Approximations of the lattice dynamics

APPROXIMATIONS OF THE LATTICE DYNAMICS

BY

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To my family.

Abstract

This investigation is devoted to the study of the Fermi-Pasta-Ulam (FPU) lattice dynamics. Approximations of the FPU lattice dynamics have been an old subject, it is believed that the *stability of the FPU traveling waves depends on the stability of the KDV solitary waves.* The key question is: *Are the traveling waves of the FPU lattice stable if the traveling waves of KDV type equation are stable?.*

We consider the FPU lattice with the nonlinear potential which leads to the generalized Korteweg-de Vries (gKDV) equation, which is known to have orbitally stable traveling waves in a subcritical case and orbitally unstable traveling waves in critical and supercritical cases. In order to pursue the question asked above, we use the energy method.

We establish that the $H^s(\mathbb{R})$ norm of the solution of the gKDV equation is bounded by a time-independent constant in the subcritical case, whereas the $H^s(\mathbb{R})$ norm grows at most exponentially in time in the critical and supercritical cases. With the help of these results, we extend the time scale for the approximation of the traveling waves of the FPU lattice by the traveling waves of the gKDV equation logarithmically in the subcritical case. In the critical and supercritical cases, we extend the time scale by a double-logarithmic factor.

Our results show that the traveling waves of the FPU lattice are stable if the solitary waves of the gKDV equation are stable in the subcritical case. On the other hand, in the critical and supercritical cases, our results are restricted to small-norm initial data, which exclude solitary waves.

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Chapter 1

Introduction

1.1 A one dimensional Fermi-Pasta-Ulam chain

To study the transfer of heat energy in solids, E. Fermi, J. Pasta, and S. Ulam carried out a series of experiments at Los Alamos Laboratory in the middle of 1950s [1]. These experiments are discussed in detail in subsequent publications [2, 3, 4]. The solid was modeled by a one dimensional chain of particles of equal masses connected by nonlinear identical springs (Figure 1.1). If the domain is unbounded, any particle can be selected as the zeroth particle, so that all other particles can be labeled with $n \in \mathbb{Z}$. The equation of motion governing the nth particle in the chain can be obtained by applying Newton's law of motion.

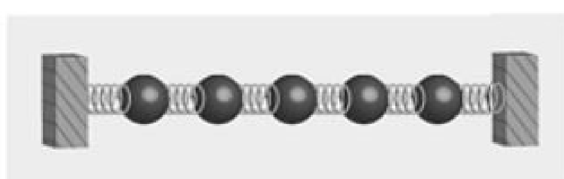

Figure 1.1: One dimensional spring mass system

Let us denote displacement produced in the nth particle by x_n. The spring connecting the nth and $(n+1)$th particle is deformed by the displacement $x_{n+1} - x_n$. Similarly, the spring connecting the nth and $(n-1)$th particles is deformed by the displacement $x_n - x_{n-1}$. Let V be the potential energy of each spring. The force exerted by each spring is $-V'$. Now applying Newton's law of motion to the nth particle we get

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}. \tag{1.1}$$

Let us now introduce variables for the relative displacement, $u_n = x_{n+1} - x_n$, $n \in \mathbb{Z}$. Equations (1.1) become $\ddot{x}_n = V'(u_n) - V'(u_{n-1})$, and $\ddot{x}_{n+1} = V'(u_{n+1}) - V'(u_n)$. Subtracting the former equation from the latter, we get

$$\ddot{u}_n = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z}. \tag{1.2}$$

The lattice equation (1.2) is referred to as Fermi-Pasta-Ulam (FPU) lattice equation. The sequence $(u_n)_{n\in\mathbb{Z}}$ is a function of time $t \in \mathbb{R}$, with values in $\mathbb{R}^{\mathbb{Z}}$ and the dot denotes the time derivative.

1.2 Some historical background

In [1], E. Fermi, J. Pasta, and S. Ulam studied the heat conduction problem modeled by the finite spring-mass system (Figure 1.1). Fermi, Pasta and Ulam considered different number of masses in their experiments, e.g., 16, 32, 64. They were expecting that upon excitation the nonlinear interactions would cause the equipartition of the energy. They carried out computer simulations, which led to a completely different outcome. The system did not approach equipartition, rather the energy returned to the originally excited mode and a few other nearby modes. This phenomenon is known as FPU recurrence.

The major development in the history of FPU system was made when N. J. Zabusky and M. D. Kruskal in [2] approximated the FPU spring-mass system by the Korteweg–de Vries (KDV) equation. They established a relation between the FPU recurrence and the nondestructive collision of solitons of the KDV equation. The authors of [2] explained FPU recurrence by decomposing the initial data into several solitons with different velocities. The lattice being of finite length reflects the solitons at the end points, which causes the solitons to collide with each other. However, during the collision solitons preserve their shapes and velocities. The authors of [2] conclude that at some later time, all the solitons arrive almost in the same phase and almost reconstruct the initial state through the nonlinear interaction.

Not everyone was convinced with the computation of FPU [1], the popular conjecture was that FPU had not run the simulations long enough and the time required to achieve equipartition of energy may be too long to observe numerically. In 1972 Los Alamos physicists J. L. Tuck and M. T. Menzel in [3] provided further insight into the FPU phenomenon with numerical simulations that revealed recurrences on long time scales. A more detailed overview of the FPU phenomenon is given by J. Ford in [4].

1.3 Formal derivation of the KDV type equation

For the class of an anharmonic potential, we will choose $V(u)$ in the form

$$V(u) = \frac{1}{2}u^2 + \frac{\epsilon^2}{p+1}u^{p+1}, \tag{1.3}$$

where ϵ is the strength of the anharmonicity and $p \geq 2,\ p \in \mathbb{N}$. The equation (1.2) can be re-written as

$$\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \epsilon^2(u_{n+1}^p - 2u_n^p + u_{n-1}^p), \quad n \in \mathbb{Z}. \tag{1.4}$$

The above FPU lattice (1.4) can be reduced to the generalized Koreweg-de Vries (gKDV) equation by using the leading order solution

$$u_n(t) = W\left(\epsilon(n-t), \epsilon^3 t\right) = W\left(\xi, \tau\right),\ \xi = \epsilon(n-t), \quad \text{and} \quad \tau = \epsilon^3 t.$$

Using formal Taylor's series, we expand u_{n+1} to get

$$\begin{aligned} u_{n+1} &= W\left(\epsilon(n-t) + \epsilon, \epsilon^3 t\right) = W\left(\xi + \epsilon, \tau\right) \\ &= W + \epsilon\, W_\xi + \frac{\epsilon^2}{2!}\, W_{\xi\xi} + \frac{\epsilon^3}{3!}\, W_{\xi\xi\xi} + \frac{\epsilon^4}{4!}\, W_{\xi\xi\xi\xi} + \mathcal{O}(\epsilon^5), \end{aligned} \tag{1.5}$$

here partial derivatives are denoted by subscripts. Now using the Taylor's series expansion for u_{n-1} we obtain

$$\begin{aligned} u_{n-1} &= W\left(\epsilon(n-t) - \epsilon, \epsilon^3 t\right) = W\left(\xi - \epsilon, \tau\right) \\ &= W - \epsilon\, W_\xi + \frac{\epsilon^2}{2!}\, W_{\xi\xi} - \frac{\epsilon^3}{3!}\, W_{\xi\xi\xi} + \frac{\epsilon^4}{4!}\, W_{\xi\xi\xi\xi} - \mathcal{O}(\epsilon^5). \end{aligned} \tag{1.6}$$

From (1.5) and (1.6) we obtain

$$u_{n+1} - 2\,u_n + u_{n-1} = \epsilon^2\, W_{\xi\xi} + \frac{1}{12}\epsilon^4 W_{\xi\xi\xi\xi} + \mathcal{O}(\epsilon^6) \tag{1.7}$$

and

$$u_{n+1}^p - 2\,u_n^p + u_{n-1}^p = \epsilon^2\, (W^p)_{\xi\xi} + \mathcal{O}(\epsilon^4). \tag{1.8}$$

Next applying the chain rule to $u_n = W(\xi, \tau)$, where $\xi = \epsilon(n-t),\ \tau = \epsilon^3 t$, we have

$$\dot{u}_n = \frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial W}{\partial \tau}\frac{\partial \tau}{\partial t} = -\epsilon W_\xi + \epsilon^3 W_\tau$$

and

$$\begin{aligned}\ddot{u}_n &= -\epsilon\left(-\epsilon W_\xi + \epsilon^3 W_\tau\right)_\xi + \epsilon^3\left(-\epsilon W_\xi + \epsilon^3 W_\tau\right)_\tau \\ &= \epsilon^2 W_{\xi\xi} - 2\,\epsilon^4 W_{\xi\tau} + \epsilon^6 W_{\tau\tau}.\end{aligned} \tag{1.9}$$

Substituting (1.7), (1.8) and (1.9) in (1.4) we obtain

$$2\,\epsilon^4 W_{\xi\tau} + \frac{1}{12}\epsilon^4 W_{\xi\xi\xi\xi} + \epsilon^4\left(W^p\right)_{\xi\xi} + \mathcal{O}(\epsilon^6) = 0.$$

Comparing the coefficients of ϵ^4, we get at the leading order,

$$2\,W_{\xi\tau} + \frac{1}{12}W_{\xi\xi\xi\xi} + \left(W^p\right)_{\xi\xi} = 0. \tag{1.10}$$

Integrating once with respect to ξ subject to the zero boundary conditions, we obtain

$$2\,W_\tau + \frac{1}{12}W_{\xi\xi\xi} + \left(W^p\right)_\xi = 0,\ \ \xi \in \mathbb{R}. \tag{1.11}$$

This equation (1.11) is called the generalized Korteweg–de Vries (gKDV) equation.

For $p = 2$, equation (1.11) reduces to the Korteweg–de Vries (KDV) equation derived by Korteweg and de Vries [5] in the context of propagation of long waves in shallow water. For $p = 3$, equation (1.11) reduces to the modified Korteweg–de Vries (mKDV) equation. The KDV and mKDV equations are studied extensively due to their relevance in a number of physical systems. It was found that these models are integrable by the inverse scattering transform method [6]. Equation (1.11) is called subcritical if $2 \le p \le 4$, critical if $p = 5$ and supercritical for $p \ge 6$, with respect to how $L^2(\mathbb{R})$ changes under a dilation.

1.4 Problem statement and motivation

Justification of the gKDV equation in the context of the FPU lattices has been a subject of interest for many authors. G. Friesecke and R.L. Pego in the series of papers [7, 8, 9, 10] justified the KDV approximation for traveling waves and proved the nonlinear stability of small amplitude solitary waves in generic FPU chains from analysis of the orbital and asymptotic stability of KDV solitons. T. Mizumachi [11, 12], A. Hoffman and C. E. Wayne [13], and G. N. Benes, A. Hoffman and C. E. Wayne [14] extended these results to the proof of asymptotic stability of several solitary waves in the FPU lattices. G. Schneider and C. E. Wayne in [15] obtained the validity of the KDV equation for time-dependent solutions on the time scale of $\mathcal{O}(\epsilon^{-3})$. J. Gaison, and S. Moskow, and J.D. Wright, and Q. Zhang in [16] generalized these results for polyatomic FPU lattices.

The approximation of the traveling waves in the FPU lattice by the KDV type equation leads to a popular belief that *The nonlinear stability of the FPU traveling waves resembles* *the orbital stability of the KDV solitary waves.* For the KDV equation, the positive traveling waves are orbitally stable for all amplitudes and the FPU traveling waves are also stable [7, 8, 9, 10]. On the other hand, there are some nonlinear potentials which may lead to the KDV type equations whose traveling waves are not stable for all amplitudes. For example, if we consider the nonlinear potential (1.3), we arrive at the generalized KDV equation (1.11), which is known to have orbitally stable traveling waves for $p = 2, 3, 4$ (subcritical case) and orbitally unstable traveling waves for $p \geq 5$ (critical and supercritical case) [17]. This leads to the question: *Are the traveling waves of the FPU lattice* (1.4) *stable if the traveling waves of the gKDV equation* (1.11) *are orbitally stable?* This is the main question addressed in this thesis.

Relevant results were obtained by E. Dumas and D. Pelinovsky in [18] on the validity of the KDV type approximation for stability theory of traveling waves in FPU lattices. In [18] an analytical technique was used to obtain the justification of the log-KDV equation in the context of FPU lattices. Nonlinear stability of all FPU traveling waves up to the time scale of $\mathcal{O}(\epsilon^{-3})$ was established, if the traveling waves satisfy the specific scaling leading to the log-KDV approximation (see Theorem 2 in [18]). Theorem 3 in [18] showed that the nonlinear stability of the FPU traveling waves on the time scale of $\mathcal{O}(\epsilon^{-3})$ may depend on the orbital stability of the traveling waves in the KDV type equation. Authors of [18] say that at first it appears that the results of Theorem 2 and Theorem 3 are in contradiction. They further claim that no contradiction arises as a matter of fact, because the energy method used in the proof of Theorems 2 and 3 gives the upper bound on the approximation errors to be exponentially growing at the time scale of $\mathcal{O}(\epsilon^{3})$. As a result, the unstable eigenvalues of the linearized gKDV equation at the traveling waves lead to exponential divergence at the time scale of $\mathcal{O}(\epsilon^{3})$, which can not be detected by the approximation results provided by Theorems 2 and 3.

