McMaster University

Department of Mathematics and Statistics

MATH 790 - Major Research Project

STUDY OF VORTICES IN TWO-DIMENSIONAL HARMONIC POTENTIALS

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Introduction

Bose-Einstein condensation (BEC) is a physical phenomenon, initially predicted by S.N.Bose and A.Einstein in 1924-25. This phenomenon occurs when a gas of bosonic atoms is cooled below a critical transition temperature and consequently all atoms condense at the lowest energetic state. At this state all atoms become absolutely identical such that no measurement can distinguish between different atoms [6]. BEC has been widely studied mathematically in resent years [5,8,10]. A nonlinear evolution equation, called the Gross-Pitaevskii equation, has been used to model this phenomenon in the mean-field approximation.

The goal of this project is to provide detailed analytical and numerical calculations of vortices in the two-dimensional harmonic potentials. The main equation to consider is the Gross-Pitaevskii equation,

$$i\epsilon u_t + \epsilon^2(u_{xx} + u_{yy}) + (1 - x^2 - y^2 - |u|^2)u = 0, \tag{0.1}$$

where ϵ is a small parameter that is inversely proportional to chemical potential and u is a wave function.

In the first section of this project, we use the method of Lyapunov-Schmidt reduction for the local bifurcation, to study the vortex solution near the bifurcation point $\epsilon = \frac{1}{4}$. We prove the birth and the persistence of the vortex solution of the stationary Gross-Pitaevskii equation for small $|\epsilon - \frac{1}{4}|$. Section 2 contains the numerical results. Numerical shooting methods are employed to approximate the vortex solution to the stationary equation in the existence interval. We also discuss numerically the convexity of the energy functional near the vortex solution. In section 3 we use calculus of variations to prove the existence of the vortex solution for all $0 < \epsilon < \frac{1}{4}$. Finally, in section 4 we prove the uniqueness of the positive vortex solution using some ODE techniques. Section 5 concludes the project with the list of open problems. Appendixes I and II give numerical codes of the MATLAB programs.

1. Bifurcations of Vortex Solutions

Let us define the Schrödinger operator $\mathcal{H}_0$ for a two-dimensional harmonic oscillator in the form:

$$\mathcal{H}_0 = -\epsilon^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + x^2 + y^2 - 1, \quad \epsilon > 0. \tag{1.1}$$

The domain of $\mathcal{H}_0$ is:

$$Dom(\mathcal{H}_0) = \left\{ f \in H^2(\mathbb{R}^2) : |x|^2 f \in L^2(\mathbb{R}^2) \right\}. \tag{1.2}$$

The stationary Gross-Pitaevskii equation can be written in the form:

$$\mathcal{H}_0 u = -|u|^2 u. \tag{1.3}$$

We are studying the bifurcations of vortex solutions when the right-hand side of the equation (cubic term) is small.

Spectrum of a harmonic oscillator.
To find the spectrum of $\mathcal{H}_0$ we write the eigenvalue equation:

$$\mathcal{H}_0 f = \lambda f, \quad f \in Dom(\mathcal{H}_0), \tag{1.4}$$

where λ stands for the eigenvalues and f for the corresponding eigenfunctions. Substituting equation (1.1) to (1.4), we write the eigenvalue equation explicitly:

$$-\epsilon^2(f_{xx} + f_{yy}) + (x^2 + y^2)f - f = \lambda f. \tag{1.5}$$

We can use the separation of variables to represent the wave function in the product form $f(x, y) = \varphi(x)\psi(y)$, where $\varphi(x)$ and $\psi(y)$ satisfy the equations:

$$-\epsilon^2\varphi'' + x^2\varphi = \mu\varphi, \quad -\epsilon^2\psi'' + y^2\psi = \nu\psi. \tag{1.6}$$

Comparing (1.5) and (1.6) we can see that the eigenvalues λ of the Schrödinger operator are found from μ and ν as follows:

$$\mu + \nu - 1 = \lambda. \tag{1.7}$$

In order to derive a formula to compute the different values of λ, we need to look at the eigenvalues of the one-dimensional harmonic oscillator. We know [9] that the eigenvalues of a harmonic oscillator are equidistant, so that

$$\mu_k = \mu_0 + kh, \quad k \in \mathbb{N}_0, \tag{1.8}$$

where μ_0 is the smallest eigenvalue and h is the distance. The set $\mathbb{N}_0$ includes all possible integers and the zero. We also know that the first two eigenfunctions are the Hermite functions in the form:

$$\varphi_0 = e^{-\alpha x^2}, \quad \varphi_1 = xe^{-\alpha x^2}, \tag{1.9}$$

where α is a parameter. By substituting (1.9) to (1.6), we can see that $\alpha = \frac{1}{2\epsilon}$, $\mu_0 = \epsilon$ and $\mu_1 = 3\epsilon$ therefore $h = 2\epsilon$ and $\mu_k = \epsilon(1 + 2k), k \in \mathbb{N}_0$ and by symmetry $\nu_m = \epsilon(1 + 2m), m \in \mathbb{N}_0$. Therefore the eigenvalues of $\mathcal{H}_0$ are known exactly:

$$\sigma(\mathcal{H}_0) = \left\{\lambda_{k,m}(\epsilon) = -1 + 2\epsilon(k + m + 1), \quad (k, m) \in \mathbb{N}_0^2\right\}. \tag{1.10}$$

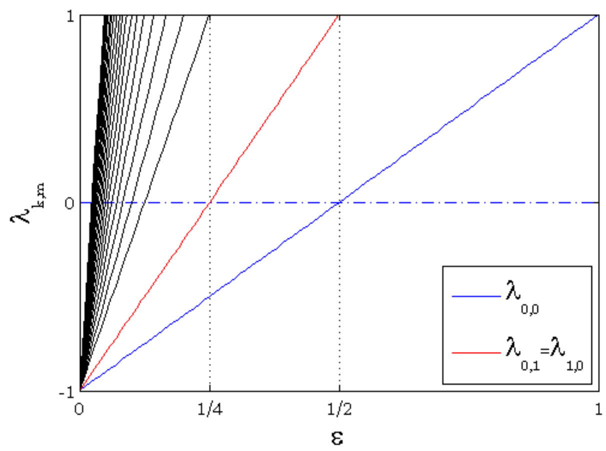

FIGURE 1

Eigenvalues $\lambda_{k,m}$ are plotted as a function of ϵ in Figure 1. When eigenvalues cross zero, this signals a bifurcation of a solution in the stationary equation (1.3). The lowest eigenvalue $\lambda_{0,0}$, crosses 0 at $\epsilon = \frac{1}{2}$ and induces a local bifurcation of the ground state. For ϵ near $\frac{1}{2}$ the ground state is a positive, radially symmetric solution that is close to the linear eigenstate $f_{0,0} = e^{-(x^2+y^2)}$ [10]. Similarly, the double eigenvalue $\lambda_{1,0} = \lambda_{0,1}$ crosses zero at $\epsilon = \frac{1}{4}$ and induces a local bifurcation of the vortex of charge one. In this section we study this local bifurcation. It can be seen in Figure 1 that there are bifurcation points of other stationary solutions which can be studied as well by similar methods.

Local bifurcation Theory.

