Stability and interactions of algebraic solitons in the massive Thirring model

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Section 1. The massive Thirring model (MTM)

The coupled mode equations

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = (\gamma_1|u|^2 + \gamma_2|v|^2)u \\ \mathrm{i}(v_t - v_x) + u = (\gamma_2|u|^2 + \gamma_1|v|^2)v \end{cases}$$

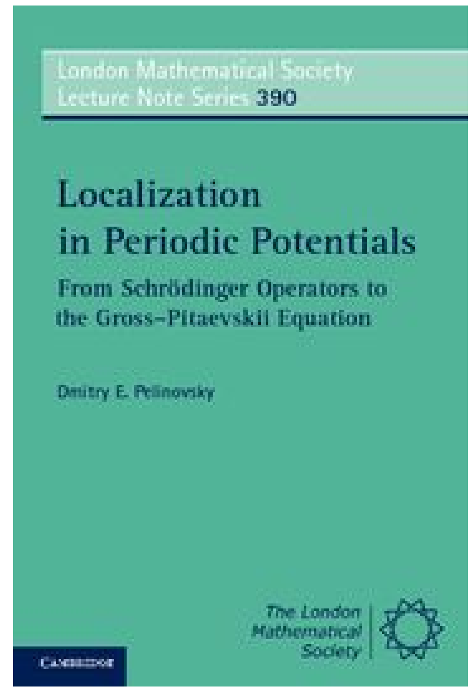
is popular model for two counter-propagating waves across the periodic systems (photonics, Bose–Einstein condensation).

For instance, the Gross–Pitaevskii equation with 2π-periodic bounded potential $V(x) : \mathbb{R} \to \mathbb{R}$,

$$i\partial_t \psi = -\partial_x^2 \psi + \varepsilon V(x)\psi \pm |\psi|^2 \psi,$$

is reduced asymptotically to the coupled mode equations as $\epsilon \to 0$ for the superposition of two harmonic waves

$$\psi(x,t) \sim \sqrt{\varepsilon} u(\varepsilon x, \varepsilon t) e^{\frac{i}{2}x + \frac{i}{4}t} + \sqrt{\varepsilon} v(\varepsilon x, \varepsilon t) e^{-\frac{i}{2}x + \frac{i}{4}t}$$

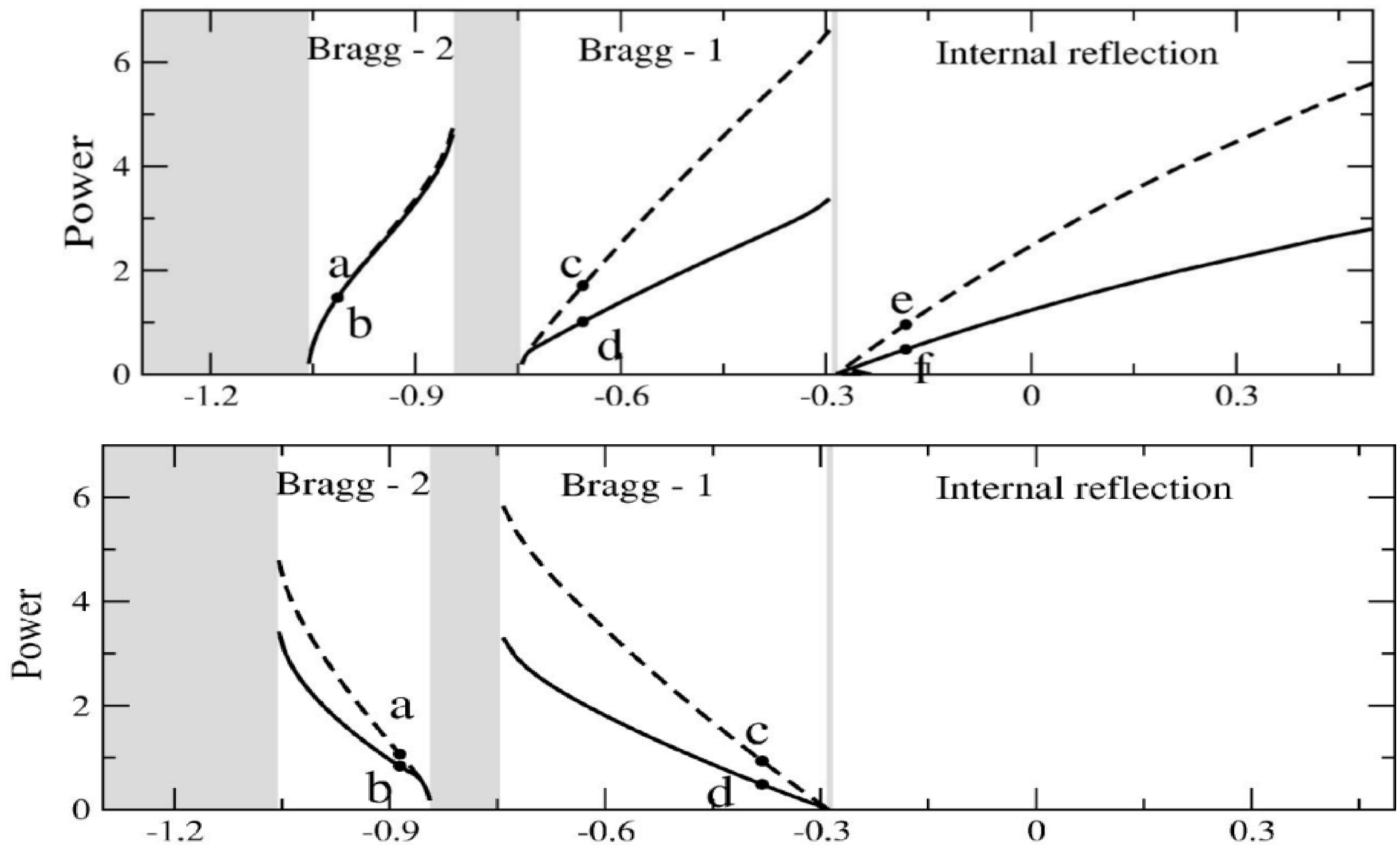


If $\gamma_1 = 0$, the system was introduced in quantum field theory by Thirring in 1958 as the relativistically invariant Dirac equation in one dimension and is now known as the MTM in laboratory coordinates.

Gap solitons

These are localized solution existing in the gap of purely continuous spectrum of

$$\mathcal{L} := -\partial_x^2 + \varepsilon V(x), \quad \mathrm{Dom}(\mathcal{L}) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}), \qquad V(x+2\pi) = V(x).$$



Algebraic solitons attain maximal power $\int_{\mathbb{R}} |\psi|^2 dx$ in each family and the solutions are expressed explicitly within the coupled-mode theory.

About exceptionality of the MTM

The MTM in laboratory coordinates

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = |v|^2 u, \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v, \end{cases}$$

is the only example of the coupled-mode theory which is relativistically invariant:

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \mapsto \begin{bmatrix} \left(\frac{1-c}{1+c}\right)^{1/4} u\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \\ \left(\frac{1+c}{1-c}\right)^{1/4} v\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \end{bmatrix}, \qquad c \in (-1,1).$$

It also has standard symmetries of translations in x, t, and $\arg(u) = \arg(v)$.

