlinear and incoherent, i.e. the complex phases of different beams are not mixed. This feature leads to further important properties of the system (1.1).

Property 1.1 The linear spectrum of the system (1.1) with $f_n \equiv 0$ is uncoupled:

$$\psi(z, x) = \sum_{n=1}^N \mathbf{e}_n \int_{-\infty}^{\infty} \alpha_n(k) e^{i(kx + \omega_n(k)z)} dk, \quad (1.4)$$

where $\omega_n = -d_n k^2 \leq 0$, $\boldsymbol{\psi} = (\psi_1, \dots, \psi_N)^T$ and \mathbf{e}_n is the unit n^{th} -component vector in \mathbb{R}^N .

Property 1.2 Any solution of the system (1.1) is translated along an N -parameter group of phase rotations:

$$\boldsymbol{\psi}_\theta(z, x) = e^{i\boldsymbol{\theta}} \boldsymbol{\psi}(z, x), \quad (1.5)$$

where $\boldsymbol{\theta} = \text{diag}(\theta_1, \dots, \theta_N)$ is a diagonal matrix.

Property 1.3 The N -parameter group of phase rotations (1.5) is associated with N conserved quantities (mode powers):

$$Q_n = \int_{\mathbb{R}} dx |\psi_n|^2(z, x), \quad (1.6)$$

where $\psi_n \in L^2(\mathbb{R})$ is assumed.

Property 1.4 When $f_n = f_n(|\psi_1|^2, \dots, |\psi_N|^2)$, any solution of the system (1.1) is translated along a one-parameter group of space translations:

$$\psi_s(z, x) = \psi(z, x - s), \quad (1.7)$$

where $s \in \mathbb{R}$.

Property 1.5 Under the condition (1.3), the system (1.1) conserves the Hamiltonian:

$$H = \int_{\mathbb{R}} dx \left[\sum_{n=1}^N d_n \left| \frac{\partial \psi_n}{\partial x} \right|^2 - U(|\psi_1|^2, \dots, |\psi_N|^2) \right] \quad (1.8)$$

and the vector field momentum associated with the symmetry (1.7):

$$P = i \int_{\mathbb{R}} dx \sum_{n=1}^N \left(\bar{\psi}_n \frac{\partial \psi_n}{\partial x} - \psi_n \frac{\partial \bar{\psi}_n}{\partial x} \right), \quad (1.9)$$

where the potential function $U = U(|\psi_1|^2, \dots, |\psi_N|^2)$ is defined as

$$f_n = \frac{\partial U}{\partial |\psi_n|^2}. \quad (1.10)$$

In this case, the system (1.1) takes the Hamiltonian form in canonical variables $\mathbf{u}(z, x)$ and $\mathbf{w}(z, x)$:

$$\frac{\partial}{\partial z} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \mathcal{J} \begin{pmatrix} \frac{\delta H}{\delta \mathbf{u}} \\ \frac{\delta H}{\delta \mathbf{w}} \end{pmatrix}, \quad \mathcal{J} = \frac{1}{2} \begin{pmatrix} \mathcal{O}_N & \mathcal{I}_N \\ -\mathcal{I}_N & \mathcal{O}_N \end{pmatrix}, \quad (1.11)$$

where $\mathbf{u}, \mathbf{w} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}^{2N}$, \mathcal{J} is the skew-symmetric matrix: $\mathcal{J}^+ = -\mathcal{J}$, \mathcal{I}_N and \mathcal{O}_N are identity and zero matrices in $\mathbb{R}^{N \times N}$, and the Hamiltonian of (1.11) follows from (1.8) with $\psi = \mathbf{u} + i\mathbf{w}$.

Remark 1.6 We shall work in Hilbert space $X(\mathbb{R}) = H^1(\mathbb{R})$ for vector complex-valued functions $\psi \in \mathbb{C}^N$, where the principal part of functionals H and P is well-defined in (1.8) and (1.9). We define the inner product for two elements $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R})$ as

$$\langle \mathbf{f} | \mathbf{g} \rangle = \int_{\mathbb{R}} dx (\bar{\mathbf{f}} \cdot \mathbf{g}) = \int_{\mathbb{R}} dx \left(\sum_{n=1}^N \bar{f}_n(x) g_n(x) \right). \quad (1.12)$$

The conserved functionals Q_n in (1.6) are positive-definite squared norms in space $L^2(\mathbb{R})$:

$$Q_n = \langle \psi_n \mathbf{e}_n | \psi_n \mathbf{e}_n \rangle. \quad (1.13)$$

2 Linearized stability problem for optical solitons

Incoherent optical solitons are defined as stationary solutions of the coupled system (1.1):

$$\psi(z, x) = e^{i(\boldsymbol{\theta} + z\boldsymbol{\beta} + (vx - v^2z)\mathcal{D}^{-1})} \boldsymbol{\Phi}(x - 2vz - s), \quad (2.1)$$

where $v, s \in \mathbb{R}$, $\boldsymbol{\theta} = \text{diag}(\theta_1, \dots, \theta_N)$, $\boldsymbol{\beta} = \text{diag}(\beta_1, \dots, \beta_N)$, $\mathcal{D} = \text{diag}(d_1, \dots, d_N)$, and $\boldsymbol{\Phi} : \mathbb{R} \rightarrow \mathbb{R}^N$. The envelope functions $\boldsymbol{\Phi} = \boldsymbol{\Phi}(x)$ satisfy the system of equations:

$$d_n \frac{d^2 \Phi_n}{dx^2} - \beta_n \Phi_n + f_n(\Phi_1^2, \dots, \Phi_N^2) \Phi_n = 0, \quad (2.2)$$

subject to zero boundary conditions at infinity:

$$\lim_{|x| \rightarrow \infty} \boldsymbol{\Phi}(x) = \mathbf{0}. \quad (2.3)$$

Assumption 2.1 *There exists an exponentially decaying solution $\boldsymbol{\Phi} \in X(\mathbb{R})$, $\boldsymbol{\Phi}(x) \in \mathbb{R}^N$ of the problem (2.2)–(2.3) in an open domain $\boldsymbol{\beta} \in \mathcal{B} \subset \mathbb{R}^N$. The stationary solution is not degenerate, i.e. $\Phi_n(x) = 0$ only in a finite number of points $x \in \mathbb{R}$. The mapping $\boldsymbol{\beta} \rightarrow \boldsymbol{\Phi}(x)$ is C^2 on $\boldsymbol{\beta} \in \mathcal{B}$.*

Lemma 2.2 *Exponentially decaying solutions of (2.2) may exist only if $\beta_n > 0$, $n = 1, \dots, N$. In the domain of existence, components of the solution $\boldsymbol{\Phi}(x)$ decay asymptotically as*

$$\lim_{x \rightarrow \pm\infty} |\Phi_n(x) e^{a_n |x|} - c_n^\pm| = 0, \quad a_n = \sqrt{\frac{\beta_n}{d_n}} (> 0), \quad (2.4)$$

where c_n^\pm are some constants such that $c_n^\pm \neq 0$.

Proof. Since $f_n(0, \dots, 0) = 0$, the system (2.2) is uncoupled in the limit (2.3) and components $\Phi_n(x)$ of the exponentially decaying solution $\boldsymbol{\Phi}(x)$ satisfy the asymptotic limit (2.4) provided that $\beta_n > 0$ (since $d_n > 0$). When $\beta_n < 0$, solutions of (2.2) are oscillatory and non-decaying in the limit (2.3). When $\beta_n = 0$, solutions of (2.2) may decay algebraically. \square

Remark 2.3 Lemma 2.2 is related to the fact that $\omega_n \leq 0$ in the linear spectrum (1.4). The spectrum of exponentially decaying stationary solutions (2.1) with $v = 0$ is isolated from the linear spectrum (1.4), when $\beta_n > 0$. Otherwise, as it happens for a system of coherently coupled NLS equations [PY02], the exponentially decaying solutions $\boldsymbol{\Phi}(x)$ become embedded into the linear spectrum (embedded solitons). Such solutions are semi-stable due to nonlinearity-induced radiative decay, even if they are linearly stable [PY02]. We notice that the algebraically decaying solutions may exist in the system (2.2) for $\beta_n = 0$ and they are embedded into the edge of the linear spectrum at $\omega_n = 0$. We do not consider algebraically decaying solutions in this paper.

Definition 2.4 Families of stationary solutions $\Phi = \Phi(x)$ are classified by the nodal index $\mathbf{i} = (i_1, \dots, i_N)$, where i_n is the number of zeros of $\Phi_n(x)$ for $x \in \mathbb{R}$. The stationary solution $\Phi(x)$ with zero nodal index is called a fundamental soliton.

Lemma 2.5 When they exist, the stationary solutions $\Phi = \Phi(x)$ are critical points of the Lyapunov functional:

$$\Lambda[\psi] = H[\psi] + \sum_{n=1}^N \beta_n Q_n[\psi], \quad (2.5)$$

where Q_n and H are given by (1.6) and (1.8).

Proof. The first variation of $\Lambda[\psi]$ vanishes if $\psi = \Phi(x)$ satisfies the system (2.2). \square

Definition 2.6 Suppose the problem (2.2)–(2.3) has a stationary solution $\Phi(x)$ in the parameter domain $\beta \in \mathcal{B}$. Define the energy surface of the stationary solutions,

$$\Lambda_s(\beta) = H_s(\beta) + \sum_{n=1}^N \beta_n Q_{ns}(\beta), \quad (2.6)$$

where $H_s(\beta) = H[\Phi]$ and $Q_{ns}(\beta) = Q_n[\Phi]$. The Hessian matrix \mathcal{U} of the energy surface $\Lambda_s(\beta)$ is a symmetric matrix with the elements

$$\mathcal{U}_{nm} = \frac{\partial^2 \Lambda_s}{\partial \beta_n \partial \beta_m}. \quad (2.7)$$

Lemma 2.7 Matrix elements of the Hessian matrix \mathcal{U} are continuous functions of β in \mathcal{B} , computed as

$$\mathcal{U}_{nm} = \frac{\partial Q_{ns}}{\partial \beta_m} = 2 \langle \Phi_n \mathbf{e}_n | \frac{\partial \Phi}{\partial \beta_m} \rangle. \quad (2.8)$$

Matrix \mathcal{U} in a domain $\beta \in \mathcal{B}$ has N real bounded eigenvalues.

Proof. It follows from Lemma 2.5 that

$$\frac{\partial \Lambda_s}{\partial \beta_n} = Q_{ns} + \left(\frac{\partial H_s}{\partial \beta_n} + \sum_{m=1}^N \beta_m \frac{\partial Q_{ms}}{\partial \beta_n} \right) = Q_{ns}. \quad (2.9)$$

If $\Phi \in L^2(\mathbb{R})$ and $\Phi(x)$ is C^2 function of β in \mathcal{B} , the second derivatives of $\Lambda_s(\beta)$ exists and equal to (2.8). Since \mathcal{U} is symmetric matrix with bounded elements, all eigenvalues of \mathcal{U} are real and bounded. \square

Definition 2.8 Denote the number of negative, zero and positive eigenvalues of \mathcal{U} as $n(\mathcal{U})$, $z(\mathcal{U})$, and $p(\mathcal{U})$, respectively, such that $n(\mathcal{U}) + z(\mathcal{U}) + p(\mathcal{U}) = N$.

Since parameters v , s , and $\boldsymbol{\theta}$ can be scaled out of the stability problem, we set them to zero in the rest of the paper. Linearization at the stationary solutions (2.1) is defined by the expansion,

$$\boldsymbol{\psi} = e^{iz\boldsymbol{\beta}} [\boldsymbol{\Phi}(x) + \mathbf{U}(z, x) + i\mathbf{W}(z, x)], \quad (2.10)$$

where $\mathbf{U}, \mathbf{W} \in \mathbb{R}^N$ are perturbations of the stationary solutions. Neglecting nonlinear terms of \mathbf{U} and \mathbf{W} , we find that the perturbation terms $\mathbf{U}(z, x)$ and $\mathbf{W}(z, x)$ satisfy the linearized system in Hamiltonian form:

$$\frac{\partial}{\partial z} \begin{pmatrix} \mathbf{U} \\ \mathbf{W} \end{pmatrix} = \mathcal{J} \begin{pmatrix} \frac{\delta h}{\delta \mathbf{U}} \\ \frac{\delta h}{\delta \mathbf{W}} \end{pmatrix}, \quad (2.11)$$

where the linearized Hamiltonian h is the second variation of the Lyapunov functional (2.5):

$$h = \frac{1}{2}\delta^2\Lambda = \langle \mathbf{U} | \mathcal{L}_1 \mathbf{U} \rangle + \langle \mathbf{W} | \mathcal{L}_0 \mathbf{W} \rangle. \quad (2.12)$$

The linearized Hamiltonian is a sum of two quadratic forms associated with the matrix Schrödinger operators \mathcal{L}_0 and \mathcal{L}_1 with the elements:

$$(\mathcal{L}_0)_{nm} = \begin{cases} -d_n \frac{d^2}{dx^2} + \beta_n - f_n(\Phi_1^2, \dots, \Phi_N^2), & m = n \\ 0, & m \neq n \end{cases} \quad (2.13)$$

$$(\mathcal{L}_1)_{nm} = \begin{cases} -d_n \frac{d^2}{dx^2} + \beta_n - f_n(\Phi_1^2, \dots, \Phi_N^2) - 2 \frac{\partial f_n}{\partial \Phi_n^2} \Phi_n^2, & m = n \\ -2 \frac{\partial f_n}{\partial \Phi_m^2} \Phi_n \Phi_m, & m \neq n \end{cases} \quad (2.14)$$

The diagonal operator \mathcal{L}_0 is a composition of N scalar Schrödinger operators. The matrix operator \mathcal{L}_1 is symmetric in the Hamiltonian case, when the functions $f_n(|\psi_1|^2, \dots, |\psi_N|^2)$ satisfy the constraints (1.3). Both quadratic forms in (2.12) are real-valued. The linearized problem (2.11) reduces to a linear eigenvalue problem after separation of variables: $\mathbf{U} = \mathbf{u}(x)e^{\lambda z}$, $\mathbf{W} = \mathbf{w}(x)e^{\lambda z}$. Eigenvalues λ are defined by the spectrum of the non-self-adjoint operator \mathcal{A} :

$$\mathcal{A} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{O}_N & \mathcal{I}_N \\ -\mathcal{I}_N & \mathcal{O}_N \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 & \mathcal{O}_N \\ \mathcal{O}_N & \mathcal{L}_0 \end{pmatrix}. \quad (2.15)$$

We use standard definitions of eigenvalues of \mathcal{A} from [HS96, Definition 1.4].

Definition 2.9 The value λ is an eigenvalue of \mathcal{A} if $\ker(\mathcal{A} - \lambda) \neq \{0\}$ and there exists a non-zero vector function $(\mathbf{u}, \mathbf{w})^T \in \ker(\mathcal{A} - \lambda)$ called an eigenvector of \mathcal{A} such that $\mathbf{u}, \mathbf{w} \in X(\mathbb{R})$. The dimension of $\ker(\mathcal{A} - \lambda)$ is called the geometric multiplicity of λ .

Definition 2.10 The discrete spectrum of \mathcal{A} , $\sigma_{\text{dis}}(\mathcal{A})$ is the set of all eigenvalues of \mathcal{A} with finite algebraic multiplicity which are isolated from the continuous spectrum of \mathcal{A} , $\sigma_{\text{con}}(\mathcal{A})$. The embedded spectrum of \mathcal{A} , $\sigma_{\text{emb}}(\mathcal{A})$ is the set of all eigenvalues with finite algebraic multiplicity which belong to the continuous spectrum of \mathcal{A} , including the boundary points. The essential spectrum of \mathcal{A} , $\sigma_{\text{ess}}(\mathcal{A})$ is $\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{con}}(\mathcal{A}) \cup \sigma_{\text{emb}}(\mathcal{A})$. The total spectrum of \mathcal{A} , $\sigma(\mathcal{A})$ is $\sigma(\mathcal{A}) = \sigma_{\text{dis}}(\mathcal{A}) \cup \sigma_{\text{ess}}(\mathcal{A})$.

The non-self-adjoint linear eigenvalue problem (2.15) is formulated as a coupled system for two symmetric matrix Schrödinger operators \mathcal{L}_0 and \mathcal{L}_1 . Some standard properties of these operators are reviewed next.

Lemma 2.11 *Let \mathcal{L} be a symmetric matrix Schrödinger operator, either \mathcal{L}_0 or \mathcal{L}_1 . Continuous spectrum of \mathcal{L} has N branches located at*

$$\sigma_{\text{con}}(\mathcal{L}) = \cup_{1 \leq n \leq N} \{\lambda \in \mathbb{R} : \lambda \geq \beta_n\}. \quad (2.16)$$

Eigenvalues of discrete and embedded spectrum of \mathcal{L} are located at:

$$\sigma_{\text{dis}}(\mathcal{L}) = \cup_m \{\lambda_m : \lambda_m \in \mathbb{R}, \lambda_m < \beta_{\min}\}, \quad (2.17)$$

$$\sigma_{\text{emb}}(\mathcal{L}) = \cup_m \{\lambda_m : \lambda_m \in \mathbb{R}, \beta_{\min} \leq \lambda_m < \beta_{\max}\}, \quad (2.18)$$

where $\beta_{\min} = \min_{1 \leq n \leq N} \beta_n$ and $\beta_{\max} = \max_{1 \leq n \leq N} \beta_n$. The algebraic multiplicity of eigenvalues coincides with their geometric multiplicity and does not exceed N .

Proof. The matrix Schrödinger operator \mathcal{L} has exponentially decaying potentials and becomes a diagonal differential operator in the limit $|x| \rightarrow \infty$. The Weyl's criterion for essential spectrum applies and leads to (2.16) [HS96, Theorem 7.2]. The spectral problem $\mathcal{L}\mathbf{u} = \lambda\mathbf{u}$ becomes asymptotically uncoupled in the limit $|x| \rightarrow \infty$ as

$$-d_n u_n''(x) + \beta_n u_n(x) = \lambda u_n(x). \quad (2.19)$$

Therefore, exponentially decaying solutions of the problem $\mathcal{L}\mathbf{u} = \lambda\mathbf{u}$ are decomposed in the limit $|x| \rightarrow \infty$ over a basis of N vector-functions $\mathbf{e}_n e^{-b_n|x|}$, $n = 1, \dots, N$, where $b_n = \sqrt{(\beta_n - \lambda)/d_n}$ such that $\text{Re}(b_n) > 0$. For $\lambda < \beta_{\min}$, all vector-functions are exponentially decaying and there exist no more than N linearly independent eigenvectors $\mathbf{u}(x)$ for some (isolated) values of λ . This contributes to eigenvalues of discrete spectrum (2.17). For $\lambda \geq \beta_{\max}$, all vector functions are non-decaying and an exponentially decaying eigenvector $\mathbf{u}(x)$ does not exist for all (continuous) values of λ . For $\beta_{\min} \leq \lambda < \beta_{\max}$, some components $u_n(x)$ are exponentially decaying, while the other components $u_n(x)$ are non-decaying but bounded. Let N_1 be the number of non-decaying components. Then, there exist N_1 branches of continuous spectrum at this value of λ , and an embedded eigenvalue in (2.18) (if it exists) corresponds to at most $N - N_1$ linearly independent exponentially decaying eigenvectors $\mathbf{u}(x)$. Furthermore, since \mathcal{L} is self-adjoint, the algebraic multiplicity of eigenvalues always coincides with their geometric multiplicity [HS96, Theorem 6.7]. \square

Proposition 2.12 *Let \mathcal{L} be a symmetric matrix Schrödinger operator, either \mathcal{L}_0 or \mathcal{L}_1 . There exists an orthonormal basis in $X(\mathbb{R})$ constructed with eigenfunctions of the continuous, discrete and embedded spectrum of \mathcal{L} such that a function $\mathbf{f} \in X(\mathbb{R})$ can be decomposed over the basis as*

$$\mathbf{f}(x) = \sum_{m=1}^M a_m \mathbf{u}_m(x) + \sum_{n=1}^N \int_{-\infty}^{\infty} b_n(k) \mathbf{u}_n(x, k) dk, \quad (2.20)$$

where $a_m = \langle \mathbf{u}_m | \mathbf{f} \rangle$, $b_n(k) = \langle \mathbf{u}_n(k) | \mathbf{f} \rangle$, and $M = \dim(\sigma_{\text{dis}}(\mathcal{L}) \cup \sigma_{\text{emb}}(\mathcal{L}))$. The quadratic form for operator \mathcal{L} is orthogonally diagonalized as follows:

$$\langle \mathbf{f} | \mathcal{L} \mathbf{f} \rangle = \sum_{m=1}^M \lambda_m |a_m|^2 + \sum_{n=1}^N \int_{-\infty}^{\infty} \lambda_n(k) |b_n(k)|^2 dk, \quad (2.21)$$

where $\lambda_n(k) = \beta_n + d_n k^2$.