An abstract introduction of the local bifurcation theory is as follow. A local bifurcation problem can be formulated as an equation:

$$F(u,\lambda) = 0, \quad F : U \times \mathbb{R} \to Z \tag{1.11}$$

where u is unknown, λ is the bifurcation parameter, U and Z are Banach spaces. The point (u_0, λ_0) is called a bifurcation point if changing the value of λ near λ_0 changes the number of solutions u to the equation (1.11). The method of Lyapunov-Schmidt Reduction is used to study the local bifurcation problem. This method is described in many texts [3,7]. The main outcome of the method is a reduction of the non-linear equation (1.11) to a finite-dimensional root-finding problem.

Let us assume that the Jacobian $D_uF(u_0,\lambda_0) := L$ is a Fredholm operator. Let $N(L) \subset U$ and $R(L) \subset Z$ denote the kernel and the range of L. Let us also assume that the $N(L) \neq \{0\}$. Note that if $N(L) = \{0\}$, then the operator L is invertible and by implicit function theorem, the solution exists uniquely in the neighborhood of u_0 in U, hence the bifurcation does not occur at (u_0, λ_0). Since L is a Fredholm operator, there exists closed complements to the kernel and range of L, so that U and Z can be decomposed to:

$$U = N(L) \oplus U_0, \tag{1.12}$$

$$Z = R(L) \oplus Z_0. \tag{1.13}$$

Let P and Q denote projection operators such that:

$$P : U \to N(L), \tag{1.14}$$

$$Q : Z \to Z_0, \tag{1.15}$$

then the problem can be projected into the form:

$$QF(Pu + (I - P)u, \lambda) = 0, \tag{1.16}$$

$$(I - Q)F(Pu + (I - P)u, \lambda) = 0, \tag{1.17}$$

where I is an identity operator. Note that $(I - Q)L(I - P)$ is invertible. If we can find the function ψ which maps $N(L) \times \mathbb{R} \to U_0$ and satisfies the equation (1.17), then we can rewrite the equation (1.16) as the bifurcation problem:

$$QF(Pu + \psi(Pu, \lambda), \lambda) = 0. \tag{1.18}$$

If $N(L)$ is finite-dimensional, the bifurcation problem (1.18) reduces to a standard root finding problem, which can be studied in many cases by using normal form expansions [3,7].

Application of the Local bifurcation Theory.

To study the stationary equation (1.3) near the point $\epsilon = \frac{1}{4}$, let $F := \mathcal{H}_0 u + |u|^2 u$ hence $D_u F(0, \epsilon) = \mathcal{H}_0$. The following theorem presents the main results of this section.

Theorem 1. Let $\mu = \frac{1}{16} - \epsilon^2$. There is $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, there exist vortex solutions of the form,

$$u = \sqrt{128\mu}(x \pm iy)e^{-2(x^2+y^2)} + \mathcal{O}_{H^2(\mathbb{R}^2)}(\sqrt{\mu^3}), \tag{1.19}$$

in the stationary equation (1.3).

To prove Theorem 1, we apply the method of Lyaponuv-Schmidt reductions. We start by defining U, Z, $N \subset U$ and $R \subset Z$ as:

$$U = Dom(\mathcal{H}_0|_{\epsilon=\frac{1}{4}}), \quad Z = L^2(\mathbb{R}^2),$$

$$N \equiv Ker(\mathcal{H}_0|_{\epsilon=\frac{1}{4}}) = Span\{f_{1,0}, f_{0,1}\}, \tag{1.20}$$

$$R \equiv Range(\mathcal{H}_0|_{\epsilon=\frac{1}{4}}) \tag{1.21}$$

where

$$f_{1,0} = xe^{-2(x^2+y^2)} \quad f_{0,1} = ye^{-2(x^2+y^2)}. \tag{1.22}$$

Furthermore we will introduce the inner product in $L^2(\mathbb{R}^2)$ as $< f, g >:= \int_{\mathbb{R}^2} \overline{f(x)}g(x)dx$. Because $\mathcal{H}_0|_{\epsilon=\frac{1}{4}}$ is self-adjoint, if $f \in N$ and $g \in R$, then $< f, g >= 0$. Using the decomposition we have:

$$\forall \varphi \in N : \quad \varphi = c_1 f_{1,0} + c_2 f_{0,1} = (c_1 x + c_2 y)e^{-2(x^2+y^2)}, \tag{1.23}$$

and

$$\forall \psi \in R : \quad < f_{1,0}, \psi >=< f_{0,1}, \psi >= 0. \tag{1.24}$$

Now let us rewrite the stationary equation (1.3) as the local bifurcation equation:

$$(\mathcal{H}_0|_{\epsilon=\frac{1}{4}} + \mu\Delta)u = -|u|^2 u. \tag{1.25}$$

Substituting the decomposition,

$$u = \sqrt{\mu}(\varphi + c_1 f_{1,0} + c_2 f_{0,1}), \quad \varphi \in R, \quad (c_1, c_2) \in \mathbb{C}^2, \tag{1.26}$$

into the equation (1.25) we obtain:

$$(\mathcal{H}_0|_{\epsilon=\frac{1}{4}} + \mu\Delta)(\varphi + c_1 f_{1,0} + c_2 f_{0,1}) = -\mu\mathcal{F}(\varphi + c_1 f_{1,0} + c_2 f_{0,1}), \tag{1.27}$$

where $\mathcal{F}(u) = |u|^2 u$. Projecting this equation to the orthogonal complement of R, we obtain:

$$c_1 < f_{1,0}, (-\Delta)f_{1,0} > + < \varphi, (-\Delta)f_{1,0} >=< f_{1,0}, \mathcal{F}(\varphi + c_1 f_{1,0} + c_2 f_{0,1}) >; \tag{1.28}$$

$$c_2 < f_{0,1}, (-\Delta)f_{0,1} > + < \varphi, (-\Delta)f_{0,1} >=< f_{0,1}, \mathcal{F}(\varphi + c_1 f_{1,0} + c_2 f_{0,1}) > . \tag{1.29}$$

Using MATLAB, the exact values of the following integrals have been computed:

$$\begin{aligned} < f_{1,0}, (-\Delta)f_{1,0} > &= 16 < f_{1,0}, (1 - |x|^2)f_{1,0} > \\ &= 16 \int\int_{\mathbb{R}^2} (1 - r^2)r^3 e^{-4r^2} dr = \frac{\pi}{4} = I; \end{aligned}$$

$$\begin{aligned} < f_{0,1}, (-\Delta)f_{1,0} > &= 16 < f_{0,1}, (1 - |x|^2)f_{1,0} > \\ &= 16 \int\int_{\mathbb{R}^2} xye^{-\frac{x^2+y^2}{\epsilon}} (1 - x^2 - y^2)dxdy = 0; \end{aligned}$$

By symmetry, we have

$$< f_{0,1}, (-\Delta)f_{0,1} >= \frac{\pi}{4} = I, \quad < f_{1,0}, (-\Delta)f_{0,1} >= 0.$$