The technique used in [18] is more general and is applicable to a large class of FPU models, which results in the generalized KDV equation with possibly large-amplitude traveling waves. One did not have to construct the two-dimensional manifold of traveling waves or to use projections and modulation equations from the theory in [8, 9, 10]. The latter theory gives a complete proof of nonlinear orbital stability of FPU traveling waves of small amplitude, but it relies on the information about the spectral and asymptotic stability of the KDV traveling waves, which is only available in the case of integrable KDV equation (1.11) with $p = 2$. The result regarding the nonlinear stability of the FPU traveling waves (Theorem 2 in [18]) does not depend on the nonlinear potential as long as some specific scaling leading to the KDV type approximation is available.

1.5 Organization of the Thesis

We consider the nonlinear potential (1.3) and approximate the traveling waves of the FPU lattice (1.4) by the traveling waves of the gKDV equation (1.11). Following the previous work in [15, 18], we show that FPU traveling waves can be approximated by the gKDV equation (1.11) up to the time scale $\mathcal{O}(\epsilon^3)$, even though the traveling waves of the gKDV equation (1.11) are orbitally unstable if $p \geq 5$.

We also give a partial answer to the question raised in the Section 1.4. We show that the traveling waves of the FPU lattice (1.4) are stable up to an extended time scale for the gKDV equation with $p = 2, 3$ (integrable subcritical cases) and for $p \geq 5$ (critical and super-critical cases), the latter results are proved under the small-norm assumption that exclude solitary waves.

The thesis is organized as follow. In Chapter 2 we discuss some properties of the gKDV equation, according to the following sections:

- In Section 2.1, the traveling solitary wave solution to the gKDV equation (1.11) is derived.
- In Section 2.2, we discuss the local and global well posedness of the gKDV equation (1.11). The discussion of global existence is further divided into the following subsections:
  - In Subsection 2.2.1, globally well posedness is carried out for the subcritical and critical gKDV equation (1.11) in $H^1(\mathbb{R})$.
  - In Subsection 2.2.2, the integrable cases of the gKDV equation (1.11) are discussed. Using conserved quantities, it is shown that the solution is globally well posed in $H^s(\mathbb{R})$ and the upper bound on the $H^s(\mathbb{R})$ norm of the solution does not depend on time τ, for every $s \in \mathbb{N}$.
  - In Subsection 2.2.3, we review a result about the global well posedness of the gKDV equation for $p \geq 5$, $p \in \mathbb{N}$, under the assumption of small initial data. We show that for $p = 5$, the $H^s(\mathbb{R})$ norm of the solution is growing at most exponentially in time τ.

In Chapter 3, we control the approximation of the traveling waves of the FPU lattice (1.2) by the traveling waves of the gKDV equation (1.11) up to the time scale of $\mathcal{O}(\epsilon^{-3})$. We also extend the time scale for the validity of KDV type approximation. Chapter 3 is further subdivided into the following sections:

- In Section 3.1, we recover the derivation of the gKDV equation formally given in Section 1.3.
- In Section 3.2, we develop some bounds on the residual terms of the KDV type approximation.

- In Section 3.3, justification of the gKDV equation on the standard time scale $\mathcal{O}(\epsilon^{-3})$ is carried out.
- In Section 3.4, justification of the gKDV equation on an extended time scale is carried out. Here we show that in the integrable cases of gKDV equation (1.11) with $p = 2, 3$, the time interval can be extended by a logarithmic factor. On the other hand, in the critical and supercritical cases, the time interval can be extended to a smaller double-logarithmic scale, under the assumption of small-norm initial data.

Chapter 4 concludes the thesis. We also discuss open problems left for further studies.

Chapter 2

Properties of the gKDV equation

2.1 Exact traveling wave solution to the gKDV equation

The generalized Korteweg–de Vries (gKDV) equation (1.11) has a traveling wave solution called the solitary wave. This solution is of the form

$$W(\xi, \tau) = W(\eta), \tag{2.1}$$

where $\eta = \xi - c\tau$ and $c > 0$ is a speed parameter. Using

$$W_\tau = -cW',\ W_\xi = W',\ W_{\xi\xi} = W'',\ W_{\xi\xi\xi} = W''',$$

where $W' = \frac{dW}{d\eta}$, we write the equation (1.11) in the form

$$-2c\,W' + \frac{1}{12}W''' + (W^p)' = 0. \tag{2.2}$$

If we integrate this equation, we find that

$$-2c\,W + \frac{1}{12}W'' + W^p = A, \tag{2.3}$$

where A is a constant of integration. Assuming that W along with all of its derivatives approaches zero as $\xi \to \pm\infty$, we set $A = 0$, so that the equation for W becomes

$$-2c\,W + \frac{1}{12}W'' + W^p = 0. \tag{2.4}$$

The equation (2.4) is an ordinary differential equation which can be solved explicitly. If we use W' as an integrating factor, we find that

$$-2c\,WW' + \frac{1}{12}W''W' + W^pW' = 0, \tag{2.5}$$

which, when integrated again with zero boundary condition, yields

$$\frac{1}{24}W'^2 + \frac{1}{p+1}W^{p+1} - cW^2 = 0. \tag{2.6}$$

Re-arranging this equation

$$W'^2 = 24\,cW^2 - \frac{24}{p+1}W^{p+1} = W^2\left(24\,c - \frac{24}{p+1}W^{p-1}\right). \tag{2.7}$$

Since we require our solution to be real valued, it follows from the last equation that

$$\left(24\,c - \frac{24}{p+1}W^{p-1}\right) \geq 0,$$

and we have

$$\frac{dW}{d\eta} = \pm W\sqrt{24\,c - \frac{24}{p+1}W^{p-1}}.$$

This separable differential equation can be solved by integration

$$\int \frac{dW}{W\sqrt{24\,c - \frac{24}{p+1}W^{p-1}}} = \pm\int d\eta. \tag{2.8}$$

Using the substitution

$$W = (c(p+1))^{\frac{1}{p-1}}\operatorname{sech}^{\frac{2}{p-1}}(\lambda), \tag{2.9}$$

we can integrate the left hand side of equation (2.8) to obtain

$$\begin{aligned}
\int \frac{dW}{W\sqrt{24\,c - \frac{24}{p+1}W^{p-1}}} &= \int \frac{(c(p+1))^{\frac{1}{p-1}}\,\frac{2}{p-1}\operatorname{sech}^{\frac{2}{p-1}}(\lambda)\,\tanh(\lambda)\,d\lambda}{(c(p+1))^{\frac{1}{p-1}}\operatorname{sech}^{\frac{2}{p-1}}(\lambda)\sqrt{24\,c - \frac{24}{p+1}c(p+1)\operatorname{sech}^2(\lambda)}} \\
&= \frac{1}{p-1}\int \frac{\tanh(\lambda)\,d\lambda}{\sqrt{6\,c}\sqrt{1-\operatorname{sech}^2(\lambda)}} \\
&= \frac{1}{(p-1)\sqrt{6c}}\int d\lambda \\
&= \frac{\lambda}{(p-1)\sqrt{6c}}.
\end{aligned}$$

Hence from equation (2.8) we have

$$\lambda = (p-1)\sqrt{6c}(\eta + B),$$

where B is an arbitrary constant of integration. Inserting the value of λ in equation (2.9), we obtain an explicit formula for W :

$$W(\eta) = (c(p+1))^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left(\sqrt{6c}(p-1)(\eta + B) \right). \tag{2.10}$$

The following graph represents W versus η for various values of p.

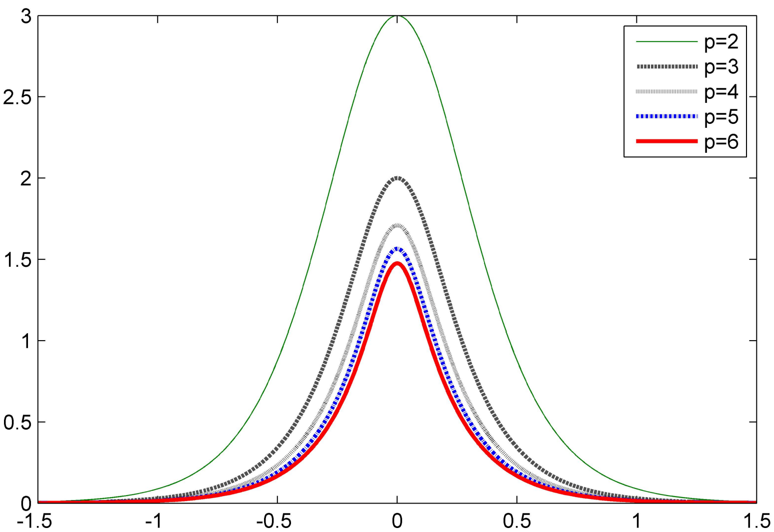

Figure 2.1: The solitary wave W for $p = 2, 3, 4, 5, 6$ and $B = 0$.

2.2 Global well posedness of the gKDV equation

In this section we discuss local and global well posedness for the gKDV equation

$$W_\tau + W_{\xi\xi\xi} + (W^p)_\xi = 0. \tag{2.11}$$

This equation is called subcritical for $p = 2, 3, 4$, critical for $p = 5$ and supercritical for $p \geq 6,\ p \in \mathbb{N}$. Local well posedness for the Cauchy problem associated with (2.11) has been studied by many authors. Below we summarize the local well posedness results.

- It was established in the work of T. Kato [19] that the gKDV equation (2.11) for any $p \geq 2$ is locally well posed in $H^s(\mathbb{R})$ with $s > \frac{3}{2}$.
- C. Kenig, G. Ponce and L. Vega in [20, 21] showed that the gKDV equation (2.11) is locally well posed in $H^s(\mathbb{R})$ with $s \geq \frac{3}{4}$ for $p = 2$, $s \geq \frac{1}{4}$ for $p = 3$, $s \geq \frac{1}{12}$ for $p = 4$, and $s \geq \frac{p-5}{2(p-1)}$ for $p \geq 5$.

The first result of global well posedness for the classical KDV equation with $p = 2$ can be traced back to the work [22] by J. L. Bona and R. Smith in 1975. Global existence of the solution W depends in a precise way on the nonlinearity. For $p = 2, 3, 4$, uniform bounds on the $H^1(\mathbb{R})$ norm can be easily obtained using the conserved quantities. However, for $p \geq 5$, conserved quantities do not provide a uniform bound on the $H^1(\mathbb{R})$ norm of the solution of the gKDV equation.

For the integrable cases $p = 2, 3$, a uniform bound on the $H^s(\mathbb{R})$ norms for any $s \in \mathbb{N}$ can also be obtained from higher order conserved quantities [23]. The global well posedness for the gKDV equation (2.11) with $p \geq 5$ in $H^s(\mathbb{R})$ with $s \geq s_p := \frac{p-5}{2(p-1)}$ was also established in [21], under the assumption of smallness of the $H^{s_p}(\mathbb{R})$-norm. It was later established in the work of Colliander et al in [24] that the gKDV equation (2.11) with $p = 2$ is globally well posed in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$ and the gKDV equation with $p = 3$ is globally well posed in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$.

The organization of this section is as follows. In subsection 2.2.1, by using appropriate inequalities, we prove that the solution to the gKDV equation (2.11) is globally well posed in $H^1(\mathbb{R})$ for $p = 2, 3, 4$, whereas the gKDV equation (2.11) for $p = 5$ is globally well posed in $H^1(\mathbb{R})$ for small $L^2(\mathbb{R})$ initial data. In subsection 2.2.2, we show for the integrable gKDV equation ($p = 2, 3$) that conserved quantities imply global existence of the solution in $H^s(\mathbb{R})$, for any $s \in \mathbb{N}$. Subsection 2.2.3 deals with the critical and supercritical gKDV equation (2.11) with $p \geq 5$, where $||W||_{H^s(\mathbb{R})}$ is growing at most exponentially.

2.2.1 Global existence in $H^1(\mathbb{R})$ ($p = 2, 3, 4, 5$).

The following theorem establishes global well posedness of the gKVD equation (2.11) in $H^1(\mathbb{R})$ for the subcritical case ($p = 2, 3, 4$) and for critical gKDV equation ($p = 5$) with small $L^2(\mathbb{R})$ initial data. We will use the following Gagliardo-Nirenberg inequality, see Appendix B.5 in [25]

$$||W||_{L^{p+1}}^{p+1} \leq C_{gn} \, ||W||_{L^2}^{\frac{p+3}{2}} \, ||W_\xi||_{L^2}^{\frac{p-1}{2}}, \tag{2.12}$$

where $C_{gn} > 0$ is Gagliardo-Nirenberg constant.

