
Dynamics of algebraic solitons in the massive Thirring model

Dmitry Pelinovsky
McMaster University, Canada

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Section 1. The massive Thirring model (MTM)

The coupled mode equations

$$\begin{cases} i(u_t + u_x) + v = (\gamma_1|u|^2 + \gamma_2|v|^2)u \\ i(v_t - v_x) + u = (\gamma_2|u|^2 + \gamma_1|v|^2)v \end{cases}$$

is a popular model for dynamics of two counter-propagating resonant waves across the periodic systems (photonics, Bose–Einstein condensation).

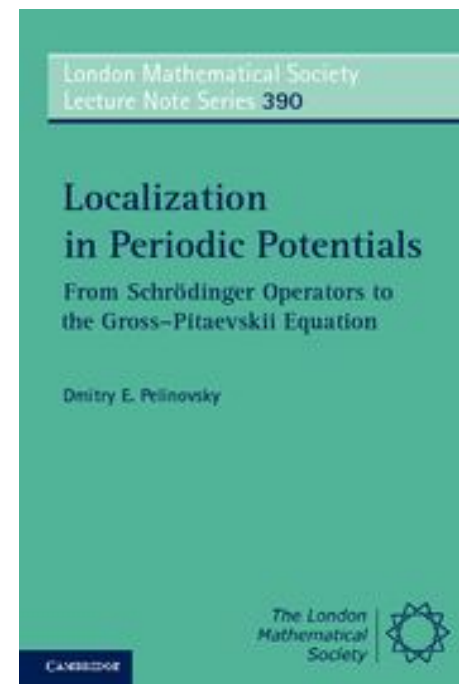
For instance, the Gross–Pitaevskii equation with 2π -periodic, bounded, real-valued potential $V(x)$:

$$i\partial_t\psi = -\partial_x^2\psi + \varepsilon V(x)\psi \pm |\psi|^2\psi,$$

is reduced asymptotically as $\varepsilon \rightarrow 0$ to the coupled mode equations for the superposition of two 2π -antiperiodic waves (under Bragg's parameteric 1 : 2 resonance)

$$\psi(x, t) \sim \sqrt{\varepsilon} \left[u(\varepsilon x, \varepsilon t) e^{\frac{i}{2}x} + v(\varepsilon x, \varepsilon t) e^{-\frac{i}{2}x} \right] e^{\frac{i}{4}t},$$

If $\gamma_1 = 0$, the system was introduced in quantum field theory by Thirring in 1958 as the relativistically invariant Dirac equation in one dimension, known now as the MTM in laboratory coordinates.