Proof. This result is central in spectral analysis of self-adjoint operators in Hilbert space $X(\mathbb{R})$ [HS96, Theorem 2.18]. For reference, we denote eigenfunctions of the discrete and embedded spectrum of \mathcal{L} as $\mathbf{u}_m(x)$, $m = 1, \dots, M$, and eigenfunction of the continuous spectrum as $\mathbf{u}_n(x, k)$, $n = 1, \dots, N$. The eigenfunctions satisfy the orthonormality relations:

$$\langle \mathbf{u}_{m'} | \mathbf{u}_m \rangle = \delta_{m', m}, \quad \langle \mathbf{u}_m | \mathbf{u}_n(k) \rangle = 0, \quad \langle \mathbf{u}_{n'}(k') | \mathbf{u}_n(k) \rangle = \delta_{n', n} \delta(k' - k) \quad (2.22)$$

and the completeness relation:

$$\delta_{i,j} \delta(x' - x) = \sum_{m=1}^M \bar{u}_{m,i}(x') u_{m,j}(x) + \sum_{n=1}^N \int_{-\infty}^{\infty} \bar{u}_{n,i}(x', k) u_{n,j}(x, k) dk. \quad (2.23)$$

□

Lemma 2.13 *The null-space of \mathcal{L}_0 has a basis of N eigenfunctions $\Phi_n(x) \mathbf{e}_n$, for $n = 1, \dots, N$. The null space of \mathcal{L}_1 has at least one eigenfunction $\Phi'(x)$.*

Proof. The eigenfunctions of the null space of \mathcal{L}_0 and \mathcal{L}_1 are related to derivatives of the soliton solutions (2.1) with respect to the translation parameters $\boldsymbol{\theta}$ and s , respectively. They can be found by direct computations. As follows from Lemma 2.11, the eigenfunctions $\Phi_n(x) \mathbf{e}_n$, for $n = 1, \dots, N$ form a basis of the null space of \mathcal{L}_0 . □

Definition 2.14 *Denote the number of negative, zero and positive eigenvalues of the discrete and embedded spectrum of operator \mathcal{L} in $X(\mathbb{R})$ as $n(\mathcal{L})$, $z(\mathcal{L})$, and $p(\mathcal{L})$, respectively. The Morse index for soliton solutions of the Hamiltonian system (2.11) is*

$$n(h) = n(\mathcal{L}_1) + n(\mathcal{L}_0). \quad (2.24)$$

Remark 2.15 According to Proposition 2.12, $n(\mathcal{L}_{0,1}) + z(\mathcal{L}_{0,1}) + p(\mathcal{L}_{0,1}) = M_{0,1} < \infty$. According to Lemma 2.13, $z(\mathcal{L}_0) = N$ and $1 \leq z(\mathcal{L}_1) \leq N$.

Lemma 2.16 *The negative index of \mathcal{L}_0 is*

$$n(\mathcal{L}_0) = \sum_{n=1}^N i_n, \quad (2.25)$$

where i_n is the nodal index (number of zeros) of $\Phi_n(x)$ for $x \in \mathbb{R}$. The negative index $n(\mathcal{L}_0)$ remains fixed in the open domain $\beta \in \mathcal{B}$.

Proof. Relation (2.25) follows from Sturm Oscillation Theorem, since \mathcal{L}_0 is a diagonal composition of scalar Schrödinger operators $(\mathcal{L}_0)_{nn}$, each has a zero bound state: $\Phi_n(x)$, such that $n((\mathcal{L}_0)_{nn}) = i_n$. Eigenfunctions $\{\Phi_n(x)\mathbf{e}_n\}_{n=1}^N$ form a basis of the null-space of \mathcal{L}_0 and therefore, the index $n(\mathcal{L}_0)$ is fixed for any continuous deformations of $\Phi(x)$ and \mathcal{L}_0 , which preserves Assumption 2.1 such that $\Phi_n(x)$ vanishes in a finite number of zeros. \square

We finish this section with some general properties of the spectrum of the linearization operator \mathcal{A} in (2.15).

Lemma 2.17 *If λ is an eigenvalue of (2.15), so are $(-\lambda)$, $\bar{\lambda}$, and $(-\bar{\lambda})$.*

Proof. This result is standard for linearized Hamiltonian systems. It follows from (2.15) that if (\mathbf{u}, \mathbf{w}) is the eigenvector for λ , then $(\mathbf{u}, -\mathbf{w})$, $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$, and $(\bar{\mathbf{u}}, -\bar{\mathbf{w}})$ are eigenvectors for $(-\lambda)$, $\bar{\lambda}$, and $(-\bar{\lambda})$, respectively. \square

Definition 2.18 *The stationary solution (2.1) is spectrally (exponentially) unstable if there exists at least one eigenvalue λ such that $\text{Re}(\lambda) > 0$. It is weakly spectrally stable if all eigenvalues λ are zero or imaginary.*

Remark 2.19 Spectral instability occurs when the eigenvalue problem (2.15) has a pair of real eigenvalues $(\lambda, -\lambda)$ or a quadruple of complex eigenvalues $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$. Weak spectral stability does not yet guarantee strong spectral stability, i.e. there may exist eigenvalues of higher algebraic multiplicity with $\text{Re}(\lambda) = 0$, that lead to linear or power growth in t . In the main part of the paper, we study the generic case, when no such eigenvalues of higher algebraic multiplicity exist in the problem (2.15). Eigenvalues of higher algebraic multiplicity with $\text{Re}(\lambda) = 0$ are studied in Section 7 as bifurcations from weak spectral stability to spectral instability.

Lemma 2.20 *Define the extended constrained function space $X_c(\mathbb{R}) = X_c^{(u)} \oplus X_c^{(w)}$, where*

$$X_c^{(u)} = \{\mathbf{u} \in X(\mathbb{R}) : \langle \Phi_n \mathbf{e}_n | \mathbf{u} \rangle = 0, n = 1, \dots, N\}, \quad (2.26)$$

$$X_c^{(w)} = \{\mathbf{w} \in X(\mathbb{R}) : \langle \Phi' | \mathbf{w} \rangle = 0\}. \quad (2.27)$$

Eigenvectors $(\mathbf{u}, \mathbf{w})^T$ of operator \mathcal{A} for $\lambda \neq 0$ belongs to the space $X_c(\mathbb{R})$.

Proof. The linear stability problem (2.15) is written as a coupled system for operators \mathcal{L}_0 and \mathcal{L}_1 :

$$\mathcal{L}_1 \mathbf{u} = -\lambda \mathbf{w}, \quad \mathcal{L}_0 \mathbf{w} = \lambda \mathbf{u}. \quad (2.28)$$

The constraints in (2.26)–(2.27) follow from the Fredholm's Alternative Theorem applied to (2.28) for $\lambda \neq 0$ with the set of eigenfunctions $\{\Phi_n(x) \mathbf{e}_n\}_{n=1}^N$ of the null space of \mathcal{L}_0 and with the eigenfunction $\Phi'(x)$ of the null space of \mathcal{L}_1 . The constraints (2.26) and (2.27) are orthogonality conditions that provide boundness of the eigenvector $(\mathbf{u}, \mathbf{w})^T$ in the limit $|x| \rightarrow \infty$. \square

Lemma 2.21 *The geometric multiplicity of the null eigenvalue of \mathcal{A} is at least $(N + 1)$ and the algebraic multiplicity of the null eigenvalue of \mathcal{A} is at least $(2N + 2)$.*

Proof. The null space of \mathcal{A} is spanned by at least $(N + 1)$ eigenvectors:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \left\{ \begin{pmatrix} \mathbf{0}_N \\ \Phi_n(x) \mathbf{e}_n \end{pmatrix}, n = 1, \dots, N; \begin{pmatrix} \Phi'(x) \\ \mathbf{0}_N \end{pmatrix} \right\}. \quad (2.29)$$

The generalized null space of \mathcal{A} includes at least $(N + 1)$ generalized eigenvectors:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \left\{ \begin{pmatrix} \frac{\partial \Phi}{\partial \beta_n} \\ \mathbf{0}_N \end{pmatrix}, n = 1, \dots, N; \begin{pmatrix} \mathbf{0}_N \\ -\frac{1}{2}x \mathcal{D}^{-1} \Phi(x) \end{pmatrix} \right\}, \quad (2.30)$$

where $\mathcal{D} = \text{diag}(d_1, \dots, d_N)$. \square

Lemma 2.22 *When $z(\mathcal{L}_1) = 1$ and $z(\mathcal{U}) = 0$, the null space of \mathcal{A} includes only $(N + 1)$ eigenvectors (2.29) and the generalized null space of \mathcal{A} includes only $(N + 1)$ generalized eigenvectors (2.30).*

Proof. When $z(\mathcal{L}_1) = 1$, the null space of \mathcal{A} has a basis of $(N + 1)$ eigenvectors (2.29), as follows from Lemma 2.13. When $z(\mathcal{U}) = 0$, the Fredholm's Alternative Theorem applied to the first N generalized eigenvectors in (2.30) fail, i.e. no second generalized eigenvectors exist. The Fredholm's Alternative Theorem always fails for the last generalized eigenvector in (2.30). \square

Corollary 2.23 *When $z(\mathcal{L}_1) = 1$ and $z(\mathcal{U}) = 0$, the generalized null space of \mathcal{A} does not intersect with the constrained function space $X_c(\mathbb{R}) = X_c^{(u)} \oplus X_c^{(w)}$. In other words, the generalized eigenvectors (2.30) do not satisfy constraints (2.26)–(2.27).*

Hypothesis 2.24 *Let $z(\mathcal{L}_1) = 1$, $z(\mathcal{U}) = 0$, and no non-zero eigenvalues λ of higher algebraic multiplicity exist in the linearization problem (2.15). The spectrum of \mathcal{A} is complete in $X_c(\mathbb{R})$ and*

the spectral decomposition for any $\mathbf{f} = (\mathbf{f}_u, \mathbf{f}_w)^T$, such that $\mathbf{f}_u \in X_c^{(u)}(\mathbb{R})$ and $\mathbf{f}_w \in X_c^{(w)}(\mathbb{R})$, takes the form:

$$\begin{pmatrix} \mathbf{f}_u(x) \\ \mathbf{f}_w(x) \end{pmatrix} = \sum_{i=1}^2 \sum_{m=1}^M a_m^{(i)} \begin{pmatrix} \mathbf{u}_m^{(i)}(x) \\ \mathbf{w}_m^{(i)}(x) \end{pmatrix} + \sum_{i=1}^2 \sum_{n=1}^N \int_{-\infty}^{\infty} b_n^{(i)}(k) \begin{pmatrix} \mathbf{u}_n^{(i)}(x, k) \\ \mathbf{w}_n^{(i)}(x, k) \end{pmatrix} dk, \quad (2.31)$$

where $\mathbf{u}_m^{(i)}(x)$, $\mathbf{w}_m^{(i)}(x)$, $m = 1, \dots, M$, $i = 1, 2$ are eigenfunctions of the discrete and embedded spectrum of \mathcal{A} and $\mathbf{u}_n^{(i)}(x, k)$, $\mathbf{w}_n^{(i)}(x, k)$, $n = 1, \dots, N$, $i = 1, 2$ are eigenfunctions of the continuous spectrum of \mathcal{A} .

Remark 2.25 As follows from Lemma 2.17, there are relations between eigenfunctions for $i = 1$ and $i = 2$: $(\mathbf{u}_m^{(2)}, \mathbf{w}_m^{(2)}) = (\mathbf{u}_m^{(1)}, -\mathbf{w}_m^{(1)})$, $m = 1, \dots, M$ and $(\mathbf{u}_n^{(2)}, \mathbf{w}_n^{(2)}) = (\mathbf{u}_n^{(1)}, -\mathbf{w}_n^{(1)})$, $n = 1, \dots, N$. When $\lambda_m \in \mathbb{C}$, the first sum in (2.31) for $i = 1$ includes also eigenvector $(\bar{\mathbf{u}}_m^{(1)}, \bar{\mathbf{w}}_m^{(1)})^T$. When $\lambda_m \in \mathbb{R}$ and $\lambda_m \in i\mathbb{R}$, eigenvectors $\mathbf{u}_m^{(1)}(x)$ are real-valued, and so are $\lambda_m \mathbf{w}_m^{(1)}(x)$.

Remark 2.26 Spectral decomposition (2.31) in the case $N = 1$ was proved in [BP93], where eigenfunctions of the discrete and continuous spectrum of \mathcal{A} were introduced and orthogonality relations between the eigenfunctions (with respect to the Dirac matrix σ_3) and the completeness relation were derived from the jump of resolvent of \mathcal{A} at the continuous spectrum. Since matrix operators \mathcal{L}_0 and \mathcal{L}_1 have the same form of diagonal positive-definite differential operators and exponentially decaying symmetric matrix potentials, we expect that Hypothesis 2.24 holds also for $N > 1$.

Remark 2.27 Assumptions of Hypothesis 2.24 are introduced to avoid Jordan block structure in the spectral decomposition (2.31), associated with eigenvalues of higher algebraic multiplicity. When the unconstrained space $X(\mathbb{R}) \oplus X(\mathbb{R})$ is used instead of $X_c^{(u)}(\mathbb{R}) \oplus X_c^{(w)}(\mathbb{R})$, the Jordan block structure always occurs for the null eigenvalue of higher algebraic multiplicity, as in Lemma 2.21. The Jordan block for the null eigenvalue was introduced in [K76] in the case $N = 1$ and $f_1 = c_1 |\psi_1|^2$. The generalized null space of \mathcal{A} disappears in the constrained function space $X_c(\mathbb{R})$, as follows from Corollary 2.23.

3 History of the stability problem and main results

Stability of solitary waves in nonlinear Schrödinger (NLS) equations was studied extensively in the recent past. The first stability-instability theorem for a scalar NLS equation (1.1) with $N = 1$ was proven by Shatah and Strauss [SS85] and Weinstein [W86]. Only positive-definite (fundamental) stationary solutions were considered in one, two, and three spatial dimensions. Fundamental solutions have the nodal index $i = 0$ and the Morse index $n(h) = 1$. A single negative eigenvalue of h does not necessary lead to spectral instability in the linearized problem (2.15) because of the constraints in (2.26)–(2.27). If $p(\mathcal{U}) = 0$ (the energy curve $\Lambda_s(\beta)$ in (2.6) is concave for the corresponding value of β), the stationary solution $\Phi(x)$ is spectrally unstable and the linearized problem

(2.15) has a single real positive eigenvalue λ . If $p(\mathcal{U}) = 1$ (the energy curve $\Lambda_s(\beta)$ is convex), the solitary wave is weakly spectrally stable and all eigenvalues λ are purely imaginary [SS85, W86].

More formal and general analysis was developed by Grillakis, Shatah and Strauss [GSS87, GSS90] by using multi-dimensional Lie groups and spectral decompositions. The following theorems were proven for an abstract Hamiltonian system with symmetries, which includes the system of coupled NLS equations (1.11).

Theorem 3.1 [GSS90] *Let $z(\mathcal{U}) = 0$, then $p(\mathcal{U}) \leq n(h)$. A stationary solution (2.1) is weakly spectrally stable if $n(h) = p(\mathcal{U})$ and it is spectrally unstable if $n(h) - p(\mathcal{U})$ is odd. The linearized problem (2.15) has at least one real positive eigenvalue λ if $n(h) - p(\mathcal{U})$ is odd.*

Theorem 3.2 [GSS90] *The linearized problem (2.15) has at most $n(h)$ unstable eigenvalues λ such that $\text{Re}(\lambda) > 0$.*

Theorem 3.3 [GSS90] *The linearized Hamiltonian h in extended constrained space $X_c(\mathbb{R}) = X_c^{(u)}(\mathbb{R}) \oplus X_c^{(w)}(\mathbb{R})$ has the negative index $\#_{<0}(h) = n(h) - p(\mathcal{U}) - z(\mathcal{U})$ and the null index $\#_{=0}(h) = z(h) + z(\mathcal{U})$.*

Theorem 3.1 is the main stability–instability theorem in [GSS90]. Theorem 3.2 is formulated in [GSS90, Theorem 5.8] for a quadrant: $\text{Re}(\lambda) < 0$, $\text{Im}(\lambda) > 0$. The method of the proof can however be applied to the left half-plane $\text{Re}(\lambda) < 0$, or equivalently, to the right half-plane $\text{Re}(\lambda) > 0$. Theorem 3.3 is formulated in [GSS90, Theorem 3.1] as a more general statement, which is equivalent to Theorem 3.3 under Assumption 2.1 ($z_0 = 0$ in notations of [GSS90]).

Theorem 3.1 generalizes stability–instability theory in finite-dimensional Hamiltonian systems with symmetries [M85]. Since the fundamental stationary solution (2.1) with $N = 1$ has always indices $n(\mathcal{L}_1) = 1$ and $n(\mathcal{L}_0) = 0$, its stability and instability is uniquely described by Theorem 3.1. However, many examples showed insufficiency of Theorem 3.1 for complete stability–instability analysis of multiple pulse solutions with $N \geq 1$. For instance, a scalar NLS equation in two dimensions has radially symmetric multiple pulses with the nodal index $i > 0$ and the Morse index $n(h) \geq 1 + 2i$. When $p(\mathcal{U}) = 1$ and $n(h) - p(\mathcal{U}) \geq 2i$ is even, Theorem 3.1 can not be applied.

While a simple application of Theorem 3.1 to the case of multicomponent stationary solutions (2.1) with $N > 1$ is given in [GSS90, Theorem 9.1], we note that Theorem 9.1 in [GSS90] derives a scalar stability criterion, computed from the minimal eigenvalue $\beta_{\min} = \min_{1 \leq n \leq N}(\beta_n)$. The scalar criterion generally fails for $N > 1$, as the Morse index $n(h)$ of multicomponent solitons (2.1) is assumed to be one in [GSS90], which does not generally hold for $N > 1$.

More special instability theorems were found by Jones [J88a, J88b] and Grillakis [G88, G90] for the scalar NLS equation (1.1) with $N = 1$. Jones [J88a, J88b] used topological and shooting methods of dynamical systems theory. When $n(\mathcal{L}_1) - p(\mathcal{U}) > n(\mathcal{L}_0)$, theorems in [J88a, J88b] predict an unstable eigenvalue, no matter whether $n(h)$ is odd or even. The results apply to

instability of radially symmetric solutions with nodal index $i > 0$ in two spatial dimensions [J88a], as well as to stability–instability of symmetric and anti-symmetric solutions in the NLS equation with x -dependent nonlinear function $f = f(x; |\psi|^2)$ [J88b].

Theorem 3.4 [J88a, J88b] *The linearized problem (2.15) with $N = 1$ has a real positive eigenvalue λ if $|n(\mathcal{L}_1) - n(\mathcal{L}_0)| > 1$.*

Grillakis [G88, G90] used theory of linear operators, orthogonal projections and quadratic forms and proved some general results for the linearized problem (2.15). In this context, the problem (2.15) is reformulated as a generalized eigenvalue problem for operators \mathcal{L}_1 and \mathcal{L}_0^{-1} . When $n(\mathcal{L}_0) = 0$, unstable eigenvalues λ may occur only as real positive eigenvalues [G88]. When $n(h) - p(\mathcal{U}) > 1$ and $n(\mathcal{L}_0) \neq 0$, complex unstable eigenvalues λ may also occur in the linearized problem (2.15) [G90].

Theorem 3.5 [G88] *Let $\#_{<0}(\mathcal{L}_1)$ and $\#_{<0}(\mathcal{L}_0)$ be the negative indices of operators \mathcal{L}_1 and \mathcal{L}_0 in $X_c^{(u)}(\mathbb{R})$. The linearized problem (2.15) has at least $|\#_{<0}(\mathcal{L}_1) - \#_{<0}(\mathcal{L}_0)|$ real positive eigenvalues λ . If $\#_{<0}(\mathcal{L}_0) = 0$, the linearized problem has exactly $\#_{<0}(\mathcal{L}_1)$ real positive eigenvalues λ .*

Theorem 3.5 is formulated in [G88, Theorem 1.2]. The theorem is more precise and general compared to Theorem 3.4, the latter takes the worst case, when $p(\mathcal{U}) = 1$ for $N = 1$. It remains unclear how Theorem 3.5, which exploits a special structure of the linearized problem (2.15), is related to general Theorem 3.1 for an abstract Hamiltonian system. It also remains unclear how the bounds on the number of unstable eigenvalues can be extended in the case of complex eigenvalues in the linearized problem (2.15).

The work of Jones [J88a, J88b] and Grillakis [G88, G90] pioneered methods of analytical (Evans) functions later developed by Alexander, Gardner, and Jones [AGJ90] and Pego and Weinstein [PW92]. The Evans function method is a convenient tool to locate unstable eigenvalues in the linearized stability problems, even if general stability–instability theorems are not available. Several methods for computations and analysis of the Evans function were proposed: Taylor expansions near the origin $\lambda = 0$ [KS98, K99], multisymplectic characterizations of the Evans function and its derivatives [BD99, BD01], bifurcation analysis for embedded and complex eigenvalues [LP98, LP00], branch point analysis for edge bifurcations [KS02], and Lyapunov–Shmidt reductions for multiple eigenvalues associated with multiple pulse solutions [S98, SS99]. Alternative analytical methods based on perturbation series expansions [PAK96, PG97, PKA98, S00] were also developed. The perturbation series expansions recover the same results as the Evans function methods but they operate with the linearized problem (2.15) directly.