Theorem 2.1. *The Cauchy problem related to the generalized KDV equation* (2.11) *is globally well posed in* $H^1(\mathbb{R})$, *for* $2 \leq p \leq 4$. *Furthermore for* $p = 5$ *the gKDV equation* (2.11) *is well posed in* $H^1(\mathbb{R})$, *with small* $L^2(\mathbb{R})$ *initial data.*

Proof. To establish the upper bound on the $H^1(\mathbb{R})$ norm, the conservation laws are used. There are two conserved quantities related to the gKDV equation (2.11), which are given by

$$\begin{aligned} \text{Mass:} \qquad & \mathcal{H}_0 = \int W^2 d\xi, \\ \text{Hamiltonian:} \qquad & \mathcal{H}_1 = \int \left(W_\xi^2 - \frac{1}{p+1} W^{p+1} \right) d\xi, \end{aligned} \tag{2.13}$$

From (2.13), we have

$$\mathcal{H}_1 \geq ||W_\xi||^2_{L^2} - \frac{1}{p+1}||W^{p+1}||_{L^1}. \tag{2.14}$$

Next, we know that $||W^{p+1}||_{L^1} = ||W||^{p+1}_{L^{p+1}}$. By using Gagliardo-Nirenberg inequality (2.12) we get

$$\mathcal{H}_1 \geq ||W_\xi||^2_{L^2} - \frac{1}{p+1}C_{gn}||W||^{\frac{p+3}{2}}_{L^2}\,||W_\xi||^{\frac{p-1}{2}}_{L^2}. \tag{2.15}$$

Putting $p = 2$ in (2.15), we have

$$\begin{aligned}
\mathcal{H}_1 &\geq ||W_\xi||^2_{L^2} - \frac{1}{3}C_{gn}||W||^{\frac{5}{2}}_{L^2}\,||W_\xi||^{\frac{1}{2}}_{L^2} \\
&= ||W_\xi||^2_{L^2} - \frac{1}{3}C_{gn}\mathcal{H}_0{}^{\frac{5}{4}}\,||W_\xi||^{\frac{1}{2}}_{L^2}.
\end{aligned}$$

Since $\mathcal{H}_0{}^{\frac{5}{4}}\,||W_\xi||^{\frac{1}{2}}_{L^2} \leq \frac{1}{2}\mathcal{H}_0{}^{\frac{5}{2}} + \frac{1}{2}||W_\xi||_{L^2}$, we obtain from the last equation:

$$\begin{aligned}
\mathcal{H}_1 &\geq ||W_\xi||^2_{L^2} - \frac{C_{gn}}{6}\mathcal{H}_0{}^{\frac{5}{2}} - \frac{C_{gn}}{6}||W_\xi||_{L^2} \\
\mathcal{H}_1 + \frac{C_{gn}}{6}\mathcal{H}_0{}^{\frac{5}{2}} + \left(\frac{C_{gn}}{12}\right)^2 &\geq ||W_\xi||^2_{L^2} - \frac{C_{gn}}{6}||W_\xi||_{L^2} + \left(\frac{C_{gn}}{12}\right)^2 \\
\mathcal{H}_1 + \frac{C_{gn}}{6}\mathcal{H}_0{}^{\frac{5}{2}} + \left(\frac{C_{gn}}{12}\right)^2 &\geq \left(||W_\xi||_{L^2} - \frac{C_{gn}}{12}\right)^2 \\
||W_\xi||_{L^2} &\leq \frac{C_{gn}}{12} + \sqrt{\mathcal{H}_1 + \frac{C_{gn}}{6}\mathcal{H}_0{}^{\frac{5}{2}} + \left(\frac{C_{gn}}{12}\right)^2}.
\end{aligned}$$

Hence, there exists a constant $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ such that for every $t \in \mathbb{R}$, we have a global bound:

$$||W||_{H^1} \leq \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1). \tag{2.16}$$

Next putting $p = 3$ in (2.15) we have

$$\begin{aligned}
\mathcal{H}_1 &\geq ||W_\xi||^2_{L^2} - \frac{1}{4}C_{gn}||W||^3_{L^2}\,||W_\xi||_{L^2} \\
&= ||W_\xi||^2_{L^2} - \frac{1}{4}C_{gn}\mathcal{H}_0{}^{\frac{3}{2}}\,||W_\xi||_{L^2},
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{H}_1 + \left(\frac{C_{gn}\mathcal{H}_0^{\frac{3}{2}}}{8}\right)^2 &\geq ||W_\xi||_{L^2}^2 - 2\frac{C_{gn}}{8}\mathcal{H}_0{}^{\frac{3}{2}}\,||W_\xi||_{L^2} + \left(\frac{C_{gn}\mathcal{H}_0^{\frac{3}{2}}}{8}\right)^2 \\
\mathcal{H}_1 + \left(\frac{C_{gn}\mathcal{H}_0^{\frac{3}{2}}}{8}\right)^2 &\geq \left(||W_\xi||_{L^2} - \frac{C_{gn}\mathcal{H}_0^{\frac{3}{2}}}{8}\right)^2 \\
||W_\xi||_{L^2} &\leq \frac{C_{gn}\mathcal{H}_0^{\frac{3}{2}}}{8} + \sqrt{\mathcal{H}_1 + \left(\frac{C_{gn}\mathcal{H}_0^{\frac{3}{2}}}{8}\right)^2}. \qquad (2.17)
\end{aligned}$$

Hence, there exists a constant $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ such that for every $t \in \mathbb{R}$, we have a global bound:

$$||W||_{H^1} \leq \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1). \qquad (2.18)$$

Similarly for $p = 4$, equation (2.15) implies

$$\begin{aligned}
\mathcal{H}_1 &\geq ||W_\xi||_{L^2}^2 - \frac{1}{5}C_{gn}||W||_{L^2}^{\frac{7}{2}}\,||W_\xi||_{L^2}^{\frac{3}{2}} \\
&= ||W_\xi||_{L^2}^2 - \frac{1}{5}C_{gn}\mathcal{H}_0{}^{\frac{7}{4}}\,||W_\xi||_{L^2}^{\frac{3}{2}}. \qquad (2.19)
\end{aligned}$$

Let us denote $||W_\xi||_{L^2}^{\frac{1}{2}} = x$ and $\frac{1}{5}C_{gn}\mathcal{H}_0{}^{\frac{7}{2}} = \alpha$, then from equation (2.19), we obtain

$$\mathcal{H}_1 \geq x^4 - \alpha\, x^3 = f(x). \qquad (2.20)$$

Now $f'(x) = 4x^3 - 3\alpha x^2 = 0$ admits a nonzero root $x_0 = \frac{3\alpha}{4}$. The function $f(x)$ is monotonically increasing for $x > x_0$. Hence, for any $\mathcal{H}_0$, $\mathcal{H}_1 > 0$, there exists a constant $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ such that for every $t \in \mathbb{R}$, we have a global bound:

$$||W||_{H^1} \leq \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1). \qquad (2.21)$$

For $p = 5$, the conserved quantities do not provide a uniform bound in the $H^1(\mathbb{R})$ norm. Inserting $p = 5$, in equation (2.15), we obtain

$$\begin{aligned}
\mathcal{H}_1 &\geq ||W_\xi||_{L^2}^2 \left(1 - \frac{1}{6}C_{gn}||W||_{L^2}^4\right) \\
&= ||W_\xi||_{L^2}^2 \left(1 - \frac{1}{6}C_{gn}\mathcal{H}_0^2\right). \qquad (2.22)
\end{aligned}$$

Now if the initial $L^2(\mathbb{R})$ data is small enough, such that $1 > \frac{1}{6}C_{gn}\mathcal{H}_0^2$ then

$$||W_\xi||_{L^2}^2 \leq \frac{\mathcal{H}_1}{(1 - \frac{1}{6}C_{gn}\mathcal{H}_0^2)}. \tag{2.23}$$

Hence, for small enough initial data in the $L^2(\mathbb{R})$ norm, there exists a constant $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$ such that for every $t \in \mathbb{R}$, we have a global bound:

$$||W||_{H^1} \leq \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1). \tag{2.24}$$

□

2.2.2 Integrable cases ($p = 2, 3$)

The generalized KDV equation (2.11) reduces to the integrable KDV equation for $p = 2$ and to the integrable mKDV equation for $p = 3$. The integrable KDV equations possess an infinite number of conserved quantities [23, 26]. To simplify computations, let us consider the rescaled versions of the KDV ($p = 2$) and mKDV ($p = 3$) equations given by

$$W_\tau + W_{\xi\xi\xi} + WW_\xi = 0, \tag{2.25}$$

and

$$W_\tau + W_{\xi\xi\xi} + W^2W_\xi = 0, \tag{2.26}$$

respectively. The following result establishes the global existence of gKDV equation for $p = 2, 3$ in $H^s(\mathbb{R})$ for every $s \in \mathbb{N}$, such that the $H^s(\mathbb{R})$ norm of the global solution is bounded from above for all times.

Theorem 2.2. *There exists a unique global solution to the KDV equation* (2.25) *and mKDV equation* (2.26) *in $H^s(\mathbb{R})$ for every $s \in \mathbb{N}$. In particular, there exists a constant C_s such that for every $t \in \mathbb{R}$,*

$$||W||_{H^s(\mathbb{R})} \leq C_s.$$

Proof. It is well known, see [26], that the KDV equation (2.25) possess the following conserved quantities in addition to the mass and Hamiltonian in (2.13):

$$\begin{aligned}
\mathcal{H}_2 &= \int \left(W_{\xi\xi}^2 - \frac{5}{3}WW_\xi^2 + \frac{5}{36}W^4 \right) d\xi, \\
\mathcal{H}_3 &= \int \left(W_{\xi\xi\xi}^2 - \frac{7}{3}WW_{\xi\xi}^2 + \frac{35}{18}W^2W_\xi^2 - \frac{7}{108}W^5 \right) d\xi, \\
\mathcal{H}_4 &= \int \left(W_{\xi\xi\xi\xi}^2 + \frac{15}{17}W_{\xi\xi}^3 - 3WW_{\xi\xi\xi}^2 - \frac{35}{36}W_\xi^4 + \frac{7}{2}W^2W_{\xi\xi}^2 - \frac{35}{18}W^3W_\xi^2 + \frac{7}{216}W^6 \right) d\xi.
\end{aligned}$$

By using Sobolev inequality, we obtain for $\mathcal{H}_2$:

$$\begin{aligned}\mathcal{H}_2 &= \|W_{\xi\xi}\|_{L^2}^2 - \frac{5}{3}\int WW_\xi^2 d\xi + \frac{5}{36}\|W\|_{L^4}^4, \\ &\geq \|W_{\xi\xi}\|_{L^2}^2 - \frac{5}{3}C\|W\|_{H^1}^3,\end{aligned}$$

so that

$$\|W_{\xi\xi}\|_{L^2}^2 \leq \mathcal{H}_2 + \frac{5}{3}C\|W\|_{H^1}^3.$$

Since $\|W\|_{H^1}$ is controlled by (2.16), there exists a constant $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2)$ such that for any $t \in \mathbb{R}$, we have a global bound:

$$\|W\|_{H^2} \leq \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2). \tag{2.27}$$

Similarly, we obtain for $\mathcal{H}_3$ with Sobolev inequality

$$\begin{aligned}\mathcal{H}_3 &= \|W_{\xi\xi\xi}\|_{L^2}^2 - \frac{7}{3}\int WW_{\xi\xi}^2 d\xi + \frac{35}{18}\int W^2W_\xi^2 d\xi - \frac{7}{108}\int W^5 d\xi \\ &\geq \|W_{\xi\xi\xi}\|_{L^2}^2 - \frac{7}{3}C_1\|W\|_{H^2}^3 - \frac{7}{108}C_2\|W\|_{H^1}^5,\end{aligned}$$

so that

$$\|W_{\xi\xi\xi}\|_{L^2}^2 \leq \mathcal{H}_3 + \frac{7}{3}C_1\|W\|_{H^2}^3 + \frac{7}{108}C_2\|W\|_{H^1}^5.$$

Since $\|W\|_{H^2}$ is now controlled by (2.27), there exists a constant $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ such that for every $t \in \mathbb{R}$, we have a global bound:

$$\|W\|_{H^3} \leq \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3),$$

The method applies to $\mathcal{H}_4$, etc, to yield a global solution in $H^4(\mathbb{R})$, etc.

For the mKDV equation (2.26), we have the following list of additional conserved quantities [26]:

$$\begin{aligned}\mathcal{H}_2 &= \int \left(W_{\xi\xi}^2 - \frac{5}{3}W^2W_\xi^2 + \frac{1}{18}W^6\right) d\xi. \\ \mathcal{H}_3 &= \int \left(W_{\xi\xi\xi}^2 + \frac{63}{54}W_\xi^4 - \frac{63}{27}W^2W_{\xi\xi}^2 + \frac{35}{18}W^4W_\xi^2 - \frac{5}{216}W^6\right) d\xi. \\ \mathcal{H}_4 &= \int \left(W_{\xi\xi\xi\xi}^2 + \frac{51}{3}W_\xi^2W_{\xi\xi}^2 + \frac{20}{3}W\,W_{\xi\xi}^3 - 3W^2W_{\xi\xi\xi}^2 - \frac{399}{54}W^2W_\xi^4 + \frac{7}{2}W^4W_{\xi\xi}^2 \right. \\ &\quad \left. - \frac{35}{18}W^6W_\xi^2 + \frac{7}{648}W^{10}\right) d\xi.\end{aligned}$$

By using the Sobolev inequality, we obtain for $\mathcal{H}_2$:

$$\begin{aligned} \mathcal{H}_2 &\geq ||W_{\xi\xi}||_{L^2}^2 - \frac{5}{3}\int W^2 W_\xi^2 d\xi \\ &\geq ||W_{\xi\xi}||_{L^2}^2 - C||W||_{H^1}^4, \end{aligned}$$

so that

$$||W_{\xi\xi}||_{L^2}^2 \leq \mathcal{H}_2 + C||W||_{H^1}^4.$$

Since $||W||_{H^1}$ is controlled by (2.18), there exists a constant $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2)$ such that for any $t \in \mathbb{R}$, we have a global bound:

$$||W||_{H^2} \leq \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2). \tag{2.28}$$

The method extends for $\mathcal{H}_3$ etc, to yield a global solution in $H^3(\mathbb{R})$, etc. □

2.2.3 Critical and supercritical cases ($p \geq 5$)

V. Martel, F. Merle and P. Raphaël showed in a series of papers [27, 28, 29, 30] that there exists a solution W to the critical gKDV equation (2.11) with $p = 5$ such that $||W||_{H^1} \to \infty$ as $\tau \uparrow T$, where $T < +\infty$. This indicates a possibility of a blow up in a finite time . On the other hand, Theorem 2.1 excludes blow up for $p = 5$ if the initial data is small in the $L^2(\mathbb{R})$ norm. Further, numerical simulations by D. B. Dix and W. R. McKinney [31] suggest blow up in a finite time for the supercritical gKDV equation (2.11) with $p \geq 6$.