MTM is integrable due to existence of the Lax pair [Mikhailov, 1976]:

$$\partial_x \varphi = L(u,v,\lambda)\varphi, \qquad \partial_t \varphi = A(u,v,\lambda)\varphi, \qquad \varphi(x,t) \in \mathbb{C}^2, \qquad \lambda \in \mathbb{C}.$$

About exceptionality of the algebraic solitons

One-soliton solutions are expressed in 1-parameter form:

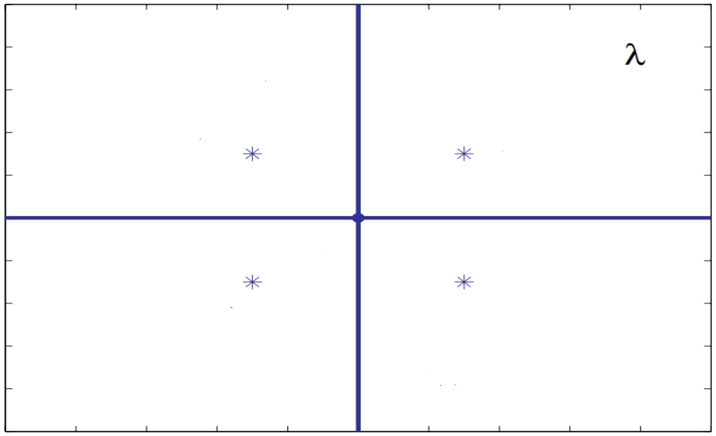
$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \sin\gamma \begin{bmatrix} \mathrm{sech}\left(x \ \sin\gamma + \frac{\mathrm{i}\gamma}{2}\right) \\ \mathrm{sech}\left(x \ \sin\gamma - \frac{\mathrm{i}\gamma}{2}\right) \end{bmatrix} e^{\mathrm{i}t\cos\gamma}, \quad \gamma \in (0,\pi)$$

with zero limit as $\gamma \to 0$ and nonzero limit as $\gamma \to \pi$:

$$\gamma = \pi : \qquad \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{1+2\mathrm{i}x} \\ \dfrac{2}{1-2\mathrm{i}x} \end{bmatrix} e^{-\mathrm{i}t}.$$

Note $\omega = \cos\gamma \in (-1,1)$ in the gap of the Dirac operator $\begin{bmatrix} \mathrm{i}\partial_x & 1 \\ 1 & -\mathrm{i}\partial_x \end{bmatrix}$.

Also note eigenvalue $\lambda = e^{\frac{i}{2}\gamma}$ in the Lax spectrum of the Lax pair.



Section 2. Stability of algebraic solitons

For the MTM with exponential solitons, several results are available:

- Scattering to solitons via IST: Villarroel (1991), Lee (1994), P. & Saalmann (2019), He, J. Liu, & Qu (preprint).
- Orbital stability in $H^1(\mathbb{R})$ via constrained minimization of higher-order energy: P. & Shimabukuro (2014)
- Orbital stability in weighted $L^2(\mathbb{R})$ spaces via Darboux transformations: Contreras, P., & Shimabukuro (2016)

In the limit of algebraic solitons, all methods fail!

- Slowly decaying potentials do not allow to develop IST.
- There is no coercivity of the second variation from higher-order energy.
- Darboux transformation become trivial and generalizations do not help: [Guo, L. Ling, & Liu (2013)]

The only available result is from Klaus, P, & Rothos (2006)

Consider the Kaup–Newell spectral problem

$$\partial_x \varphi = \begin{bmatrix} -i\lambda^2 & \lambda w(x) \\ -\lambda \bar{w}(x) & i\lambda^2 \end{bmatrix} \varphi,$$

where $|w(x)| = |u(x)| = |v(x)|$ in relation to the MTM. Assume that

$$|w(x)| \sim \frac{b}{|x|} \quad \text{as } |x| \to \infty \quad \text{for some } \; b > 0.$$

- $\lambda_0 = i$ is an embedded eigenvalue only if $b > \frac{1}{2}$. [$b = 1$ for algebraic soliton.]
- If $\lambda_0 = i$ is an embedded eigenvalue, then its geometric multiplicity is one. Its algebraic multiplicity is $N+1$ only if $b > N + \frac{1}{2}$. [No examples were given.]
- Let w_0 be the algebraic soliton and $\epsilon w_1(x)$ be a perturbation with fixed profile w_1. For every $\epsilon \neq 0$, there exists a simple eigenvalue of the Lax spectrum in each quadrant of $\mathbb{C}$ independently of the sign of ϵ provided that w_1 satisfies a non-degeneracy condition. [This suggests stability of an algebraic soliton.]

Why is this surprising?

The Gardner (modified KdV) equation

$$u_t + 12uu_x + 6u^2u_x + u_{xxx} = 0$$

also has the algebraic soliton

$$u_0(x) = -\frac{4}{1+4x^2}$$

associated with the Zakharov–Shabat spectral problem

$$\partial_x \varphi = \begin{bmatrix} i\lambda & -1-u(x) \\ 1+u(x) & -i\lambda \end{bmatrix} \varphi.$$

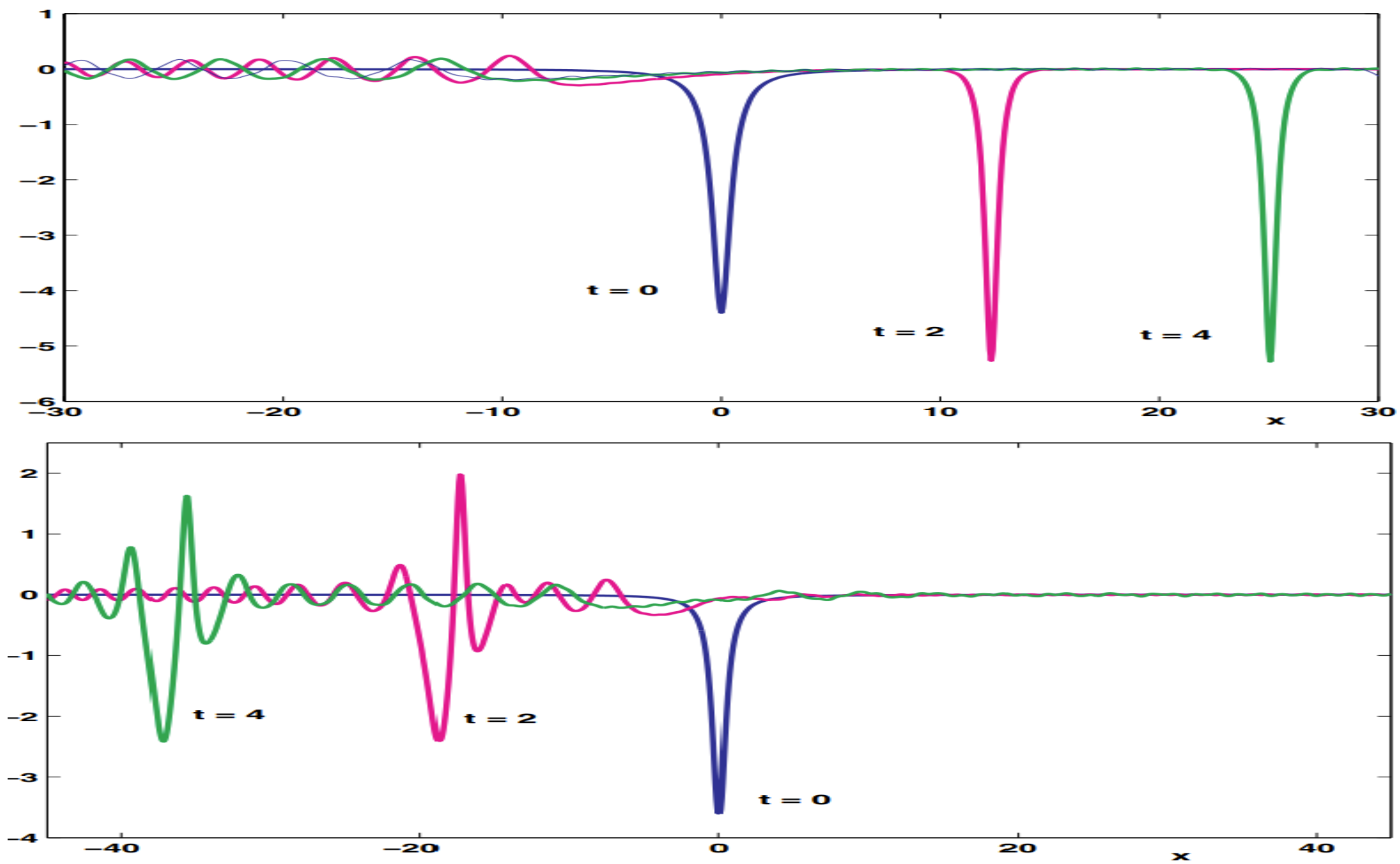
However, the algebraic soliton is a nonzero minimum of conserved momentum $Q(u) = \int_{\mathbb{R}} u^2 dx$ among exponential solitons and hence it is nonlinearly unstable.