Analytical methods based on Evans function expansions or perturbation series expansions focus on local analysis of transitions to instability. They do not generally recover complete stability–instability theory. Only numerical analysis may give complete information on number, type, and location of unstable eigenvalues in a complex plane of λ . Accurate numerical computations are

performed by integrating the Evans function in the first quadrant of the complex plane of λ and by using the winding number theorem [PW02, BDG01, PS03]. However, numerical algorithms based on integration in the complex plane of λ are computationally inefficient. They are typically slow and memory consuming, especially for the coupled system (1.1) with large $N > 1$.

In this paper we revisit the problem of linearized stability and instability of stationary solutions (2.1) in the coupled NLS equations (1.1) with $N \geq 1$. We develop two new methods of analysis: (i) negative eigenvalues of a constrained linear problem are counted from finite-dimensional (matrix) algebra and (ii) the negative subspace of a linear differential matrix operator with positive continuous spectrum is proved to be invariant in two diagonal representations. The first method develops the work of Vakhitov and Kolokolov [VK73], where eigenvalues of a constrained Sturm–Liouville operator were traced by means of a scalar analytical function. Preliminary results of the use of this method were published in [PK00]. The second method develops the Sylvester’s Inertial Theorem for quadratic forms associated with finite-dimensional (matrix) operators [G61]. The new methods of analysis reported here complete the fifteen years of history in development of stability–instability theory of optical solitons, with the following main results.

Theorem 3.6 (Negative index of constrained operators) *Operator \mathcal{L}_1 in the constrained space $X_c^{(u)}(\mathbb{R})$ has exactly $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$ negative eigenvalues and $\#_{=0}(\mathcal{L}_1) = z(\mathcal{L}_1) + z(\mathcal{U})$ zero eigenvalues. Operator \mathcal{L}_0 in the constrained space $X_c^{(u)}(\mathbb{R})$ has exactly $\#_{<0}(\mathcal{L}_0) = n(\mathcal{L}_0)$ negative eigenvalues and no zero eigenvalues, i.e. $\#_{=0}(\mathcal{L}_0) = 0$.*

Corollary 3.7 *Let $z(\mathcal{U}) = 0$. Then, $p(\mathcal{U}) \leq n(\mathcal{L}_1)$.*

Theorem 3.8 (Closure relation for negative index) *Let $z(\mathcal{L}_1) = 1$, $z(\mathcal{U}) = 0$, and Hypothesis 2.24 be satisfied. Suppose the linearized stability problem (2.15) has N_{real} real positive eigenvalues λ , $2N_{\text{comp}}$ complex eigenvalues λ with $\text{Re}(\lambda) > 0$, $2N_{\text{imag}}^-$ imaginary eigenvalues λ such that $\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle < 0$, and no non-zero eigenvalues of higher algebraic multiplicity. Dimension of the negative subspace of the linearized Hamiltonian h in $X_c(\mathbb{R})$ is invariant as*

$$\#_{<0}(h) = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0) = N_{\text{real}} + 2N_{\text{comp}} + 2N_{\text{imag}}^-. \quad (3.1)$$

Theorem 3.9 (Bounds on unstable eigenvalues) *Let $z(\mathcal{L}_1) = 1$, $z(\mathcal{U}) = 0$, Hypothesis 2.24 be satisfied, and no non-zero eigenvalues λ of higher algebraic multiplicity exist in (2.15). The linearized problem (2.15) has N_{unst} unstable eigenvalues λ such that $\text{Re}(\lambda) > 0$, where the number N_{unst} is bounded in the interval:*

$$|n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)| \leq N_{\text{unst}} \leq |n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0)|. \quad (3.2)$$

The problem has N_{real} real positive eigenvalues λ , such that $|n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)| \leq N_{\text{real}} \leq N_{\text{unst}}$ and $2N_{\text{comp}} = N_{\text{unst}} - N_{\text{real}}$ complex eigenvalues λ with $\text{Re}(\lambda) > 0$, such that $0 \leq N_{\text{comp}} \leq \min(n(\mathcal{L}_0), n(\mathcal{L}_1) - p(\mathcal{U}))$.

Corollary 3.10 *Let $z(\mathcal{U}) = 0$. When $n(\mathcal{L}_0) = 0$, the linearized problem (2.15) has exactly $N_{\text{real}} = n(\mathcal{L}_1) - p(\mathcal{U})$ real positive eigenvalues λ . When $n(\mathcal{L}_1) = p(\mathcal{U})$, the linearized problem (2.15) has exactly $N_{\text{real}} = n(\mathcal{L}_0)$ real positive eigenvalues λ . If both $n(\mathcal{L}_0) = 0$ and $n(\mathcal{L}_1) = p(\mathcal{U})$, the soliton solution (2.1) is weakly spectrally stable.*

Theorem 3.6 decomposes general Theorem 3.3 in the case, when h is a sum of two quadratic forms for \mathcal{L}_1 and \mathcal{L}_0 as in (2.12) (see also Remark 3.11). As a result, the upper bound on $p(\mathcal{U})$ of Theorem 3.1 is improved as $p(\mathcal{U}) \leq n(\mathcal{L}_1) \leq n(h)$, as in Corollary 3.7. Also the stability criterion of Theorem 3.1 decomposes into two conditions: $n(\mathcal{L}_1) = p(\mathcal{U})$ and $n(\mathcal{L}_0) = 0$, as in Corollary 3.10.

Theorem 3.8 gives a precise statement of the closure relation between indices $n(\mathcal{L}_0)$, $n(\mathcal{L}_1)$, $p(\mathcal{U})$, $z(\mathcal{U})$, N_{real} , N_{comp} , and N_{imag}^- . This theorem generalizes earlier results for $\#_{<0}(\mathcal{L}_1) = \#_{<0}(\mathcal{L}_0)$ formulated in [G88, Theorem 1.3] (when $N_{\text{comp}} = N_{\text{imag}}^- = 0$) and in [G90, Theorem 2.3] (when $N_{\text{real}} = N_{\text{imag}}^- = 0$).

Theorem 3.9 is a corollary of Theorems 3.5, 3.6, and 3.8. The lower bound in (3.2) is identical to that in Theorem 3.5 in view of Theorem 3.6. The upper bound in (3.2) improves Theorem 3.2. Theorem 3.9 also agrees with the instability criterion of Theorem 3.1. Let $z(\mathcal{U}) = 0$ and $m = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0)$ be odd. Then $|n(\mathcal{L}_1) - p(\mathcal{U}) - n(\mathcal{L}_0)| = |m - 2n(\mathcal{L}_0)| > 0$ and $N_{\text{unst}} > 0$. Therefore, Theorem 3.9 also guarantees instability for odd m , as Theorem 3.1.

Not only Theorems 3.6, 3.8, and 3.9 generalize the known stability–instability theorems, with new methods of analysis. The closure relation (3.1) allows us to locate all unstable eigenvalues of the linearization problem (2.15) with a new computational algorithm.

1. Compute indices $n(\mathcal{L}_0)$ and $n(\mathcal{L}_1)$ from the spectrum of symmetric matrix Schrödinger operators \mathcal{L}_0 and \mathcal{L}_1 . Check that $z(\mathcal{L}_1) = 1$.
2. Compute index $p(\mathcal{U})$ from the spectrum of symmetric Hessian matrix \mathcal{U} . Check that $z(\mathcal{U}) = 0$.
3. Solve the linear eigenvalue problem (2.15) for real positive λ : $0 < \lambda < \Lambda$, where Λ is a large positive number. Count the number N_{real} of real unstable eigenvalues λ and check that no non-zero eigenvalue exists with $\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle = 0$.
4. Solve the linear eigenvalue problem (2.15) for imaginary positive values of λ : $0 < \text{Im}(\lambda) < \beta_{\text{max}}$, where $\beta_{\text{max}} = \max_n(\beta_n)$. Count the number N_{imag}^- of imaginary (isolated or embedded) eigenvalues λ , such that $\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle < 0$ and check that no non-zero eigenvalue exists with $\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle = 0$.
5. When Hypothesis 2.24 is valid, the number of unstable eigenvalues is $N_{\text{unst}} = N_{\text{real}} + 2N_{\text{comp}}$, where

$$2N_{\text{comp}} = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0) - N_{\text{real}} - 2N_{\text{imag}}^-. \quad (3.3)$$

The new algorithm is more efficient computationally compared with the numerical algorithm based on the winding number theorem for Evans functions [PW02, BDG01]. Indeed, steps (1),

(3), and (4) of the new algorithm require computing eigenvalues λ along the four line segments, while the winding number theorem requires computing eigenvalues λ along a contour in the first quadrant of the complex plane λ . Two problems in step (1) are self-adjoint, two other problems in steps (3) and (4) are not self-adjoint but real-valued. Hessian matrix \mathcal{U} in step (2) is computed from numerical or analytical dependences $Q_{ns}(\beta)$ of the stationary solutions (2.1).

The new algorithm is also effective for analytical (asymptotic) studies near instability bifurcations. Suppose that operators \mathcal{L}_0 and \mathcal{L}_1 in the linearized problem (2.15) change according to a continuous deformation. If the deformation does not change the number $\#_{<0}(h)$, the total number of eigenvalues $N = N_{\text{real}} + 2N_{\text{comp}} + 2N_{\text{imag}}^-$ is invariant, and no new eigenvalues can emerge from resonances in the positive-definite continuous spectrum. If the deformation changes the number $\#_{<0}(h)$, the total number $N = N_{\text{real}} + 2N_{\text{comp}} + 2N_{\text{imag}}^-$ is adjusted accordingly.

When \mathcal{L}_1 and \mathcal{L}_0 are matrix operators (e.g. in the finite-difference approximation on a finite interval), Theorems 3.6, 3.8, and 3.9 reduce to stability-instability theorems for critical points in finite-dimensional Hamiltonian systems with symmetry constraints [M85]. Constraints in (2.26) and (2.27) imply that the perturbation terms $\mathbf{U}(z, x)$ and $\mathbf{W}(z, x)$ in (2.10) do not change the conserved quantities Q_n and P in the linear expansion of (1.6) and (1.9). When $n(\mathcal{L}_0) = 0$, the quadratic form $\langle \mathbf{W} | \mathcal{L}_0 \mathbf{W} \rangle$ in (2.12) is equivalent to a positive-definite kinetic energy, while the quadratic form $\langle \mathbf{U} | \mathcal{L}_1 \mathbf{U} \rangle$ in (2.12) is equivalent to a sign-indefinite potential energy. The Morse index of \mathcal{L}_1 under constraints (2.26) is $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$. When $n(\mathcal{L}_0) = 0$, the Morse index defines uniquely the unstable subspace of the linearized system according to Corollary 3.10. When $n(\mathcal{L}_0) > 0$ and both \mathcal{L}_0 and \mathcal{L}_1 are not positive definite, complex instabilities may occur and they are defined by Theorems 3.8 and 3.9.

Remark 3.11 The constraint (2.27) does not lead to a non-trivial contribution to Theorems 3.6, 3.8, and 3.9. This constraint may mislead understanding general stability-instability Theorems 3.1, 3.2, and 3.3 in applications to the coupled NLS equations (1.1). We show here that the constraint (2.27) does not affect positive and zero eigenvalues of the Hessian matrix of stationary solutions (2.1). It follows from (2.1)–(2.2) that parameter v can be scaled out of the stability problem. A general family of v -dependent stationary solutions is defined from (2.1) as:

$$\psi(z, x) = e^{iz\omega} \Psi(x - 2vz - s), \quad (3.4)$$

where $\omega = \text{diag}(\omega_1, \dots, \omega_N)$. The v -dependent stationary solutions are critical points of the Lyapunov functional in the form:

$$\Lambda[\psi] = H[\psi] + \sum_{n=1}^N \omega_n Q_n[\psi] + vP[\psi], \quad (3.5)$$

where Q_n and P are given by (1.6) and (1.9). The general Hessian matrix \mathbf{U}_H has the structure:

$$\mathbf{U}_H = \begin{bmatrix} \mathcal{U} & \frac{\partial \mathbf{Q}_s}{\partial v} \\ \frac{\partial \mathbf{Q}_s^T}{\partial v} & \frac{\partial P_s}{\partial v} \end{bmatrix}, \quad (3.6)$$

where $\mathbf{Q}_s = (Q_{1s}, \dots, Q_{Ns})^T$, $Q_{ns}(\boldsymbol{\omega}, v) = Q_n[\Psi]$, and $P_s(\boldsymbol{\omega}, v) = P[\Psi]$. It follows from (2.1), (2.2), and (3.4) that a transformation

$$\Psi_n(x) = \Phi_n(x) e^{id_n^{-1}vx}, \quad \omega_n = \beta_n + \frac{v^2}{d_n}, \quad (3.7)$$

expresses $Q_{ns}(\boldsymbol{\omega}, v)$, $P_s(\boldsymbol{\omega}, v)$ as functions of $\boldsymbol{\beta}$:

$$\frac{\partial Q_{ns}}{\partial v}(\boldsymbol{\omega}, v) = - \sum_{m=1}^N \frac{2v}{d_m} \frac{\partial Q_{ms}}{\partial \beta_n}(\boldsymbol{\beta}), \quad (3.8)$$

$$\frac{\partial P_s}{\partial v}(\boldsymbol{\omega}, v) = - \sum_{n=1}^N \frac{2}{d_n} Q_{ns}(\boldsymbol{\beta}) + \sum_{n=1}^N \sum_{m=1}^N \frac{4v^2}{d_n d_m} \frac{\partial Q_{ms}}{\partial \beta_n}(\boldsymbol{\beta}). \quad (3.9)$$

As a result, the quadratic form for $\mathbf{x} \in \mathbb{R}^{N+1}$ transforms to a quadratic form for $\mathbf{y} \in \mathbb{R}^N$ as

$$\langle \mathbf{x}, \mathcal{U}_H \mathbf{x} \rangle_{\mathbb{R}^{N+1}} = \langle \mathbf{y}, \mathcal{U} \mathbf{y} \rangle_{\mathbb{R}^N} - \left(\sum_{n=1}^N \frac{2Q_{ns}}{d_n} \right) x_{N+1}^2, \quad (3.10)$$

where

$$y_n = x_n - \frac{2v}{d_n} x_{N+1}, \quad n = 1, \dots, N,$$

and we have used the standard vector product:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{R}^N} = \sum_{n=1}^N \bar{a}_n b_n. \quad (3.11)$$

The additional eigenvalue for x_{N+1} is always negative, i.e. $p(\mathcal{U}_H) = p(\mathcal{U})$ and $z(\mathcal{U}_H) = z(\mathcal{U})$.

4 Eigenvalues of constrained spectral problems for \mathcal{L}_1 and \mathcal{L}_0

Here we prove Theorem 3.6 with a new method of analysis, when eigenvalues of a constrained spectral problem are counted from matrix algebra. This method was announced in [PK00] as a generalization of the Vakhitov–Kolokolov method [VK73] for a matrix case. Similar studies of a scalar constrained spectral problem were reported in [BP93, G88].

We start with formulation of the problem. Given the properties of the spectrum of \mathcal{L}_1 and \mathcal{L}_0 in $X(\mathbb{R})$, we study properties of the spectrum of operators \mathcal{L}_1 and \mathcal{L}_0 in the constrained function space $X_c^{(u)}$, defined by (2.26). The constrained space is an orthogonal complement of the null space of \mathcal{L}_0 in $X(\mathbb{R})$. The spectrum of \mathcal{L}_1 and \mathcal{L}_0 is complete in $X_c^{(u)}$ due to the general result in [HS96, Proposition 2.7].

Proposition 4.1 [HS96] *Let M be a closed subspace of a Hilbert space \mathcal{H} and M^\perp be the orthogonal complement of M in \mathcal{H} , such that $M^\perp = \{x \in \mathcal{H} : \langle x | m \rangle = 0 \ \forall m \in M\}$. The subset M^\perp is a closed subspace of \mathcal{H} and is therefore a Hilbert space.*

Lemma 4.2 *The constrained spectral problem for \mathcal{L}_0 ,*

$$\mathcal{L}_0 \mathbf{u} = \lambda \mathbf{u}, \quad \mathbf{u} \in X_c^{(u)}(\mathbb{R}), \quad (4.1)$$

has exactly $\#_{<0}(\mathcal{L}_0) = n(\mathcal{L}_0)$ negative eigenvalues.

Proof. Eigenfunctions of \mathcal{L}_0 for non-zero eigenvalues λ are all orthogonal to the eigenfunctions $\Phi_n(x)\mathbf{e}_n$, $n = 1, \dots, N$ of the null space of \mathcal{L}_0 . Therefore, they are all belong to $X_c^{(u)}(\mathbb{R})$. \square

Lemma 4.3 *The constrained spectral problem for \mathcal{L}_1 takes the form:*

$$\mathcal{L}_1 \mathbf{u} = \lambda \mathbf{u} - \sum_{m=1}^N \nu_m \Phi_m(x) \mathbf{e}_m, \quad \mathbf{u} \in X_c^{(u)}(\mathbb{R}), \quad (4.2)$$

where ν_1, \dots, ν_N are Lagrange multipliers. When $\ker(\mathcal{L}_1 - \lambda) = \{0\}$, the constrained spectral problem has a solution if and only if there exists a non-zero solution of the homogeneous linear system for ν_1, \dots, ν_N :

$$\sum_{m=1}^N \langle \Phi_n \mathbf{e}_n | (\lambda - \mathcal{L}_1)^{-1} \Phi_m \mathbf{e}_m \rangle \nu_m = 0, \quad n = 1, \dots, N. \quad (4.3)$$

Proof. When λ is not in the spectrum of \mathcal{L}_1 in $X(\mathbb{R})$, the operator $(\mathcal{L}_1 - \lambda)$ is invertible in $X(\mathbb{R})$ and the orthogonality conditions in (2.26) result in the matrix system (4.3). \square

Remark 4.4 In the rest of this section, we denote negative eigenvalues of the discrete spectrum of \mathcal{L}_1 in $X(\mathbb{R})$ as λ_{-k} with orthonormal eigenfunctions $\mathbf{u}_{-k}(x)$, accounting their multiplicity. We can order the negative eigenvalues λ_{-k} from the minimal eigenvalue $\lambda_{-n(\mathcal{L}_1)}$ to the maximal negative eigenvalue $\lambda_{-1} < 0$. We also denote positive eigenvalues of the discrete, embedded and continuous spectrum of \mathcal{L}_1 in $X(\mathbb{R})$ as λ_k with orthonormal eigenfunctions $\mathbf{u}_k(x)$. The spectral decomposition (2.20) can be written as a sum of three terms $\sum_{\lambda_{-k} < 0}$, $\sum_{\lambda_{-k} = 0}$, and $\sum_{\lambda_{-k} > 0}$, where $\sum_{\lambda_{-k} < 0}$ contains $n(\mathcal{L}_1)$ terms from negative discrete spectrum of \mathcal{L}_1 , $\sum_{\lambda_{-k} = 0}$ contains $z(\mathcal{L}_1)$ terms from null space of \mathcal{L}_1 , and $\sum_{\lambda_k > 0}$ includes $p(\mathcal{L}_1)$ terms from positive discrete and embedded spectrum of \mathcal{L}_1 and N integral terms from the continuous spectrum of \mathcal{L}_1 .