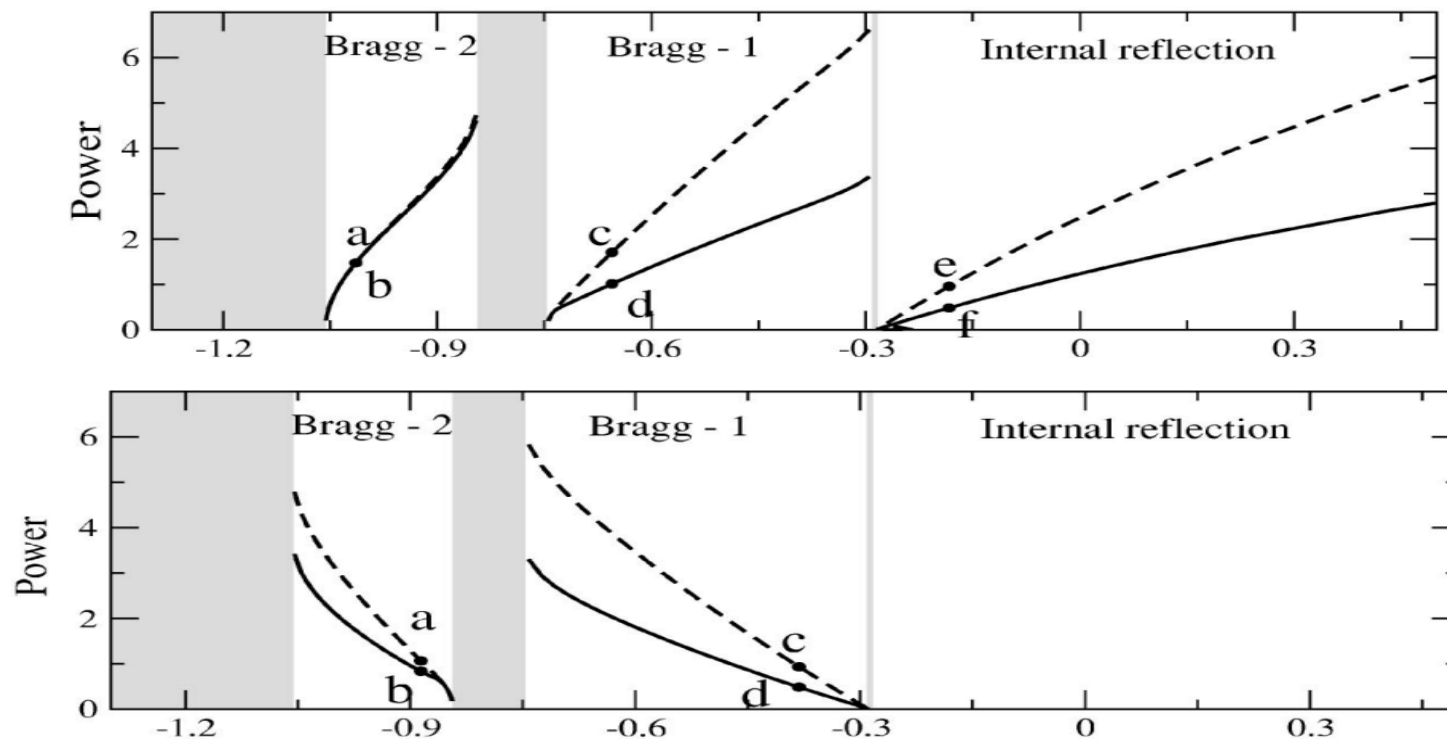


Gap solitons

For every $\gamma_1, \gamma_2 > 0$, there exist spatially decaying (localized) solutions called gap solitons (solitary waves) in the gaps of purely continuous spectrum of

$$\mathcal{L} := -\partial_x^2 + \varepsilon V(x), \quad \text{Dom}(\mathcal{L}) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}), \quad V(x + 2\pi) = V(x).$$

[Y.Kivshar, D.P., A. Sukhorukov (2004)] [A. Pankov (2005)] [M. Weinstein (2006)]



The decay rate is exponential except for the end points in each finite gap. Algebraic solitons attain the maximal power $\int_{\mathbb{R}} |\psi|^2 dx$ in each family and the solutions are expressed explicitly within the coupled-mode theory.

About exceptionality of the MTM

The MTM in laboratory coordinates

$$\begin{cases} i(u_t + u_x) + v = |v|^2 u, \\ i(v_t - v_x) + u = |u|^2 v, \end{cases}$$

is the only example of the coupled-mode system which is relativistically invariant:

$$\begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} \mapsto \begin{bmatrix} \left(\frac{1-c}{1+c}\right)^{1/4} u\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \\ \left(\frac{1+c}{1-c}\right)^{1/4} v\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \end{bmatrix}, \quad c \in (-1, 1).$$

It also has standard symmetries of translations in x , t , and $\arg(u) = \arg(v)$.

MTM is integrable due to existence of the Lax pair [Mikhailov, 1976]:

$$\partial_x \varphi = L(u, v, \lambda) \varphi, \quad \partial_t \varphi = A(u, v, \lambda) \varphi, \quad \varphi(x, t) \in \mathbb{C}^2, \quad \lambda \in \mathbb{C}.$$

See also [Orfanidis, 1976], [Kaup & Newell, 1977], [Barashenkov & Getmanov, 1989], [Villaroel, 1991], [Lee, 1994], [Zhou, 1995].