The following result of C. Kenig, G. Ponce, and L. Vega [21] eliminates blow up in a finite time for small-norm initial data. First we introduce the following notations. Let $1 \leq p, q \leq \infty$ and $f : \mathbb{R} \times [-\tau, \tau] \to \mathbb{R}$. Define

$$||f||_{L_\xi^p L_\tau^q} \equiv \left(\int_{-\infty}^{\infty} \left(\int_{-\tau}^{\tau} |f(\xi, \tau)|^q d\tau \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}$$

and

$$||f||_{L_\tau^q L_\xi^p} \equiv \left(\int_{-\tau}^{\tau} \left(\int_{-\infty}^{\infty} |f(\xi, \tau)|^p d\xi \right)^{\frac{q}{p}} d\tau \right)^{\frac{1}{q}}.$$

Theorem 2.3. *Let $p = 5$. There exists $\delta > 0$ such that for any initial $W_0 \in L^2(\mathbb{R})$ with*

$$||W_0||_{L^2} < \delta,$$

there exists a unique strong solution W of the Cauchy problem related to the gKDV equation (2.11) *satisfying*

$$W \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L^\infty(\mathbb{R}; L^2(\mathbb{R})),$$

and

$$\left|\left|\frac{\partial W}{\partial \xi}\right|\right|_{L^\infty_\xi L^2_\tau} \leq D < \infty. \tag{2.29}$$

Similar results also remain true for $p \geq 6$, see [21]. Our next result shows that the upper bound on the solution W in the $H^s(\mathbb{R})$ norm depends on τ and grows as $t \to \infty$. It is important for justification analysis to understand the growth rate of $||W||_{H^s(\mathbb{R})}$. First, we note the following Gronwall's inequality, see Appendix B.6 in [25].

Lemma 2.1. *Let $C \geq 0$, $k(t)$ be a given continuous non-negative function for all $t \geq 0$, and $y(t)$ be a continuous function satisfying the integral inequality*

$$0 \leq y(t) \leq C + \int_0^t k(t')y(t')dt'.$$

Then

$$y(t) \leq C\, e^{\int_0^t k(t')dt'} \text{ for all } t \geq 0. \tag{2.30}$$

Theorem 2.4. *For $p = 5$. Under the assumption of Theorem 2.3, the upper bound for the $H^s(\mathbb{R})$ norm of the solution W of the gKDV equation (2.11), depends on τ and grows at most exponentially as $\tau \to \infty$, that is, there exists a constant $c_s > 0$ and $k_s > 0$ such that*

$$||W||_{H^s(\mathbb{R})} \leq c_s e^{k_s \int_0^\tau ||W_\xi||_{L^\infty} d\tau'}. \tag{2.31}$$

Proof. For $p = 5$, we differentiate the gKDV equation (2.11) twice with respect to ξ, multiply by $W_{\xi\xi}$ and then integrate with respect to ξ, to obtain

$$\begin{aligned}&\int W_{\tau\xi\xi}W_{\xi\xi}d\xi + \int W_{\xi^5}W_{\xi\xi}d\xi + 60\int W^2W_\xi^3W_{\xi\xi}d\xi + 60\int W^3W_\xi W_{\xi\xi}^2 d\xi \\ &+5\int W^4 W_{\xi\xi\xi}W_{\xi\xi}d\xi = 0.\end{aligned} \tag{2.32}$$

Using integration by parts and zero boundary conditions, it can be shown that the term $\int W_{\xi^5}W_{\xi\xi}d\xi = 0$. Similarly, using integration by parts and zero boundary conditions, the term $5\int W^4W_{\xi\xi\xi}W_{\xi\xi}d\xi = -10\int W^3W_\xi W_{\xi\xi}^2 d\xi$. Hence from (2.32), we have

$$\frac{1}{2}\frac{d}{d\tau}\int W_{\xi\xi}^2 d\xi - 30\int WW_\xi^5 d\xi + 50\int W^3W_\xi W_{\xi\xi}^2 d\xi = 0. \tag{2.33}$$

Let $Q_2 = ||W_{\xi\xi}||_{L^2}$. Using Holder inequality, we rewrite (2.33) as follows:

$$Q_2\frac{dQ_2}{d\tau} \leq 30||W||_{L^\infty}||W_\xi||_{L^\infty}||W_\xi||_{L^4}^4 + 50||W||_{L^\infty}^3||W_\xi||_{L^\infty}\, Q_2^2.$$

By Gagliardo-Nirenberg inequality (2.12), we have

$$||W||_{L^4}^4 \leq C_{gn}||W_\xi||_{L^2}^3||W_{\xi\xi}||_{L^2}^2,$$

so that

$$\frac{dQ_2}{d\tau} \leq 30C_{gn}||W||_{L^\infty}||W_\xi||_{L^2}^3||W_\xi||_{L^\infty} + 50||W||_{L^\infty}^3||W_\xi||_{L^\infty}\, Q_2 \tag{2.34}$$

Thanks to Theorem 2.1, there exists a constant $k_1,\ k_2 > 0$ such that

$$30C_{gn}||W||_{L^\infty}||W_\xi||_{L^2}^3 \leq k_1,\ \ 50||W||_{L^\infty}^3 \leq k_2.$$

Therefore, we have

$$\frac{d}{d\tau}\left(e^{-k_2\int_0^\tau ||W_\xi||_{L^\infty}d\tau'}Q_2(\tau)\right) \leq k_1||W_\xi||_{L^\infty}e^{-k_2\int_0^\tau ||W_\xi||_{L^\infty}d\tau'}. \tag{2.35}$$

Now using Gronwall's inequality (2.30), we get

$$Q_2(\tau) \leq \left(\frac{k_1}{k_2} + Q_2(0)\right)e^{k_2\int_0^\tau ||W_\xi||_{L^\infty}d\tau'}$$

which yields the bound (2.31) for $s = 2$ and large τ.

Next we show that $||W||_{H^3(\mathbb{R})}$ is also growing at the same rate. Again we differentiate the gKDV equation (2.11) three times with respect to ξ, multiply by $W_{\xi\xi\xi}$ and then integrate with respect to ξ, to obtain

$$\begin{aligned}&\int W_{\tau\xi\xi\xi}W_{\xi\xi\xi}d\xi + \int W_{\xi^6}W_{\xi\xi\xi}d\xi + 120\int WW_\xi^4W_{\xi\xi\xi}d\xi + 360\int W^2W_\xi^2W_{\xi\xi}W_{\xi\xi\xi}d\xi\\ &+ 60\int W^3W_{\xi\xi}^2W_{\xi\xi\xi} + 80\int W^3W_\xi W_{\xi\xi\xi}^2d\xi + 5\int W^4W_{\xi^4}W_{\xi\xi\xi}d\xi = 0.\end{aligned} \tag{2.36}$$

Using integration by parts and zero boundary condition, for W and all of its derivatives, it can be shown that the term $\int W_{\xi^6}W_{\xi\xi\xi}d\xi = 0$. Similarly, using integration by parts and zero boundary conditions, we arrive at

$$\begin{aligned}60\int W^3W_{\xi\xi}^2W_{\xi\xi\xi}d\xi &= -15\int WW_\xi^4W_{\xi\xi\xi}d\xi + 60\int W^2W_\xi^2W_{\xi\xi}W_{\xi\xi\xi}d\xi\\ 5\int W^4W_{\xi^4}W_{\xi\xi\xi}d\xi &= -10\int W^3W_\xi W_{\xi\xi\xi}^2d\xi.\end{aligned}$$

From (2.36), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int W_{\xi\xi\xi}^2 d\xi &\leq 105\int W W_\xi^4 W_{\xi\xi\xi} d\xi + 420\int W^2 W_\xi^2 W_{\xi\xi} W_{\xi\xi\xi} d\xi \\ &+ 70\int W^3 W_\xi W_{\xi\xi\xi}^2 d\xi.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}||W_{\xi\xi\xi}||_{L^2}^2 &\leq 105||W W_\xi^3||_{L^\infty}||W_\xi||_{L^2}||W_{\xi\xi\xi}||_{L^2} + 420||W^2 W_\xi^2||_{L^\infty}||W_{\xi\xi}||_{L^2}||W_{\xi\xi\xi}||_{L^2} \\ &+ 70||W^3 W_\xi||_{L^\infty}||W_{\xi\xi\xi}||_{L^2}^2. \qquad (2.37)\end{aligned}$$

Let $Q_3 = ||W_{\xi\xi\xi}||_{L^2}$. Then, the differential inequality (2.37) can be written as

$$\begin{aligned}Q_3\frac{dQ_3}{d\tau} &\leq 105||W W_\xi^3||_{L^\infty}||W_\xi||_{L^2} Q_3 + 420||W^2 W_\xi^2||_{L^\infty} Q_2\, Q_3 \\ &+ 70||W^3 W_\xi||_{L^\infty} Q_3^2,\end{aligned}$$

so that

$$\begin{aligned}\frac{dQ_3}{d\tau} &\leq 105||W W_\xi^3||_{L^\infty}||W_\xi||_{L^2} + 420||W^2 W_\xi^2||_{L^\infty} Q_2 + 70||W^3 W_\xi||_{L^\infty} Q_3 \\ &\leq Const_1 + Const_2||W_\xi||_{L^\infty}\, Q_2^2 + Const_3||W_\xi||_{L^\infty} Q_3, \qquad (2.38)\end{aligned}$$

thanks to Sobolev's inequality and previous bounds. Using a similar technique as explained above, we obtain (2.31) for $s = 3$. The same method applies to $||W||_{H^4(\mathbb{R})}$, etc. □

Chapter 3

Approximations of the Fermi-Pasta-Ulam lattice dynamics

In this chapter, the relation between the FPU lattice equation (1.4) and the gKDV equation (1.11) is established. The FPU equation (1.4) can be written as the FPU system,

$$\begin{cases} \dot{u}_n = q_{n+1} - q_n, \\ \dot{q}_n = u_n - u_{n-1} + \epsilon^2 \left(u_n^p - u_{n-1}^p\right), \end{cases} \quad n \in \mathbb{Z}. \tag{3.1}$$

Any solution $(u,q) \in C^1(\mathbb{R}, l^2(\mathbb{Z}))$ of the FPU system (3.1) provides a $C^2(\mathbb{R}, l^2(\mathbb{Z}))$ solution u to the FPU equation (1.4). The FPU lattice system (3.1) admits the conserved energy

$$H := \frac{1}{2} \sum_{n\in\mathbb{Z}} \left(q_n^2 + u_n^2 + \frac{2\epsilon^2}{p+1} u_n^{p+1} \right). \tag{3.2}$$

In the next result, we prove local (in time) well-posedness of the Cauchy problem associated with the FPU system (3.1) in the sequence space $l^2(\mathbb{Z})$. In what follows, the sequences $\{u_n(t)\}_{n\in\mathbb{Z}}$, $\{q_n(t)\}_{n\in\mathbb{Z}}$ in the space $l^2(\mathbb{Z})$ are denoted by $\mathbf{u}(t)$ and $\mathbf{q}(t)$ respectively. To prove the local existence result, we recall the statement of the Banach fixed point theorem.

Definition 3.1. *Let M be a closed non-empty set in Banach space X. The operator $A : M \to M$ is called a contraction if there is $0 \le q < 1$ such that*

$$\forall x, y \in M : \ ||Ax - Ay||_X \le q||x - y||_X.$$

Theorem 3.1. (**Banach fixed point theorem**) *Let M be a closed non-empty set in a Banach space X and let $A : M \to M$ be a contraction operator. Then, there exists a unique fixed point of A in M, that is, there exists a unique $x \in M$ such that $A(x) = x$.*

By using Banach fixed point theorem, we establish local well-posedness of the Cauchy problem for the FPU system (3.1).

Theorem 3.2. *There exists a local solution $(\boldsymbol{u}(t), \boldsymbol{q}(t)) \in C^1\left([-t_0, t_0], l^2(\mathbb{Z})\right)$ for some*

$t_0 > 0$, *to the Cauchy problem associated with the FPU system* (3.1) *for initial data* $(\boldsymbol{u}_0, \boldsymbol{q}_0) \in l^2(\mathbb{Z})$ *such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *and* $\boldsymbol{q}(0) = \boldsymbol{q}_0$.

Proof. To prove the local well-posedness of the FPU system (3.1), we write the system in the integral form as

$$\begin{cases} \mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \nabla_+ \mathbf{q}(s)ds, \\ \mathbf{q}(t) = \mathbf{q}(0) + \int_0^t \nabla_- \mathbf{u}(s)ds + \epsilon^2 \int_0^t \nabla_- \mathbf{u}^p(s)ds, \end{cases} t \in [-t_0, t_0], \tag{3.3}$$

where $(\nabla_+ \mathbf{q})_n = \mathbf{q}_{n+1} - \mathbf{q}_n$ and $(\nabla_- \mathbf{u})_n = \mathbf{u}_n - \mathbf{u}_{n-1}$. The space $X = L^\infty([-t_0, t_0], l^2(\mathbb{Z}))$ is a Banach space. We consider an open ball $B_\delta(\mathbf{0}) \subset X$, with radius $\delta > 0$ and centered at $\mathbf{0}$. It is obvious that $||\nabla_\pm \mathbf{q}||_{l^2} \leq 2||\mathbf{q}||_{l^2}$.