Projecting (1.27) to R, we get:

$$P_0(\mathcal{H}_0|_{\epsilon=\frac{1}{4}} + \mu\Delta)P_0\varphi + \mu(\Delta)(c_1 f_{1,0} + c_2 f_{0,1}) = -\mu P_0 \mathcal{F}(\varphi + c_1 f_{1,0} + c_2 f_{0,1}). \tag{1.30}$$

where P_0 is the orthogonal projection operator which can be defined:

$$P_0 : L^2 \to R \subset L^2; \quad P_0(c_1 f_{1,0} + c_2 f_{0,1}) = 0; \quad P_0(\varphi) = \varphi. \tag{1.31}$$

Since $P_0\mathcal{H}_0|_{\epsilon=\frac{1}{4}}P_0$ is invertible, the Implicit Function Theorem says that for any $c_1, c_2 \in \mathbb{C}$ and small μ, there exists a unique solution of (1.30) such that $\|\varphi\|_{H^2} = \mathcal{O}(\mu)$ as $\mu \to 0$ if $c_1, c_2 = \mathcal{O}(1)$. Substituting this solution to the system (1.28)-(1.29) and truncating at the leading orders in μ, we obtain the truncated normal form equations:

$$Ic_1 = J_1|c_1|^2 c_1 + J_2(2|c_2|^2 c_1 + {c_2}^2 \bar{c_1}) \tag{1.32}$$

$$Ic_2 = J_1|c_2|^2 c_2 + J_2(2|c_1|^2 c_2 + {c_1}^2 \bar{c_2}) \tag{1.33}$$

where

$$J_1 =< f_{1,0}^2, f_{0,1}^2 >= \int\int_{\mathbb{R}}^2 x^4 e^{-8(x^2+y^2)} dxdy = \frac{3\pi}{2048} \tag{1.34}$$

$$J_2 =< f_{1,0}^2, f_{0,1}^2 >= \int\int_{\mathbb{R}}^2 x^2 y^2 e^{-8(x^2+y^2)} dxdy = \frac{\pi}{2048} \tag{1.35}$$

For simplicity we denote $J_2 = J$ and $J_1 = 3J$.
We are now looking for all acceptable solutions of the truncated system (1.32) and (1.33).

Vortex Solution. Let $c_2 = \pm ic_1$ then $Ic_1 = (J_1 + J_2)|c_1|^2 c_1 = 4J|c_1|^2 c_1$. If we assume that $c_1 \neq 0$ then we have:

$$c_1 = \left(\frac{I}{4J}\right)^{\frac{1}{2}} = \sqrt{128}; \quad c_2 = \pm i\sqrt{128}. \tag{1.36}$$

The approximation (1.26) can now be written as:

$$u = \sqrt{128\mu}(x \pm iy)e^{-2(x^2+y^2)} + \mathcal{O}_{H^2(\mathbb{R}^2)}(\mu^{\frac{3}{2}}) \tag{1.37}$$

Dipole Solution. Let $c_1, c_2 \neq 0$ and $c_2 \neq \pm ic_1$, if we multiply both sides of the equation (1.32) by $\bar{c_1}$ we will have:

$$I|c_1|^2 = 3J|c_1|^4 + J(2|c_2|^2|c_1|^2 + {c_2}^2 \bar{c_1}^2) \tag{1.38}$$

which leads to the conclusion that $\mathrm{Im}(\bar{c_1}^2 {c_2}^2) = 0$. The vortex solution (1.36) satisfies this condition. The other solution with $c_1 \in \mathbb{R}$ should have $c_2 \in \mathbb{R}$, in which case the equation (1.32) leads to:

$${c_1}^2 + {c_2}^2 = \frac{I}{3J}. \tag{1.39}$$

Note that a particular solution is:

$$c_1 = \left(\frac{I}{3J}\right)^{\frac{1}{2}} = \sqrt{\frac{512}{3}}; \quad c_2 = 0. \tag{1.40}$$

The general solution for this family can be derived using the polar coordinates and the equation (1.26) in the following form:

$$u = \sqrt{\frac{512\mu}{3}}(x\cos(\alpha) + y\sin(\alpha))e^{-2(x^2+y^2)} + \mathcal{O}_{H^2(\mathbb{R}^2)}(\mu^{\frac{3}{2}}), \tag{1.41}$$

where α is an arbitrary parameter. So far we have identified all possible solutions of the truncated system (1.32) and (1.33). To be able to use Implicit Function Theorem and discuss the existence of the solution near bifurcation point ($\epsilon = \frac{1}{4}$), we need to check the invertibility condition of the Jacobian of this truncated system at each solution.

Persistence of Vortex and Dipole Solutions.

It has been shown that the vortex solutions satisfy the reduction $c_2 = \pm i c_1$, or more precisely,

$$c_1 = a_1, \quad c_2 = i a_2, \quad a_1 = a_2 = \sqrt{\frac{I}{4J}} = \sqrt{128}. \tag{1.42}$$

Rewriting the system (1.32)-(1.33) in terms a_1 and a_2 we obtain:

$$g_1 = (I - 3J{a_1}^2 - J{a_2}^2)a_1 = 0, \tag{1.43}$$

$$g_2 = (I - 3J{a_2}^2 - J{a_1}^2)a_2 = 0. \tag{1.44}$$

Then the Jacobian operator becomes:

$$D_{\overrightarrow{a}}(\overrightarrow{g}) = \begin{bmatrix} I - 9J{a_1}^2 - J{a_2}^2 & -2Ja_1a_2 \\ -2Ja_1a_2 & I - 9J{a_2}^2 - J{a_1}^2 \end{bmatrix} \tag{1.45}$$

substituting a_1 and a_2 using (1.42) the determinant of the Jacobian operator becomes:

$$det(D_{\overrightarrow{a}}(\overrightarrow{g}))|_{\overrightarrow{a}=\overrightarrow{a_1}} = 32a_1^2a_2^2J^2 \neq 0. \tag{1.46}$$

Therefore, the matrix (1.45) is invertible at the solution (1.42) and by Implicit Function Theorem, there exists a unique vortex solution near the bifurcation point $\epsilon = \frac{1}{4}$ for $\epsilon < \frac{1}{4}$.
It has been shown that the dipole solutions satisfy the reduction $c_1, c_2 \in \mathbb{R}$ or more precisely:

$$c_1 = a_1, \quad c_2 = a_2, \quad {a_1}^2 + {a_2}^2 = \frac{I}{3J} = \frac{512}{3}. \tag{1.47}$$

Rewriting the system (1.32)-(1.33) in terms a_1 and a_2, we obtain:

$$g_1 = (I - 3J{a_1}^2 - 3J{a_2}^2)a_1 = 0, \tag{1.48}$$

$$g_2 = (I - 3J{a_2}^2 - 3J{a_1}^2)a_2 = 0. \tag{1.49}$$

The Jacobian operator becomes:

$$D_{\overrightarrow{a}}(\overrightarrow{g}) = \begin{bmatrix} -6J{a_1}^2 & -6Ja_1a_2 \\ -6Ja_1a_2 & -6J{a_2}^2 \end{bmatrix} = -6J \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \tag{1.50}$$

The Jacobian Matrix has $Rank = 1$ and can not be invertible. Although we had found the solution for this family -(1.41)- but Implicit Function Theorem could not be applied and the existence and uniqueness of the solution near $\epsilon = \frac{1}{4}$ can not be concluded without computation of the higher orders of the perturbation theory.