Lemma 4.5 *Let $z(\mathcal{L}_1) = 1$. When $\ker(\mathcal{L}_1 - \lambda) = \{0\}$ in $X(\mathbb{R})$, the constrained problem (4.2) has a solution if and only if the matrix $\mathcal{A}(\lambda)$ has a zero eigenvalue, where the matrix elements of $\mathcal{A}(\lambda)$ are:*

$$\mathcal{A}_{nm}(\lambda) = \sum_{\lambda_{-k} < 0} \frac{\langle \Phi_n \mathbf{e}_n | \mathbf{u}_{-k} \rangle \langle \mathbf{u}_{-k} | \Phi_m \mathbf{e}_m \rangle}{\lambda - \lambda_{-k}} + \sum_{\lambda_k > 0} \frac{\langle \Phi_n \mathbf{e}_n | \mathbf{u}_k \rangle \langle \mathbf{u}_k | \Phi_m \mathbf{e}_m \rangle}{\lambda - \lambda_k}. \quad (4.4)$$

Proof. Solutions of the constrained problem (4.2) in $X(\mathbb{R})$ can be decomposed over a basis of orthogonal eigenfunctions $\mathbf{u}_k(x)$ in $X(\mathbb{R})$ similar to (2.20). If $z(\mathcal{L}_1) = 1$, the decomposition takes the form:

$$\mathbf{u}(x) = \sum_{m=1}^N \nu_m \left[\sum_{\lambda_{-k} < 0} \frac{\langle \mathbf{u}_{-k} | \Phi_m \mathbf{e}_m \rangle}{\lambda - \lambda_{-k}} \mathbf{u}_{-k}(x) + \frac{\langle \mathbf{u}_0 | \Phi_m \mathbf{e}_m \rangle}{\lambda} \mathbf{u}_0(x) + \sum_{\lambda_k > 0} \frac{\langle \mathbf{u}_k | \Phi_m \mathbf{e}_m \rangle}{\lambda - \lambda_k} \mathbf{u}_k(x) \right], \quad (4.5)$$

where the middle zero term vanishes since $\mathbf{u}_0(x) = \Phi'(x)$ and $\langle \Phi'(x) | \Phi_m \mathbf{e}_m \rangle = 0$. If $\mathbf{u} \in X_c^{(u)}(\mathbb{R})$, the constraints (2.26) must be satisfied for the solution $\mathbf{u}(x)$ in (4.5). The constraints are satisfied if and only if the homogeneous linear system $\mathcal{A}(\lambda)\boldsymbol{\nu} = \mathbf{0}$ has a non-zero solution for $\boldsymbol{\nu}$, where $\mathcal{A}(\lambda)$ is given by (4.4). \square

Lemma 4.6 *The matrix eigenvalue problem $\mathcal{A}(\lambda)\boldsymbol{\nu} = \alpha(\lambda)\boldsymbol{\nu}$ with $\lambda \in \mathbb{R}$, $\lambda \leq 0$ has N real eigenvalues $\alpha_1(\lambda), \dots, \alpha_N(\lambda)$, which are meromorphic functions of λ .*

Proof. The matrix $\mathcal{A}(\lambda)$ has N real eigenvalues $\alpha(\lambda)$ for $\lambda \leq 0$ since it is Hermitian for $\lambda \in \mathbb{R}$, such that $\mathcal{A}_{nm}(\lambda) = \bar{\mathcal{A}}_{mn}(\lambda)$. Coefficients of $\mathcal{A}(\lambda)$ have pole singularities at $\lambda = \lambda_{-k}$ for $\lambda \leq 0$, unless $\langle \Phi_n \mathbf{e}_n | \mathbf{u}_{-k} \rangle = 0$, $n = 1, \dots, N$. Since $\Phi_n \in L^2(\mathbb{R})$, $\mathbf{u} \in X(\mathbb{R})$, and $\langle \Phi_n \mathbf{e}_n | \mathbf{u} \rangle < \infty$, the series for $\mathcal{A}_{nm}(\lambda)$ are bounded and converge for $\lambda \neq \lambda_{-k}$. In the limit $\lambda \rightarrow -\infty$, $\mathcal{A}_{nm}(\lambda)$ converges to zero uniformly. As a result, all eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ are meromorphic functions for $\lambda \leq 0$, which may have only pole singularities at $\lambda = \lambda_{-k}$. \square

Lemma 4.7 *Eigenvalues $\alpha_1(\lambda), \dots, \alpha_N(\lambda)$ of the matrix $\mathcal{A}(\lambda)$ are decreasing functions of λ for $\lambda \leq 0$, $\lambda \neq \lambda_{-k}$. All eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ are negative for $\lambda < \lambda_{-n}(\mathcal{L}_1)$.*

Proof. For Hermitian matrices, the set of eigenvalues $\{\alpha_n(\lambda)\}_{n=1}^N$ corresponds to the set of orthonormal eigenvectors $\{\boldsymbol{\nu}^{(n)}\}_{n=1}^N$ such that $\langle \boldsymbol{\nu}^{(n')}, \boldsymbol{\nu}^{(n)} \rangle_{\mathbb{R}^N} = \delta_{n',n}$, where $\langle \mathbf{f}, \mathbf{g} \rangle_{\mathbb{R}^N}$ is given by (3.11). We construct quadratic forms associated to the pair $(\alpha_n, \boldsymbol{\nu}^{(n)})$, $n = 1, \dots, N$:

$$\alpha_n(\lambda) = \langle \boldsymbol{\nu}^{(n)}, \mathcal{A}(\lambda)\boldsymbol{\nu}^{(n)} \rangle_{\mathbb{R}^N}, \quad \alpha'_n(\lambda) = \langle \boldsymbol{\nu}^{(n)}, \mathcal{A}'(\lambda)\boldsymbol{\nu}^{(n)} \rangle_{\mathbb{R}^N}. \quad (4.6)$$

Computing the derivative of $\mathcal{A}'(\lambda)$, we rewrite the second equality in (4.6) as

$$\alpha'_n(\lambda) = - \left(\sum_{\lambda_{-k} < 0} \frac{b_{-k}}{(\lambda - \lambda_{-k})^2} + \sum_{\lambda_k > 0} \frac{b_k}{(\lambda - \lambda_k)^2} \right) = -\langle \mathbf{u} | \mathbf{u} \rangle < 0, \quad (4.7)$$

where

$$b_{\pm k} = \left| \sum_{m=1}^N \langle \Phi_m \mathbf{e}_m | \mathbf{u}_{\pm k} \rangle \nu_m^{(n)} \right|^2 \geq 0, \quad (4.8)$$

and the last equality in (4.7) follows from representation (4.5) in $X(\mathbb{R})$ for $\lambda \neq \lambda_{\pm k}$. As a result, all eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ are decreasing functions of λ for $\lambda \leq 0$, $\lambda \neq \lambda_{-k}$. In order to prove that all eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ are negative for $\lambda < \lambda_{-n}(\mathcal{L}_1)$, we find from (4.3) and (4.4) that

$$\lim_{\lambda \rightarrow -\infty} (\lambda \mathcal{A}_{nm}(\lambda)) = \langle \Phi_n \mathbf{e}_n | \Phi_m \mathbf{e}_m \rangle = Q_{ns} \delta_{n,m}, \quad (4.9)$$

where $Q_{ns}(\boldsymbol{\beta}) = Q_n[\boldsymbol{\Phi}]$ is defined by (1.13). It follows from the first equality in (4.6) and (4.9) that

$$\lim_{\lambda \rightarrow -\infty} \lambda \alpha_n(\lambda) = Q_{ns}(\boldsymbol{\beta}) > 0, \quad n = 1, \dots, N,$$

and $\lim_{\lambda \rightarrow -\infty} \alpha_n(\lambda) = -0$, $n = 1, \dots, N$. Since eigenvalues $\alpha(\lambda)$ are continuous and decreasing for $\lambda < \lambda_{-n}(\mathcal{L}_1)$, they remain negative for all values of $\lambda < \lambda_{-n}(\mathcal{L}_1)$. \square

Lemma 4.8 *Let λ_{-k} be an eigenvalue of the discrete spectrum of \mathcal{L}_1 in $X(\mathbb{R})$ of multiplicity q_{-k} such that q_{-k}^{\parallel} eigenfunctions $\mathbf{u}_{-k}(x)$ belong to the constrained space $X_c^{(u)}(\mathbb{R})$, while $q_{-k}^{\perp} = (q_{-k} - q_{-k}^{\parallel})$ eigenfunctions $\mathbf{u}_{-k}(x)$ belong to the orthogonal complement of $X_c^{(u)}(\mathbb{R})$ in $X(\mathbb{R})$. There exist $(N - q_{-k}^{\perp})$ eigenvalues $\alpha_n(\lambda)$ that are continuous at $\lambda = \lambda_{-k}$ and q_{-k}^{\perp} eigenvalues $\alpha_n(\lambda)$ that have infinity discontinuities, jumping from negative infinity for $\lambda = \lambda_{-k} - 0$ to positive infinity for $\lambda = \lambda_{-k} + 0$.*

Proof. In the limit $\lambda \rightarrow \lambda_{-k}$, we find that

$$\lim_{\lambda \rightarrow \lambda_{-k}} (\lambda - \lambda_{-k}) \mathcal{A}_{nm}(\lambda) = \sum_{r=1}^{q_{-k}} \langle \Phi_n \mathbf{e}_n | \mathbf{u}_{-k_r} \rangle \langle \mathbf{u}_{-k_r} | \Phi_m \mathbf{e}_m \rangle = \sum_{r=1}^{q_{-k}^{\perp}} \langle \Phi_n \mathbf{e}_n | \mathbf{u}_{-k_r} \rangle \langle \mathbf{u}_{-k_r} | \Phi_m \mathbf{e}_m \rangle. \quad (4.10)$$

Denote $\mathcal{B}_{-k} = \lim_{\lambda \rightarrow \lambda_{-k}} (\lambda - \lambda_{-k}) \mathcal{A}(\lambda)$. The quadratic form $\langle \boldsymbol{\nu}, \mathcal{B}_{-k} \boldsymbol{\nu} \rangle$ is diagonalized in normal variables,

$$x_r = \sum_{m=1}^N \langle \mathbf{u}_{-k_r} | \Phi_m \mathbf{e}_m \rangle \nu_m,$$

such that $\langle \boldsymbol{\nu}, \mathcal{B}_{-k} \boldsymbol{\nu} \rangle = \sum_{r=1}^{q_{-k}^{\perp}} |x_r|^2$. Therefore, the matrix \mathcal{B}_{-k} has exactly q_{-k}^{\perp} positive eigenvalues and $(N - q_{-k}^{\perp})$ zero eigenvalues. Positive eigenvalues of \mathcal{B}_{-k} correspond to q_{-k}^{\perp} eigenvalues $\alpha_n(\lambda)$ jumping from negative infinity for $\lambda = \lambda_{-k} - 0$ to positive infinity for $\lambda = \lambda_{-k} + 0$. Zero eigenvalues of \mathcal{B}_{-k} correspond to $(N - q_{-k}^{\perp})$ eigenvalues $\alpha_n(\lambda)$ that are continuous and have convergent Taylor series at $\lambda = \lambda_{-k}$. \square

Lemma 4.9 *At $\lambda = 0$, there exist $p(\mathcal{U})$ positive, $z(\mathcal{U})$ zero, and $n(\mathcal{U})$ negative eigenvalues $\alpha_n(0)$, $n = 1, \dots, N$.*

Proof. At $\lambda = 0$, the constrained eigenvalue problem (4.2) has an exact solution in $X(\mathbb{R})$:

$$\mathbf{u}_{\lambda=0}(x) = \sum_{m=1}^N \nu_m \frac{\partial \Phi(x)}{\partial \beta_m} + c_0 \Phi'(x), \quad (4.11)$$

where c_0 is not defined and

$$\mathcal{L}_1 \frac{\partial \Phi}{\partial \beta_m} = -\Phi_m(x) \mathbf{e}_m. \quad (4.12)$$

Substituting (4.11) into (2.26), we find that $\mathcal{A}(0) = \frac{1}{2}\mathcal{U}$, where \mathcal{U} is defined in (2.8). \square

Lemma 4.10 *Let $z(\mathcal{L}_1) = 1$. The constrained spectral problem (4.2) has exactly $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$ negative eigenvalues λ and $\#_{=0}(\mathcal{L}_0) = 1 + z(\mathcal{U})$ zero eigenvalues.*

Proof. We consider eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ as meromorphic functions of λ for $\lambda \leq 0$. Starting with small negative values at $\lambda \rightarrow -\infty$, eigenvalues $\alpha_n(\lambda)$, $n = 1, \dots, N$ decrease as λ increases toward $n(\mathcal{L}_1)$ pole singularities at $\lambda = \lambda_{-k}$. At each pole singularity, q_{-k}^\perp eigenvalues $\alpha(\lambda)$ jump and pop up to the positive half-plane. The total number of jumping eigenvalues for $\lambda < 0$ is $\sum_{\lambda_{-k} < 0} q_{-k}^\perp$. Only jumping eigenvalues may cross the value $\alpha(\lambda) = 0$, which corresponds to a negative eigenvalue λ of the constrained problem (4.2) in space $X_c^{(u)}(\mathbb{R})$. We find the total number of zeros of $\alpha(\lambda)$ at $\lambda \leq 0$ from Lemma 4.9 as

$$\sum_{\lambda_{-k} < 0} q_{-k}^\perp - p(\mathcal{U}).$$

At each $\lambda = \lambda_{-k}$, there are q_{-k}^\parallel eigenfunctions $\mathbf{u}_{-k}(x)$ that lie in the constrained space $X_c^{(u)}(\mathbb{R})$. Therefore, the total number of eigenvalues λ in $X_c^{(u)}(\mathbb{R})$ for $\lambda \leq 0$ is

$$\sum_{\lambda_{-k}} q_{-k}^\perp - p(\mathcal{U}) + \sum_{\lambda_{-k}} q_{-k}^\parallel = \sum_{\lambda_{-k}} q_{-k} - p(\mathcal{U}) = n(\mathcal{L}_1) - p(\mathcal{U}).$$

Subtracting the number $z(\mathcal{U})$ of zero eigenvalues at $\lambda = 0$, we prove the lemma. \square

Corollary 4.11 *The following bounds on numbers $p(\mathcal{U})$, $n(\mathcal{U})$ and $n(\mathcal{L}_1)$ are satisfied:*

$$p(\mathcal{U}) \leq \sum_{\lambda_{-k} < 0} q_{-k}^\perp \leq n(\mathcal{L}_1), \quad n(\mathcal{U}) \geq N - n(\mathcal{L}_1). \quad (4.13)$$

Lemma 4.12 *Let \mathcal{U} be the Hessian matrix with bounded eigenvalues. Eigenfunctions of the null space of \mathcal{L}_1 in $X(\mathbb{R})$ belong to $X_c^{(u)}(\mathbb{R})$ for any $1 \leq z(\mathcal{L}_1) \leq N$.*

Proof. Suppose there exists an eigenfunction $\mathbf{u}_0(x)$ of the null space of \mathcal{L}_1 in $X(\mathbb{R})$ such that it does not belong to $X_c^{(u)}(\mathbb{R})$. It follows from Lemma 4.8 that there exists an eigenvalue $\alpha_n(\lambda)$ that diverges at $\lambda \rightarrow -0$. It contradicts Lemma 4.9 since all eigenvalues of \mathcal{U} are bounded. \square

Theorem 3.6 is proved with Lemma 4.2, 4.10, and 4.12.

5 Stability-instability analysis with diagonalization theorems

Here we prove Theorems 3.8 and 3.9 with a new method of analysis, when spectrum of the linearized problem (2.15) for the non-self-adjoint operator \mathcal{A} is controlled by the spectrum of constrained eigenvalue problems for the self-adjoint operators \mathcal{L}_0 and \mathcal{L}_1 . Similar methods were developed in [G88, G90] but matrix algebra was not previously employed in spectral analysis.

Lemma 5.1 *The non-zero spectrum of \mathcal{A} in $X_c(\mathbb{R})$ maps to the non-zero spectrum of the problem,*

$$\mathcal{L}_1 \mathbf{u} = \gamma \mathcal{L}_0^{-1} \mathbf{u}, \quad \mathbf{u} \in X_c^{(u)}(\mathbb{R}), \quad \gamma = -\lambda^2. \quad (5.1)$$

Proof. Eigenfunctions $\{\Phi_n(x) \mathbf{e}_n\}_{n=1}^N$ form a basis for the null space of \mathcal{L}_0 . The operator \mathcal{L}_0 is invertible on functions $\mathbf{u}(x)$ in the constrained function space $X_c^{(u)}(\mathbb{R})$. It follows from (2.28) that $\mathbf{w} = \lambda \mathcal{L}_0^{-1} \mathbf{u}$ and the problem (2.28) is equivalent to (2.15) for any $\lambda \neq 0$. Two eigenfunctions of \mathcal{A} in $X_c(\mathbb{R})$ corresponds to a single eigenfunction of (5.1) in $X_c^{(u)}(\mathbb{R})$. \square

Corollary 5.2 *Let $\gamma = \gamma_m$ be a non-zero eigenvalue of (5.1) with the eigenfunction $\mathbf{u} = \mathbf{u}_m(x)$ in $X_c^{(u)}(\mathbb{R})$, such that*

$$\langle \mathbf{u}_m | \mathcal{L}_1 \mathbf{u}_m \rangle = \gamma_m \langle \mathbf{u}_m | \mathcal{L}_0^{-1} \mathbf{u}_m \rangle. \quad (5.2)$$

Eigenvalue γ_m is real if either \mathcal{L}_1 or \mathcal{L}_0^{-1} is positive definite.

Remark 5.3 The problem (5.1) is a classical problem of simultaneous diagonalization of two self-adjoint operators \mathcal{L}_1 and \mathcal{L}_0^{-1} . Each operator can be orthogonally diagonalized due to Proposition 2.12. However, the orthogonal diagonalizations (2.20)–(2.21) are relevant for (5.1) only if the operators \mathcal{L}_1 and \mathcal{L}_0^{-1} commute, such that there exists a common basis for $\langle \mathbf{f} | \mathcal{L}_1 \mathbf{f} \rangle$ and $\langle \mathbf{f} | \mathcal{L}_0^{-1} \mathbf{f} \rangle$ [G61]. Operators \mathcal{L}_1 and \mathcal{L}_0^{-1} do not commute since $\mathcal{L}_0 \mathcal{L}_1 \neq \mathcal{L}_1 \mathcal{L}_0$. Therefore, eigenfunctions of \mathcal{L}_1 and \mathcal{L}_0^{-1} in $X_c^{(u)}(\mathbb{R})$ do not produce any eigenfunctions of (5.1). Moreover, complex eigenvalues and multiple eigenvalues with higher algebraic multiplicity may occur in the problem (5.1).

Lemma 5.4 *The spectrum of (5.1) is real if $\Phi(x)$ is a fundamental soliton, such that $\Phi_n(x) > 0$, $n = 1, \dots, N$ for $x \in \mathbb{R}$, and $n(\mathcal{L}_0) = \sum_{n=1}^N i_n = 0$. Moreover, the positive-definite operator \mathcal{L}_0 can*

be factorized as $\mathcal{L}_0 = \mathcal{S}^+ \mathcal{S}$, where \mathcal{S} has a diagonal form with the elements:

$$\mathcal{S}_{nn} = \sqrt{d_n} \left(\frac{\Phi'_n(x)}{\Phi_n(x)} - \frac{d}{dx} \right). \quad (5.3)$$

Proof. The factorization formula (5.3) follows from explicit computations:

$$\mathcal{S}_{nn}^+ \mathcal{S}_{nn} = d_n \left(\frac{\Phi''_n}{\Phi_n} - \frac{d^2}{dx^2} \right) = (\mathcal{L}_0)_{nn}.$$

Using the transformation $\mathbf{u} = \mathcal{S}^+ \mathbf{v}$, we rewrite (5.1) in the form

$$\mathcal{S} \mathcal{L}_1 \mathcal{S}^+ \mathbf{v} = \gamma \mathbf{v}.$$

Since $\mathcal{S} \mathcal{L}_1 \mathcal{S}^+$ is a self-adjoint operator, all eigenvalues γ are real. It is also clear from (5.3) that the operators \mathcal{S}^+ has empty null space, and the transformation $\mathbf{u} = \mathcal{S}^+ \mathbf{v}$ is invertible. \square

Corollary 5.5 *Hypothesis 2.24 is satisfied if $\Phi(x)$ is a fundamental soliton.*

Lemma 5.6 *Let $\gamma = \gamma_k = \gamma_{Rk} + i\gamma_{Ik}$ be a complex eigenvalue of (5.1) such that $\gamma_{Rk}, \gamma_{Ik} \neq 0$, with a complex-valued eigenfunction $\mathbf{u} = \mathbf{u}_k(x) = \mathbf{u}_{Rk}(x) + i\mathbf{u}_{Ik}(x)$. A linear combination of two real-valued eigenfunctions $\mathbf{f}(x) = a_k \mathbf{u}_{Rk}(x) + b_k \mathbf{u}_{Ik}(x)$ diagonalizes the quadratic forms $\langle \mathbf{f} | \mathcal{L}_1 \mathbf{f} \rangle$ and $\langle \mathbf{f} | \mathcal{L}_0^{-1} \mathbf{f} \rangle$ with respect to Jordan blocks,*

$$\langle \mathbf{f} | \mathcal{L}_1 \mathbf{f} \rangle = \mathbf{a}_k^T \hat{\gamma}_k \hat{l}_k \mathbf{a}_k, \quad \langle \mathbf{f} | \mathcal{L}_0^{-1} \mathbf{f} \rangle = \mathbf{a}_k^T \hat{l}_k \mathbf{a}_k, \quad (5.4)$$

where $\mathbf{a}_k = (a_k, b_k)^T$,

$$\hat{\gamma}_k = \begin{pmatrix} \gamma_{Rk} & -\gamma_{Ik} \\ \gamma_{Ik} & \gamma_{Rk} \end{pmatrix}, \quad \hat{l}_k = \begin{pmatrix} l_{Rk} & l_{Ik} \\ l_{Ik} & -l_{Rk} \end{pmatrix}, \quad (5.5)$$

and

$$l_{Rk} = \langle \mathbf{u}_{Rk} | \mathcal{L}_0^{-1} \mathbf{u}_{Rk} \rangle = -\langle \mathbf{u}_{Ik} | \mathcal{L}_0^{-1} \mathbf{u}_{Ik} \rangle, \quad (5.6)$$

$$l_{Ik} = \langle \mathbf{u}_{Ik} | \mathcal{L}_0^{-1} \mathbf{u}_{Rk} \rangle = \langle \mathbf{u}_{Rk} | \mathcal{L}_0^{-1} \mathbf{u}_{Ik} \rangle. \quad (5.7)$$

Proof. Since the quadratic forms $\langle \mathbf{u}_k | \mathcal{L}_1 \mathbf{u}_k \rangle$ and $\langle \mathbf{u}_k | \mathcal{L}_0^{-1} \mathbf{u}_k \rangle$ in (5.2) are real-valued, the eigenvalues γ_k can be complex only if

$$\langle \mathbf{u}_k | \mathcal{L}_1 \mathbf{u}_k \rangle = \langle \mathbf{u}_k | \mathcal{L}_0^{-1} \mathbf{u}_k \rangle = 0. \quad (5.8)$$

The zero inner product (5.8) for \mathcal{L}_0^{-1} results in relations (5.6) and (5.7). The Jordan blocks (5.4)–(5.5) follow then from direct computations. \square

Lemma 5.7 Let $\gamma = \gamma_m$ be a real eigenvalue of (5.1) with a single real-valued eigenvector $\mathbf{u} = \mathbf{u}_m(x)$ in $X_c^{(u)}(\mathbb{R})$. The eigenvalue $\gamma = \gamma_m$ is a multiple eigenvalue of higher algebraic multiplicity if and only if

$$l_m = \langle \mathbf{u}_m | \mathcal{L}_0^{-1} \mathbf{u}_m \rangle = 0. \quad (5.9)$$

Proof. The eigenvalue $\gamma = \gamma_m$ is a degenerate eigenvalue of higher algebraic multiplicity if and only if the derivative problem,

$$\mathcal{L}_1 \mathbf{u}'_m = \gamma_m \mathcal{L}_0^{-1} \mathbf{u}'_m + \mathcal{L}_0^{-1} \mathbf{u}_m, \quad (5.10)$$

has a solution in $X_c^{(u)}(\mathbb{R})$. Using the Fredholm Alternative Theorem with $\mathbf{u}_m(x)$, we arrive to the condition (5.9). \square

Remark 5.8 A complex eigenvalue $\gamma = \gamma_k = \gamma_{Rk} + i\gamma_{Ik}$, such that $\gamma_{Rk}, \gamma_{Ik} \neq 0$, with a single complex-valued eigenvector $\mathbf{u} = \mathbf{u}_k(x) = \mathbf{u}_{Rk}(x) + i\mathbf{u}_{Ik}(x)$ is a multiple eigenvalue of higher algebraic multiplicity if and only if $l_{Rk} = l_{Ik} = 0$ in (5.6)–(5.7). In the rest of this section, we consider the generic case when multiple eigenvalues of higher algebraic multiplicity do not occur in the spectrum of the problem (5.1).