About exceptionality of the algebraic solitons

One-soliton solutions are expressed in 1-parameter form:

$$\begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = \sin \gamma \begin{bmatrix} \operatorname{sech} \left(x \sin \gamma + \frac{i\gamma}{2} \right) \\ \operatorname{sech} \left(x \sin \gamma - \frac{i\gamma}{2} \right) \end{bmatrix} e^{it \cos \gamma}, \quad \gamma \in (0, \pi)$$

with zero limit as $\gamma \rightarrow 0$ and nonzero limit as $\gamma \rightarrow \pi$:

$$\gamma = \pi : \quad \begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = \begin{bmatrix} \frac{2}{1 + 2ix} \\ \frac{2}{1 - 2ix} \end{bmatrix} e^{-it}, \quad (u, v) \in L^2(\mathbb{R}).$$

The frequency of standing waves $\omega = \cos \gamma \in (-1, 1)$ takes values in the gap of the Dirac operator $\mathcal{D} := \begin{bmatrix} i\partial_x & 1 \\ 1 & -i\partial_x \end{bmatrix}$ in $L^2(\mathbb{R})$, where $\sigma(\mathcal{D}) = (-\infty, -1] \cup [1, \infty)$.

The algebraic soliton ($\gamma = \pi$) attains the maximal power $Q(u, v) = \int_{\mathbb{R}} (|u|^2 + |v|^2) dx = 4\pi$ along the family.

Section 2. Stability of algebraic solitons

For stability of exponential solitons, several analytical results are available:

- Orbital stability in $H^1(\mathbb{R})$ via constrained minimization of energy:
P. & Shimabukuro (2014)
- Orbital stability in $L^2(\mathbb{R})$ via Bäcklund transformation:
Contreras, P., & Shimabukuro (2016)
- Asymptotic stability via IST:
P. & Saalman (2019), He, Liu, & Qu (2024)

In the limit of algebraic solitons, all methods fail!

- There is no coercivity of the second variation from higher-order energy.
- Bäcklund transformation become trivial and generalizations do not help:
[Guo, Ling, & Liu (2013)]
- Solvability of the IST is not justified for slowly decaying potentials.

The only available result is from Klaus, P, & Rothos (2006)

Consider the Kaup–Newell spectral problem

$$\partial_x \varphi = \begin{bmatrix} -i\lambda^2 & \lambda w(x) \\ -\lambda \bar{w}(x) & i\lambda^2 \end{bmatrix} \varphi,$$

where $|w(x)| = |u(x)| = |v(x)|$ is defined from the solution of the MTM.

Assume that

$$|w(x)| \sim \frac{b}{|x|} \quad \text{as } |x| \rightarrow \infty \quad \text{for some } b > 0.$$

- $\lambda_0 = i$ is an embedded eigenvalue only if $b > \frac{1}{2}$. [*$b = 1$ for algebraic soliton.*]
- If $\lambda_0 = i$ is an embedded eigenvalue, then its geometric multiplicity is one. Its algebraic multiplicity is $N + 1$ only if $b > N + \frac{1}{2}$. [*No examples were given.*]
- Let w_0 be the algebraic soliton and $\epsilon w_1(x)$ be a perturbation with fixed profile w_1 . For every $\epsilon \neq 0$, there exists a simple eigenvalue of the Lax spectrum in each quadrant of $\mathbb{C}$ independently of the sign of ϵ provided that w_1 satisfies a non-degeneracy condition. [*This suggests stability of an algebraic soliton.*]

Why is this surprising?

The Gardner (modified KdV) equation

$$u_t + 12uu_x + 6u^2u_x + u_{xxx} = 0$$

also has the algebraic soliton

$$u_0(x) = -\frac{4}{1 + 4x^2}.$$

It is associated with the embedded eigenvalue $\lambda = i$ in the Zakharov–Shabat spectral problem