First we show that the right hand side of the integral equation (3.3) defines a Lipschitz continuous map in the ball $B_\delta(\mathbf{0})$ with $\delta > 0$. Indeed, we have

$$\begin{aligned} ||\mathbf{u}(0) + \int_0^t \nabla_+ \mathbf{q}(s)ds||_X &= \sup_{t\in[-t_0,t_0]} \left|\left|\mathbf{u}(0) + \int_0^t \nabla_+ \mathbf{q}(s)ds\right|\right|_{l^2} \\ &\leq ||\mathbf{u}(0)||_{l^2} + \sup_{t\in[-t_0,t_0]} \left|\left|\int_0^t \nabla_+ \mathbf{q}(s)ds\right|\right|_{l^2} \\ &\leq ||\mathbf{u}(0)||_{l^2} + 2\int_0^{t_0} ||\mathbf{q}(s)||_{l^2}\, ds \\ &\leq ||\mathbf{u}(0)||_{l^2} + 2t_0 \, ||\mathbf{q}(t)||_X \, , \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &|| \ \mathbf{q}(0) + \int_0^t \nabla_- \mathbf{u}(s)ds + \epsilon^2 \int_0^t \nabla_- \mathbf{u}^p(s)ds\Big|\Big|_X \\ \leq \ & ||\mathbf{q}(0)||_{l^2} + \sup_{t\in[-t_0,t_0]} \left|\left|\int_0^t \nabla_- \mathbf{u}(s)ds\right|\right|_{l^2} + \epsilon^2 \sup_{t\in[-t_0,t_0]} \left|\left|\int_0^t \nabla_- \mathbf{u}^p(s)ds\right|\right|_{l^2} \\ \leq \ & ||\mathbf{q}(0)||_{l^2} + 2\int_0^{t_0} ||\mathbf{u}(s)||_{l^2}\, ds + 2\epsilon^2 \int_0^{t_0} ||\mathbf{u}^p(s)||_{l^2}\, ds \\ \leq \ & ||\mathbf{q}(0)||_{l^2} + 2t_0||\mathbf{u}(t)||_X + 2\epsilon^2 t_0 ||\mathbf{u}(t)||_X^p. \end{aligned} \tag{3.5}$$

In the last inequality we used the Banach algebra property of $X = L^\infty([-t_0, t_0], l^\infty(\mathbb{Z}))$. From the bounds (3.4) and (3.5) we conclude that for every $\delta > 0$ there is a sufficiently small $t_0 > 0$ such that the operator $\mathcal{A}$ maps $B_\delta(\mathbf{0})$ to itself, where $\mathcal{A}$ represents the right hand side of the integral equation (3.3).

Next we show that by choosing sufficiently small $t_0 > 0$, the Lipzchitz constant corresponding to the map $\mathcal{A} : B_\delta(\mathbf{0}) \to B_\delta(\mathbf{0})$ can be made smaller then unity, which results in a contraction of $\mathcal{A}$ in $B_\delta(\mathbf{0})$. We obtain

$$\begin{aligned}
\left|\left|\mathcal{A}\mathbf{u}' - \mathcal{A}\mathbf{u}''\right|\right|_X &= \sup_{t\in[-t_0,t_0]} \left|\left|\int_0^t \nabla_+ \mathbf{q}'(s)ds - \int_0^t \nabla_+\mathbf{q}''(s)ds\right|\right|_{l^2} \\
&= \sup_{t\in[-t_0,t_0]} \left|\left|\int_0^t \nabla_+ \left(\mathbf{q}' - \mathbf{q}''\right)(s)ds\right|\right|_{l^2} \\
&\leq 2\int_0^{t_0} \left|\left|\left(\mathbf{q}' - \mathbf{q}''\right)(s)\right|\right|_{l^2} ds \\
&\leq 2t_0 \left|\left|\mathbf{q}' - \mathbf{q}''\right|\right|_X \\
&= k_1 \left|\left|\mathbf{q}' - \mathbf{q}''\right|\right|_X ,
\end{aligned} \tag{3.6}$$

where $k_1 = 2t_0$, so that if $t_0 < \frac{1}{2}$ then $k_1 < 1$. Also we have

$$\begin{aligned}
\left|\left|\mathcal{A}\mathbf{q}' - \mathcal{A}\mathbf{q}''\right|\right|_X &= \sup_{t\in[-t_0,t_0]} \left|\left|\int_0^t \nabla_- \left(\mathbf{u}' - \mathbf{u}''\right)(s)ds + \epsilon^2\int_0^t \nabla_-(\mathbf{u}'^p - \mathbf{u}''^p)(s)ds\right|\right|_{l^2} \\
&\leq 2\int_0^{t_0} \left|\left|\left(\mathbf{u}' - \mathbf{u}''\right)(s)\right|\right|_{l^2} ds + 2\epsilon^2\int_0^{t_0} \left|\left|(\mathbf{u}'^p - \mathbf{u}''^p)(s)\right|\right|_{l^2} ds \\
&\leq 2t_0 \left|\left|\left(\mathbf{u}' - \mathbf{u}''\right)(s)\right|\right|_X + 2\epsilon^2 t_0 \left|\left|(\mathbf{u}'^p - \mathbf{u}''^p)(s)\right|\right|_X \\
&\leq 2t_0\left(1 + \epsilon^2\left|\left|(\mathbf{u}'^{p-1} + ... + \mathbf{u}''^{p-1})\right|\right|_X\right)\left|\left|\mathbf{u}' - \mathbf{u}''\right|\right|_X \\
&\leq k_2 \left|\left|\mathbf{u}' - \mathbf{u}''\right|\right|_X ,
\end{aligned} \tag{3.7}$$

where $k_2 = 2t_0\left(1 + \epsilon^2 p\delta^{p-1}\right)$, so that if $t_0 < \frac{1}{2(1+\epsilon^2 p\delta^{p-1})}$ then $k_2 < 1$. From (3.6) and (3.7) we conclude that if

$$t_0 \leq \min\left(\frac{1}{2}, \frac{1}{2\left(1 + \epsilon^2 p\delta^{p-1}\right)}\right)$$

then $k = \max(k_1, k_2) < 1$, showing that $\mathcal{A} : B_\delta(\mathbf{0}) \to B_\delta(\mathbf{0})$ is a contraction. By the Banach fixed point theorem, there exists a unique fixed point $(\mathbf{u}, \mathbf{q})$ in $B_\delta(\mathbf{0})$ such that

$$\mathcal{A}(\mathbf{u}, \mathbf{q}) = (\mathbf{u}, \mathbf{q}).$$

Since $\nabla_\pm$ are bounded operators, the integral of a continuous function gives a continuously differentiable function. Hence, $(\mathbf{u}(t), \mathbf{q}(t)) \in C^1([-t_0, t_0], l^2(\mathbb{Z}))$. □

Remark 3.1. *If p is an odd integer, then the local solution $(\boldsymbol{u}(t), \boldsymbol{q}(t)) \in C\left([-t_0, t_0], l^2(\mathbb{Z})\right)$ can be continued globally in time by using the energy conservation* (3.2).

3.1 Derivation of the gKDV equation and residual terms

We shall recover the derivation of the gKDV equation (1.11). In our present treatment, we work with the FPU lattice system (3.1). Let us consider the decomposition

$$u_n(t) = W(\epsilon(n-t), \epsilon^3 t) + \mathcal{U}_n(t), \;\; q_n = P_\epsilon(\epsilon(n-t), \epsilon^3 t) + \mathcal{P}_n(t), \;\; n \in \mathbb{Z}, \tag{3.8}$$

where $W(\xi, \tau)$ is a smooth solution to the gKDV equation (1.11). Therefore, W is independent of ϵ. On the other hand, $P_\epsilon(\xi, \tau)$ depends on ϵ. This function can be found from the first equation of system (3.1), which is rewritten as

$$P_\epsilon(\xi + \epsilon, \tau) - P_\epsilon(\xi, \tau) = -\epsilon \partial_\xi W(\xi, \tau) + \epsilon^3 \partial_\tau W(\xi, \tau). \tag{3.9}$$

We look for an approximate solution P_ϵ to this equation, under the form

$$P_\epsilon := P^0 + \epsilon P^1 + \epsilon^2 P^2 + \epsilon^3 P^3, \tag{3.10}$$

with functions P^j decaying to zero as $\xi \to \infty$. Inserting (3.10) into (3.9), we get

$$\begin{aligned} &\epsilon \partial_\xi P_\epsilon(\xi, \tau) + \frac{\epsilon^2}{2!} \partial_{\xi\xi} P_\epsilon(\xi, \tau) + \frac{\epsilon^3}{3!} \partial_{\xi\xi\xi} P_\epsilon(\xi, \tau) + \frac{\epsilon^4}{4!} \partial_{\xi\xi\xi\xi} P_\epsilon(\xi, \tau) + ... \\ &= -\epsilon \partial_\xi W(\xi, \tau) + \epsilon^3 \partial_\tau W(\xi, \tau). \end{aligned} \tag{3.11}$$

Collecting the terms in powers of ϵ, we get

$$\begin{aligned} \mathcal{O}(\epsilon): &\qquad \partial_\xi P^0 = -\partial_\xi W \\ \mathcal{O}(\epsilon^2): &\qquad \partial_\xi P^1 + \frac{1}{2} \partial_{\xi\xi} P^0 = 0 \\ \mathcal{O}(\epsilon^3): &\qquad \partial_\xi P^2 + \frac{1}{2} \partial_{\xi\xi} P^1 + \frac{1}{6} \partial_{\xi\xi\xi} P^0 = \frac{-1}{24} \partial_{\xi\xi\xi} W - \frac{1}{2} \partial_\xi W^p \\ \mathcal{O}(\epsilon^4): &\qquad \partial_\xi P^3 + \frac{1}{2} \partial_{\xi\xi} P^2 + \frac{1}{6} \partial_{\xi\xi\xi} P^1 + \frac{1}{24} \partial_{\xi\xi\xi\xi} P^0 = 0. \end{aligned}$$

These equations are satisfied when

$$\begin{aligned} P^0 &= -W \\ P^1 &= \frac{1}{2} \partial_\xi W \\ P^2 &= \frac{-1}{8} \partial_{\xi\xi} W - \frac{1}{2} W^p \\ P^3 &= \frac{1}{48} \partial_{\xi\xi\xi} W + \frac{1}{4} \partial_\xi W^p. \end{aligned}$$

When a solution W to the gKDV equation (1.11) is given, we can define

$$P_\epsilon = -W + \frac{\epsilon}{2} W_\xi - \frac{\epsilon^2}{8} W_{\xi\xi} - \frac{\epsilon^2}{2} W^p + \frac{\epsilon^3}{48} W_{\xi\xi\xi} + \frac{\epsilon^3}{4} p W^{p-1} W_\xi. \tag{3.12}$$

By construction, the pair of functions (W, P_ϵ) solves the first equation in system (3.1) up to the $\mathcal{O}(\epsilon^4)$ terms. Substituting the decomposition (3.8) into the FPU lattice system (3.1), we obtain the evolutionary problem for the error terms

$$\begin{cases} \dot{\mathcal{U}}_n = \mathcal{P}_{n+1} - \mathcal{P}_n + Res_n^1, \\ \dot{\mathcal{P}}_n = \mathcal{U}_n - \mathcal{U}_{n-1} + p\epsilon^2 \left(W(\epsilon(n-t), \epsilon^3 t))^{p-1}\mathcal{U}_n - W(\epsilon(n-1-t), \epsilon^3 t)^{p-1}\mathcal{U}_{n-1}\right) \\ \qquad + \mathcal{R}_n(W, \mathcal{U})(t) + Res_n^2(t), \end{cases} \tag{3.13}$$

where

$$\begin{aligned} \mathcal{R}_n(W,\mathcal{U})(t) \;&:=\; \epsilon^2 \left[\{W(\epsilon(n-t), \epsilon^3 t) + \mathcal{U}_n(t)\}^p - W(\epsilon(n-t), \epsilon^3 t)^p - pW(\epsilon(n-t), \epsilon^3 t)^{p-1}\mathcal{U}_n\right] \\ &- \;\epsilon^2[\{W(\epsilon(n-1-t), \epsilon^3 t) + \mathcal{U}_{n-1}(t)\}^p - W(\epsilon(n-1-t), \epsilon^3 t)^p \\ &- \;pW(\epsilon(n-1-t), \epsilon^3 t)^{p-1}\mathcal{U}_{n-1}], \end{aligned}$$

and

$$\begin{aligned} Res_n^1 \;&= P_\epsilon(\epsilon(n+1-t), \epsilon^3 t) - P_\epsilon(\epsilon(n-t), \epsilon^3 t) + \epsilon\partial_\xi W(\epsilon(n-t), \epsilon^3 t) - \epsilon^3\partial_\tau W(\epsilon(n-t), \epsilon^3 t), \\ Res_n^2 \;&= \epsilon\partial_\xi P_\epsilon(\epsilon(n-t), \epsilon^3 t) - \epsilon^3\partial_\tau P_\epsilon(\epsilon(n-t), \epsilon^3 t) + W(\epsilon(n-t), \epsilon^3 t) - W(\epsilon(n-1-t), \epsilon^3 t) \\ &\quad + \epsilon^2[W(\epsilon(n-t), \epsilon^3 t)^p - W(\epsilon(n-1-t), \epsilon^3 t)^p]. \end{aligned}$$

3.2 Bounds on residual terms

In this section we will develop bounds on the residual terms. To establish these bounds we recall the following result, proved in Lemma 5.1 in [18].