Lemma 5.9 Suppose that the problem (5.1) has no multiple eigenvalues of higher algebraic multiplicity in $X_c^{(u)}(\mathbb{R})$. Eigenfunctions $\mathbf{u}_m(x)$ for real eigenvalues γ_m , $(\mathbf{u}_{Rk}(x), \mathbf{u}_{Ik}(x))$ for complex eigenvalues $\gamma_{Rk} + i\gamma_{Ik}$ and $\mathbf{u}_n(x; k)$ for N branches of continuous spectrum $\gamma \geq \beta_n^2$, $n = 1, \dots, N$ satisfy the orthogonality relations with respect to operators \mathcal{L}_0^{-1} ,

$$\langle \mathbf{u}_{m'} | \mathcal{L}_0^{-1} \mathbf{u}_m \rangle = l_m \delta_{m',m}, \quad \langle \mathbf{u}_{n'}(k') | \mathcal{L}_0^{-1} \mathbf{u}_n(k) \rangle = l_n(k) \delta_{n',n} \delta(k' - k), \quad (5.11)$$

and

$$\begin{aligned} \langle \mathbf{u}_{Rk'} | \mathcal{L}_0^{-1} \mathbf{u}_{Rk} \rangle &= -\langle \mathbf{u}_{Ik'} | \mathcal{L}_0^{-1} \mathbf{u}_{Ik} \rangle = l_{Rk} \delta_{k',k}, \\ \langle \mathbf{u}_{Ik'} | \mathcal{L}_0^{-1} \mathbf{u}_{Rk} \rangle &= \langle \mathbf{u}_{Rk'} | \mathcal{L}_0^{-1} \mathbf{u}_{Ik} \rangle = l_{Ik} \delta_{k',k}, \end{aligned} \quad (5.12)$$

where $l_m \neq 0$, $l_{Rk}, l_{Ik} \neq 0$, and $l_n(k) > 0$.

Proof. Orthogonality relations (5.11) for eigenfunctions $\mathbf{u}_m(x)$ of the discrete and embedded (real) spectrum of (5.1) in $X_c^{(u)}(\mathbb{R})$ follow from the identity:

$$(\gamma_{m'} - \gamma_m) \langle \mathbf{u}_{m'} | \mathcal{L}_0^{-1} \mathbf{u}_m \rangle = 0. \quad (5.13)$$

Orthogonality relations (5.12) for eigenfunctions $\mathbf{u}_{Rk}(x)$, $\mathbf{u}_{Ik}(x)$ of the discrete (complex) spectrum of (5.1) follow from similar standard relations. Coefficients l_m , l_{Rk} , and l_{Ik} are non-zero due to Lemma 5.7, since no multiple eigenvalues of higher algebraic multiplicity are supposed to exist in

(5.1). Orthogonality relations for eigenfunctions $\mathbf{u}_n(x; k)$ of the continuous spectrum of (5.1) in $X_c^{(u)}(\mathbb{R})$ follow from Wronskian identity:

$$\begin{aligned} & (\gamma_{n'}(k') - \gamma_n(k)) \langle \mathbf{u}_{n'}(k') | \mathcal{L}_0^{-1} \mathbf{u}_n(k) \rangle = \\ & \lim_{L \rightarrow \infty} \sum_{m=1}^N d_m \left(\bar{u}_{n',m}(x, k') \frac{du_{n,m}(x, k)}{dx} - \frac{d\bar{u}_{n',m}(x, k')}{dx} u_{n,m}(x, k) \right) \Big|_{x=-L}^{x=L}, \end{aligned} \quad (5.14)$$

where $\gamma_n(k) = (\beta_n + d_n k^2)^2$. Spectral densities $l_n(k)$ are positive since the linearized Hamiltonian in (2.12) is positive at eigenfunctions of the continuous spectrum of (5.1) [C01]. Let $a_{\pm}(k) = \lim_{x \rightarrow \pm\infty} u_{n,n}(x, k) e^{ikx}$ in a diagonal representation of the eigenfunctions $\mathbf{u}_n(x, k)$. Then, spectral densities $l_n(k)$ can be computed from (5.14) in an explicit form:

$$l_n(k) = \frac{\pi(|a_+(k)|^2 + |a_-(k)|^2)}{2(\beta_n + d_n k^2)} > 0. \quad (5.15)$$

□

Corollary 5.10 *The set of eigenfunctions $\{\mathbf{u}_m(x), (\mathbf{u}_{Rk}(x), \mathbf{u}_{Ik}(x)), \mathbf{u}_n(x; k)\}$ is also orthogonal with respect to operators \mathcal{L}_1 .*

Proposition 5.11 *Let Hypothesis 2.24 be satisfied. There exists a basis in $X_c^{(u)}(\mathbb{R})$, which orthogonally diagonalizes both quadratic forms $\langle \mathbf{u} | \mathcal{L}_1 \mathbf{u} \rangle$ and $\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle$, with respect to operators \mathcal{L}_0^{-1} and Jordan blocks (5.5).*

Proof. The constrained Hilbert space $X_c^{(u)}(\mathbb{R})$ is complete according to Proposition 4.1. Any functions $\mathbf{f}_u \in X_c^{(u)}(\mathbb{R})$, $\mathbf{f}_w \in X_c^{(w)}(\mathbb{R})$ can be expanded according to the spectral decomposition (2.31) of Hypothesis 2.24. If $\mathbf{f}_u \in \mathbb{R}^N$, the real-valued decomposition (2.31) takes the form,

$$\mathbf{f}_u(x) = \sum_k [a_k \mathbf{u}_{Rk}(x) + b_k \mathbf{u}_{Ik}(x)] + \sum_m c_m \mathbf{u}_m(x) + \sum_{n=1}^N \int_{-\infty}^{\infty} d_n(k) \mathbf{u}_n(x; k) dk. \quad (5.16)$$

The real-valued quadratic forms for \mathcal{L}_1 and \mathcal{L}_0^{-1} are simultaneously diagonalized according to explicit computations,

$$\langle \mathbf{f}_u | \mathcal{L}_1 \mathbf{f}_u \rangle = \sum_k \mathbf{a}_k^T \hat{\gamma}_k \hat{l}_k \mathbf{a}_k + \sum_m \gamma_m l_m |c_m|^2 + \sum_{n=1}^N \int_{-\infty}^{\infty} \gamma_n(k) l_n(k) |d_n(k)|^2 dk, \quad (5.17)$$

$$\langle \mathbf{f}_u | \mathcal{L}_0^{-1} \mathbf{f}_u \rangle = \sum_k \mathbf{a}_k^T \hat{l}_k \mathbf{a}_k + \sum_m l_m |c_m|^2 + \sum_{n=1}^N \int_{-\infty}^{\infty} l_n(k) |d_n(k)|^2 dk, \quad (5.18)$$

where $\gamma_n(k) = (\beta_n + d_n k^2)^2$, $\mathbf{a}_k = (a_k, b_k)^T$ and the Jordan blocks $\hat{\gamma}_k$ and \hat{l}_k are defined by (5.5). □

Remark 5.12 Eigenvalues γ of the diagonalization problem (5.1) correspond to three different types of eigenvalues λ of the linear stability problem (2.15). When $\gamma = \gamma_m > 0$, the linear problem (2.15) has two pure imaginary eigenvalues λ which are weakly spectrally stable. When $\gamma = \gamma_m < 0$, the linear problem (2.15) has two real eigenvalues λ , including an unstable (real and positive) eigenvalue. When $\gamma = \gamma_k$ is complex, the linear problem (2.15) has four complex eigenvalues λ , including two unstable eigenvalues with $\text{Re}(\lambda) > 0$. We trace the unstable eigenvalues λ of the stability problem (2.15) from negative and complex eigenvalues γ of the diagonalization problem (5.1). The unstable subspace of (2.15) is controlled by finite-dimensional negative subspaces of operators \mathcal{L}_1 and \mathcal{L}_0 .

Proposition 5.13 *Let \mathcal{L} be a symmetric matrix Schrödinger operator in Hilbert space $X_c^{(u)}(\mathbb{R})$, either \mathcal{L}_1 or \mathcal{L}_0^{-1} , which has a continuous spectrum bounded from below by $\lambda \geq \lambda_0 \geq 0$ and a finite-dimensional discrete spectrum for $\lambda_{\min} < \lambda < \lambda_0$ with negative index $\#_{<0}(\mathcal{L})$. If there exist another basis that diagonalizes the quadratic form $\langle \mathbf{f}_u | \mathcal{L} \mathbf{f}_u \rangle$ with respect to operators \mathcal{L} and Jordan blocks (5.5), the negative index of the quadratic form $\langle \mathbf{f}_u | \mathcal{L} \mathbf{f}_u \rangle$ remains invariant as $\#_{<0}(\mathcal{L})$.*

Proof. Suppose the operator \mathcal{L} is orthogonally diagonalized by the basis $S_u = E_u^- \wedge E_u^+$, where E_u^- is the negative subspace spanned by eigenfunctions $\{\mathbf{u}_m^-\}_{m=1}^{M_u}$ such that $\lambda_m = \langle \mathbf{u}_m^- | \mathcal{L} \mathbf{u}_m^- \rangle < 0$ and $\#_{<0}(S_u) = M_u$, and E_u^+ is the non-negative (discrete and continuous) subspace spanned by eigenfunctions \mathbf{u}^+ such that $\langle \mathbf{u}^+ | \mathcal{L} \mathbf{u}^+ \rangle \geq 0$. Suppose the operator \mathcal{L} is orthogonally diagonalized with respect to operators \mathcal{L} and Jordan blocks (5.5) by another basis $S_v = E_v^c \wedge E_v^- \wedge E_v^+$, where E_v^- is the real negative subspace spanned by eigenfunctions $\{\mathbf{v}_m^-\}_{m=1}^{M_v}$, such that $l_m = \langle \mathbf{v}_m^- | \mathcal{L} \mathbf{v}_m^- \rangle < 0$, E_v^c is the complex subspace spanned by eigenfunctions $\{\mathbf{v}_{Rk}, \mathbf{v}_{Ik}\}_{k=1}^{K_v}$ such that $l_{Rk} = \langle \mathbf{v}_{Rk} | \mathcal{L} \mathbf{v}_{Rk} \rangle$, $l_{Ik} = \langle \mathbf{v}_{Ik} | \mathcal{L} \mathbf{v}_{Rk} \rangle$, and E_v^+ is the real non-negative (discrete and continuous) subspace spanned by eigenfunctions \mathbf{v}^+ such that $\langle \mathbf{v}^+ | \mathcal{L} \mathbf{v}^+ \rangle \geq 0$. The Jordan block \hat{l}_k in (5.5) has one positive and one negative eigenvalue, $l_{\pm k} = \pm \sqrt{l_{Rk}^2 + l_{Ik}^2}$, such that $\#_{<0}(S_v) = M_v + K_v$. The eigenvectors $\mathbf{v}_{Rk}(x)$ and $\mathbf{v}_{Ik}(x)$ can be orthogonalized with respect to operator \mathcal{L} in linear combinations:

$$\mathbf{v}_{\pm k} = l_{Ik} \mathbf{v}_{Rk}(x) + (l_{\pm k} - l_{Rk}) \mathbf{v}_{Ik}(x).$$

Suppose that $(M_v + K_v) > M_u$ and show that this is false. The case $(M_v + K_v) < M_u$ can be treated similarly. Consider a function $\mathbf{g}_u(x)$ given by

$$\mathbf{g}_u(x) = \sum_{k=1}^{K_v} a_k \mathbf{v}_{-k}(x) + \sum_{m=1}^{M_v} c_m \mathbf{v}_m^-(x) + \sum c_+ \mathbf{u}^+(x), \quad (5.19)$$

where the last sum includes both summation over discrete spectrum and integration over the continuous spectrum of \mathcal{L} . The eigenfunctions $\mathbf{v}_{-k}(x)$ and $\mathbf{v}_m^-(x)$ can be decomposed over the basis of S_u :

$$\mathbf{v}_{-k}(x) = \sum_{j=1}^{M_u} \alpha_{kj} \mathbf{u}_j^-(x) + \sum \alpha_k \mathbf{u}^+(x), \quad (5.20)$$

$$\mathbf{v}_m^-(x) = \sum_{j=1}^{M_u} \gamma_{mj} \mathbf{u}_j^-(x) + \sum \gamma_m \mathbf{u}^+(x). \quad (5.21)$$

Therefore, the function $\mathbf{g}_u(x)$ is decomposed as:

$$\mathbf{g}_u(x) = \sum_{j=1}^{M_u} \left(\sum_{k=1}^{K_v} \alpha_{kj} a_k + \sum_{m=1}^{M_v} \gamma_{mj} c_m \right) \mathbf{u}_j^-(x) + \sum \left(c_+ + \sum_{k=1}^{K_v} \alpha_k a_k + \sum_{m=1}^{M_v} \gamma_m c_m \right) \mathbf{u}^+(x). \quad (5.22)$$

Consider a particular case $\mathbf{g}_u(x) = \mathbf{0}$. Since the set S_u is complete, then

$$c_+ + \sum_{k=1}^{K_v} \alpha_k a_k + \sum_{m=1}^{M_v} \gamma_m c_m = 0, \quad (5.23)$$

$$\sum_{k=1}^{K_v} \alpha_{kj} a_k + \sum_{m=1}^{M_v} \gamma_{mj} c_m = 0, \quad j = 1, \dots, M_u. \quad (5.24)$$

The second system (5.24) is under-determined, i.e. $(M_v + K_v - M_u)$ unknowns (a_k, c_m) are arbitrary. Therefore, there exists a non-zero solution of (5.23)–(5.24) such that $(a_k, c_m) \neq 0$ for some values of (k, m) . As a result, a non-zero vector $\mathbf{s}_u(x)$ is defined by (5.19) with $\mathbf{g}_u(x) = \mathbf{0}$:

$$\mathbf{s}_u(x) = \sum_{k=1}^{K_v} a_k \mathbf{v}_{-k}(x) + \sum_{m=1}^{M_v} c_m \mathbf{v}_m^-(x) = - \sum c_+ \mathbf{u}^+(x).$$

Therefore, the quadratic form $\langle \mathbf{s}_u | \mathcal{L} \mathbf{s}_u \rangle$ can be bounded by two contradictory ways:

$$\begin{aligned} \langle \mathbf{s}_u | \mathcal{L} \mathbf{s}_u \rangle &= - \sum_{k=1}^{K_v} 2 (l_{Rk}^2 + l_{Ik}^2) \left(\sqrt{l_{Rk}^2 + l_{Ik}^2} + l_{Rk} \right) |a_k|^2 + \sum_{m=1}^{M_v} l_m |c_m|^2 < 0, \\ \langle \mathbf{s}_u | \mathcal{L} \mathbf{s}_u \rangle &= \sum \langle \mathbf{u}^+ | \mathcal{L} \mathbf{u}^+ \rangle |c_+|^2 \geq 0. \end{aligned}$$

The contradiction is resolved if and only if $(M_v + K_v) = M_u$. □

Remark 5.14 Proposition 5.13 generalizes the inertial theorem for quadratic forms associated with finite-dimensional (matrix) operators [G61]. We use this result in order to determine sharp bounds on the number of negative and complex eigenvalues γ of the diagonalization problem (5.1) from the number of negative eigenvalues of the self-adjoint operators \mathcal{L}_1 and \mathcal{L}_0^{-1} in the constrained Hilbert space $X_c^{(u)}(\mathbb{R})$.

Corollary 5.15 *Suppose the diagonalization problem (5.1) has N_{comp} complex eigenvalues $\gamma = \gamma_k$ and no degenerate eigenvalues of higher algebraic multiplicity. Suppose the constrained eigenvalue problems (4.1) and (4.2) have $\#_{<0}(\mathcal{L}_0)$ and $\#_{<0}(\mathcal{L}_1)$ negative eigenvalues λ . Then $N_{\text{comp}} \leq \min(\#_{<0}(\mathcal{L}_0), \#_{<0}(\mathcal{L}_1))$ and there exists $(\#_{<0}(\mathcal{L}_1) - N_{\text{comp}})$ eigenfunctions $\mathbf{u}_m(x)$ in the problem (5.1) such that $\langle \mathbf{u}_m | \mathcal{L}_1 \mathbf{u}_m \rangle < 0$ and $(\#_{<0}(\mathcal{L}_0) - N_{\text{comp}})$ eigenfunctions $\mathbf{u}_m(x)$ such that $\langle \mathbf{u}_m | \mathcal{L}_0^{-1} \mathbf{u}_m \rangle < 0$.*

Remark 5.16 The constrained problems (4.1) and (4.2) have a common set of eigenfunctions if operators \mathcal{L}_1 and \mathcal{L}_0 commute. Let $\lambda = \xi_m$ be an eigenvalue of (4.1) and $\lambda = \eta_m$ be an eigenvalue of (4.2), with the same eigenfunction $\mathbf{u}_m(x)$. There exists an eigenvalue $\gamma = \gamma_m$ of the problem (5.1), such that

$$\gamma_m = \eta_m \xi_m = \eta_m \frac{\langle \mathbf{u}_m | \mathcal{L}_0 \mathbf{u}_m \rangle}{\langle \mathbf{u}_m | \mathbf{u}_m \rangle} = \eta_m \frac{\langle \mathbf{u}_m | \mathbf{u}_m \rangle}{\langle \mathbf{u}_m | \mathcal{L}_0^{-1} \mathbf{u}_m \rangle}. \quad (5.25)$$

This formula was assumed in [PK00] to approximate $\lambda^2 = -\gamma_m$ from the given solution of the constrained problem (4.2). The approximation formula (5.25) is not exact for non-commuting operators \mathcal{L}_1 and \mathcal{L}_0 , as in our case.

We now prove Theorems 3.8, 3.9, and Corollary 3.10. It follows from Corollary 5.15 that

$$\#_{<0}(\mathcal{L}_1) = N_{\text{comp}} + \#_{<0}(\gamma_m l_m), \quad (5.26)$$

$$\#_{<0}(\mathcal{L}_0) = N_{\text{comp}} + \#_{<0}(l_m). \quad (5.27)$$

Taking the sum of (5.26) and (5.27), we find that

$$\#_{<0}(\mathcal{L}_1) + \#_{<0}(\mathcal{L}_0) = 2N_{\text{comp}} + \#_{<0}(\gamma_m l_m) + \#_{<0}(l_m) = 2N_{\text{comp}} + 2N_{\text{imag}}^- + N_{\text{real}},$$

which is the closure relation (3.1) in Theorem 3.8 (with the account of Theorem 3.6).

Taking the difference of (5.26) and (5.27), we find that

$$|\#_{<0}(\mathcal{L}_1) - \#_{<0}(\mathcal{L}_0)| = |\#_{<0}(\gamma_m l_m) - \#_{<0}(l_m)| \leq N_{\text{real}} \leq N_{\text{unst}},$$

which is the lower bound (3.2) in Theorem 3.9 (with the account of Theorem 3.6). The upper bound in (3.2) is a corollary of Theorem 3.8.