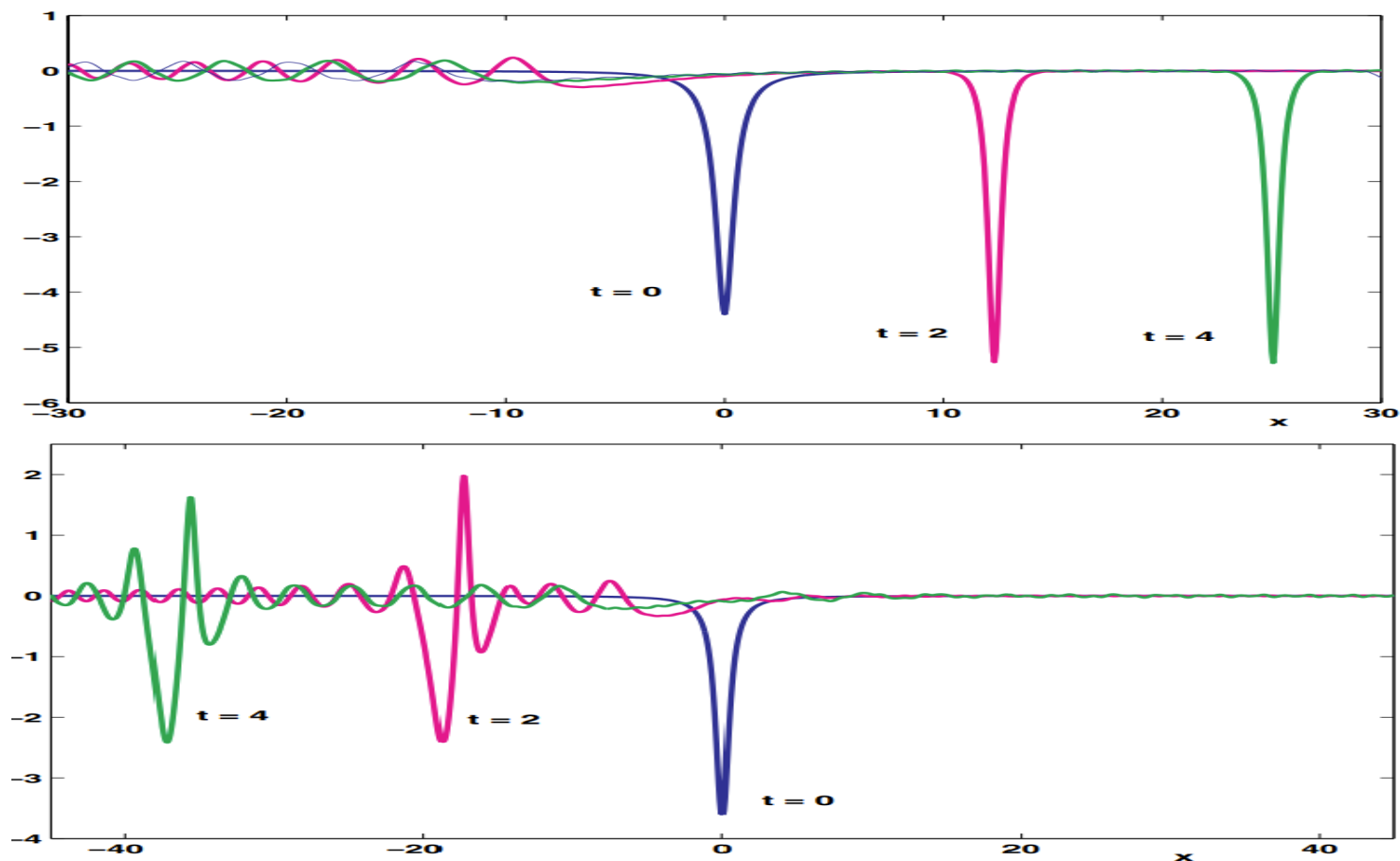
$$\partial_x \varphi = \begin{bmatrix} i\lambda & -1 - u(x) \\ 1 + u(x) & -i\lambda \end{bmatrix} \varphi.$$

Since the algebraic soliton is a nonzero minimum of conserved momentum $Q(u) = \int_{\mathbb{R}} u^2 dx$ among exponential solitons, it is nonlinearly unstable.

Instability for similar mKdV and NLS models was shown in analysis papers: Fukaya & Hayashi (2021), Kfoury, Le Coz & Tsai (2022).