Instability for similar mKdV and NLS models was shown in analysis papers: Fukaya & Hayashi (2021), Kfoury, Le Coz & Tsai (2022).

More results on instability of algebraic solitons in mKdV

- $\lambda_0 = i$ is an embedded eigenvalue of the Zakharov–Shabat spectral problem.

- Let $\epsilon u_1(x)$ be a perturbation with fixed profile u_1. For every $\epsilon \neq 0$, we have either an eigenvalue $\lambda = i + i\mathcal{O}(|\epsilon|^{2/3})$ or a symmatric pair of eigenvalues $\lambda = i \pm \mathcal{O}(|\epsilon|^{2/3})$ depending on the sign of ϵ. [Klaus, P, & Rothos (2006)]



More results on instability of algebraic solitons in mKdV

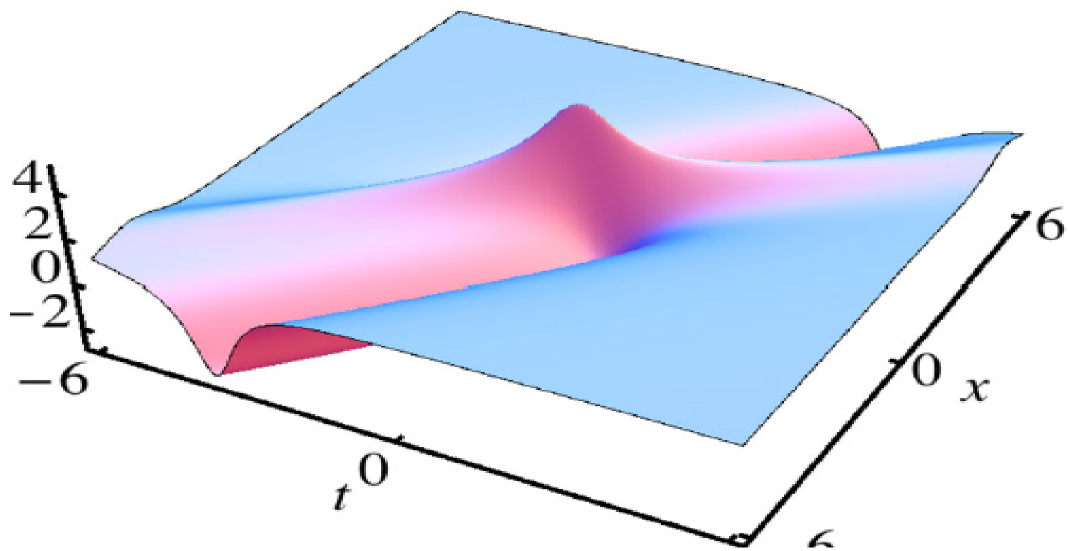
The Gardner equation

$$u_t + 12uu_x + 6u^2u_x + u_{xxx} = 0$$

has a hierarchy of rational solutions:

$$u_1(x) = -\frac{4}{1+4x^2}, \quad u_2(x,t) = \frac{P_4(x,t)}{P_6(x,t)}, \quad \ldots.$$

Chowdury, Ankiewicz & Akhmediev (2016)
Xing, Wang, Mihalache, Porsezian, & He (2017)



This solutions illustrated directly the instability of algebraic solitons.

Section 3. Rational solutions of MTM

Consider the MTM in laboratory coordinates

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = |v|^2 u, \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v. \end{cases}$$

Rational solution on nonzero background (rogue waves) were already constructed:
Guo, Wang, Cheng, & J.He (2017)
Ye, Bu, Pan, Chen, Mihalche, & Baronio (2021)
Chen, Yang, & Feng (2023).

Surprising, construction of algebraic solitons on the zero background has never been done for MTM, although it was done for derivative NLS [Wang & Wu (2022)]

We have constructed the second-order rational solution
Jiaqi Han, Cheng He, & D.P., PRE 110 (2024) 034202

Construction of the second-order rational solution

We use the bilinear formulation of the MTM from [Chen & Feng (2023)]:

$$u = \frac{g}{\bar{f}}, \qquad v = \frac{h}{f},$$

where

$$\left.\begin{array}{r} \mathrm{i}f(g_t + g_x) - \mathrm{i}g(f_t + f_x) + h\bar{f} = 0, \\ \mathrm{i}\bar{f}(h_t - h_x) - \mathrm{i}h(\bar{f}_t - \bar{f}_x) + gf = 0, \\ \mathrm{i}\bar{f}(f_x + f_t) - \mathrm{i}f(\bar{f}_t + \bar{f}_x) - |h|^2 = 0, \\ \mathrm{i}f(\bar{f}_t - \bar{f}_x) - \mathrm{i}\bar{f}(f_t - f_x) - |g|^2 = 0. \end{array}\right\}$$

- Exponential 2-soliton solutions with 8 parameters.
- A smart parameterization respecting Lorentz and translational symmetries for two eigenvalues $\lambda_1 = \delta_1 e^{i\gamma_1}$ and $\lambda_2 = \delta_2 e^{i\gamma_2}$ of the Kaup–Newell problem.
- A limit to algebraic 2-soliton solutions with 6 parameters with

$$\delta_1 \neq \delta_2, \qquad \gamma_1, \gamma_2 \to \pi.$$

- A limit to the algebraic double-soliton with 5 parameters as $\delta_1 \to \delta_2$.

The algebraic double-soliton

It is expressed as the second-order rational solution

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{4(-3+6\mathrm{i}x-12x^2-8\mathrm{i}x^3-12t(2x-\mathrm{i})-\mathrm{i}\beta)}{3+24\mathrm{i}x-24x^2+32\mathrm{i}x^3-16x^4+48t^2+2\beta(2x-\mathrm{i})} \\ \dfrac{4(-3-6\mathrm{i}x-12x^2+8\mathrm{i}x^3+12t(2x+\mathrm{i})+\mathrm{i}\beta)}{3-24\mathrm{i}x-24x^2-32\mathrm{i}x^3-16x^4+48t^2+2\beta(2x+\mathrm{i})} \end{bmatrix} e^{-\mathrm{i}t},$$

where $\beta \in \mathbb{R}$ is a parameter in addition to $c \in (-1,1)$ and $x_0, t_0, \theta_0 \in \mathbb{R}$.

The existence of the second-order rational solution suggestes the existence of a hierarchy of rational solutions in the form:

$$u_1(x,t) = \frac{2}{1+2ix}e^{-it}, \quad u_2(x,t) = \frac{P_3(x,t)}{P_4(x,t)}e^{-it}, \quad \dots.$$

This is classified as Open Problem I.

Properties of the algebraic double-soliton

1. $u(\cdot,t), v(\cdot,t) \in C^\omega(\mathbb{R})$ for every $t \in \mathbb{R}$ and $\beta \in \mathbb{R}$
2. $Q(u,v) = \int_{\mathbb{R}} (|u|^2 + |v|^2) dx = 8\pi = 2Q_{AS}$.