At least $|\#_{<0}(\mathcal{L}_1) - \#_{<0}(\mathcal{L}_0)|$ eigenvalues γ are negative. The remaining $2 \min(\#_{<0}(\mathcal{L}_1), \#_{<0}(\mathcal{L}_0))$ eigenvalues γ contribute to either complex (N_{comp}), real negative (N_{real}) or real positive (N_{imag}^-) eigenvalues of the diagonalization problem (5.1). If $\#_{<0}(\mathcal{L}_0) = 0$ or $\#_{<0}(\mathcal{L}_1) = 0$, then $N_{\text{comp}} = N_{\text{imag}}^- = 0$, and all unstable (real positive) eigenvalues λ are given by $\max(\#_{<0}(\mathcal{L}_1), \#_{<0}(\mathcal{L}_0))$ negative eigenvalues γ .

6 Example of two coupled NLS equations

Here we apply the computational algorithm described in Section 3 to analysis of stability and instability of optical solitons in a system of two incoherently coupled NLS equations,

$$\begin{aligned} i\psi_{1z} + \psi_{1xx} + (|\psi_1|^2 + \chi|\psi_2|^2) \psi_1 &= 0, \\ i\psi_{2z} + \psi_{2xx} + (\chi|\psi_1|^2 + |\psi_2|^2) \psi_2 &= 0. \end{aligned} \quad (6.1)$$

The system is a particular example of (1.1) with $N = 2$, $d_1 = d_2 = 1$, and

$$U = \frac{1}{2}|\psi_1|^4 + \chi|\psi_1|^2|\psi_2|^2 + \frac{1}{2}|\psi_2|^4. \quad (6.2)$$

The system (6.1) has several families of stationary solutions (2.1) with different values of the nodal index $\mathbf{i} = (i_1, i_2)^T$ [Y97]. We consider only a particular n -family of multiple pulse solutions with nodal index $\mathbf{i} = (0, n)^T$, which is locally close to the fundamental NLS soliton, $\Phi_{\text{NLS}}(x) = (\Phi_{\text{NLS}}, 0)^T$, where

$$\Phi_{\text{NLS}}(x) = \sqrt{\beta_1} \Phi^{(0)}(X), \quad X = \sqrt{\beta_1} x, \quad \Phi^{(0)}(X) = \sqrt{2} \operatorname{sech}(X). \quad (6.3)$$

Existence of such solutions has been studied asymptotically and numerically [Y97, PY00]. We will show here that the negative index of the linearized Hamiltonian h in (2.12) at $\epsilon = 0$ is exactly $2n$ and there are exactly $2n$ complex unstable eigenvalues λ in the stability problem (2.15) for $\epsilon \neq 0$, under some constraints.

At $\epsilon = 0$, the matrix operators \mathcal{L}_0 and \mathcal{L}_1 in (2.13) and (2.14) become decoupled as $\mathcal{L}_0 = \beta_1 \operatorname{diag}(L_0, L_s)$ and $\mathcal{L}_1 = \beta_1 \operatorname{diag}(L_1, L_s)$, where L_0 , L_1 , and L_s are scalar Schrödinger operators in variable $X = \sqrt{\beta_1} x$:

$$L_0 = -\frac{d^2}{dX^2} + 1 - 2 \operatorname{sech}^2(X), \quad (6.4)$$

$$L_1 = -\frac{d^2}{dX^2} + 1 - 6 \operatorname{sech}^2(X), \quad (6.5)$$

$$L_s = -\frac{d^2}{dX^2} + \lambda_n^2(\chi) - 2\chi \operatorname{sech}^2(X), \quad (6.6)$$

where $\beta_2 = \beta_1 \lambda_n^2(\chi)$ at $\epsilon = 0$, and $\lambda_n^2(\chi)$ is given by (6.8) below.

Lemma 6.1 *Let $\chi_n = n(n+1)/2$ and $C_n = C_n(\chi)$ be defined by (6.14) below. There exists some $R > 0$ if $\chi > \chi_n$ and $C_n(\chi) \neq 0$ such that the n -family of stationary solutions $\Phi(x) = (\Phi_1, \Phi_2)^T$ with the nodal index $\mathbf{i} = (0, n)^T$ bifurcates from $\Phi_{\text{NLS}} = (\Phi_{\text{NLS}}, 0)^T$ in the one-sided domain \mathcal{B}_n :*

$$\mathcal{B}_n = \left\{ \frac{\beta_2}{\beta_1} : 0 < \left| \frac{\beta_2}{\beta_1} - \lambda_n^2(\chi) \right| < R, \quad \operatorname{sign} \left(\frac{\beta_2}{\beta_1} - \lambda_n^2(\chi) \right) = \operatorname{sign}(C_n(\chi)) \right\}, \quad (6.7)$$

where

$$\lambda_n(\chi) = \frac{\sqrt{1 + 8\chi} - (2n + 1)}{2}. \quad (6.8)$$

Proof. Define the perturbation series expansions of stationary solutions $\Phi = (\Phi_1, \Phi_2)^T$ in variable $X = \sqrt{\beta_1} x$,

$$\begin{aligned} \Phi_1(x) &= \sqrt{\beta_1} \left[\Phi^{(0)}(X) + \epsilon^2 \Phi^{(2)}(X) + \mathcal{O}(\epsilon^4) \right], \\ \Phi_2(x) &= \sqrt{\beta_1} \left[\epsilon \Phi^{(1)}(X) + \epsilon^3 \Phi^{(3)}(X) + \mathcal{O}(\epsilon^5) \right], \end{aligned} \quad (6.9)$$

where ϵ is a function of β_2/β_1 , defined by the expansion,

$$\frac{\beta_2}{\beta_1} = \lambda_n^2(\chi) + \epsilon^2 C_n(\chi) + \mathcal{O}(\epsilon^4). \quad (6.10)$$

Direct computations from (2.2), (6.9), and (6.10) result in a set of linear problems in powers of ϵ ,

$$L_s \Phi^{(1)} = 0, \quad (6.11)$$

$$L_1 \Phi^{(2)} = \chi \Phi^{(0)} \left(\Phi^{(1)} \right)^2, \quad (6.12)$$

$$L_s \Phi^{(3)} = -C_n(\chi) \Phi^{(1)} + 2\chi \Phi^{(0)} \Phi^{(1)} \Phi^{(2)} + \left(\Phi^{(1)} \right)^3, \quad (6.13)$$

where L_1 and L_s are given by (6.5) and (6.6). The problem (6.11) always has a bound state $\Phi^{(1)} = \Phi_n^{(1)}(X)$ that satisfies the boundary conditions (2.3) [Y97]. When $n = 0$ and $\chi > 0$, the bound state $\Phi_0^{(1)} = \text{sech}^s(X)$, $s = \lambda_0(\chi)$ is a fundamental soliton. When $n > 0$ and $\chi > \chi_n = n(n+1)/2$, the bound state $\Phi_n^{(1)}(X)$ has exactly n nodes for $x \in \mathbb{R}$. The problem (6.12) has a bounded solution $\Phi^{(2)}(X)$, since the right-hand-side $\chi \Phi^{(0)} \left(\Phi_n^{(1)} \right)^2$ is orthogonal to the kernel of the operator \mathcal{L}_1 , which is $\Phi^{(0)'}(x)$. The problem (6.13) has a bounded solution if and only if the right-hand-side is orthogonal to the kernel of \mathcal{L}_s , which is $\Phi_n^{(1)}(x)$. The orthogonality condition defines the parameter $C_n(\chi)$ in the form:

$$C_n(\chi) = \frac{\left\langle \left(\Phi_n^{(1)} \right)^2 \mid \left(2\chi \Phi^{(0)} \Phi^{(2)} + \left(\Phi_n^{(1)} \right)^2 \right) \right\rangle}{\left\langle \Phi_n^{(1)} \mid \Phi_n^{(1)} \right\rangle}, \quad (6.14)$$

where the inner product (1.12) is used for scalar functions in $L^2(\mathbb{R})$. The condition $C_n(\chi) \neq 0$ gives the necessary condition of existence of the n -family of soliton solutions in the one-sided domain (6.7). The regular perturbation series expansions (6.9) and (6.10) converges in a local neighborhood of $\Phi_{\text{NLS}}(X)$ and $\lambda_n^2(\chi)$ as ϵ is sufficiently small for $C_n(\chi) \neq 0$. \square

Remark 6.2 The n -family of stationary solutions $\Phi = (\Phi_1, \Phi_2)^T$ bifurcates from $\Phi_{\text{NLS}} = (\Phi_{\text{NLS}}, 0)^T$ due to existence of the bound state $\Phi_n^{(1)}(X)$ of the Schrödinger operator \mathcal{L}_s . More general families of multiple pulse solutions were classified for $n = 0, 1, 2$ in [Y97]. It was shown numerically that the n -family of stationary solutions $\Phi = (\Phi_1, \Phi_2)^T$ exists in the interval $\lambda_n^2(\chi) < \beta_2/\beta_1 < 1$ for $\chi_n < \chi < \chi_{n+1}$ and in the interval $1 < \beta_2/\beta_1 < \lambda_n^2(\chi)$ for $\chi > \chi_{n+1}$. This numerical result suggests that $C_n(\chi) > 0$ for $\chi_n < \chi < \chi_{n+1}$ and $C_n(\chi) < 0$ for $\chi > \chi_{n+1}$. The case $\chi = \chi_{n+1}$, where $C_n(\chi_{n+1}) = 0$, is special and is excluded from Lemma 6.1. We apply the computational stability-instability algorithm for a generic case $C_n(\chi) \neq 0$ and evaluate indices $n(\mathcal{L}_1)$, $n(\mathcal{L}_0)$, $p(\mathcal{U})$, $z(\mathcal{U})$, N_{real} , and N_{comp} in the one-sided neighborhood \mathcal{B}_n , given by (6.7).

Lemma 6.3 *In the domain \mathcal{B}_n , the negative and zero indices of \mathcal{L}_1 and \mathcal{L}_0 in constrained function space $X_c^{(u)}(\mathbb{R})$ are*

$$\#_{<0}(\mathcal{L}_1) = n, \quad \#_{<0}(\mathcal{L}_0) = n, \quad \#_{=0}(\mathcal{L}_1) = 1, \quad \#_{=0}(\mathcal{L}_0) = 0. \quad (6.15)$$

Proof. Operator L_0 in (6.4) has a single zero bound state, operator L_1 in (6.5) has a single negative and single zero bound states, and operator L_s in (6.6) has n negative and single zero bound states.

Therefore, in the limit $\epsilon \rightarrow 0$, we have $n(\mathcal{L}_0) = 0 + n = n$, $z(\mathcal{L}_0) = 1 + 1 = 2$, $n(\mathcal{L}_1) = 1 + n$ and $z(\mathcal{L}_1) = 1 + 1 = 2$. It follows from Lemma 2.16 that $n(\mathcal{L}_0) = n$ and $z(\mathcal{L}_0) = 2$ for any $\epsilon \neq 0$. On the other hand, the double zero eigenvalue of \mathcal{L}_1 may split into two eigenvalues for $\epsilon \neq 0$, one of each is always zero as $z(\mathcal{L}_1) \geq 1$. We trace the other zero eigenvalue for $\epsilon \neq 0$ by the regular perturbation series,

$$\mathbf{u}(x) = \begin{bmatrix} 0 \\ \Phi_n^{(1)}(X) \end{bmatrix} + \epsilon \begin{bmatrix} u^{(1)}(X) \\ 0 \end{bmatrix} + \epsilon^2 \begin{bmatrix} 0 \\ u^{(2)}(X) \end{bmatrix} + \mathcal{O}(\epsilon^3) \quad (6.16)$$

and

$$\frac{\lambda}{\beta_1} = \epsilon^2 \lambda_2 + \mathcal{O}(\epsilon^4).$$

Corrections of the perturbation series (6.16) satisfy a set of linear non-homogeneous equations following from the problem $\mathcal{L}_1 \mathbf{u} = \lambda \mathbf{u}$ as

$$L_1 u^{(1)} = 2\chi \Phi^{(0)} \left(\Phi_n^{(1)} \right)^2, \quad (6.17)$$

$$L_s u^{(2)} = (\lambda_2 - C_n(\chi)) \Phi_n^{(1)} + 2\chi \Phi^{(0)} \Phi_n^{(1)} \left(u^{(1)} + \Phi^{(2)} \right) + 3 \left(\Phi_n^{(1)} \right)^3. \quad (6.18)$$

It follows from (6.12) and (6.17) that $u^{(1)} = 2\Phi^{(2)}$. Solutions of (6.18) exist if and only if the right-hand-side of (6.18) is orthogonal to $\Phi_n^{(1)}(X)$. Using (6.14), we find that $\lambda_2 = -2C_n(\chi)$. Therefore, for $\epsilon \neq 0$, the second zero eigenvalue of \mathcal{L}_1 becomes negative if $C_n(\chi) > 0$. Thus, we have in the domain (6.7) for sufficiently small non-zero ϵ : $n(\mathcal{L}_1) = 2 + n$ if $C_n(\chi) > 0$ and $n(\mathcal{L}_1) = 1 + n$ if $C_n(\chi) < 0$.

The indices $p(\mathcal{U})$ and $z(\mathcal{U})$ for the Hessian matrix \mathcal{U} are computed from (2.8) and (6.9):

$$\begin{aligned} \mathcal{U}_{1,1} &= \frac{\partial Q_{1s}}{\partial \beta_1} = \frac{2}{\sqrt{\beta_1}} + 2\sqrt{\beta_1} \langle \Phi^{(0)} | \Phi^{(2)} \rangle \frac{\partial \epsilon^2}{\partial \beta_1} + \mathcal{O}(\epsilon^2), \\ \mathcal{U}_{1,2} &= \frac{\partial Q_{1s}}{\partial \beta_2} = 2\sqrt{\beta_1} \langle \Phi^{(0)} | \Phi^{(2)} \rangle \frac{\partial \epsilon^2}{\partial \beta_2} + \mathcal{O}(\epsilon^2), \\ \mathcal{U}_{2,1} &= \frac{\partial Q_{2s}}{\partial \beta_1} = \sqrt{\beta_1} \langle \Phi_n^{(1)} | \Phi_n^{(1)} \rangle \frac{\partial \epsilon^2}{\partial \beta_1} + \mathcal{O}(\epsilon^2), \\ \mathcal{U}_{2,2} &= \frac{\partial Q_{2s}}{\partial \beta_2} = \sqrt{\beta_1} \langle \Phi_n^{(1)} | \Phi_n^{(1)} \rangle \frac{\partial \epsilon^2}{\partial \beta_2} + \mathcal{O}(\epsilon^2). \end{aligned}$$

It follows from (6.10) that

$$\frac{\partial \epsilon^2}{\partial \beta_2} = \frac{1}{\beta_1 C_n(\chi)} + \mathcal{O}(\epsilon^2) \quad (6.19)$$

and

$$\det(\mathcal{U}) = \frac{2 \langle \Phi_n^{(1)} | \Phi_n^{(1)} \rangle}{\beta_1 C_n(\chi)} + \mathcal{O}(\epsilon^2). \quad (6.20)$$

The matrix \mathcal{U} has two non-zero eigenvalues in the limit $\epsilon \rightarrow 0$ if and only if $C_n(\chi) \neq 0$, i.e. $z(\mathcal{U}) = 0$. It follows from (6.20) that $p(\mathcal{U}) = 2$ for $C_n(\chi) > 0$ and $p(\mathcal{U}) = 1$ for $C_n(\chi) < 0$. Using Theorem

3.6, we conclude that $\#_{<0}(\mathcal{L}_1) = n$ and $\#_{<0}(\mathcal{L}_0) = n$ in the constrained space $X_c^{(u)}(\mathbb{R})$ for any $\epsilon \neq 0$ in the domain \mathcal{B}_n . \square

Corollary 6.4 *In the domain \mathcal{B}_n , the fundamental soliton ($n = 0$) is weakly spectrally stable, while the n -family of stationary solutions may have at most N_{unst} real or complex unstable eigenvalues, where $0 \leq N_{\text{unst}} \leq 2n$.*

Lemma 6.5 *Let $g_{k,n}^\pm$ be defined by (6.29). In the domain \mathcal{B}_n , the n -family of stationary solutions $\Phi(x) = (\Phi_1, \Phi_2)^T$ is spectrally unstable for $n > 0$ and the linearized problem (2.15) has exactly n pairs of complex eigenvalues λ with $\text{Re}(\lambda) > 0$ if $|g_{k,n}^+|^2 + |g_{k,n}^-|^2 \neq 0$ for any $0 \leq k < n$.*

Proof. The linearized problem (2.15) has exactly n embedded eigenvalues at $\epsilon = 0$, such that $\text{Re}(\lambda) = 0$ and $|\text{Im}(\lambda)| > \beta_{\min} = \min(\beta_1, \beta_2)$ [PY00]. This follows from decoupling the linearized problem (2.15) at $\epsilon = 0$ into two systems,

$$\beta_1 L_1 u_1 = -\lambda w_1, \quad \beta_1 L_0 w_1 = \lambda u_1 \quad (6.21)$$

and

$$\beta_1 L_s(u_2 \pm iw_2) = \pm i\lambda(u_2 \pm iw_2). \quad (6.22)$$

The first problem (6.21) has the continuous spectrum in the domain $\text{Re}(\lambda) = 0$ and $|\text{Im}(\lambda)| \geq \beta_1$ and the zero eigenvalue $\lambda = 0$ of algebraic multiplicity 4 and geometric multiplicity 2 [KS98]. The second problem (6.22) has the continuous spectrum in the domain $\text{Re}(\lambda) = 0$ and $|\text{Im}(\lambda)| \geq \beta_1 \lambda_n^2(\chi)$ and $2n + 2$ isolated eigenvalues in the points $\lambda = \pm i\beta_1(\lambda_k^2 - \lambda_n^2)$, where $k = 0, 1, \dots, (n-1), n$. It follows from (6.8) that for $\chi > \chi_n$:

$$\lambda_k^2 - \lambda_n^2 = (n-k)(2\lambda_n(\chi) + (n-k)) > (n-k)^2 > 1, \quad 0 \leq k < n. \quad (6.23)$$

Therefore, $2n$ isolated eigenvalues of the problem (6.22) belong to the continuous spectrum of the problem (6.21). We show that all embedded eigenvalues of the linearized stability problem (2.15) bifurcate in the complex plane of λ for $\epsilon \neq 0$ under the constraint $|g_{k,n}^+|^2 + |g_{k,n}^-|^2 \neq 0$ for any $0 \leq k < n$. We trace a particular embedded eigenvalue $\lambda = i\beta_1(\lambda_k^2 - \lambda_n^2)$, $0 \leq k < n$ with the regular perturbation series in the domain \mathcal{B}_n ,

$$\mathbf{u}(x) = \begin{bmatrix} 0 \\ \Phi_k^{(1)}(X) \end{bmatrix} + \epsilon \begin{bmatrix} u_k^{(1)}(X) \\ 0 \end{bmatrix} + \epsilon^2 \begin{bmatrix} 0 \\ u_k^{(2)}(X) \end{bmatrix} + \mathcal{O}(\epsilon^3), \quad (6.24)$$

$$\mathbf{w}(x) = \begin{bmatrix} 0 \\ -i\Phi_k^{(1)}(X) \end{bmatrix} + \epsilon \begin{bmatrix} w_k^{(1)}(X) \\ 0 \end{bmatrix} + \epsilon^2 \begin{bmatrix} 0 \\ w_k^{(2)}(X) \end{bmatrix} + \mathcal{O}(\epsilon^3), \quad (6.25)$$

and

$$\frac{\lambda}{\beta_1} = \lambda_k^{(0)} + \epsilon^2 \lambda_k^{(2)} + \mathcal{O}(\epsilon^4), \quad \lambda_k^{(0)} = i(\lambda_k^2 - \lambda_n^2). \quad (6.26)$$

Corrections of the perturbation series (6.24) and (6.25) satisfy a set of linear non-homogeneous equations following from the linearized problem (2.15):

$$\begin{aligned} L_1 u_k^{(1)} + \lambda_k^{(0)} w_k^{(1)} &= 2\chi \Phi^{(0)} \Phi_n^{(1)} \Phi_k^{(1)}, \\ L_0 w_k^{(1)} - \lambda_k^{(0)} u_k^{(1)} &= 0 \end{aligned} \quad (6.27)$$

and

$$\begin{aligned} L_s u_k^{(2)} + \lambda_k^{(0)} w_k^{(2)} &= - \left(C_n(\chi) - 2\chi \Phi^{(0)} \Phi^{(2)} - 3 \left(\Phi_n^{(1)} \right)^2 \right) \Phi_k^{(1)} + 2\chi \Phi^{(0)} \Phi_n^{(1)} u_k^{(1)} + i\lambda_k^{(2)} \Phi_k^{(1)}, \\ L_s w_k^{(2)} - \lambda_k^{(0)} u_k^{(2)} &= i \left(C_n(\chi) - 2\chi \Phi^{(0)} \Phi^{(2)} - \left(\Phi_n^{(1)} \right)^2 \right) \Phi_k^{(1)} + \lambda_k^{(2)} \Phi_k^{(1)}. \end{aligned} \quad (6.28)$$