More results on instability of algebraic solitons in mKdV

- $\lambda_0 = i$ is an embedded eigenvalue of the Zakharov–Shabat spectral problem.
- Let $\epsilon u_1(x)$ be a perturbation with fixed profile u_1 . For every $\epsilon \neq 0$, we have either a simple eigenvalue $\lambda = i + i\mathcal{O}(|\epsilon|^{2/3})$ or a symmetric pair of eigenvalues $\lambda = i \pm \mathcal{O}(|\epsilon|^{2/3})$ depending on the sign of ϵ . [P & Grimshaw (1997)]



More results on instability of algebraic solitons in mKdV

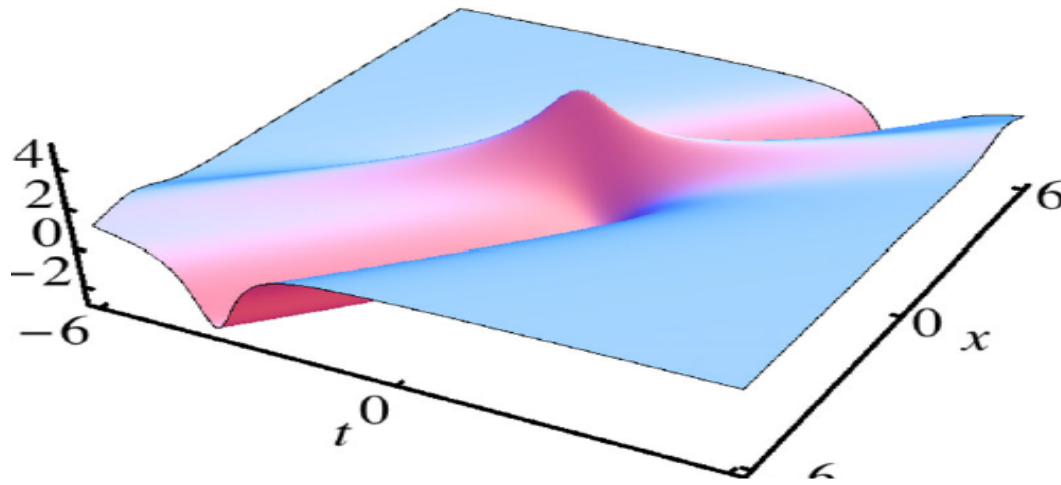
The instability of algebraic solitons can be shown from the rational solutions of the Gardner equation

$$u_t + 12uu_x + 6u^2u_x + u_{xxx} = 0.$$

A hierarchy of rational solutions is available in the form:

$$u_1(x) = -\frac{4}{1+4x^2}, \quad u_2(x, t) = \frac{P_4(x, t)}{P_6(x, t)}, \quad \dots$$

[Chowdury, Ankiewicz & Akhmediev (2016)], [Xing *et al.* (2017)],
[Bilman, Miller, Young (2025)]



The solution u_2 suggests the instability of u_1 : $\|u_2(\cdot, t) - u_1\|_{L^2}$ grows in time t .

Section 3. Rational solutions of MTM

Consider the MTM in laboratory coordinates

$$\begin{cases} i(u_t + u_x) + v = |v|^2 u, \\ i(v_t - v_x) + u = |u|^2 v. \end{cases}$$

Rational solutions on nonzero background (rogue waves) were already constructed:

Guo, Wang, Cheng, & He (2017)

Ye, Bu, Pan, Chen, Mihalche, & Baronio (2021)

Chen, Yang, & Feng (2023).

Surprisingly, rational solutions on the zero background have not been constructed for MTM.

We have constructed the second-order rational solution to the MTM:

Jiaqi Han, Cheng He, & D.P. (2024)

Construction of the second-order rational solution

We use the bilinear formulation of the MTM from [Chen & Feng (2023)]:

$$u = \frac{g}{\bar{f}}, \quad v = \frac{h}{f},$$

where

$$\left. \begin{aligned} \mathrm{i}f(g_t + g_x) - \mathrm{i}g(f_t + f_x) + h\bar{f} &= 0, \\ \mathrm{i}\bar{f}(h_t - h_x) - \mathrm{i}h(\bar{f}_t - \bar{f}_x) + gf &= 0, \\ \mathrm{i}\bar{f}(f_x + f_t) - \mathrm{i}f(\bar{f}_t + \bar{f}_x) - |h|^2 &= 0, \\ \mathrm{i}f(\bar{f}_t - \bar{f}_x) - \mathrm{i}\bar{f}(f_t - f_x) - |g|^2 &= 0. \end{aligned} \right\}$$