2. Numerical Results for Vortex Solution

2.1. **Shooting Method.**

So far we have derived the vortex solution analytically near the bifurcation point $\epsilon = \frac{1}{4}$. By the local bifurcation method, we showed that the vortex solutions exist for $\epsilon < \frac{1}{4}$. The approximate solution (1.37) can be written in polar coordinate (r, θ) as:

$$u(x, y) \simeq \sqrt{128\mu} r e^{-2r^2+i\theta}. \tag{2.1}$$

In this section we want to compute the solution numerically using shooting method. To do so, let us consider the equation (1.3), for vortex solutions $u(x, y) = \phi(r)e^{i\theta}$. The function $\phi(r)$ satisfies an ordinary differential equation:

$$-\epsilon^2 \left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi \right) + (r^2 - 1)\phi = -\phi^3. \tag{2.2}$$

Because $r = 0$ is a regular singular point of the differential equation (2.2), the first step is to derive the approximate solution for the equation using Frobenius series.
For the second order ODE of the form:

$$u'' + \frac{p(r)}{r}u' + \frac{q(r)}{r^2}u = 0, \tag{2.3}$$

the Frobenius series takes the form $u(r) = \sum_{k=0}^{\infty} a_{k+l}r^{k+l}$ where $(a_l \neq 0)$. To find l, we need to solve the indicial equation, which for the equation (2.3) has the general form of

$$l(l-1) + p(0)l + q(0) = 0.$$

In our case, $p(0) = 1$ and $q(0) = -1$, so that $l = 1$ is one of the solutions of the indicial equation. Substituting $\phi = a_1r + a_2r^2 + a_3r^3 + a_4r^4 + \mathcal{O}(r^5)$ to the equation (2.2) we obtain $a_1 = s$, $a_2 = 0$, $a_3 = \frac{-s}{8\epsilon^2}$ and $a_4 = 0$ so that the Frobenius series is written as:

$$\phi(r) = sr - \frac{s}{8\epsilon^2}r^3 + \mathcal{O}(r^5). \tag{2.4}$$

We would like to use the shooting method starting with the boundary conditions at $r = 0$: $\phi(0) = 0$ and $\phi'(0) = s$, where s is the shooting parameter.

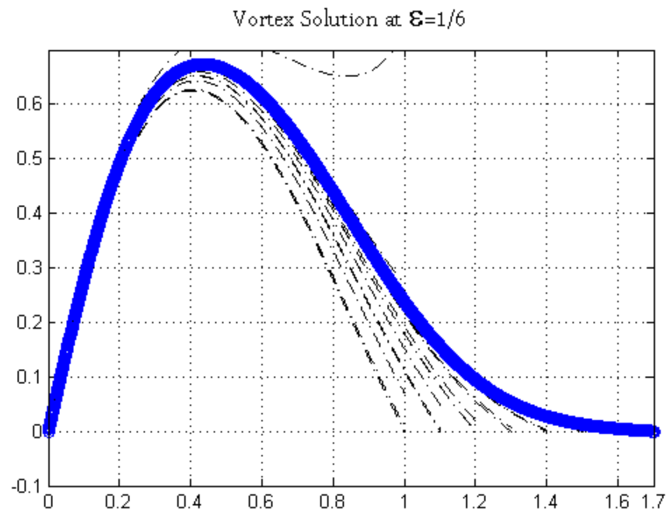

FIGURE 2. Vortex solution for $\epsilon = \frac{1}{6}$. The shooting method starts from $s = 2.5$ and after some iterations converges to $s = 2.9016$.

The second step is to break the second-order ODE to two first-order equations. If we let $U = \phi(r)$ and $V = \phi'(r)$, the equation (2.2) becomes:

$$\begin{cases} U' = V; \\ V' = \frac{1}{\epsilon^2}[-\frac{\epsilon^2}{r}V + \frac{\epsilon^2}{r^2}U + (r^2-1)U + U^3]. \end{cases} \tag{2.5}$$

Using Heun's method as the ODE solver from $U(h) = sh - \frac{s}{8\epsilon^2}h^3$ and $V(h) = s - \frac{3sh^2}{8\epsilon^2}$, where the step-size h is small, we compute the solution for all $r \in [0, 1.7]$. The Matlab function Shooting.m [Appendix 1] which is a modification of the code shooting-nonlinear-ODE.m from the text [4] finds the appropriate s for different values of ϵ. Note that the shooting method doesn't converge for the values of ϵ close to 0 and $\frac{1}{4}$.

The results for $\epsilon = \frac{1}{6}$ are shown in Figure 2. The starting value for s is 2.5. It can be seen that after several iterations (black dashed lines) the solutions converge to the blue line where s=2.9016. Note that the larger we can make the length of the interval, the more accurate results we will obtain. However due to the term $(r^2-1)\phi$ in the equation (2.2) the shooting method fails to converge for $r > 1.7$ and sufficiently small ϵ. Therefore we just approximate the vortex solution on the interval [0,1.7] subjected to the Dirichlet boundary condition $\phi(1.7) = 0$.

2.2. **Convexity of the energy functional.**

In this section we would like to see numerically that the energy functional is not convex near the vortex solution. Straightforward calculations show that the stationary equation (1.3) is the

Euler-Lagrange equation of the energy functional:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[\epsilon^2|\nabla u|^2 + (|x|^2 - 1)|u|^2\right] dx + \frac{1}{4}\int_{\mathbb{R}^2} |u|^4 dx, \tag{2.6}$$

If we let $\psi \in C_0^\infty(\mathbb{R}^2)$ to be a perturbation, the above statement can be expressed as:

$$\frac{d}{dt}E(u + t\psi)|_{t=0} = 0, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

Furthermore expanding the energy functional (2.6) up to the quadratic order leads to:

$$E(u + t\psi) - E(u) = \frac{1}{4}t^2 < \begin{bmatrix} \psi \\ \overline{\psi} \end{bmatrix}, \mathcal{H}\begin{bmatrix} \psi \\ \overline{\psi} \end{bmatrix} >_{L^2} + \mathcal{O}(t^3), \tag{2.8}$$

where $\mathcal{H}$ can be derived as:

$$\mathcal{H} = \begin{bmatrix} -\epsilon^2\Delta + |x|^2 - 1 + 2|u|^2 & u^2 \\ \overline{u}^2 & -\epsilon^2\Delta + |x|^2 - 1 + 2|u|^2 \end{bmatrix}. \tag{2.9}$$

To study the convexity near the vortex solution we can substitute $u(x, y) = \phi(r)e^{i\theta}$ to the matrix Schrödinger operator (2.9). Let us expand ψ and the complex conjugate $\overline{\psi}$ by using the Fourier series in θ:

$$\psi = \sum_{n\in\mathbb{N}} V_n e^{in\theta}, \tag{2.10}$$

$$\overline{\psi} = \sum_{n\in\mathbb{N}} W_n e^{in\theta}. \tag{2.11}$$

We also define the Laplace operator for the n-th azimuthal mode by:

$$\Delta_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}. \tag{2.12}$$

We know [8] that, each pair of $(V_n, W_{n-2})_{n\in\mathbb{N}}$ is decoupled from other pairs of Fourier coefficients. We also know [10] that the matrix Schrödinger operator acting on (V_1, W_{-1}) and (V_n, W_{n-2}) for $n \geq 3$ and $n \leq -1$ is positive. Therefore, to study the convexity of the quadratic form of $E(u)$, it suffices to look at the matrix Schrödinger operator acting on (V_2, W_0) and (V_0, W_{-2}). In what follows, we consider the eigenvalue problem in the form:

$$H_2 \begin{bmatrix} V_2 \\ W_0 \end{bmatrix} = \gamma \begin{bmatrix} V_2 \\ W_0 \end{bmatrix}, \tag{2.13}$$

where

$$H_2 = \begin{bmatrix} -\epsilon^2\Delta_2 + r^2 - 1 + 2\phi^2 & \phi^2 \\ \phi^2 & -\epsilon^2\Delta_0 + r^2 - 1 + 2\phi^2 \end{bmatrix}. \tag{2.14}$$