Since the eigenvalue $\lambda_k^{(0)} = i(\lambda_k^2 - \lambda_n^2)$ belongs to the continuous spectrum of the problem (6.27) for $0 \leq k < n$, the correction terms $(u_k^{(1)}, w_k^{(1)})$ have non-vanishing tails in the limit $|X| \rightarrow \infty$. We add Sommerfeld radiation conditions to uniquely determine the correction terms $(u_k^{(1)}, w_k^{(1)})$:

$$\lim_{X \rightarrow \pm\infty} \begin{pmatrix} u_k^{(1)} \\ w_k^{(1)} \end{pmatrix} e^{\pm i\kappa_{k,n} X} = \begin{pmatrix} 1 \\ i \end{pmatrix} g_{k,n}^{\pm}, \quad \lim_{X \rightarrow \pm\infty} \begin{pmatrix} u_k^{(1)} \\ w_k^{(1)} \end{pmatrix} e^{\mp i\kappa_{k,n} X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (6.29)$$

where $g_{k,n}^{\pm}$ are some constants and $\kappa_{k,n} = \sqrt{\lambda_k^2 - \lambda_n^2} - 1$ (> 0). Since correction terms $\Phi^{(0)}$, $\Phi_n^{(1)}$, and $\Phi^{(2)}$ are all real-valued, we find from (6.27) that

$$\begin{aligned} 2\chi \text{Im} \langle u_k^{(1)} | \Phi^{(0)} \Phi_n^{(1)} \Phi_k^{(1)} \rangle &= \frac{1}{2i} \left(\langle u_k^{(1)} | L_1 u_k^{(1)} \rangle + \langle w_k^{(1)} | L_0 w_k^{(1)} \rangle - \langle L_1 u_k^{(1)} | u_k^{(1)} \rangle - \langle L_0 w_k^{(1)} | w_k^{(1)} \rangle \right) \\ &= 2\kappa_{k,n} \left(|g_{k,n}^+|^2 + |g_{k,n}^-|^2 \right) \geq 0. \end{aligned} \quad (6.30)$$

Solutions of the problem (6.28) are bounded at infinity if and only if the right-hand-side of (6.28) is orthogonal to $\Phi_k^{(1)}(X)$. The orthogonality condition defines $\lambda_k^{(2)}$ as

$$i\lambda_k^{(2)} \langle \Phi_k^{(1)} | \Phi_k^{(1)} \rangle + \chi \langle \Phi^{(0)} \Phi_n^{(1)} \Phi_k^{(1)} | u_k^{(1)} \rangle = \langle \left(\Phi_k^{(1)} \right)^2 | \left(C_n(\chi) - 2\chi \Phi^{(0)} \Phi^{(2)} - 2 \left(\Phi_n^{(1)} \right)^2 \right) \rangle, \quad (6.31)$$

such that

$$\text{Re} \left(\lambda_k^{(2)} \right) = \chi \frac{\text{Im} \langle u_k^{(1)} | \Phi^{(0)} \Phi_n^{(1)} \Phi_k^{(1)} \rangle}{\langle \Phi_k^{(1)} | \Phi_k^{(1)} \rangle} = \frac{\kappa_{k,n} \left(|g_{k,n}^+|^2 + |g_{k,n}^-|^2 \right)}{\langle \Phi_k^{(1)} | \Phi_k^{(1)} \rangle} \geq 0. \quad (6.32)$$

Under the constraint $|g_{k,n}^+|^2 + |g_{k,n}^-|^2 \neq 0$ for $0 \leq k < n$, all embedded eigenvalues $\lambda = \beta_1 \lambda_k^{(0)}$, $0 \leq k < n$ become unstable complex eigenvalue λ of the linearized problem (2.15) such that $\text{Re}(\lambda) > 0$ in the domain \mathcal{B}_n . Convergence of perturbation series expansions (6.24)–(6.26) can be proved with the Gap Lemma [KS98, KS02] for analytical continuations of the Evans function associated with the linearized problem (2.15) for exponentially decaying potentials. \square

Remark 6.6 Embedded eigenvalues λ of the linearized problem (2.15) do not have to bifurcate to the complex plane of λ . The example of such persistent embedded eigenvalues is given by a system of two coupled NLS equations (1.1) with $N = 2$, $d_1 = d_2 = 1$, and $U = U(|\psi_1|^2 + |\psi_2|^2)$. Due to additional symmetry of the potential function U , the linearized problem (2.15) with stationary solution $\Phi = (\Phi_1, \Phi_2)^T$ for any $\beta = \text{diag}(\beta_1, \beta_2)$ always has a particular exact solution,

$$\mathbf{u} = \begin{pmatrix} -\Phi_2 \\ \Phi_1 \end{pmatrix} \quad \mathbf{w} = \mp i \begin{pmatrix} \Phi_2 \\ \Phi_1 \end{pmatrix} \quad \lambda = \pm i(\beta_1 - \beta_2). \quad (6.33)$$

If $\beta_1 \neq \beta_2$ and $|\beta_2 - \beta_1| \geq \beta_{\min} = \min(\beta_1, \beta_2)$, the pair of eigenvalues $\lambda = \pm i(\beta_1 - \beta_2)$ belong to the embedded spectrum of the operator \mathcal{A} . The embedded eigenvalues trap the negative index of the linearized Hamiltonian h in (2.12) if

$$\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle = -\frac{\langle \Phi_1 | \Phi_1 \rangle - \langle \Phi_2 | \Phi_2 \rangle}{\beta_1 - \beta_2} < 0. \quad (6.34)$$

When

$$U = \frac{1}{s} (|\psi_1|^2 + |\psi_2|^2) - \frac{1}{s^2} \log(1 + s(|\psi_1|^2 + |\psi_2|^2)), \quad (6.35)$$

the system of coupled NLS equations (1.1) describes optical solitons in photorefractive saturable crystals with saturation parameter s [DN02]. It was shown numerically [OK99] that the multiple pulse solutions $\Phi = (\Phi_1, \Phi_2)^T$ with nodal index $\mathbf{i} = (0, n)^T$ bifurcate from single-component solutions $\Phi = (\Phi_1, 0)^T$ for $n = 1, 2$ and $s < 1$ in the domain $\beta_{\text{bif}} < \beta_2/\beta_1 < 1$, where $\beta_{\text{bif}} > 0$. At the bifurcation point, the eigenvalues $\lambda = \pm i(\beta_1 - \beta_2)$ always correspond to the negative value of $\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle$ in (6.34). In addition, these eigenvalues are embedded into continuous spectrum of the problem (2.15) if $\beta_2/\beta_1 \leq 0.5$. Since the embedded eigenvalues are preserved in the embedded spectrum of \mathcal{A} due to the exact solution (6.33) and trap the negative index of the linearized Hamiltonian h in (2.12) with $2N_{\text{imag}}^- = 2$, the stationary solutions $\Phi = (\Phi_1, \Phi_2)^T$ with nodal index $\mathbf{i} = (0, 1)^T$ and Morse index $\#_{<0}(h) = 2$ are weakly spectrally stable in the neighborhood of the bifurcation point $\beta_2/\beta_1 = \beta_{\text{bif}}$, in agreement with numerical results in [OK99]. The stationary solutions $\Phi = (\Phi_1, \Phi_2)^T$ with nodal index $\mathbf{i} = (0, 2)^T$ and Morse index $\#_{<0}(h) = 4$ are also weakly spectrally stable in the neighborhood of the bifurcation point $\beta_2/\beta_1 = \beta_{\text{bif}}$, according to numerical results in [OK99], because two eigenvalues $\lambda = \pm i(\beta_1 - \beta_2)$ with negative values of h are preserved by (6.33) and the other two eigenvalues λ with negative values of h are not embedded into continuous spectrum of \mathcal{A} , such that $2N_{\text{imag}}^- = 4$.

7 Bifurcations of unstable eigenvalues

Bifurcations of unstable eigenvalues may occur in the linearized stability problem (2.15), when operators \mathcal{L}_1 and \mathcal{L}_0 change along to a continuous deformation and support non-generic eigenvalues λ such that $\text{Re}(\lambda) = 0$. Three non-generic cases are excluded from Theorems 3.8 and 3.9: (i) $z(\mathcal{L}_1) \neq 1$, (ii) $z(\mathcal{U}) \neq 0$, and (iii) non-zero multiple eigenvalues λ of higher algebraic multiplicity.

In a generic case, the index $\#_{<0}(h) = n(\mathcal{L}_1) - p(\mathcal{U}) + n(\mathcal{L}_0)$ changes in the first two cases, but it remains constant in the third case. We classify the three non-generic cases as (i) zero eigenvalue bifurcation, (ii) symmetry bifurcation, and (iii) Hamiltonian Hopf bifurcation. Here we derive necessary conditions, when new unstable eigenvalues with $\text{Re}(\lambda) > 0$ bifurcate from the non-generic eigenvalues with $\text{Re}(\lambda) = 0$.

We note that Theorems 3.8 and 3.9 exclude also the cases when Hypothesis 2.24 is not satisfied. This restriction may exclude the cases when new unstable eigenvalues bifurcate from resonances of the continuous spectrum of operator \mathcal{A} , defined as zeros of the Evans function [KS98]. It was recently proved in [KS03] that eigenvalues bifurcating from resonances of the continuous spectrum of operators with exponentially decaying potentials have the same signature of the linearized Hamiltonian, as the spectral density of eigenfunctions of the continuous spectrum. Since the continuous spectrum of \mathcal{A} is positive definite as in Lemma 5.9 and the unstable eigenvalues have negative indices of the linearized Hamiltonian, we conjecture that no unstable eigenvalues can bifurcate from resonances of the continuous spectrum in the linearized stability problem (2.15). This issue is left open for further rigorous studies.

(i) Zero eigenvalue bifurcation

When $z(\mathcal{L}_1) \neq 1$, the null space of operator \mathcal{L}_1 is spanned by $(z(\mathcal{L}_1) - 1)$ eigenfunctions $\mathbf{u}_0(x)$ and by the eigenfunction $\Phi'(x)$. According to Lemma 4.12, the eigenfunctions of the null space of \mathcal{L}_1 belong to the constrained space $X_c^{(u)}(\mathbb{R})$ for any $z(\mathcal{L}_1) \geq 1$, if \mathcal{U} is bounded. The eigenfunction $\Phi'(x)$ always persists in the null space of \mathcal{L}_1 , since it is related to the translational symmetry (1.7) of the system (1.1). However, additional eigenfunctions $\mathbf{u}_0(x)$ of the null space of \mathcal{L}_1 , if they exist, generally move from the null space of \mathcal{L}_1 , under a continuous deformation of \mathcal{L}_1 . If the deformation changes the index $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U})$, new unstable eigenvalues λ may arise in the linearized problem (2.15). Here we study the characteristic features of this bifurcation. We consider bifurcation of co-dimension one, when $z(\mathcal{L}_1) = 2$.

Proposition 7.1 *Let ϵ be the bifurcation parameter, such that, at $\epsilon = 0$, there exists non-zero eigenfunction $\mathbf{u}_0 \in X_c^{(u)}(\mathbb{R})$, $\mathbf{u}_0 \neq \Phi'(x)$, and $\mathcal{L}_1 \mathbf{u}_0 = \mathbf{0}$. Assume that $\mathcal{L}_1(\epsilon)$ and $\mathcal{L}_0(\epsilon)$ depend analytically on ϵ . There exists $\epsilon_* > 0$ such that the linearized problem (2.15) has a real positive eigenvalue λ if $l_0 = \langle \mathbf{u}_0 | \mathcal{L}_0^{-1}(0) \mathbf{u}_0 \rangle \neq 0$ and $\delta l_1 = \langle \mathbf{u}_0 | \mathcal{L}'_1(0) \mathbf{u}_0 \rangle \neq 0$, in the domain:*

$$\mathcal{D}_0 = \{\epsilon : 0 < |\epsilon| < \epsilon_*, \quad \text{sign}(\epsilon) = -\text{sign}(l_0 \delta l_1)\}.$$

Proof. We expand solutions of (5.1) in power series of ϵ :

$$\mathbf{u}(x) = \mathbf{u}_0(x) + \epsilon \mathbf{u}_1(x) + \mathcal{O}(\epsilon^2), \quad \gamma = \epsilon \gamma_1 + \mathcal{O}(\epsilon^2). \quad (7.1)$$

The function $\mathbf{u}_1(x)$ solves the non-homogeneous problem in $X_c^{(u)}(\mathbb{R})$:

$$\mathcal{L}_1(0) \mathbf{u}_1 + \mathcal{L}'_1(0) \mathbf{u}_0 = \gamma_1 \mathcal{L}_0^{-1}(0) \mathbf{u}_0. \quad (7.2)$$

Using the Fredholm Alternative Theorem, we find from (7.2) that $\delta l_1 = \gamma_1 l_0$. If $\delta l_1 \neq 0$, $l_0 \neq 0$, and $\text{sign}(\epsilon) = -\text{sign}(l_0 \delta l_1)$, the eigenvalue γ is negative in the first order of ϵ . The power series expansions (7.1) converge as they represent regular perturbation theory for linear operators with exponentially decaying potentials [K76]. \square

Corollary 7.2 *Let \mathcal{L}_0 be positive definite, such that $l_0 > 0$. Bifurcation of a new negative eigenvalue λ in the constrained problem (4.2) always results in a new negative eigenvalue γ of the problem (5.1), such that*

$$\lim_{\epsilon \rightarrow 0} \frac{\gamma}{\lambda} = \frac{\langle \mathbf{u}_0 | \mathbf{u}_0 \rangle}{\langle \mathbf{u}_0 | \mathcal{L}_0^{-1}(0) \mathbf{u}_0 \rangle}. \quad (7.3)$$

Remark 7.3 The zero eigenvalue bifurcation may occur only if $N > 1$ in the system (1.1). This bifurcation is generic when the system (1.1) is a perturbation of uncoupled NLS equations, with $f_n = F_n(|\psi_n|^2) + \epsilon G_n(|\psi_1|^2, \dots, |\psi_N|^2)$.

(ii) Symmetry bifurcation

When $z(\mathcal{U}) \neq 0$, the null space of the constrained problem (4.2) is not empty. If a deformation changes the negative index $\#_{<0}(\mathcal{L}_1) = n(\mathcal{L}_1) - p(\mathcal{U}) - z(\mathcal{U})$, new unstable eigenvalues λ may arise in the linearized problem (2.15). Here we study the characteristic features of this bifurcation. We consider bifurcation of co-dimension one, when $z(\mathcal{U}) = 1$.

Proposition 7.4 *Let ϵ be the bifurcation parameter, such that, at $\epsilon = 0$, there exists non-zero eigenvector $\boldsymbol{\nu}_0 \in \mathbb{R}^N$, $\mathcal{U}\boldsymbol{\nu}_0 = \mathbf{0}$, and*

$$\mathcal{L}_1 \mathbf{u}_0 = - \sum_{n=1}^N (\nu_0)_n \Phi_n(x) \mathbf{e}_n, \quad \mathbf{u}_0 \in X_c^{(u)}(\mathbb{R}). \quad (7.4)$$

Assume that $\mathcal{L}_1(\epsilon)$, $\mathcal{L}_0(\epsilon)$, and $\mathcal{U}(\epsilon)$ depend analytically on ϵ . There exists $\epsilon_ > 0$ such that the linearized problem (2.15) has a real positive eigenvalue λ if $l_0 = \langle \mathbf{u}_0 | \mathcal{L}_0^{-1}(0) \mathbf{u}_0 \rangle \neq 0$ and $\delta u = \langle \boldsymbol{\nu}_0, \mathcal{U}'(0) \boldsymbol{\nu}_0 \rangle_{\mathbb{R}^N} \neq 0$, in the domain:*

$$\mathcal{D}_0 = \{ \epsilon : 0 < |\epsilon| < \epsilon_*, \quad \text{sign}(\epsilon) = -\text{sign}(l_0 \delta u) \}.$$

Proof. If there exists $\boldsymbol{\nu}_0 \in \mathbb{R}^N$, such that $\mathcal{U}(0)\boldsymbol{\nu}_0 = \mathbf{0}$, the problem (4.2) has a general solution in $X_c^{(u)}(\mathbb{R})$ for $\gamma = 0$:

$$\mathbf{u}_0(x) = \sum_{n=1}^N (\nu_0)_n \frac{\partial \Phi(x)}{\partial \beta_n} + c_0 \Phi'(x), \quad (7.5)$$

where c_0 is not defined. We expand solutions of (5.1) in power series of ϵ :

$$\mathbf{u}(x) = \mathbf{u}_0(x) + \epsilon \mathbf{u}_1(x) + \mathcal{O}(\epsilon^2), \quad \gamma = \epsilon \gamma_1 + \mathcal{O}(\epsilon^2). \quad (7.6)$$

The function $\mathbf{u}_1(x)$ solves the non-homogeneous problem:

$$\mathcal{L}_1(0)\mathbf{u}_1 + \mathcal{L}'_1(0)\mathbf{u}_0 = \gamma_1\mathcal{L}_0^{-1}(0)\mathbf{u}_0 - \mathcal{L}_0^{-1}\mathcal{L}'_0(0)\mathcal{L}_1(0)\mathbf{u}_0, \quad (7.7)$$

subject to the constraints, following from the power series expansions of (2.26):

$$\langle \Phi_n(0)\mathbf{e}_n | \mathbf{u}_1 \rangle + \langle \Phi'_n(0)\mathbf{e}_n | \mathbf{u}_0 \rangle = 0., \quad (7.8)$$

where $\Phi'(0)$ stands for derivative of $\Phi(\epsilon)$ in ϵ . Using (4.12) and $\mathcal{L}_0\Phi_n\mathbf{e}_n = \mathbf{0}$ for any ϵ , we derive the following relations:

$$\mathcal{L}_1(0)\frac{\partial\Phi'(0)}{\partial\beta_n} + \mathcal{L}'_1(0)\frac{\partial\Phi}{\partial\beta_n} = -\Phi'_n(0)\mathbf{e}_n, \quad (7.9)$$

$$\mathcal{L}_0(0)\Phi'_n(0)\mathbf{e}_n + \mathcal{L}'_0(0)\Phi_n\mathbf{e}_n = 0. \quad (7.10)$$

Using (7.8), (7.10), and the Fredholm Alternative Theorem, we find from (7.7) that

$$\langle \mathbf{u}_0 | \mathcal{L}'_1(0)\mathbf{u}_0 \rangle = \gamma_1\langle \mathbf{u}_0 | \mathcal{L}_0^{-1}(0)\mathbf{u}_0 \rangle - 2\sum_{n=1}^N(\nu_0)_n\langle \Phi'_n(0)\mathbf{e}_n | \mathbf{u}_0 \rangle. \quad (7.11)$$

Using (7.9), we find from (7.11) that

$$\gamma_1\langle \mathbf{u}_0 | \mathcal{L}_0^{-1}(0)\mathbf{u}_0 \rangle = \sum_{n=1}^N\sum_{m=1}^N\nu_{0n}\nu_{0m}\frac{\partial}{\partial\epsilon}\langle \Phi_m(\epsilon)\mathbf{e}_m | \frac{\partial\Phi(\epsilon)}{\partial\beta_n} \rangle \Big|_{\epsilon=0} = \frac{1}{2}\langle \boldsymbol{\nu}_0, \mathcal{U}'(0)\boldsymbol{\nu}_0 \rangle_{\mathbb{R}^N}, \quad (7.12)$$

and $\delta u = 2\gamma_1 l_0$. When $\delta u \neq 0$, $l_0 \neq 0$, and $\text{sign}(\epsilon) = -\text{sign}(l_0\delta u)$, the eigenvalue γ is negative in the first order of ϵ . \square

Corollary 7.5 *Let \mathcal{L}_0 be positive definite, such that $l_0 > 0$. Bifurcation of a new negative eigenvalue of \mathcal{U} always results in a new negative eigenvalue γ of the problem (5.1).*

Remark 7.6 The symmetry bifurcations have been analyzed in [PG97, S00, PK00]. We notice that the only difference between symmetry and zero eigenvalue bifurcations is the origin of a new negative eigenvalue λ of the operator \mathcal{L}_1 in the constrained space $X_c^{(u)}(\mathbb{R})$.

(iii) Hamiltonian Hopf bifurcation

When the linearized stability problem (2.15) has non-zero multiple eigenvalues λ_0 of higher algebraic multiplicity, the corresponding eigenfunctions $\mathbf{u}_0(x)$ satisfy the constraint $l_0 = \langle \mathbf{u}_0 | \mathcal{L}_0^{-1}\mathbf{u}_0 \rangle = 0$, according to Lemma 5.7. The negative index of the quadratic forms $\langle \mathbf{f} | \mathcal{L}_1\mathbf{f} \rangle$ and $\langle \mathbf{f} | \mathcal{L}_0^{-1}\mathbf{f} \rangle$ collapses for such multiple eigenvalues. If a continuous deformation destroys multiple eigenvalues, new unstable eigenvalues λ may arise in the linearized problem (2.15). Here we study the characteristic features of this bifurcation in the case, when the multiple eigenvalues λ_0 are purely real or purely

imaginary. More general types of the bifurcation for complex λ can be studied with similar methods. We consider bifurcation of co-dimension one, when the non-zero multiple eigenvalue $\lambda = \lambda_0$ has algebraic multiplicity two.