For the proofs, the bilinear formulation is very useful:

$$|u|^2 + |v|^2 = \frac{|g|^2 + |h|^2}{|f|^2} = 2\mathrm{i}\left(\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}}\right),$$

where

$$f = 16x^4 + 32\mathrm{i}x^3 + 24x^2 + 24\mathrm{i}x - 3 - 48t^2 - 2\beta(2x + \mathrm{i})$$

satisfies

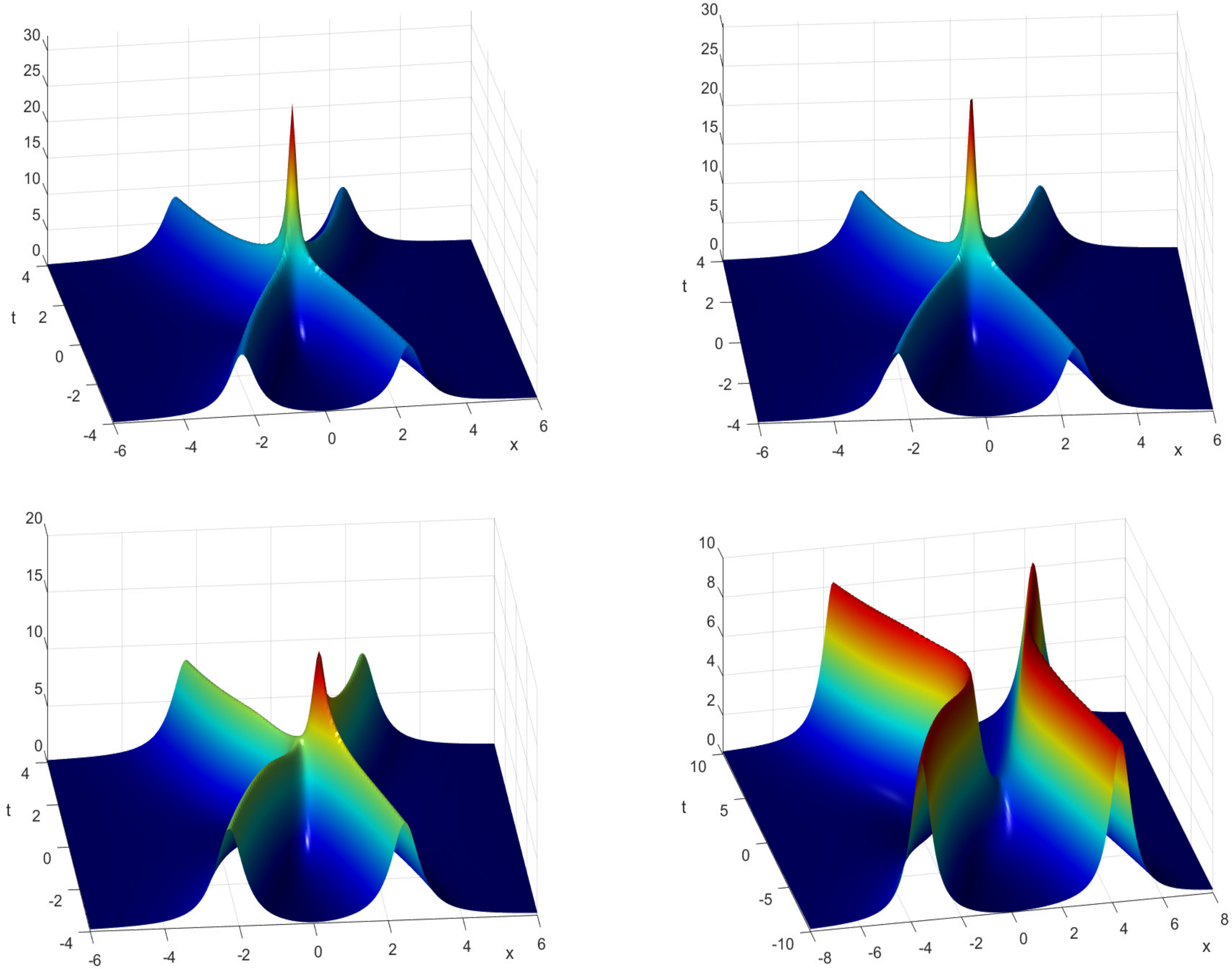
- f has no zeros on $\mathbb{R}$ in x,
- f has one root in $\mathbb{C}_+$ and three roots in $\mathbb{C}_-$,
- $f_x/f - \bar{f}_x/\bar{f} = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$.

By the argument principle,

$$Q(u,v) = 4\pi(N_- - N_+) = 8\pi.$$

Properties of the algebraic double-soliton

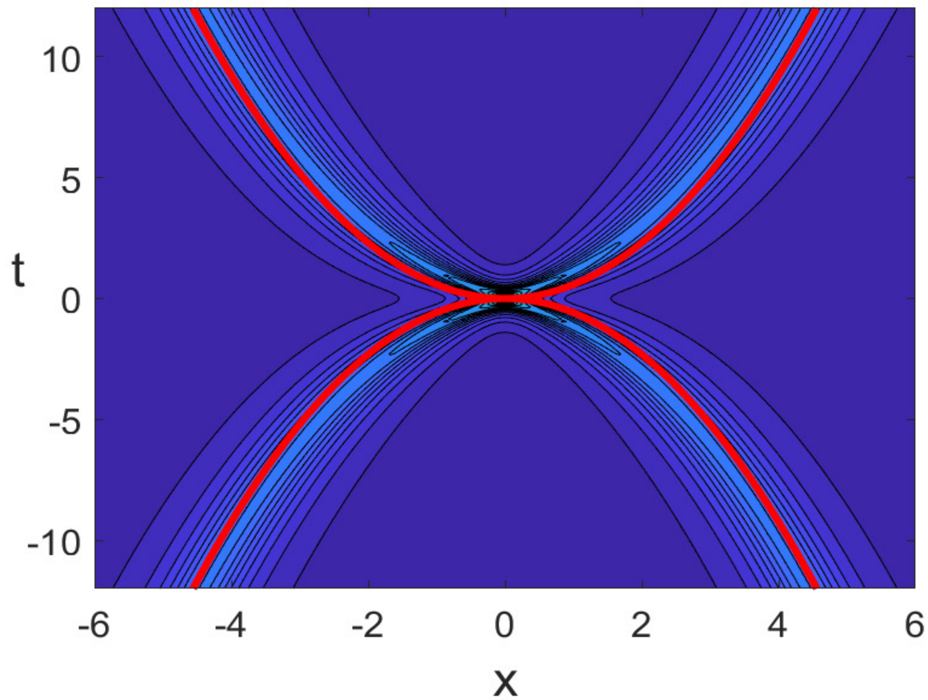
3. The solution suggests slow scattering of two identical algebraic solitons.



This suggests orbital stability of a single algebraic soliton as Open Problem II.

Properties of the algebraic double-soliton

4. The two solitons move along the parabolas $x^2 = \pm 3t$.



Properties of the algebraic double-soliton

5. The solution corresponds to the double embedded eigenvalue at $\lambda_0 = i$.
Zhi-Qiang Li, D.P., S. Tian (in progress)

An eigenvector satisfies

$$\partial_x \varphi_0 = L(u, v, \lambda_0)\varphi_0, \quad \partial_t \varphi_0 = A(u, v, \lambda_0)\varphi_0.$$

A generalized eigenvector satisfies

$$\partial_x \varphi_1 = L(u, v, \lambda_0)\varphi_1 + \partial_\lambda L(u, v, \lambda_0)\varphi_0, \quad \partial_t \varphi_1 = A(u, v, \lambda_0)\varphi_1 + \partial_\lambda A(u, v, \lambda_0)\varphi_0.$$

With explicit computations, we obtain

$$u \sim \frac{b = 2}{|x|}, \qquad \varphi_0 \sim \frac{1}{|x|^2}, \qquad \varphi_1 \sim \frac{1}{|x|} \quad \text{as} \quad |x| \to \infty,$$

in agreement with Klaus, P. & Rothos (2006).