Lemma 2 in [10] states that for all values of $\epsilon < \frac{1}{4}$ near $\epsilon = \frac{1}{4}$, the spectral problem (2.13) has exactly one negative eigenvalue. To show the result numerically, let us write $V_2 = V$ and $W_0 = W$, then the eigenvalue problem (2.13) can be written as:

$$[-\epsilon^2\Delta_2 + r^2 - 1 + 2\phi^2]V + \phi^2 W = \gamma V, \tag{2.15}$$

$$\phi^2 V + [-\epsilon^2\Delta_0 + r^2 - 1 + 2\phi^2]W = \gamma W. \tag{2.16}$$

We will use finite difference method to solve the system of linear equations (2.15)-(2.16), by using central differences:

$$\left.\frac{dx}{dr}\right|_{r=r_k} = \frac{x_{k+1} - x_{k-1}}{2h} \tag{2.17}$$

$$\left.\frac{d^2x}{dr^2}\right|_{r=r_k} = \frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} \tag{2.18}$$

where h is the step size. Using (2.17) and (2.18), equations (2.15) and (2.16) become:

$$-\frac{\epsilon^2}{h^2}[(1+\frac{h}{2r_k})V_{k+1} - (2+\frac{4h^2}{r_k^2})V_k + (1-\frac{h}{2r_k})V_{k-1}] + (r_k^2 - 1 + 2\phi_k^2)V_k + \phi_k^2 W_k = \gamma V_k, \tag{2.19}$$

$$-\frac{\epsilon^2}{h^2}[(1+\frac{h}{2r_k})W_{k+1} - 2W_k + (1-\frac{h}{2r_k})W_{k-1}] + (r_k^2 - 1 + 2\phi_k^2)W_k + \phi_k^2 V_k = \gamma W_k, \tag{2.20}$$

where $1 < k < N-1$ and $N = kh$ is the length of interval. Note that we still need some information about the boundary conditions.

Let us set the Dirichlet boundary conditions at $r = 1.7$: $V(1.7) = W(1.7) = 0$. To setup boundary conditions at $r = 0$, we use the Frobenius series:

$$V(r) = a_2 r^2 + \mathcal{O}(r^3), \tag{2.21}$$

$$W(r) = b_0 + b_2 r^2 + \mathcal{O}(r^3), \tag{2.22}$$

where b_0 and a_2 are non-zero. Therefore, $V(0) = 0$. To start the iterations, we need to find a condition for $W(0)$. The condition can be obtained by substituting the equation (2.22) to (2.15) at the point $r = 0$ as,

$$-\epsilon^2 4b_2 - b_0 = \gamma b_0, \tag{2.23}$$

where $b_0 = W_0$ and $b_2 = \frac{1}{2}W''(0)$. Using (2.18) we approximate $W''(0) = \frac{W_1 - 2W_0 + W_{-1}}{h^2}$. Since equation (2.22) suggests that $W'(0) = 0$ therefore by (2.17), we have $W_1 = W_{-1}$. Then $b_2 = \frac{W_1 - W_0}{h^2}$ and we can write:

$$-\frac{4\epsilon^2}{h^2}(W_1 - W_0) - W_0 = \gamma W_0. \tag{2.24}$$

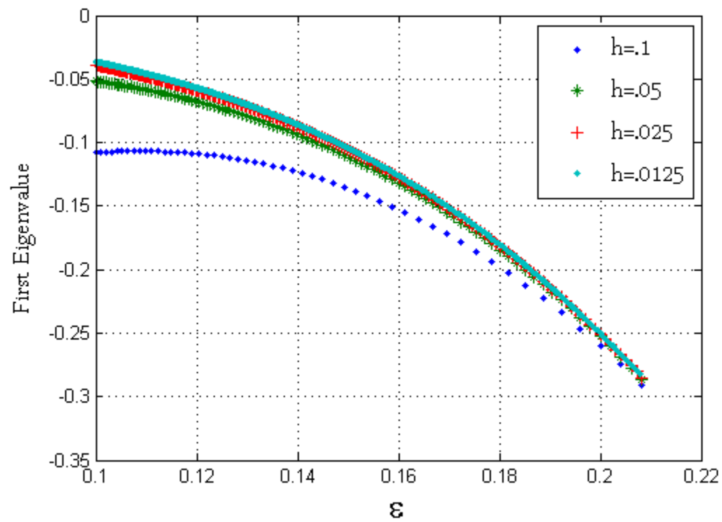

FIGURE 3. The negative eigenvalue of the spectral problem (2.13) vs ϵ.

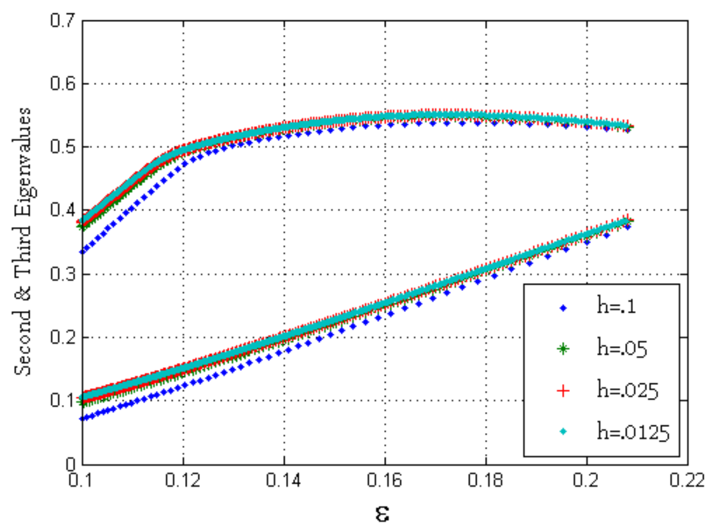

FIGURE 4. Two positive eigenvalues of the spectral problem (2.13) vs ϵ.

We now have enough information to solve the system of equations (2.19)-(2.20), complemented with the boundary conditions $V_0 = V_N = 0$, $W_N = 0$, and W_0 to be uniquely specified by the equation (2.24). The MATLAB function eigenvalues.m [Appendix II] computes the eigenvalues of the coefficient matrix representing the operator H_2 that we just derived.

Figure 3 shows how the negative eigenvalue of the spectral problem (2.13) depends on ϵ. Figure 4 displays the two smallest positive eigenvalues of the spectral problem (2.13) versus ϵ.