The Hamiltonian Hopf bifurcation does not occur, when the multiple eigenvalue λ_0 has equal geometric and algebraic multiplicities. In the latter case, the quadratic forms for \mathcal{L}_1 and \mathcal{L}_0^{-1} are non-zero for linearly independent eigenfunctions corresponding to λ_0 and therefore the negative index of the quadratic forms $\langle \mathbf{f} | \mathcal{L}_1 \mathbf{f} \rangle$ and $\langle \mathbf{f} | \mathcal{L}_0^{-1} \mathbf{f} \rangle$ does not collapse. Multiple eigenvalue λ with linearly independent eigenvectors split into distinct eigenvalues, once the degeneracy is broken, such that no unstable eigenvalues bifurcate in this case.

Proposition 7.7 *Let ϵ be the bifurcation parameter, such that, at $\epsilon = 0$, there exist non-zero eigenfunctions $\mathbf{u}_0 \in X_c^{(u)}(\mathbb{R})$, $\mathbf{u}'_0 \in X_c^{(u)}(\mathbb{R})$, such that $\mathcal{L}_1 \mathbf{u}_0 = \gamma_0 \mathcal{L}_0^{-1} \mathbf{u}_0$, $\mathcal{L}_1 \mathbf{u}'_0 = \gamma_0 \mathcal{L}_0^{-1} \mathbf{u}'_0 + \mathcal{L}_0^{-1} \mathbf{u}_0$, and $l_0 = \langle \mathbf{u}_0 | \mathcal{L}_0^{-1} \mathbf{u}_0 \rangle = 0$. Assume that $\mathcal{L}_1(\epsilon)$ and $\mathcal{L}_0(\epsilon)$ depend analytically on ϵ . There exists $\epsilon_* > 0$ such that the linearized problem (2.15) has two complex eigenvalues λ with $\text{Re}(\lambda) > 0$ if $l'_0 = \langle \mathbf{u}_0 | \mathcal{L}_0^{-1}(0) \mathbf{u}'_0 \rangle \neq 0$ and $\delta h = \langle \mathbf{u}_0 | (\mathcal{L}'_1(0) - \gamma_0 \mathcal{L}_0^{-1'}(0)) \mathbf{u}_0 \rangle \neq 0$, in the domain:*

$$\mathcal{D}_0 = \{ \epsilon : 0 < |\epsilon| < \epsilon_*, \quad \text{sign}(\epsilon) = -\text{sign}(l'_0 \delta h) \}.$$

Proof. We expand solutions of (5.1) in power series of $\epsilon^{1/2}$:

$$\mathbf{u}(x) = \mathbf{u}_0(x) + \epsilon^{1/2} \gamma_1 \mathbf{u}'_0(x) + \epsilon \mathbf{u}_2(x) + \mathcal{O}(\epsilon^{3/2}), \quad \gamma = \gamma_0 + \epsilon^{1/2} \gamma_1 + \epsilon \gamma_2 + \mathcal{O}(\epsilon^{3/2}). \quad (7.13)$$

The function $\mathbf{u}_2(x)$ solves the non-homogeneous problem in $X_c^{(u)}(\mathbb{R})$:

$$\mathcal{L}_1(0) \mathbf{u}_2 + \mathcal{L}'_1(0) \mathbf{u}_0 = \gamma_0 \mathcal{L}_0^{-1}(0) \mathbf{u}_2 + \gamma_0 \mathcal{L}_0^{-1'}(0) \mathbf{u}_0 + \gamma_1^2 \mathcal{L}_0^{-1}(0) \mathbf{u}'_0 + \gamma_2 \mathcal{L}_0^{-1}(0) \mathbf{u}_0. \quad (7.14)$$

Using the Fredholm Alternative Theorem, we find from (7.14) that $\delta h = \gamma_1^2 l'_0$. Since $(\gamma - \gamma_0)^2 = \epsilon \gamma_1^2 + \mathcal{O}(\epsilon^{3/2})$, the eigenvalues γ bifurcate into the complex plane if $l'_0 \neq 0$, $\delta h \neq 0$, and $\text{sign}(\epsilon) = -\text{sign}(l'_0 \delta h)$. \square

Corollary 7.8 *Let $l'_0 \neq 0$ and $\delta h \neq 0$. There exists $\epsilon_* > 0$ such that the problem (5.1) has two real eigenvalues γ in the domain,*

$$\mathcal{D}_0 = \{ \epsilon : 0 < |\epsilon| < \epsilon_*, \quad \text{sign}(\epsilon) = \text{sign}(l'_0 \delta h) \}.$$

Remark 7.9 Hamiltonian Hopf bifurcations may occur either for two purely imaginary eigenvalues λ or for two real eigenvalues λ , when either $\gamma_0 > 0$ or $\gamma_0 < 0$. Characteristic features of the complex bifurcations were analyzed in [G90] and recently in [ABP99, S01].

8 Transverse stability-instability theorems

Stability–instability analysis based on simultaneous diagonalization of two linear operators in a constrained function space can be applied to stability problems of stationary solutions in other Hamiltonian dynamical systems, such as coupled Klein–Gordon equations and coupled Korteweg–de Vries equations. We show here that similar analysis is applied also to problems of transverse stability and instability of incoherent optical solitons (2.1).

Transverse (symmetry-breaking) instabilities may occur when the stationary solutions in (z, x) are perturbed in another spatial dimension, say in y [KP00]. The system of coupled NLS equations (1.1) can be rewritten in three spatial dimensions (z, x, y) as:

$$i\frac{\partial\psi_n}{\partial z} + d_n\left(\frac{\partial^2\psi_n}{\partial x^2} + \frac{\partial^2\psi_n}{\partial y^2}\right) + f_n(|\psi_1|^2, \dots, |\psi_N|^2)\psi_n = 0, \quad n = 1, \dots, N. \quad (8.1)$$

We assume again that parameters d_n are all positive. Linearization of the stationary solutions (2.1) for incoherent optical solitons is defined by the expansion,

$$\psi = e^{i\beta z} [\Phi(x) + \mathbf{U}(z, x, y) + i\mathbf{W}(z, x, y)], \quad (8.2)$$

where $\mathbf{U}, \mathbf{W} \in \mathbb{R}^N$ are perturbations of the stationary solutions. Separating the variables (z, x, y) as $\mathbf{U} = \mathbf{u}(x) e^{\lambda z + ipy}$, $\mathbf{W} = \mathbf{w}(x) e^{\lambda z + ipy}$, we arrive to the linear eigenvalue problem,

$$(\mathcal{L}_1 + p^2\mathcal{D}) \mathbf{u} = -\lambda\mathbf{w}, \quad (\mathcal{L}_0 + p^2\mathcal{D}) \mathbf{w} = \lambda\mathbf{u}, \quad (8.3)$$

where p is a real parameter and $\mathcal{D} = \text{diag}(d_1, \dots, d_N)$. Eigenvalues λ and eigenvectors $(\mathbf{u}, \mathbf{w})^T$ of the linearized problem (8.3) depend on parameter p .

Lemma 8.1 *There exist exactly $n(\mathcal{L}_{1,0})$ negative eigenvalues λ of the problem*

$$\mathcal{L}_{1,0}\mathbf{u} = \lambda\mathcal{D}\mathbf{u}, \quad (8.4)$$

in function space $\mathbf{u} \in X(\mathbb{R})$.

Proof. Since \mathcal{D} is positive definite, all eigenvalues λ in (8.4) are real. The spectrum of (8.4) is complete in $X(\mathbb{R})$. Proposition 5.13 suggests that the negative index of quadratic forms $\langle \mathbf{f} | \mathcal{L}_{1,0} \mathbf{f} \rangle$ is invariant in any diagonal representation. The problem (8.4) defines an orthogonal basis in $X(\mathbb{R})$ with respect to the weighted inner product $\langle \mathbf{u}_m | \mathcal{D} \mathbf{u}_m \rangle > 0$. Since $\#_{<0}(\mathcal{L}_{1,0}) = n(\mathcal{L}_{1,0})$ in $X(\mathbb{R})$, the problem (8.4) has exactly $n(\mathcal{L}_{1,0})$ negative eigenvalues λ . \square

Corollary 8.2 *Let $\lambda_m^{(n)}$ be negative eigenvalues of the operator $(\mathcal{L}_0)_{nn}$ for $m = 1, \dots, n(\mathcal{L}_0)$. Eigenvalues of the problem (8.4) for a diagonal operator \mathcal{L}_0 are $\lambda = -|\lambda_m^{(n)}|/d_n$.*

Lemma 8.3 *Let the problems (8.4) have negative eigenvalues λ at $\lambda = -\{\Lambda_{1,m}^2\}_{m=1}^{n(\mathcal{L}_1)}$ and $\lambda = -\{\Lambda_{0,m}^2\}_{m=1}^{n(\mathcal{L}_0)}$. Let p^2 belong to the domain*

$$\mathcal{D}_{n_1, n_0} = \{p^2 \in \mathbb{R}_+ : \Lambda_{1, n_1}^2 < p^2 < \Lambda_{1, (n_1+1)}^2, \quad \Lambda_{0, n_0}^2 < p^2 < \Lambda_{0, (n_0+1)}^2\}, \quad (8.5)$$

where $0 \leq n_1 \leq n(\mathcal{L}_1)$, $0 \leq n_0 \leq n(\mathcal{L}_0)$, such that $\Lambda_{1,0}^2 = \Lambda_{0,0}^2 \equiv 0$, and $\Lambda_{1, (n(\mathcal{L}_1)+1)}^2 = \Lambda_{0, (n(\mathcal{L}_0)+1)}^2 \equiv \infty$. In the domain \mathcal{D}_{n_1, n_0} , there exist exactly $(n(\mathcal{L}_1) - n_1)$ negative eigenvalues of the problem

$$(\mathcal{L}_1 + p^2 \mathcal{D}) \mathbf{u} = \lambda \mathbf{u}, \quad (8.6)$$

and exactly $(n(\mathcal{L}_0) - n_0)$ negative eigenvalues of the problem,

$$(\mathcal{L}_0 + p^2 \mathcal{D}) \mathbf{u} = \lambda \mathbf{u}, \quad (8.7)$$

in function space $\mathbf{u} \in X(\mathbb{R})$.

Proof. The result follows from continuity of eigenvalues λ of the uncoupled problems (8.6) and (8.7) with respect to parameter p^2 in the domain $0 \leq p^2 < \infty$. Each negative eigenvalue $\lambda = \lambda(p^2)$ of operator $(\mathcal{L}_{1,0} + p^2 \mathcal{D})$ is an increasing function of p^2 if \mathcal{D} is positive definite, since

$$\lambda'(p^2) = \frac{\langle \mathbf{u} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathcal{D} \mathbf{u} \rangle} > 0. \quad (8.8)$$

When p^2 increases, eigenvalues $\lambda(p^2)$ pass through the zero value at the boundaries between domains \mathcal{D}_{n_1, n_0} in (8.5), and the number of negative eigenvalues of (8.6)–(8.7) reduces according to the multiplicity of eigenvalues $\lambda = -\Lambda_{1, n_1}^2$ and $\lambda = -\Lambda_{0, n_0}^2$ in (8.4). \square

Proposition 8.4 *Let Hypothesis 2.24 be satisfied for the linearized problem (8.3) such that no multiple eigenvalues of higher algebraic multiplicity exists in domain \mathcal{D}_{n_1, n_0} , defined by (8.5). In the domain \mathcal{D}_{n_1, n_0} , the linearized problem (8.3) has N_{unst} unstable eigenvalues $\lambda = \lambda(p)$ such that $\text{Re}(\lambda) > 0$, where*

$$|n(\mathcal{L}_1) - n(\mathcal{L}_0) - n_1 + n_0| \leq N_{\text{unst}} \leq |n(\mathcal{L}_1) + n(\mathcal{L}_0) - n_1 - n_0|. \quad (8.9)$$

There are N_{real} real positive eigenvalues λ , such that $|n(\mathcal{L}_1) - n(\mathcal{L}_0) - n_1 + n_0| \leq N_{\text{real}} \leq N_{\text{unst}}$ and $2N_{\text{comp}} = N_{\text{unst}} - N_{\text{real}}$ complex eigenvalues λ with $\text{Re}(\lambda) > 0$, such that $0 \leq N_{\text{comp}} \leq \min(n(\mathcal{L}_0) - n_0, n(\mathcal{L}_1) - n_1)$.

Proof. The linearization problem (8.3) can be rewritten in the form of a diagonalization problem,

$$(\mathcal{L}_1 + p^2 \mathcal{D}) \mathbf{u} = \gamma (\mathcal{L}_0 + p^2 \mathcal{D})^{-1} \mathbf{u}, \quad \gamma = -\lambda^2. \quad (8.10)$$

If $p^2 > 0$ and $p^2 \neq \Lambda_{0,m}^2$ for any $m = 1, \dots, n(\mathcal{L}_0)$, the operator $(\mathcal{L}_0 + p^2 \mathcal{D})$ is invertible and the diagonalization problem (8.10) is defined in function space $\mathbf{u} \in X(\mathbb{R})$. Proposition 5.13 applies with $\#_{<0}(\mathcal{L}_1 + p^2 \mathcal{D}) = n(\mathcal{L}_1) - n_1$ and $\#_{<0}(\mathcal{L}_0 + p^2 \mathcal{D}) = n(\mathcal{L}_0) - n_0$ in the domain \mathcal{D}_{n_1, n_0} . Proposition 8.4 is then equivalent to Theorem 3.9. \square

Proposition 8.5 *Let $z(\mathcal{L}_1) = 1$, $z(\mathcal{U}) = 0$, and Hypothesis 2.24 be satisfied. Suppose the linearized problem (2.15) has N_{unst} unstable eigenvalues and no degenerate eigenvalues with $\langle \mathbf{u} | \mathcal{L}_0^{-1} \mathbf{u} \rangle = 0$. There exists $p_*^2 > 0$ such that the linearized problem (8.3) has exactly \hat{N}_{unst} unstable eigenvalues in the domain $0 < p^2 < p_*^2$, where $\hat{N}_{\text{unst}} = N_{\text{unst}} + p(\mathcal{U})$. The new $p(\mathcal{U})$ unstable eigenvalues λ are all real and positive.*

Proof. Lemma 2.22 suggests that the linearized problem (8.3) with $p^2 = 0$ has $N + 1$ double zero eigenvalues in function space $X(\mathbb{R})$. The symmetry-breaking perturbation with $p^2 > 0$ split these double eigenvalues into pairs of real or imaginary eigenvalues $\lambda(p)$ of the linearized problem (8.3). We show that $p(\mathcal{U})$ double eigenvalues split into pairs of real eigenvalues λ . We expand solutions of (8.3) into power series of p as follows,

$$\mathbf{u} = p\lambda_1 \sum_{n=1}^N c_n \frac{\partial \Phi}{\partial \beta_n} + \mathcal{O}(p^3), \quad (8.11)$$

$$\mathbf{w} = \sum_{n=1}^N c_n \Phi_n(x) \mathbf{e}_n + p^2 \mathbf{w}_2(x) + \mathcal{O}(p^4), \quad (8.12)$$

where $\lambda = p\lambda_1 + \mathcal{O}(p^3)$. The function $\mathbf{w}_2(x)$ satisfies the non-homogeneous linear problem,

$$\mathcal{L}_0 \mathbf{w}_2 = \lambda_1^2 \sum_{n=1}^N c_n \frac{\partial \Phi}{\partial \beta_n} - \sum_{n=1}^N c_n d_n \Phi_n(x) \mathbf{e}_n. \quad (8.13)$$

Using the Fredholm Alternative Theorem, we find that $\mathbf{c} = (c_1, \dots, c_N)^T$ satisfies the generalized eigenvalue problem,

$$\lambda_1^2 \mathcal{U} \mathbf{c} = 2\mathcal{D}\mathcal{Q}_s \mathbf{c}, \quad (8.14)$$

where $\mathcal{Q}_s = \text{diag}(Q_{1s}, \dots, Q_{ns})$ and \mathcal{U} is the Hessian matrix in (2.8). Sylvester's Inertial Theorem suggests that the linear system (8.14) has exactly $p(\mathcal{U})$ positive eigenvalues and $n(\mathcal{U})$ negative eigenvalues λ_1^2 , since $\mathcal{D}\mathcal{Q}_s$ is positive-definite. Therefore, positive eigenvalues of \mathcal{U} are related to new unstable (real and positive) eigenvalues $\lambda = \lambda(p)$ in the linearization problem (8.3) for sufficiently small values of $p^2 > 0$, in addition to N_{unst} unstable eigenvalues $\lambda(p)$ existing in the limit $p^2 \rightarrow 0$ with $\text{Re}(\lambda) > 0$. \square

Remark 8.6 Proposition 8.5 agrees with Proposition 8.4 for $n_1 = 0$ and $n_0 = 0$. We also notice that the $(N+1)$ -th double zero eigenvalue with the eigenfunction $\mathbf{u} = \Phi'(x)$, $\mathbf{w} = \mathbf{0}$ splits into a pair of imaginary eigenvalues for $p^2 > 0$. This property corresponds to Remark 3.11, since the translational symmetry (1.7) does not change the index $p(\mathcal{U}_H)$.

Example 8.7 A particular n -family of stationary solutions $\Phi = (\Phi_1, \Phi_2)$ of the coupled NLS equations (6.1) has $N_{\text{unst}} = 2n$ complex unstable eigenvalues in the problem (8.3) for $p^2 = 0$,

according to Lemma 6.5. Lemma 6.3 suggests that $p(\mathcal{U}) = 2$ for $C_n(\chi) > 0$ and $p(\mathcal{U}) = 1$ for $C_n(\chi) < 0$. The fundamental soliton with $n = 0$ is weakly spectrally stable with respect to symmetry-preserving perturbations with $p^2 = 0$. It is however spectrally unstable with respect to symmetry-breaking perturbations with $p^2 > 0$, such that the linearized problem (8.3) has two real positive eigenvalues for $C_0(\chi) > 0$ and one real positive eigenvalue for $C_0(\chi) < 0$. It was shown numerically in [Y97] that $C_0(\chi) > 0$ for $0 < \chi < 1$ and $C_0(\chi) < 0$ for $\chi > 1$.

Example 8.8 The hyperbolic NLS equation takes the form,

$$i\psi_z + \psi_{xx} - \psi_{yy} + |\psi|^2\psi = 0. \quad (8.15)$$

Although the hyperbolic NLS equation (8.15) does not satisfy the condition that \mathcal{D} be positive, we can still apply the stability–instability theorems in the domain $0 < p^2 < \beta$, where $\beta > 0$ is parameter of the stationary solutions (6.3) such that $\psi(z, x) = \sqrt{2\beta} \operatorname{sech}(\sqrt{\beta}x) e^{i\beta z}$. In this case, $n(\mathcal{L}_1) = 1$, $n(\mathcal{L}_0) = 0$, and the zero eigenvalues of \mathcal{L}_1 and \mathcal{L}_0 become negative eigenvalues λ in the uncoupled problems (8.6) and (8.7) for $0 < p^2 < \beta$. Proposition 8.4 applies with $n_1 = -1$ and $n_0 = -1$ and suggests that there are $1 \leq N_{\text{unst}} \leq 3$ unstable eigenvalues in the linearized problem (8.3) for $0 < p^2 < \beta$. Indeed, it was shown in [P01] that there is one real positive eigenvalue $\lambda(p)$ in the domain $0 < p^2 < p_{\text{thr}}^2$, where $p_{\text{thr}}^2 < \beta$, and three unstable ($N_{\text{real}} = 1$, $2N_{\text{comp}} = 2$) eigenvalues $\lambda(p)$ for $p_{\text{thr}}^2 < p^2 < \beta$. The stability–instability analysis breaks in the domain $p^2 \geq \beta$, where the negative space of operators $(\mathcal{L}_1 + p^2\mathcal{D})$ and $(\mathcal{L}_0 + p^2\mathcal{D})$ becomes infinite-dimensional.

9 Summary

We have revisited analysis of linearized stability and instability in a general Hamiltonian system of N incoherently coupled NLS equations. We have proved the most general stability–instability theorems that define existence, type and number of unstable eigenvalues of the linearized stability problem. With the use of these theorems, a new computational algorithm can be developed to compute number $2N_{\text{comp}}$ of complex eigenvalues as a difference of the negative index $\#_{<0}(h)$ of the constrained linearized Hamiltonian h and numbers N_{real} and $2N_{\text{imag}}^-$ of real positive and imaginary eigenvalues of the linearized problem. All examples were studied analytically with the use of regular perturbation series. The examples have illustrated validity of the stability–instability theorems. Numerical codes can be developed in further studies in order to trace unstable eigenvalues beyond the limits of perturbation series and also to clarify linearized stability of all families of multiple pulse solutions of N incoherently coupled NLS equations.

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