Lemma 3.1. *There exists $C > 0$ such that for all $X \in H^1$ and $\epsilon \in (0, 1]$,*

$$||x||_{l^2} \le C\epsilon^{-\frac{1}{2}}||X||_{H^1} \tag{3.14}$$

where $x_n := X(\epsilon n)$, $n \in \mathbb{Z}$.

We also use the following result from Chapter 4 [32],

Lemma 3.2. *Suppose $f : [a, b] \to \mathbb{R}$ is of class C^n, and $f^{(n+1)}$ is integrable. Then $\forall x \in [a, b]$ we have $f(x) = P_n(x) + R_n(x)$ where*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} \tag{3.15}$$

and

$$R_n(x) = \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-r)^n f^{(n+1)}(a + r(x-a))dr. \tag{3.16}$$

The next result provide bounds on $\mathcal{R}$, Res^1 and Res^2 in the time evolutionary system (3.13).

Lemma 3.3. *Let $W \in C\left([-\tau_0, \tau_0], H^6(\mathbb{R})\right)$ be a solution to the gKDV equation* (1.11) *for some $\tau_0 > 0$. Then, there exist positive constants C_W and $C_{W,\mathcal{U}}$, such that for all $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$ and $\epsilon \in (0,1]$,*

$$\left|\left|Res^1\right|\right|_{l^2} + \left|\left|Res^2\right|\right|_{l^2} \leq C_W \epsilon^{\frac{9}{2}}, \tag{3.17}$$

and

$$||\mathcal{R}(W, \mathcal{U})||_{l^2} \leq \epsilon^2 C_{W,\mathcal{U}} \, ||\mathcal{U}||^2_{l^2}, \tag{3.18}$$

where C_W and $C_{W,\mathcal{U}}$ are constant proportional to $||W||_{H^6} + ||W||^p_{H^6}$ and $||W||^{p-2}_{H^6} + ||\mathcal{U}||^{p-2}_{l^2}$ respectively.

Proof. To prove the bound (3.17), we use (3.12) in

$$Res^1_n = P_\epsilon(\epsilon(n+1-t), \epsilon^3 t) - P_\epsilon(\epsilon(n-t), \epsilon^3 t) + \epsilon\partial_\xi W(\epsilon(n-t), \epsilon^3 t) - \epsilon^3 \partial_\tau W(\epsilon(n-t), \epsilon^3 t).$$

By expanding the resulting expression with Taylor's theorem, we obtain

$$\begin{aligned} Res^1_n &= \epsilon^5 \Bigg(\frac{1}{480} W_{\xi^5}(\xi,\tau) + \frac{1}{24} p(p-1)(p-2) W(\xi,\tau)^{p-3} W_\xi(\xi,\tau)^3 \\ &\quad + \frac{1}{8} p(p-1) W(\xi,\tau)^{p-2} W_\xi(\xi,\tau) W_{\xi\xi}(\xi,\tau) + \frac{1}{24} p W(\xi,\tau)^{p-1} W_{\xi\xi\xi}(\xi,\tau) \Bigg) + \mathcal{O}(\epsilon^6), \end{aligned}$$

which can be written in the closed form (3.16) as follows:

$$\begin{aligned} Res^1_n &= \frac{1}{480}\epsilon^5 \int_0^1 (1-r)^4 \partial_{\xi\xi\xi\xi\xi\xi} W\left(\epsilon(n-t+r), \epsilon^3 t\right) dr \\ &+ \frac{1}{24}\epsilon^5 \int_0^1 (1-r)^2 \partial_{\xi\xi\xi} W^p\left(\epsilon(n-t+r), \epsilon^3 t\right) dr. \end{aligned}$$

Now using the bound (3.14) we get

$$\begin{aligned} \left|\left|Res^1\right|\right|_{l^2} &\leq \frac{1}{480}\epsilon^5 \int_0^1 (1-r)^4 ||\partial_{\xi\xi\xi\xi\xi\xi} W||_{l^2} dr + \frac{1}{24}\epsilon^5 \int_0^1 (1-r)^2 ||\partial_{\xi\xi\xi} W^p||_{l^2} dr \\ &\leq \frac{C}{2400}\epsilon^{\frac{9}{2}} ||\partial_{\xi\xi\xi\xi\xi\xi} W||_{H^1} + \frac{C}{72}\epsilon^{\frac{9}{2}} ||\partial_{\xi\xi\xi} W^p||_{H^1} \\ &\leq C_1 \epsilon^{\frac{9}{2}} \left(||W||_{H^6} + ||W||^p_{H^6}\right). \end{aligned}$$

Next, we expand the expression

$$\begin{aligned} Res^2_n &= \epsilon\partial_\xi P_\epsilon(\epsilon(n-t), \epsilon^3 t) - \epsilon^3 \partial_\tau P_\epsilon(\epsilon(n-t), \epsilon^3 t) + W(\epsilon(n-t), \epsilon^3 t) - W(\epsilon(n-1-t), \epsilon^3 t) \\ &+ \epsilon^2 (W(\epsilon(n-t), \epsilon^3 t)^p - W(\epsilon(n-1-t), \epsilon^3 t)^p), \end{aligned}$$

in Taylor's series. The coefficients of ϵ^0, ϵ^1, ϵ^2 vanish, whereas the coefficient of ϵ^3 also vanishes by using the fact that W is a solution to the gKDV equation (1.11). The coefficient of ϵ^4 also vanishes in the following computation:

$$\begin{aligned}
& \frac{1}{48}W_{\xi\xi\xi\xi} + \frac{1}{4}p(p-1)W^{p-2}W_{\xi\xi} + \frac{1}{4}pW^{p-1}W_\xi^2 - \frac{1}{2}pW^{p-1}W_{\xi\xi} - \frac{1}{2}p(p-1)W^{p-2}W_\xi^2 - \frac{1}{2}W_{\xi\tau} \\
= & \frac{-1}{48}W_{\xi\xi\xi\xi} - \frac{1}{4}p(p-1)W^{p-2}W_{\xi\xi} - \frac{1}{4}pW^{p-1}W_\xi^2 - \frac{1}{2}W_{\xi\tau} \\
= & \frac{-1}{4}\partial_\xi\left(2W_\tau + \frac{1}{12}W_{\xi\xi\xi} + pW^{p-1}W_\xi\right) \\
= & 0.
\end{aligned}$$

Finally, the coefficient of $\mathcal{O}(\epsilon^5)$ is given by

$$\begin{aligned}
& \frac{1}{5!}W_{\xi^5}(\xi,\tau) + \frac{p}{3!}W(\xi,\tau)^{p-1}W_{\xi\xi\xi}(\xi,\tau) + \frac{p(p-1)}{2}W_\xi(\xi,\tau)W_{\xi\xi}(\xi,\tau)W(\xi,\tau)^{p-2} \\
& + \frac{p(p-1)(p-2)}{3!}W(\xi,\tau)^{p-3}W_\xi(\xi,\tau)^3 + \frac{1}{8}\partial_\tau\left(W_{\xi\xi}(\xi,\tau)\right) + \frac{1}{2}\partial_\tau\left(W(\xi,\tau)^p\right).
\end{aligned}$$

Now using the bound (3.14), and following the same lines as for Res_n^1, we get

$$\left|\left|Res^2\right|\right|_{l^2} \leq C_2\epsilon^{\frac{9}{2}}\left(||W||_{H^6} + ||W||_{H^6}^p\right).$$

If $W \in C([-\tau_0,\tau_0],H^6(\mathbb{R}))$, then there exists a positive constant $C_W = C_3\left(||W||_{H^6} + ||W||_{H^6}^p\right)$, with $C_3 = \max(C_1,C_2)$, such that the bound (3.17) holds.

Next, we prove the bound (3.18). To do so, we write $\mathcal{R}_n(W,\mathcal{U})(t)$ in the form

$$\begin{aligned}
\mathcal{R}_n(W,\mathcal{U})(t) &= \epsilon^2\left(\sum_{k=2}^{p}\binom{p}{k}W(\epsilon(n-t),\epsilon^3 t)^{p-k}\mathcal{U}_n(t)^k\right) \\
&\quad -\epsilon^2\left(\sum_{k=2}^{p}\binom{p}{k}W(\epsilon(n-1-t),\epsilon^3 t)^{p-k}\mathcal{U}_{n-1}(t)^k\right).
\end{aligned} \tag{3.19}$$

Interpolating between the end point terms, we obtain for some $C > 0$:

$$\begin{aligned}
||\mathcal{R}(W,\mathcal{U})(t)||_{l^2} &\leq C\,\epsilon^2\left(||W(\epsilon(n-t),\epsilon^3 t)^{p-2}\mathcal{U}_n^2(t)||_{l^2}^2 + ||\mathcal{U}^p||_{l^2}\right) \\
&\leq C\,\epsilon^2||W(\epsilon(.-t),\epsilon^3 t)||_{L^\infty}^{p-2}||\mathcal{U}(t)||_{l^2}^2 + C\,\epsilon^2||\mathcal{U}||_{l^\infty}^{p-2}||\mathcal{U}||_{l^2}^2 \\
&\leq C\,\epsilon^2\left(||W||_{H^6}^{p-2} + ||\mathcal{U}||_{l^2}^{p-2}\right)||\mathcal{U}||_{l^2}^2,
\end{aligned}$$

which proves the bound (3.18) with $C_{W,\mathcal{U}} = C\left(||W||_{H^6}^{p-2} + ||\mathcal{U}||_{l^2}^{p-2}\right)$. □

3.3 Justification of gKDV equation on the standard time scale

We now formulate the main justification result. It is proved by using the estimates developed in Section 3.2 and the energy method.

Theorem 3.3. *Let $W \in C([-\tau_0, \tau_0], H^6(\mathbb{R}))$ be a solution to the gKDV equation* (1.11) *for any $\tau_0 > 0$. Then there exists positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{in,\epsilon}, q_{in,\epsilon}) \in l^2(\mathbb{Z})$ are given such that*

$$||u_{in,\epsilon} - W(\epsilon\cdot, 0)||_{l^2} + ||q_{in,\epsilon} - P_\epsilon(\epsilon\cdot, 0)||_{l^2} \le \epsilon^{\frac{3}{2}}, \tag{3.20}$$

the unique solution (u_ϵ, q_ϵ) to the FPU lattice equation (3.1) *with initial data $(u_{in,\epsilon}, q_{in,\epsilon})$ belongs to $C^1([-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}], l^2(\mathbb{Z}))$ and satisfy for every $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$:*

$$||u_\epsilon(t) - W(\epsilon(\cdot - t), \epsilon^3 t)||_{l^2} + ||q_\epsilon(t) - P_\epsilon(\epsilon(\cdot - t), \epsilon^3 t)||_{l^2} \le C_0\epsilon^{\frac{3}{2}}. \tag{3.21}$$

Proof. For ϵ_0 small enough and for the initial data (u_{in}, q_{in}) satisfying the bound (3.20), we can define a local in time solution (u, q) to the FPU lattice system (3.1), which is decomposed according to (3.8). Let us define for a fixed $C > 0$:

$$\mathcal{T}_C := \sup\left\{T \in [0, \tau_0\epsilon^{-3}] : \quad \mathcal{Q}(t) \le C\,\epsilon, \;\; t \in [-T, T]\right\}, \tag{3.22}$$

where $\mathcal{Q} = E^{\frac{1}{2}}$ and E is the energy like quantity given by

$$E(t) := \frac{1}{2}\sum_{n\in\mathbb{Z}} \left[\mathcal{P}_n^2 + \mathcal{U}_n^2 + \epsilon^2 pW(\epsilon(n-t), \epsilon^3 t)^{p-1}\mathcal{U}_n^2(t)\right]. \tag{3.23}$$

We prove that for fixed $C > 0$ and ϵ_0 small enough, we have $\mathcal{T}_C = \tau_0\epsilon^{-3}$. From (3.23), we can write

$$\begin{aligned} 2E(t) &= \left(||\mathcal{P}(t)||_{l^2}^2 + ||\mathcal{U}(t)||_{l^2}^2 + \epsilon^2\sum_{n\in\mathbb{Z}} pW(\epsilon(n-t), \epsilon^3 t)^{p-1}\mathcal{U}_n(t)^2\right) \\ &\ge ||\mathcal{P}(t)||_{l^2}^2 + ||\mathcal{U}(t)||_{l^2}^2\left(1 - \epsilon^2||pW(\epsilon(\cdot - t), \epsilon^3 t)^{p-1}||_{L^\infty}\right), \end{aligned}$$

hence for $\epsilon_0 < \min\left(1, ||2pW(\epsilon(\cdot - t))^{p-1}||_{L^\infty}^{-\frac{1}{2}}\right)$, and $\epsilon \in (0, \epsilon_0)$, we have

$$||\mathcal{P}||_{l^2}^2 + ||\mathcal{U}||_{l^2}^2 \le 4E(t), \;\; t \in (0, \mathcal{T}_C). \tag{3.24}$$

Taking the derivative of E with respect to t, we obtain

$$\begin{aligned} \frac{dE}{dt} = \sum_{n\in\mathbb{Z}} \; &[\mathcal{P}_n\mathcal{R}_n(W,\mathcal{U}) + \mathcal{P}_n Res_n^2(t) + \mathcal{U}_n(t)Res_n^1(t) + \epsilon^2 pW^{p-1}(\epsilon(n-t), \epsilon^3 t)\mathcal{U}_n(t)Res_n^1(t) \\ &+ \tfrac{\epsilon^2}{2}p(p-1)W^{p-2}(\epsilon(n-t), \epsilon^3 t)(-\epsilon\partial_\xi + \epsilon^3\partial_\tau)W(\epsilon(n-t), \epsilon^3 t)\mathcal{U}_n^2(t)]. \end{aligned}$$