The eigenvalues have been computed for different step size. Just like any other numerical computation we need to check the accuracy of the solution. In our case the degree of accuracy is quadratic (i.e: $E(h) = ch^2$) and as we would expect, decreasing the step size by half makes the results more accurate quadratically.

To study the eigenvalues in Figures 3 and 4, let $\epsilon = \frac{1}{4}$, when (2.1) suggests that $\phi = 0$. Now the matrix operator (2.14) can be written as:

$$H_2 = \begin{bmatrix} \mathcal{H}_0|_{n=2} & 0 \\ 0 & \mathcal{H}_0|_{n=0} \end{bmatrix}, \tag{2.25}$$

where $\mathcal{H}_0|_n$ represent restriction of the Schrödinger operator $\mathcal{H}_0$ in (1.1) to the space of functions $f = \phi(r)e^{in\theta}$. Recall from (1.10) that the eigenvalues of the operator $\mathcal{H}_0$ are $\lambda_{k,m}(\epsilon) = -1 + 2\epsilon(k+m+1)$. Therefore for $H_0|_{n=0}$ we can write $\lambda(\epsilon) = -1+2\epsilon(2l+1)$ and for $H_0|_{n=2}$ we have $\lambda(\epsilon) = -1 + 2\epsilon(2l+2+1)$, where l is any natural number including zero. Hence the matrix operator has two series of eigenvalues namely:

$$\lambda|_{n=0} = \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots \right\}, \tag{2.26}$$

$$\lambda|_{n=2} = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right\}. \tag{2.27}$$

We conclude that at $\epsilon = \frac{1}{4}$, two smallest positive eigenvalues are double eigenvalues and have the value $\frac{1}{2}$, where as the negative eigenvalue is $-\frac{1}{2}$. It can be seen in Figure 4 that, the double eigenvalues split by decreasing the value of ϵ toward zero. The main issue in both Figures 3 and 4 is that we can not compute the eigenvalues for the the whole interval $[0, \frac{1}{4}]$. The main source of this problem, is the poor approximation of vortex solution computed in section 2.1.

3. Existence of the Vortex Solution

To prove the existence of the vortex solution, we first set the following notations. We denote by $H^1(\mathbb{R}^2)$ the Hilbert space of the square integrable functions on $\mathbb{R}^2$ with square integrable first derivative. If $u(x,y) = v(r)e^{i\theta}$ and $v(r)$ is radially symmetric, where (r,θ) are the polar coordinates, we say that $v \in H^1_r(\mathbb{R}_+)$.

Now we recall the equation (2.2):

$$\epsilon^2 \left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi \right) + (1 - r^2 - \phi^2)\phi = 0, \tag{3.1}$$

The energy functional for the vortex solution is:

$$E_1(v) = \int_0^\infty \left[\epsilon^2 \left(\frac{dv}{dr} \right)^2 + \frac{\epsilon^2}{r^2}v^2 + (r^2-1)v^2 + \frac{1}{2}v^4 \right] rdr = I(v) + J(v), \tag{3.2}$$

where:

$$I(v) = \int_0^\infty \left[\epsilon^2 \left(\frac{dv}{dr} \right)^2 + \frac{\epsilon^2}{r^2} v^2 + (r^2 - 1) v^2 \right] r dr, \tag{3.3}$$

$$J(v) = \frac{1}{2} \int_0^\infty v^4 r dr. \tag{3.4}$$

Let us denote the Schrödinger operator $\mathcal{H}_0$ in (1.1), restricted to the space of vortex function $u(x, y) = v(r)e^{i\theta}$ by $\mathcal{H}_0|_{n=1}$. Then $I(v)$ can be written in quadratic form associated with $\mathcal{H}_0|_{n=1}$ as, $I(v) =< v, \mathcal{H}_0|_{n=1} v >$. Now Figure 1 suggests that $I(v)$ is positive for any $\epsilon > \frac{1}{4}$ - note that the eigenvalues of the operator are positive in this interval- hence zero solution is the global minimizer of $E_1(v)$ with $E_1(0) = 0$ for $\epsilon > \frac{1}{4}$. However, for $\epsilon < \frac{1}{4}$, $I(v)$ is non-positive and hence, a global minimizer of $E_1(v)$ denoted by ϕ yields $E_1(\phi) < 0$. Note that if ϕ is a global minimizer of $E_1(v)$, then ϕ is a solution of the Euler-Lagrange equation (3.1).

Theorem 2. For all $0 < \epsilon < \frac{1}{4}$, the energy functional (3.2) has a nonzero global minimizer ϕ, which is a solution to the stationary equation (3.1).

To prove existence of a global minimizer of $E_1(v)$, Theorem 2 of chapter 8 in [1] suggests that we have to check if $E_1(v)$ satisfies the coercivity condition, also if the Lagrangian is convex in the variable v' [1]. If we write the Lagrangian as $L(v', v, r) = \epsilon^2 (v')^2 + \frac{\epsilon^2}{r^2} v^2 + (r^2 - 1)v^2 + \frac{1}{2} v^4$, then convexity condition in variable v' is satisfied trivially. The coercivity condition for the energy functional makes sure that the energy functional attains it's infimum. The coercivity condition is also trivially satisfied because L grows like $\epsilon^2 (v')^2$ as $|v'| \to \infty$ and like $\frac{1}{2} v^4$ as $|v| \to \infty$.

Now for $\epsilon < \frac{1}{4}$, the first eigenvalue of $\mathcal{H}_0|_{n=1}$ is negative. Therefore, there exists a positive constant a such that:

$$I(v) =< v, \mathcal{H}_0|_{n=1} v > \geq -a \|v\|_{L^2}^2 \tag{3.5}$$

On the other hand, $J(v) = \frac{1}{2} \|v\|_{L^4}^4$, so that we can bound $E_1(v)$ from below:

$$E_1(v) \geq -a \|v\|_{L^2}^2 + \frac{1}{2} \|v\|_{L^4}^4, \quad \forall v \in H_r^1(\mathbb{R}_+). \tag{3.6}$$

To find the a minimum of the righthand side of the equation (3.6), let us write $v_\lambda = \lambda v_0$ with $\|v_0\|_{H_r^1} = 1$ so that $\lambda = \|v_\lambda\|_{H_r^1} > 0$. Then, the equation (3.6) can be written based on v_0 as:

$$E_1(v) \geq -a \lambda^2 \|v_0\|_{L^2}^2 + \frac{1}{2} \lambda^4 \|v_0\|_{L^4}^4. \tag{3.7}$$

Minimizing $f(\lambda) = -a\lambda^2\|u_0\|^2_{L^2} + \frac{1}{2}\lambda^4\|u\|^4_{L^4}$, with respect to λ, we find the minimum of f at $\lambda_0^2 = a\frac{\|u\|^2_{L^2}}{\|u\|^4_{L^4}}$. Therefore, we have,

$$E_1(u) \geq -\frac{1}{2}a^2\frac{\|u_0\|^4_{L^2}}{\|u_0\|^4_{L^4}}. \tag{3.8}$$