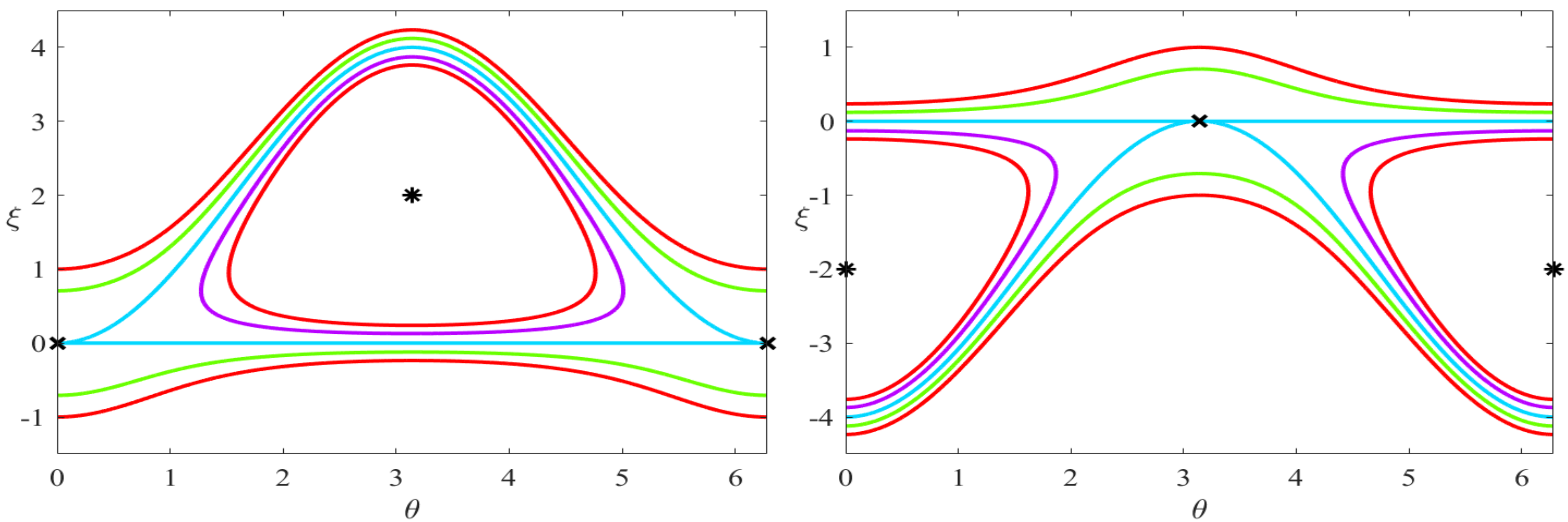
Inverse scattering with algebraic potentials is classified as Open Problem III.

Section 4. Algebraic solitons in the limit of periodic solutions

Here is the phase plane for the standing wave solutions

$$u(x,t) = U(x)e^{-i\omega t}, \qquad v(x,t) = \bar{U}(x)e^{-i\omega t},$$

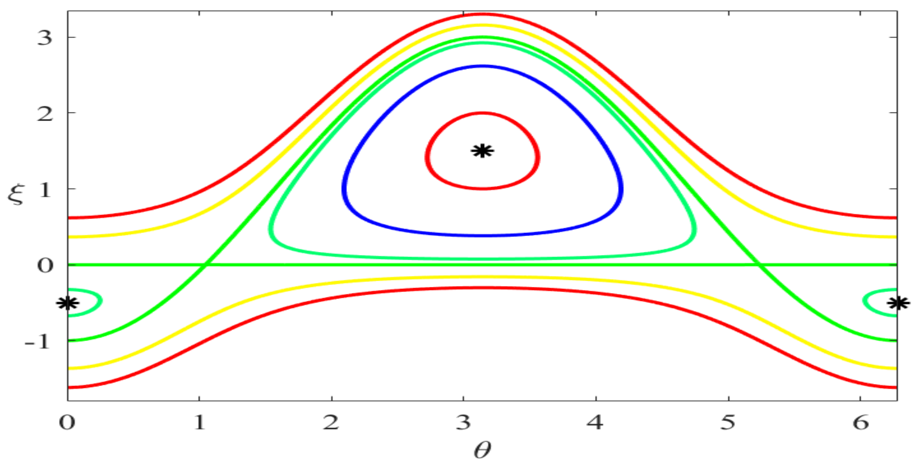
with $U(x) = \sqrt{\xi(x)}e^{\frac{i}{2}\theta(x)}$.



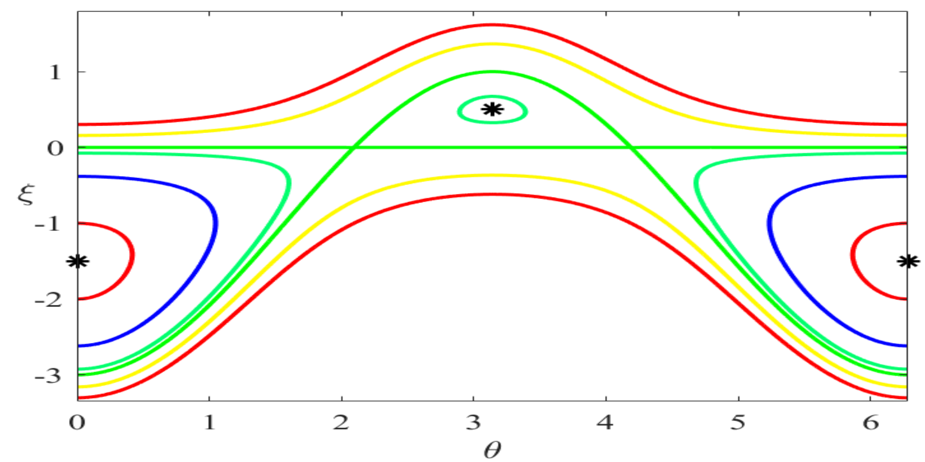
(a) $\omega = -1$. (b) $\omega = 1$.

Shikun Cui, D.P. (submitted)

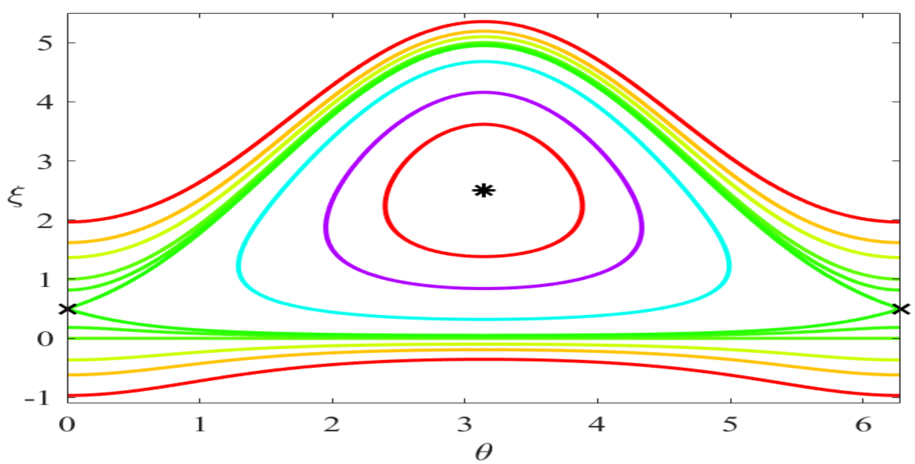
Algebraic soliton at $\omega = -1$ is located in between exponential soliton for $\omega \in (-1, 1)$ and solitons on the constant background for $\omega < -1$.