Then using Cauchy-Schwartz inequality, we estimate

$$\left|\frac{dE}{dt}\right| \leq ||\mathcal{P}||_{l^2}||\mathcal{R}(W,\mathcal{U})||_{l^2} + ||\mathcal{P}||_{l^2}||Res^2(t)||_{l^2} + \frac{3}{2}||\mathcal{U}(t)||_{l^2}||Res^1(t)||_{l^2} + \epsilon^3\tilde{C}_W||\mathcal{U}(t)||^2_{l^2},$$

where $\tilde{C}_W = C||W||^{p-2}_{H^6}\left(||W||_{H^6} + ||W||^p_{H^6}\right)$. Using Lemma 3.2 and inequality (3.24), we can write

$$\begin{aligned} \left|\frac{dE}{dt}\right| &\leq 8C_{W,\mathcal{U}}\epsilon^2 E^{\frac{3}{2}} + 5C_W\epsilon^{\frac{9}{2}}E^{\frac{1}{2}} + 4\tilde{C}_W\epsilon^3 E, \qquad (3.25)\\ &\leq 2\,\hat{C}_{W,\mathcal{U}}E^{\frac{1}{2}}\left(\epsilon^{\frac{9}{2}} + \epsilon^3 E^{\frac{1}{2}} + \epsilon^2 E\right). \end{aligned}$$

where $\hat{C}_{W,\mathcal{U}} = \max\left(4C_{W,\mathcal{U}}, \frac{5}{2}C_W, 2\tilde{C}_W\right)$. Choosing $\mathcal{Q} = E^{\frac{1}{2}}$, the energy balance equation can be written in the form

$$\begin{aligned} \left|\frac{d\mathcal{Q}}{dt}\right| &\leq \hat{C}_{W,\mathcal{U}}\left(\epsilon^{\frac{9}{2}} + \epsilon^3\mathcal{Q} + \epsilon^2\mathcal{Q}^2\right),\\ &\leq \hat{C}_{W,\mathcal{U}}\left(\epsilon^{\frac{9}{2}} + (1+C)\epsilon^3\mathcal{Q}\right). \end{aligned}$$

The latter inequality remains true as long as $\mathcal{T}_C$ is defined by (3.22) so that $||\mathcal{U}||_{l^2} \leq C\epsilon$, for some $C > 0$. Next, $\mathcal{Q}(t)$ can be written for $t > 0$ as

$$\begin{aligned} \mathcal{Q}(t) &= \mathcal{Q}(0) + \int_0^t \mathcal{Q}'(s)ds,\\ &\leq \mathcal{Q}(0) + \hat{C}_{W,\mathcal{U}}\int_0^t\left(\epsilon^{\frac{9}{2}} + (1+C)\epsilon^3\mathcal{Q}(s)\right)ds,\\ &= \mathcal{Q}(0) + \hat{C}_{W,\mathcal{U}}\epsilon^{\frac{9}{2}}t + \hat{C}_{W,\mathcal{U}}(1+C)\int_0^t \epsilon^3\mathcal{Q}(s)ds. \end{aligned}$$

By using Gronwall's inequality (2.30), we obtain

$$\mathcal{Q}(t) \leq \left(\mathcal{Q}(0) + \hat{C}_{W,\mathcal{U}}\epsilon^{\frac{9}{2}}|t|\right)e^{(1+C)\epsilon^3\hat{C}_{W,\mathcal{U}}|t|}.$$

The initial bound (3.20) ensures that $\mathcal{Q}(0) \leq C_0\epsilon^{\frac{3}{2}}$. Hence

$$\mathcal{Q}(t) \leq \left(C_0 + \hat{C}_{W,\mathcal{U}}\tau_0\right)\epsilon^{\frac{3}{2}}e^{(1+C)\hat{C}_{W,\mathcal{U}}\tau_0}, \quad t \in (-\mathcal{T}_C, \mathcal{T}_C). \qquad (3.26)$$

Finally, choosing ϵ_0 sufficiently small such that the bound $||\mathcal{U}(t)||_{l^2} \leq C\,\epsilon$ is preserved, which is always possible because $\epsilon^{\frac{3}{2}} << \epsilon$, we obtain a continuation of the local solution to the time scale $\mathcal{T}_C = \tau_0\epsilon^{-3}$. □

3.4 Justification of gKDV equation on the extended time scale

In this section, we extend the justification analysis to a longer time interval for solution of the gKDV equation (1.11). We consider two cases here: the integrable case with $p = 2, 3$ and the critical and supercritical case with $p \geq 5$. In the first case, we show that the time interval of the solution of the gKDV equation (1.11) can be extended by a logarithmic factor. In the second case, we show that the time interval can only be extended by a double logarithmic factor. In both the cases, the time τ_0 used in Section 3.3 becomes ϵ-dependent.

Case 1: Integrable gKDV $(p = 2, 3)$

Recall from Chapter 2 (Theorem 2.2) that the upper bound on the $H^6(\mathbb{R})$ norm of the solution to the gKDV equation (1.11) with $p = 2, 3$ can be bounded by a time-independent constant. Therefore,

$$\delta = \sup_{\tau\in[-\tau_0,\tau_0]} ||W(t)||_{H^6} \tag{3.27}$$

is independent of τ_0. The following theorem specifies the extension of the time scale by a logarithmic factor, where the gKDV equation (1.11) with $p = 2, 3$ can be justified.

Theorem 3.4. *Let $W \in C(\mathbb{R}, H^6(\mathbb{R}))$ be a global solution to the gKDV equation (1.11) with $p = 2, 3$. For fixed $r \in \left(0, \frac{1}{2}\right)$, there exists positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{in,\epsilon}, q_{in,\epsilon}) \in l^2(\mathbb{Z})$ are given such that*

$$||u_{in,\epsilon} - W(\epsilon\cdot, 0)||_{l^2} + ||q_{in,\epsilon} - P_\epsilon(\epsilon\cdot, 0)||_{l^2} \leq \epsilon^{\frac{3}{2}}, \tag{3.28}$$

the unique solution (u_ϵ, q_ϵ) to the FPU lattice equation (3.1) with initial data $(u_{in,\epsilon}, q_{in,\epsilon})$ belongs to $C^1\left(\left[-\frac{r|\log(\epsilon)|}{k_0}\epsilon^{-3}, \frac{r|\log(\epsilon)|}{k_0}\epsilon^{-3}\right], l^2(\mathbb{Z})\right)$, where an ϵ-independent k_o is defined by (3.33) below and (u_ϵ, q_ϵ) satisfy

$$||u_\epsilon(t) - W(\epsilon(\cdot - t), \epsilon^3 t)||_{l^2} + ||q_\epsilon(t) - P_\epsilon(\epsilon(\cdot - t), \epsilon^3 t)||_{l^2} \leq C_0 \epsilon^{\frac{3}{2}-r}, \tag{3.29}$$

for every $t \in \left[-\frac{r|\log(\epsilon)|}{k_0}\epsilon^{-3}, \frac{r|\log(\epsilon)|}{k_0}\epsilon^{-3}\right]$.

Proof. By Theorem 3.2, for ϵ_0 small enough and for the initial data $(u_{in,\epsilon}, q_{in,\epsilon})$ satisfying the bound (3.28), we can define local in time solution of the FPU system (3.1), which is decomposed according to equation (3.8). Let us define for $C > 0$ and $r \in \left(0, \frac{1}{2}\right)$

$$T_{C,r} := \sup\left\{T \in [0, \tau_0(\epsilon)\epsilon^{-3}] : \mathcal{Q}(t) \leq C\epsilon^{\frac{3}{2}-r}, t \in [-T, T]\right\}, \tag{3.30}$$

where $\tau_0(\epsilon)$ is defined by (3.36) below. By defining an energy like quantity (3.23) and following the same agrement as in the proof of Theorem 3.3, see the estimate (3.25), we arrive at the following equation

$$\left|\frac{d\mathcal{Q}}{dt}\right| = C_{W,\mathcal{U}}\epsilon^2\mathcal{Q}^2 + \hat{C}_W\epsilon^3\mathcal{Q} + C_W\epsilon^{\frac{9}{2}}, \tag{3.31}$$

where $\mathcal{Q} = E^{\frac{1}{2}}$. From Lemma 3.2, we know that the constants C_W and $C_{W,\mathcal{U}}$ are proportional to $||W||_{H^6} + ||W||_{H^6}^p$ and $||W||_{H^6}^{p-2} + ||\mathcal{U}||_{l^2}^{p-2}$, respectively. It was also established in the proof of Theorem 3.3 that the constant $\hat{C}_W$ is proportional to $||W||_{H^6}^{p-1} + ||W||_{H^6}^{2p-2}$. Using equation (3.27) and the estimate $||\mathcal{U}||_{l^2} \leq 2\mathcal{Q}$ in equation (3.31), we obtain

$$\left|\frac{d\mathcal{Q}}{dt}\right| \leq C_1\left(\delta^{p-1} + \delta^{2p-2} + \frac{1}{\epsilon}(\delta^{p-2} + \mathcal{Q}^{p-2})\mathcal{Q}\right)\epsilon^3\mathcal{Q} + C_2\left(\delta + \delta^p\right)\epsilon^{\frac{9}{2}}, \tag{3.32}$$

where C_1 and C_2 are positive constants. Since δ is independent of τ, there exists an ϵ-independent $k_0 > 0$ such that

$$C_1\left(\delta^{p-1} + \delta^{p-2} + \frac{1}{\epsilon}(\delta^{p-2} + \mathcal{Q}^{p-2})\mathcal{Q}\right) \leq k_0, \tag{3.33}$$

as long as $t \in [-T_{C,r}(\epsilon), T_{C,r}(\epsilon)]$. Using inequality (3.33) in the inequality (3.32), we obtain

$$\left|\frac{d\mathcal{Q}}{dt}\right| \leq \epsilon^3 k_0\mathcal{Q} + C_\delta\epsilon^{\frac{9}{2}}, \tag{3.34}$$

where C_δ depends on δ. From (3.34), we obtain

$$\frac{d}{dt}\left(e^{-\epsilon^3 k_0 t}\,\mathcal{Q}(t)\right) \leq C_\delta\epsilon^{\frac{9}{2}}e^{-\epsilon^3 k_0 t}.$$

Integrating this inequality from 0 to t, we arrive at

$$\mathcal{Q}(t) \leq \left(\mathcal{Q}(0) + \frac{C_\delta}{k_0}\epsilon^{\frac{3}{2}}\right)e^{k_0\tau_0(\epsilon)}, \tag{3.35}$$

where we used the restriction $\epsilon^3 t \leq \tau_0(\epsilon)$. To achieve the required extension of the time interval, we assume that

$$e^{k_0\tau_0(\epsilon)} = \frac{\mu}{\epsilon^r}, \tag{3.36}$$

where μ is a fixed constant and so is $r \in \left(0, \frac{1}{2}\right)$. Hence, we obtain

$$\tau_0(\epsilon) = \frac{r|\log(\epsilon)|}{k_0} + \mathcal{O}(1) \text{ as } \epsilon \to 0. \tag{3.37}$$

The initial condition (3.28) ensures that $\mathcal{Q}(0) \leq C_0\epsilon^{\frac{3}{2}}$ for some constant C_0. Using the bound (3.35) and equation (3.36), we arrive at

$$\mathcal{Q}(t) \leq C\epsilon^{\frac{3}{2}-r}, \tag{3.38}$$

where $C = C_0 + \frac{C_\delta}{k_0}$, which provides the required bound (3.29) for the time span (3.30) with $T_{C,r} = \tau_0\epsilon^{-3}$. □

Remark 3.2. *Since k_0 is ϵ-independent, equation* (3.37) *implies that*

$$\tau_0(\epsilon) \to \infty \text{ as } \epsilon \to 0.$$

Case 2: Critical and supercritical gKDV ($p \geq 5$)

From Theorem 2.3 in Chapter 2, we recall that the gKDV equation (1.11) with $p = 5$ is globally well posed under the assumption of $L^2(\mathbb{R})$ small initial data. From Theorem 2.4, we recall that the norm $||W||_{H^6(\mathbb{R})}$ of the solution W to the gKDV equation (1.11) grows exponentially in $\int_0^\tau ||W_\xi||_{L^\infty} d\tau'$. The same results also remain true for $p \geq 6$ [20, 21]. The following theorem specifies the extension of the time scale by a double-logarithmic factor, where the gKDV equation (1.11) with $p \geq 5$ can be justified. We assume that there exist C_s and k_s such that

$$\delta(\tau_0) = \sup_{\tau\in[-\tau_0,\tau_0]} ||W(\cdot,\tau)||_{H^6} \leq C_s e^{k_s\tau_0}. \tag{3.39}$$

Theorem 3.5. *Let $W \in C(\mathbb{R}, H^6(\mathbb{R}))$ be a global solution to the gKDV equation* (1.11) *for $p = 5$ satisfying the bound* (3.39)*. For fixed $r \in \left(0, \frac{1}{2}\right)$ there exist positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{in,\epsilon}, q_{in,\epsilon}) \in l^2(\mathbb{Z})$ are given such that*

$$||u_{in,\epsilon} - W(\epsilon\cdot, 0)||_{l^2} + ||q_{in,\epsilon} - P_\epsilon(\epsilon\cdot, 0)||_{l^2} \leq \epsilon^{\frac{3}{2}}, \tag{3.40}$$

the unique solution (u_ϵ, q_ϵ) to the FPU lattice equation (3.1) *with initial data $(u_{in,\epsilon}, q_{in,\epsilon})$ belongs to $C^1\left(\left[-\frac{1}{2k_s(p-1)}\log(|\log(\epsilon)|)\epsilon^{-3}, \frac{1}{2k_s(p-1)}\log(|\log(\epsilon)|)\epsilon^{-3}\right], l^2(\mathbb{Z})\right)$, where ϵ-independent k_s is defined by* (3.39)*, and satisfy*

$$||u_\epsilon(t) - W(\epsilon(\cdot - t), \epsilon^3 t)||_{l^2} + ||q_\epsilon(t) - P_\epsilon(\epsilon(\cdot - t), \epsilon^3 t)||_{l^2} \leq C_0\epsilon^{\frac{3}{2}-r}, \tag{3.41}$$

for every $t \in \left[-\frac{1}{2k_s(p-1)}\log(|\log(\epsilon)|)\epsilon^{-3}, \frac{1}{2k_s(p-1)}\log(|\log(\epsilon)|\right]$.