In space $H^1_r(\mathbb{R}_+)$, the lower bound is not bounded from below. Therefore, we truncate $\mathbb{R}_+$ on a bounded interval $[0, R]$, subjected to the Dirichlet boundary condition $v|_{r=R} = 0$. Using Cauchy-Schwarz Inequality we can write:

$$\|v_0\|^2_{L^2} = \int_0^R v_0^2 r dr \leq \left(\int_0^R v_0^4 r dr\right)^{\frac{1}{2}} \left(\int_0^R r dr\right)^{\frac{1}{2}} = \left(\frac{1}{2}R^2\right)^{\frac{1}{2}} \|v_0\|^2_{L^4} = \frac{R}{\sqrt{2}}\|v_0\|^2_{L^4}. \tag{3.9}$$

Substituting $\|u_0\|^2_{L^2}$ from (3.9), the equation (3.8) becomes:

$$E_1(u) \geq -\frac{a^2R^2}{4}. \tag{3.10}$$

Therefore $E_1(v)$ is bounded from below for a finite $R > 0$ and it attains it's infimum for any v in the energy space denoted by:

$$X_R = \left\{v \in H^1_r(0, R) : rv \in L^2_r(0, R)\right\}. \tag{3.11}$$

Note that X_R is embedded to $L^4_r(0, R)$ compactly. Then by Theorem 2 of chapter 8 in [1], there exist a global minimizer for the energy functional (3.2). This global minimizer is a truncation of the vortex solution $v = \phi$ on the bounded interval $[0, R]$.
Note that the compactness of the embedding of X_R into $L^4_r(0, R)$ holds only for $0 < R < \infty$. To obtain the vortex solution on $\mathbb{R}_+$, we need to use arguments similar to those used in the proof of the ground (vortex-free) state (Theorem 2.1) in [5].

4. Uniqueness of the Vortex Solution

In this part we want to prove the uniqueness of the vortex solution for $0 < \epsilon < \frac{1}{4}$. The idea of this proof relies on the same arguments as in the proof of Proposition 1.1 for ground (vortex-free) states in [2].

Theorem 3. Assume that for any fixed $0 < \epsilon < \frac{1}{4}$, there exists a positive vortex solution ϕ to the stationary equation (3.1). Then the vortex solution ϕ is unique.

Before starting the proof of the theorem, let us remind that, by solving the indicial equation in section 2.1, we saw that the solution to the equation (3.1) can be written in the form:

$$\phi(r) = rW(r), \tag{4.1}$$

where $W(r)$ is a power series solution of the differential equation:

$$\epsilon^2(rW'' + 3W') + (1 - r^2 - r^2W^2)rW = 0. \tag{4.2}$$

Hence by using Frobenius Theorem, we have power series solutions in the form $W(r) = \sum_{k=0}^{\infty} a_k r^k$. Substituting the power series into (4.2), we see that for any fixed $\epsilon \in (0, \frac{1}{4})$, $a_1 = 0$, therefore $\phi(r) = a_0 r + \mathcal{O}(r^3)$ as $r \to 0$. Let us now assume that two positive vortex solutions $v(r)$ and $u(r)$ exist. By the above arguments the vortex solutions admit power series as $r \to 0$:

$$v(r) = ar + \mathcal{O}(r^3), \tag{4.3}$$

$$u(r) = br + \mathcal{O}(r^3). \tag{4.4}$$

Let us define $\rho(r) = \frac{u(r)}{v(r)}$, then:

$$\rho(r) = \frac{b}{a} + \mathcal{O}(r^2). \tag{4.5}$$

Therefore, $\rho(0) = \frac{b}{a}$ and $\rho'(0) = 0$. We can now rewrite the equation (3.1) for the solutions u and v. Multiplying both sides of the equations by the terms vr and ur respectively and subtracting the results from each other, we will obtain:

$$\epsilon^2[r(vu'' - uv'') + (vu' + uv')] = ruv(u^2 - v^2). \tag{4.6}$$

Using ρ, we can rewrite this equation as

$$\epsilon^2 \frac{d}{dr}(rv^2\rho') = rv^4\rho(\rho^2 - 1). \tag{4.7}$$

If $\rho(0) = 1$ and $\rho'(0) = 0$ then we conclude that $\rho(r) = 1$ for all $r \in \mathbb{R}_+$ that is, $u(r) = v(r)$ and the uniqueness is proved. Let us assume that $0 < \rho(0) < 1$, then the righthand side of the equation (4.7) is negative at $r = 0$, therefore $rv^2\frac{d\rho}{dr}$ is a decreasing function for $r > 0$ locally near $r = 0$. Let us define r_0 to be:

$$r_0 = \inf\{r > 0, \rho'(r) = 0\}, \tag{4.8}$$

since $\rho' < 0$ on the interval $(0, r_0)$, then ρ is decreasing on that interval and for every r, $0 < \rho(r) < \rho(0) < 1$. We can then conclude that $v^4\rho(\rho^2 - 1) < 0$ hence $rv^2\frac{d\rho}{dr}$ is decreasing on the whole interval $(0, r_0)$. Note that r_0 can not be finite due to the fact that, $r_0 v(r_0)^2 \frac{d\rho(r_0)}{dr} < 0$

contradicts the definition of r_0 in (4.8) and hence $r_0 = \infty$. Since $0 \leq \rho(\infty) < \rho(1) \leq 1$, then we can write:

$$-1 \leq \rho(\infty) - \rho(1) = \int_1^\infty \rho'(r)dr = \int_1^\infty rv^2 \frac{d\rho}{dr}\frac{1}{rv^2}dr \leq v^2(1)\rho'(1)\int_1^\infty \frac{dr}{rv^2(r)} < 0.$$

Then in particular using the facts that $\rho'(0) < 0$, $|\rho'(1)| < \infty$ and $v \in L_r^2(\mathbb{R}_+) \cap L_r^\infty(\mathbb{R}_+)$ we have:

$$\int_1^\infty \frac{dr}{rv^2(r)} < \infty, \quad \int_1^\infty rv^2 dr < \infty.$$

Using Cauchy-Schwarz Inequality we can write:

$$\infty = \int_1^\infty 1dr = \int_1^\infty \sqrt{rv^2(r)}\frac{1}{\sqrt{rv^2(r)}}dr \leq \left(\int_1^\infty rv(r)^2 dr\right)^{\frac{1}{2}} \left(\int_1^\infty \frac{dr}{rv(r)^2}\right)^{\frac{1}{2}} < \infty.$$

which is a contradiction. Therefore, $\rho(0)$ can not be different from 1, and the uniqueness of the solutions with $\phi(r) = a_0 r + \mathcal{O}(r^3)$ is proved.

5. Open Problems.

There are several problems which can be considered for further work. The first problem is in section 1, where we derived equations for vortex and dipole solutions at the bifurcation point, using the method of Lyapunov-Schmidt reduction. Although we proved the persistence of the vortex solution, we were not able to conclude on the persistence of the dipole solution. One can study the persistence of dipole solution using higher orders of perturbation theory.

The second problem that we can address is in section 2, where we used a numerical method to approximate eigenvalues of the matrix Schrödinger operator. As we mentioned before, the results are not very accurate near $\epsilon = \frac{1}{4}$, where the eigenvalues and vortex are known from analytical calculations. The challenge is to make the numerical results more accurate. It might be helpful to try different methods such as Runge-Kutta method, as the ODE solver, in order to get more accurate results.