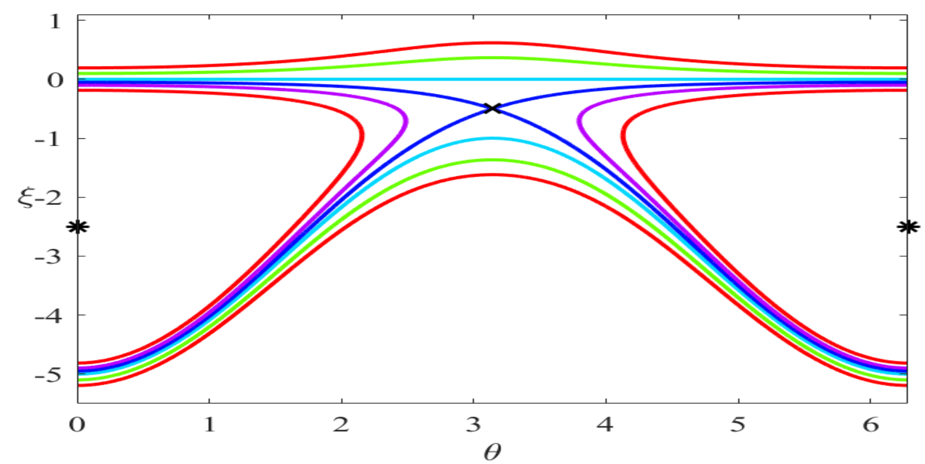
(c) $\omega \in (-1, 0)$.



(d) $\omega \in (0, 1)$.



(e) $\omega < -1$.



(f) $\omega > 1$.

Squared eigenfunction relation

Linear stability is solved by the squared eigenfunctions [Kaup–Lakoba (1996)]

$$\Lambda = \pm i\sqrt{P(\lambda)}, \quad P(\lambda) = \frac{1}{4}\left(\lambda^2 + \frac{1}{\lambda^2} - 2\omega\right)^2 - b,$$

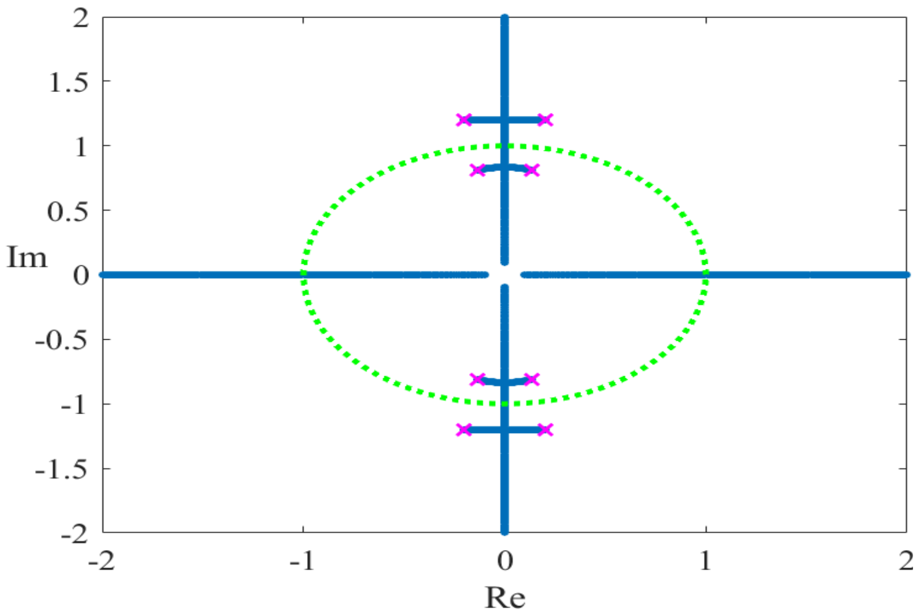
where b is the conserved energy for the spatial profile $U(x)$ and $\lambda \in \mathbb{C}$ belongs to the Lax spectrum with bounded eigenfunctions

$$\partial_x \varphi = L(u,v,\lambda)\varphi, \qquad \varphi \in L^\infty(\mathbb{R},\mathbb{C}^2)$$

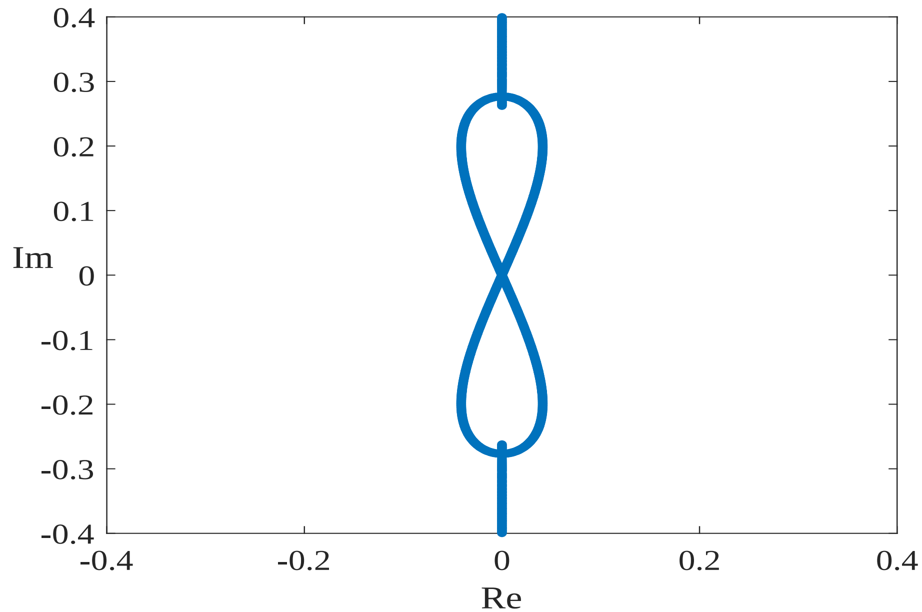
We proved some properties of roots of $P(\lambda)$ and the Lax spectrum $\Sigma \subset \mathbb{C}$:

- If $\lambda \in \Sigma$, then so are $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$, and $1/\lambda$.
- Roots of $P(\lambda)$ are given by $\{\pm\lambda_1, \pm\lambda_2, \pm\lambda_1^{-1}, \pm\lambda_2^{-2}\}$ with $\lambda_2 = \bar{\lambda}_1$ if $\lambda_1 \in \mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R} \cup \mathbb{S}^1)$.
- $\Lambda \in i\mathbb{R}$ if $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$ between roots of $P(\lambda)$ or $\arg(\lambda) = \pm\frac{\pi}{4}$ between roots of $P(\lambda)$.