Proof. By Theorem 3.2, for ϵ_0 small enough and for the initial data $(u_{in,\epsilon}, q_{in,\epsilon})$ satisfying the bound (3.40), we can define local in time solution of the FPU system (3.1), which is decomposed according to equation (3.8). Let us define for fixed $C > 0$ and $r \in \left(0, \frac{1}{2}\right)$:

$$T_{C,r} := \sup\left\{T \in [0, \tau_0(\epsilon)\epsilon^{-3}] : \mathcal{Q}(t) \leq C\epsilon^{\frac{3}{2}-r}, t \in [-T, T]\right\}, \tag{3.42}$$

where $\tau_0(\epsilon)$ is defined by (3.47) below. By defining an energy like quantity (3.23) and following the same lines as in the proof of the Theorem 3.4, we arrive at the equation (3.32). Inequality (3.39) ensures that there exists a constant $\mathcal{C}_s > 0$ such that

$$C_1 \left(\delta^{p-1} + \delta^{2p-2} + \frac{1}{\epsilon} \left(\delta^{p-2} + \mathcal{Q}^{p-2} \right) \mathcal{Q} \right) \leq \mathcal{C}_s e^{2(p-1)k_s \tau}, \tag{3.43}$$

as long as $t \in [-T_{C,r}(\epsilon), T_{C,r}(\epsilon)]$. Using inequality (3.43) into the inequality (3.32), we obtain

$$\left| \frac{d\mathcal{Q}}{dt} \right| \leq \mathcal{C}_s \epsilon^3 e^{2(p-1)k_s \epsilon^3 t} \mathcal{Q}(t) + \tilde{\mathcal{C}} \epsilon^{\frac{9}{2}} e^{pk_s \epsilon^3 t}. \tag{3.44}$$

From (3.44), we obtain

$$\frac{d}{dt} \left(e^{-\frac{\mathcal{C}_s}{2(p-1)k_s} (e^{2(p-1)k_s \epsilon^3 t} - 1)} \mathcal{Q}(t) \right) \leq \tilde{\mathcal{C}} \epsilon^{\frac{9}{2}} \left(e^{pk_s \epsilon^3 t} \right) e^{-\frac{\mathcal{C}_s}{2(p-1)k_s} (e^{2(p-1)k_s \epsilon^3 t} - 1)}.$$

Integrating this inequality from 0 to t, we obtain

$$\mathcal{Q}(t) \leq \left(\mathcal{Q}(0) + \tilde{\mathcal{C}} \epsilon^{\frac{9}{2}} \int_0^t e^{-\frac{\mathcal{C}_s}{2(p-1)k_s} (e^{2(p-1)k_s \epsilon^3 t} - 1)} dt \right) e^{\frac{\mathcal{C}_s}{2(p-1)k_s} (e^{2(p-1)k_s \tau_0} - 1)}. \tag{3.45}$$

After integration, this bound can be further simplified as

$$\mathcal{Q}(t) \leq \left(\mathcal{Q}(0) + \tilde{\tilde{\mathcal{C}}} \epsilon^{\frac{3}{2}} \right) e^{\frac{\mathcal{C}_s}{2(p-1)k_s} (e^{2(p-1)k_s \tau_0} - 1)}, \tag{3.46}$$

where we used the restriction $\epsilon^3 t \leq \tau_0(\epsilon)$. To achieve the required extension of the time interval, we assume that

$$e^{\frac{\mathcal{C}_s}{2(p-1)k_s} (e^{2(p-1)k_s \tau_0} - 1)} = \frac{\mu}{\epsilon^r}, \tag{3.47}$$

where μ is fixed and so is $r \in \left(0, \frac{1}{2}\right)$. The above equation can be solved by taking log of both sides and then simplifying as

$$e^{2(p-1)k_s \tau_0} = \frac{2r(p-1)k_s}{\mathcal{C}_s} |\log(\epsilon)| + \mathcal{O}(1) \ \text{ as } \ \epsilon \to 0,$$

so that

$$\tau_0(\epsilon) = \frac{1}{2k_s(p-1)} \log|\log(\epsilon)| + \mathcal{O}(1) \ \text{ as } \ \epsilon \to 0. \tag{3.48}$$

The initial condition (3.40) ensures that $\mathcal{Q}(0) \leq C_0 \epsilon^{\frac{3}{2}}$ for some constant C_0. Using (3.46) and (3.47), we arrive at

$$\mathcal{Q}(t) \leq C\, \epsilon^{\frac{3}{2}-q}, \tag{3.49}$$

where $C = C_0 + \tilde{\tilde{\mathcal{C}}}$, which provides the required bound (3.41) for the time interval (3.42) with $T_{C,r} = \tau_0(\epsilon)\epsilon^{-3}$. □

Remark 3.3. *Since k_s is ϵ-independent, equation* (3.48) *implies that*

$$\tau_0 \to \infty \;\; as \;\; \epsilon \to 0.$$

Chapter 4

Conclusion

In this thesis, we established the following results.

- In Theorem 2.2, we showed that the upper bound on the $H^s(\mathbb{R})$ norm of the solution of the gKDV equation (2.11) with $p = 2, 3$ does not depend on time for any $s \in \mathbb{N}$.
- In Theorem 2.4, we showed that the upper bound on the $H^s(\mathbb{R})$ norm of the solution of the gKDV equation (2.11) with $p = 5$ grows at most exponentially, for any $s \geq 2,\ s \in \mathbb{N}$.
- In Theorem 3.5, we approximated dynamics of the FPU lattice (3.1) with solutions of the gKDV equation (1.11) for $p = 2, 3$ on the logarithmically extended time scale.
- In Theorem 3.6, we approximated dynamics of the FPU lattice (3.1) with solutions of the gKDV equation (1.11) for $p = 5$ on the double-logarithmically extended time scale.

We expect that our results will help in better understanding the approximations of the lattice dynamics. Based on our results we claim the following dynamical properties:

- Solitary waves of the FPU lattice (3.1) with $p = 2, 3$ can be approximated by the stable solitary waves of the gKDV equation (1.11) with $p = 2, 3$ on an extended time interval up to $\mathcal{O}(|\log(\epsilon)|\epsilon^{-3})$.
- Dynamics of small-norm solutions to the FPU lattice (3.1) with $p = 5$ can be approximated by the global small-norm solutions to the gKDV equation (1.11) with $p = 5$ on an extended time interval up to $\mathcal{O}(\log|\log(\epsilon)|\epsilon^{-3})$.

Finally we present open problems, which are left for further studies:

- We are not able to extend the time scale of the gKDV equation (1.11) with $p = 4$ by a logarithmic factor. The difficulty is that we are unable to find a suitable energy estimate on the growth of $||W||_{H^6}$.

- We are not able to show that the exponential growth (3.39) can be deduced from the bound (2.31) in Theorem 2.4.
- Another problem is that the result of Theorem 3.6 for $p = 5$ excludes solitary waves, because the initial data has small $L^2(\mathbb{R})$ norm.

Bibliography

[1] J. Pasta E. Fermi and S. Ulam. Studies of nonlinear problems I. *Lectures in Applied Mathematics,* Report LA-1940. Los Alamos: Los Alamos Scientific Laboratory, pages 15–143, (1955).

[2] N. J. Zabusky and M. D. Kruskal. Interaction of solitons in a collisionless plasma and the recurrence of initial states. *Physical Review Letters*, 15:240–243, (1965).

[3] J. L. Tuck and M. T. Menzel. The superperiod of the nonlinear weighted string (FPU) problem. *Advances in Mathematics*, 9:399–407, (1972).

[4] J. Ford. The Fermi-Pasta-Ulam problem: paradox turns discovery. *Physics Reports*, 213:271–310, (1992).

[5] D. J. Korteweg and G. de Vries. On the change of form of long waves advancing in a canal, and on a new type of long stationary waves. *Philosophical Magazine*, 39:422–443, (1895).

[6] R. M. Miura. The Korteweg-de Vries equations: A survey of results. *SIAM review*, 18:412–459, (1976).

[7] G. Friesecke and R.L. Pego. Solitary waves on FPU lattices : I. qualitative properties, renormalization and continuum limit. *Nonlinearity*, 12:1601–1627, (1999).

[8] G. Friesecke and R.L. Pego. Solitary waves on FPU lattices : II. linear implies nonlinear stability. *Nonlinearity*, 15:1343–1359, (2002).

[9] G. Friesecke and R.L. Pego. Solitary waves on FPU lattices : III. howland-type floquet theory. *Nonlinearity*, 17:207–227, (2004).

[10] G. Friesecke and R.L. Pego. Solitary waves on FPU lattices : IV. proof of stability at low energy. *Nonlinearity*, 17:229–251, (2004).

[11] T. Mizumachi. Asymptotic stability of lattice solitons in the energy space. *Communications in Mathematical Physics*, 288:125–144, (2009).

[12] T. Mizumachi. Asymptotic stability of N-solitary waves of the FPU lattices. *Archive for Rational Mechanics and Analysis*, 207:393–457, (2013).

[13] A. Hoffman and C.E. Wayne. Asymptotic two-soliton solutions in the Fermi-Pasta-Ulam model. *J. Dynam. Differential Equations*, 21:343–351, (2009).

[14] G.N. Benes, A. Hoffman, and C.E. Wayne. Asymptotic stability of the Toda m-soliton. *Journal of Mathematical Analysis and Applications*, 386:445–460, (2012).

[15] G. Schneider and C.E. Wayne. Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi-Pasta-Ulam model. *International Conference on Differential Equations (Berlin, 1999)*, 1:555–568, (2000).

[16] J. Gaison, S. Moskow, J.D. Wright, and Q. Zhang. Approximation of polyatomic FPU lattices by KDV equations. *Multiscale Modeling and Simulation*, 12:953–995, (2014).

[17] J. Angulo Pava. *Nonlinear dispersive equations. Existence and stability of solitary and periodic travelling wave solutions*. Mathematical Surveys and Monographs, 156, (2009).

[18] E. Dumas and D. Pelinovsky. Justification of the log-KdV equation in granular chains: the case of precompression. SIAM *Journal on Mathematical Analysis*, 46:4075–4103, (2014).

[19] T. Kato. On the Cauchy problem for the (generalized) Korteweg-de Vries equation. *Studies in Applied Mathematics*, 8:93–128, (1983).

[20] C. Kenig, G. Ponce, and L. Vega. Well-posedness of the initial value problem for the Korteweg-de Vries equation. *Journal of the American Mathematical Society*, 4:323–347, (1991).

[21] C. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Communications on Pure and Applied Mathematics*, 46:527–620, (1993).

[22] J.L. Bona and R. Smith. The initial value problem for the Korteweg-de Vries equation. *Royal Society of London*, 278:555–601, (1975).

[23] J. Bona, Y. Liu, and N. V. Nguyen. Stability of solitary waves in higher-order Sobolev spaces. *Communications in Mathematical Sciences*, 2:35–52, (2004).

[24] J. Colliander, M. Keel, G. Staffilani, H. Takota, and T. Tao. Sharp global well posedness for KDV and modified KDV on $\mathbb{R}$ and $\mathbb{T}$. *J. Amer. Math. Soc.*, 16:705–749, (2003).

[25] D. Pelinovsky. *Localization in periodic potentials : from Schrödinger operators to the Gross-Pitaevskii equation*. London Mathematical Society Lecture Note Series, Cambridge University press, (2011).

[26] R.M. Miura, C.S. Gardner, and M.D. Kruskal. Korteweg-de Vries equation and generalizations II. existence of conservation laws and constants of motion. *J. Math. Phys.*, 9:1204–1209, (1968).

[27] F. Merle. Existence of blow-up solutions in the energy space for the critical generalized KdV equation. *J. Amer. Math. Soc*, 14:555–578, (2001).

[28] V. Martel and F. Merle. Blow up in finite time and dynamics of blow up solutions for the L^2-critical generalized KdV equation. *J. Amer. Math. Soc*, 15:617–664, (2002).

[29] V. Martel and F. Merle. Nonexistence of blow-up solution with minimal L^2-mass for the critical gKdV equation. *Duke Mathematical Journal*, 115:385–408, (2002).

[30] V. Martel, F. Merle, and P. Raphaël. Blow up for the critical generalized Korteweg-de Vries equation. I: Dynamics near the soliton. *Acta Mathematica*, 212:59–140, (2014).

[31] D. B. Dix and W. R. McKinney. Numerical computations of self-similar blow-up solutions of the generalized Korteweg-de Vries equation. *Differential Integral Equations*, 11:679–723, (1998).

[32] M. Lovric. *Vector calculus*. John Wiley And Sons, (1997).