The third problem is in section 3, where we used calculus of variations to prove the existence of the vortex solution. In the proof of existence, we were forced to truncate $\mathbb{R}_+$ to a bounded interval $[0, R]$ to get compactness arguments. In general, the existence should hold on the unbounded domain $\mathbb{R}_+$. Additionally we need to prove that the global minimizer of the energy functional is strictly positive on $\mathbb{R}_+$.

Appendix I: Shooting.m

```
format('longG');
h = 0.01;
s =  1.810223;
for j=4.81:h:10
for g=0:.1:.7
x = 0 : h : 1+g;
ep=1/j;
ep2=ep^2;
m = 0; ss = 0;
while (abs(s-ss) > 10^(-10)) && (abs(s-ss) < 100) && (m < 1000)
 ss = s; y(1) = 0; u(1) = s;         % (y,u) - components of U(x)
 y(2) = s*x(2) - s*x(2)^3/(8*ep2);   % Taylor series approximation
 u(2) = s - 3*s*x(2)^2/(8*ep2);      % for the first step
 z(1) = 0; v(1) = 1;                 % (z,v) - components of V(x)
 z(2) = x(2) - x(2)^3/(8*ep2);
 v(2) = 1-3*x(2)^2/(8*ep2);
  for k = 2 : length(x)-1             % iterations of the ODE solver
   yp = y(k) + h*u(k);
   up = u(k)-(h/ep2)*((ep2*u(k)/x(k))-(y(k)*(ep2)/x(k)^2)-(((x(k)^2)-1)*y(k))-y(k)^3);
   y(k+1) = y(k)+0.5*h*(u(k)+up);
   u(k+1) = u(k)-(0.5*h/ep2)*((ep2*u(k)/x(k)-(y(k)*ep2/x(k)^2)-(((x(k)^2)-1)*y(k))-...
       y(k)^3)+((ep2*up/x(k+1))-(yp*ep2/x(k+1)^2)-(((x(k+1)^2)-1)*yp)-yp^3));
   zp = z(k) + h*v(k);
   vp = v(k)-h/ep2*(ep2*v(k)/x(k)-(ep2/x(k)^2+(x(k)^2-1))*z(k)-3*y(k)^2*z(k));
   z(k+1) = z(k) + 0.5*h*(v(k)+vp);
   v(k+1) = v(k)-0.5*h/ep2*((ep2*v(k)/x(k)-((ep2/x(k)^2)+(x(k)^2-1))*z(k)-3*y(k)^...
       2*z(k))+(ep2*vp/x(k+1)-((ep2/x(k+1)^2)+(x(k+1)^2-1))*zp-3*yp^2*zp));
  end
 s = ss - y(length(x))/z(length(x));
 m = m+1;
end
end
if (abs(s-ss) < 100) && (m < 1000)
```

```
    fprintf('epsilon is %d\n',j);
    fprintf('the value of s is %d\n',s);
    fprintf('The shooting method converges in %d iterations\n',m);
    disp('.........')
else
    s
    disp('The shooting method fails.');pause
end
end
```

Appendix II: eigenvalues.m

```
st=200; %number of steps
L=10;   %length of interval
h=L/st; %step size for interval (0,1)
neig=[];%first eigenvalue
eig1=[];%Second eigenvalue
eig2=[];%Third eigenvalue
nep=[];%epsilon
a=0;
s =1.810223;
for j=4.8:h:10;
for g=0:.1:.7
x = 0:h:1+g;
y=zeros(1,st);
ep=1/j;
ep2=ep^2;
m = 0; ss = 0;
while (abs(s-ss) > 10^(-10)) && (abs(s-ss) < 100) && (m < 1000)
 ss = s; y(1) = 0; u(1) = s;          % (y,u) - components of u(x)
 y(2) = s*x(2) - s*x(2)^3/(8*ep2);   % Taylor series approximation
 u(2) = s - 3*s*x(2)^2/(8*ep2);      % for the first step
 z(1) = 0; v(1) = 1;                 % (z,v) - components of v(x)
 z(2) = x(2) - x(2)^3/(8*ep2);
 v(2) = 1-3*x(2)^2/(8*ep2);
```

```
  for k = 2 : length(x)-1             % iterations of the ODE solver
     yp = y(k) + h*u(k);
     up = u(k) - h/ep2*(ep2*u(k)/x(k)-(y(k)*ep2/x(k)^2)-(x(k)^2-1)*y(k)-y(k)^3);
     y(k+1) = y(k)+0.5*h*(u(k)+up);
     u(k+1) = u(k)-0.5*h/ep2*((ep2*u(k)/x(k)-(y(k)*ep2/x(k)^2)-(x(k)^2-1)*y(k)...
         -y(k)^3)+(ep2*up/x(k+1)-yp*ep2/x(k+1)^2-((x(k+1)^2-1)*yp)-yp^3));
     zp=z(k) + h*v(k);
     vp=v(k)-h/ep2*(ep2*v(k)/x(k)-((ep2/x(k)^2)+(x(k)^2-1))*z(k)-3*y(k)^2*z(k));
     z(k+1)= z(k)+0.5*h*(v(k)+vp);
     v(k+1)=v(k)-0.5*h/ep2*((ep2*v(k)/x(k)-(ep2/x(k)^2+(x(k)^2-1))*z(k)-3*y(k)^2*...
         z(k))+(ep2*vp/x(k+1)-((ep2/x(k+1)^2)+(x(k+1)^2-1))*zp-3*yp^2*zp));
  end
 s = ss - y(length(x))/z(length(x));
 m = m+1;
end
end
a=a+1;
yy(a,:)=y;
end
a=0;
for j=5:h:10
a=a+1;
ep=1/j;
nep=[nep,ep];
ep2=ep^2;
ri=@(n) n*h;
b11=[];
for i=1:st-1
v11(i)=2*ep2/h^2+4*ep2/ri(i)^2+ri(i)^2-1+2*yy(a,i+1)^2;
v12(i)=-ep2/h^2-ep2/(2*h*ri(i));
v21(i)=-ep2/h^2+ep2/(2*h*ri(i));
w11(i)=2*ep2/h^2+ri(i)^2-1+2*yy(a,i+1)^2;
w12(i)=-ep2/h^2-ep2/(2*h*ri(i));
w21(i)=-ep2/h^2+ep2/(2*h*ri(i));
b11(i)=yy(a,i+1)^2;
```

```
end
v=diag(v11)+diag(v12(1:end-1),1)+diag(v21(2:end),-1);
w=diag(w11)+diag(w12(1:end-1),1)+diag(w21(2:end),-1);
b11=diag(b11);
w0H=[-1+4*ep2/h^2,-4*ep2/h^2,zeros(1,st-2)];
w0V=[-ep2/h^2*(1-h/(2*ri(1))),zeros(1,st-2)];
A=[v,zeros(1,st-1)',b11;zeros(1,st-1),w0H;b11,w0V',w];
d=sort(eig(A));
neig=[neig,d(1,1)];eig1=[eig1,d(2,1)];eig2=[eig2,d(3,1)];
end
figure; plot(nep,neig,'.');
 xlabel('Epsilon'); ylabel('First Eigenvalue'); grid
 figure;  plot(nep,eig1,'.',nep,eig2,'+');hold on;
 xlabel('Epsilon'); ylabel('Second Eigenvalue & Third Eigenvalue'); grid
```

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