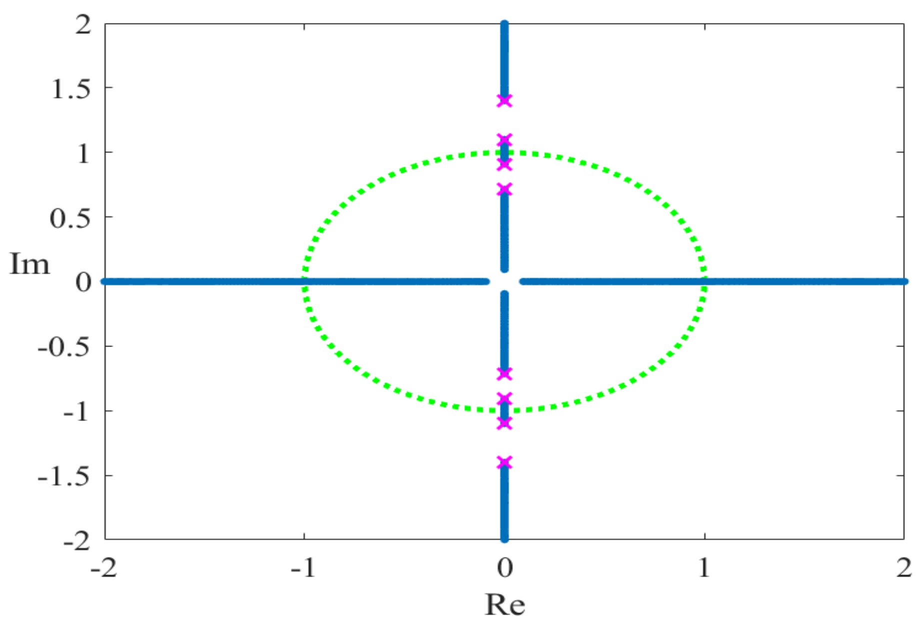
Algebraic soliton between unstable and stable branches



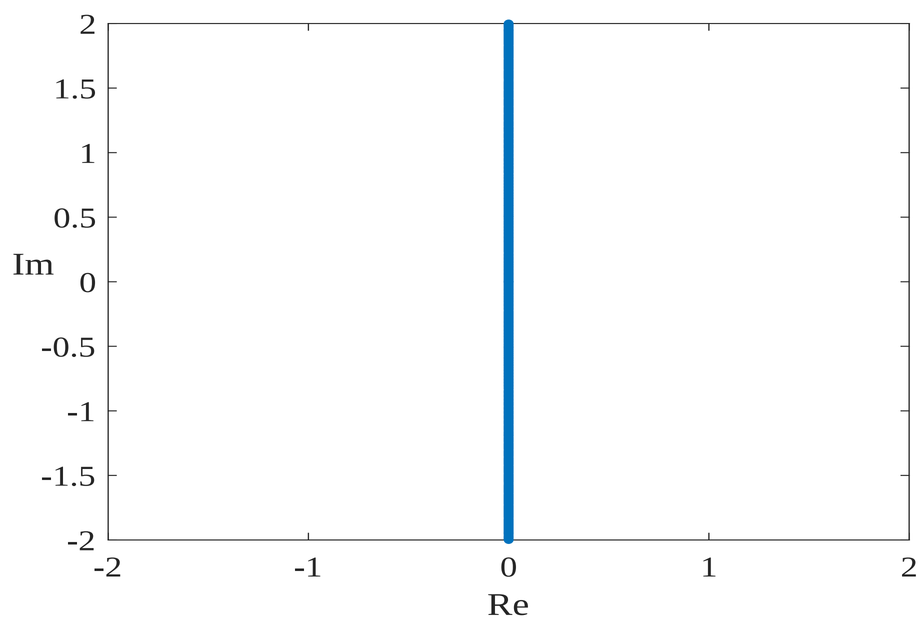
(g) Lax spectrum in λ-plane.



(h) Stability spectrum in Λ-plane.



(i) Lax spectrum in λ-plane.



(j) Stability spectrum in Λ-plane.

Comprehensive study of spectral stability for periodic waves

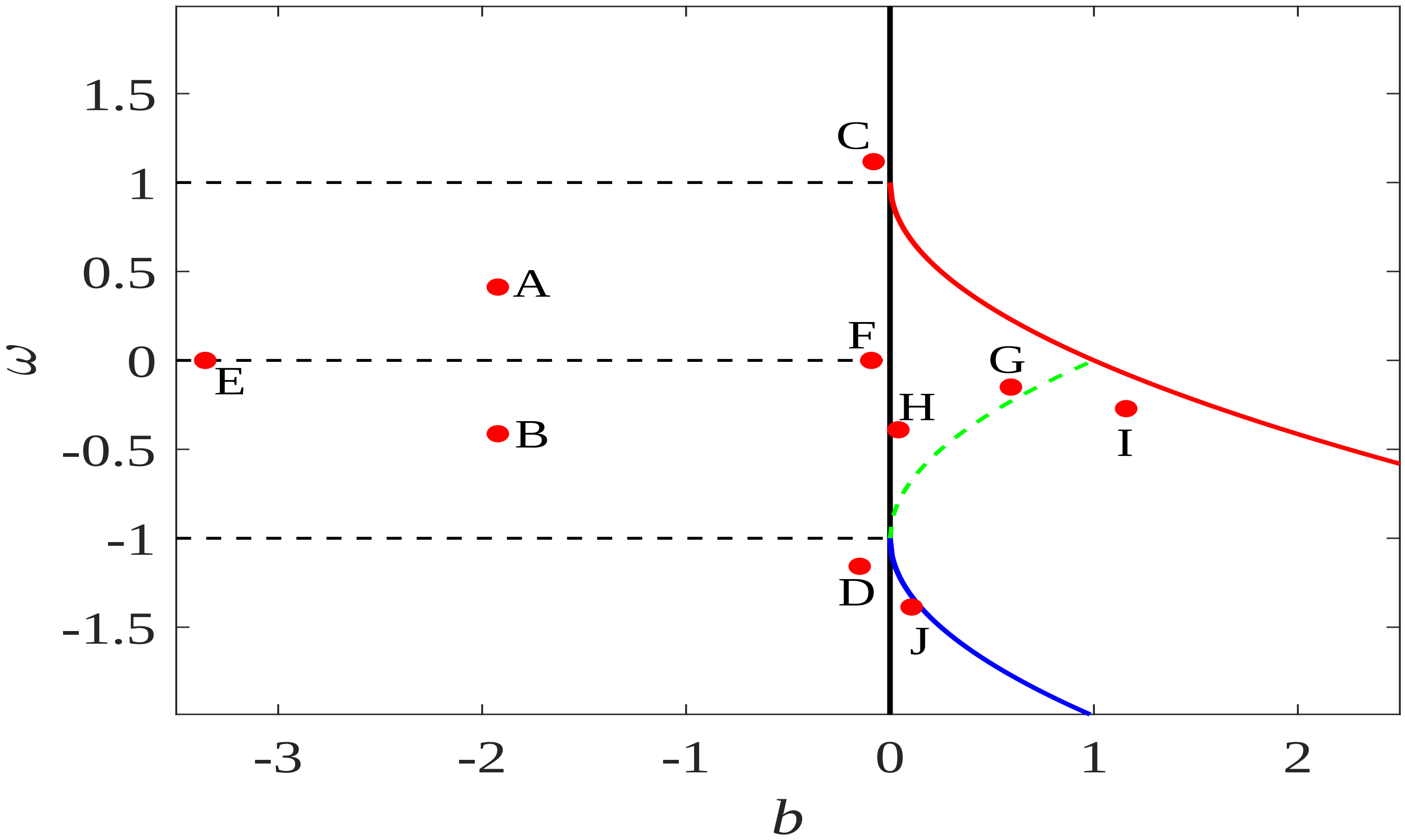
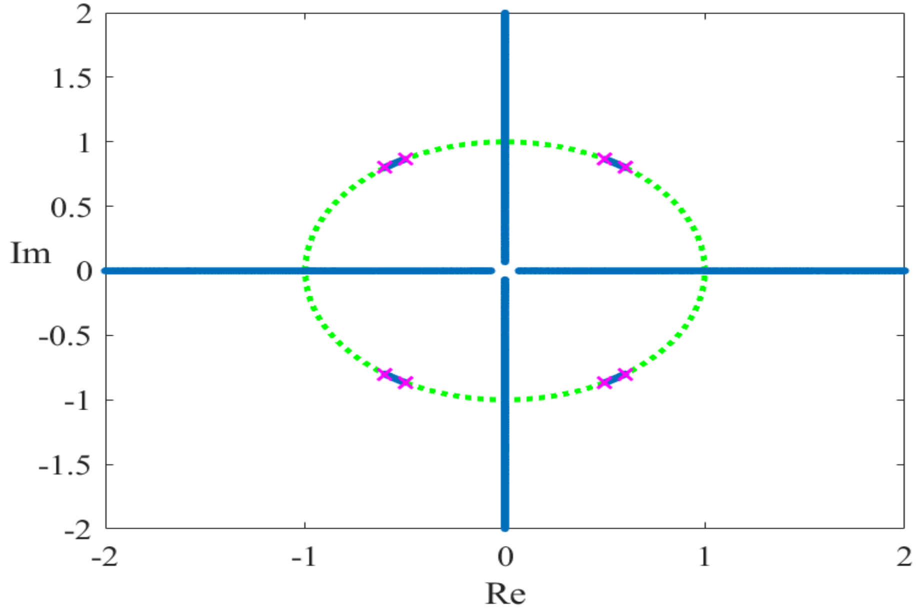


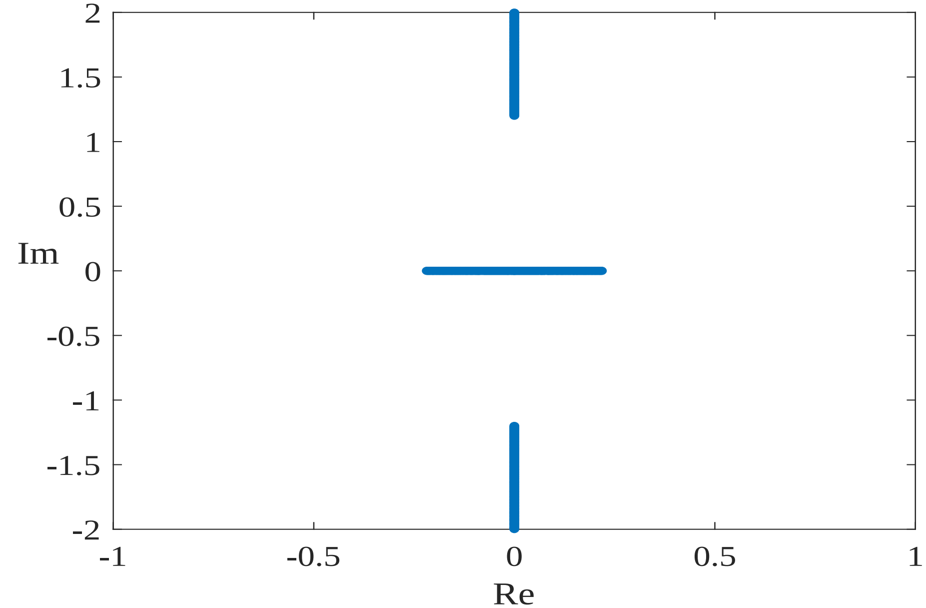
Figure 1: Sample points in each region on the (b, ω) plane.

The previous two figures correspond to points D and J near the algebraic soliton with $\omega = -1$ and $b = 0$.

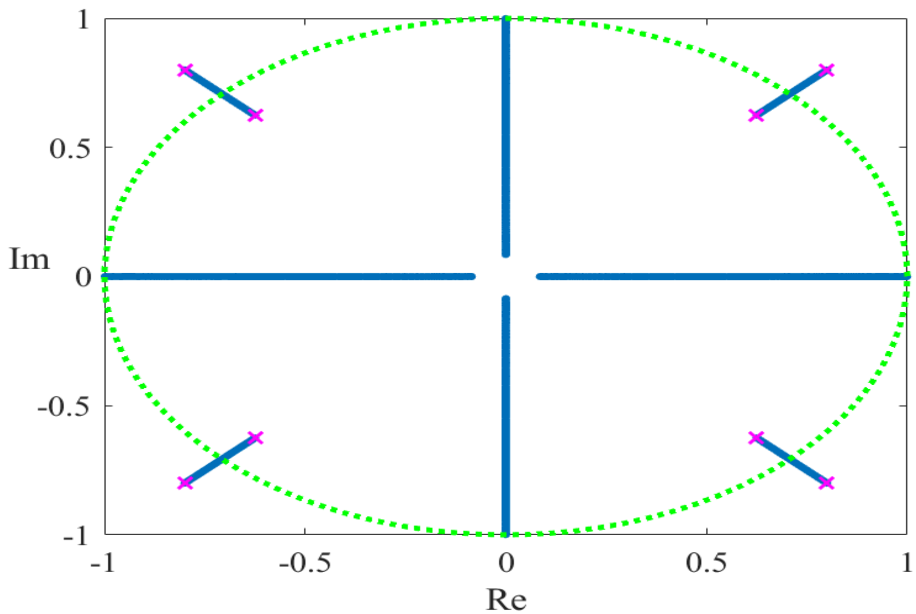
Points H and F near the exponential soliton with $\omega = 0$



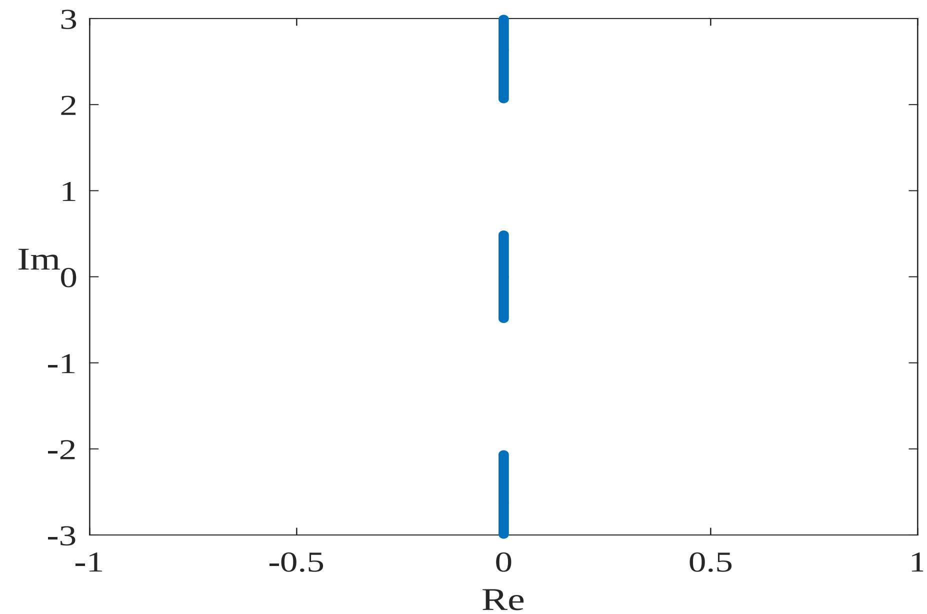
(a) Lax spectrum in λ-plane.



(b) Stability spectrum in Λ-plane.



(c) Lax spectrum in λ-plane.



(d) Stability spectrum in Λ-plane.

Section 5. Conclusion

We have considered algebraic solitons in the MTM. This opens road to solutions of the following open problems.

- Open Problem I: Construct a hierarchy of rational solutions to the MTM.
- Open Problem II: Prove the nonlinear stability of a single algebraic soliton.
- Open Problem III: Extend the IST on algebraically decaying potentials.
- Open Problem IV: Prove location of the Lax spectrum between roots of $P(\lambda)$ analytically rather than numerically.

Many thanks for your attention! Questions?