- The exponential 2-solitons are obtained with 8 parameters.
- Four parameters yields two eigenvalues $\lambda_1 = \delta_1 e^{i\gamma_1}$ and $\lambda_2 = \delta_2 e^{i\gamma_2}$ in the first quadrant of $\mathbb{C}$. Four more parameters are translational parameters.
- A limit to the algebraic 2-soliton solutions yields a soliton with 6 parameters:

$$\delta_1 \neq \delta_2, \quad \gamma_1, \gamma_2 \rightarrow \pi.$$

- The limit $\delta_1 \rightarrow \delta_2$ gives the algebraic double-soliton with 5 parameters.

The algebraic double-soliton

The algebraic double-soliton is expressed as the second-order rational solution

$$\begin{bmatrix} u(x, t) \\ v(x, t) \end{bmatrix} = \begin{bmatrix} \frac{4(-3 + 6ix - 12x^2 - 8ix^3 - 12t(2x - i) - i\beta)}{3 + 24ix - 24x^2 + 32ix^3 - 16x^4 + 48t^2 + 2\beta(2x - i)} \\ \frac{4(-3 - 6ix - 12x^2 + 8ix^3 + 12t(2x + i) + i\beta)}{3 - 24ix - 24x^2 - 32ix^3 - 16x^4 + 48t^2 + 2\beta(2x + i)} \end{bmatrix} e^{-it},$$

where $\beta \in \mathbb{R}$ is a parameter in addition to $c \in (-1, 1)$ and $x_0, t_0, \theta_0 \in \mathbb{R}$.

The existence of the second-order rational solution suggests the existence of a hierarchy of rational solutions in the form:

$$u_1(x, t) = \frac{2}{1 + 2ix} e^{-it}, \quad u_2(x, t) = \frac{P_3(x, t)}{P_4(x, t)} e^{-it}, \quad \dots$$

[Baofeng Feng, Jiaqi Han, Cheng He, & D.P. (2025) in progress]

Properties of the algebraic double-soliton

1. $u(\cdot, t), v(\cdot, t) \in C^\omega(\mathbb{R})$ for every $t \in \mathbb{R}$ and $\beta \in \mathbb{R}$
2. $Q(u, v) = \int_{\mathbb{R}} (|u|^2 + |v|^2) dx = 8\pi = 2Q_{AS}$.

For the proofs, the bilinear formulation is very useful:

$$|u|^2 + |v|^2 = \frac{|g|^2 + |h|^2}{|f|^2} = 2i \left(\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}} \right),$$

where

$$f = 16x^4 + 32ix^3 + 24x^2 + 24ix - 3 - 48t^2 - 2\beta(2x + i)$$

satisfies

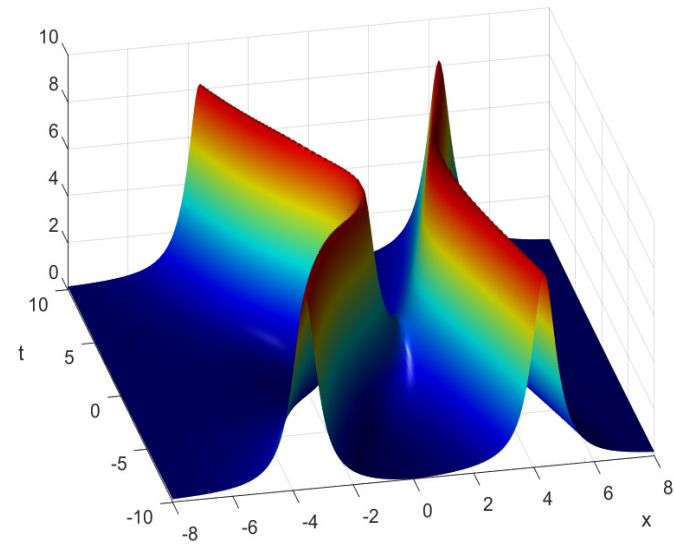
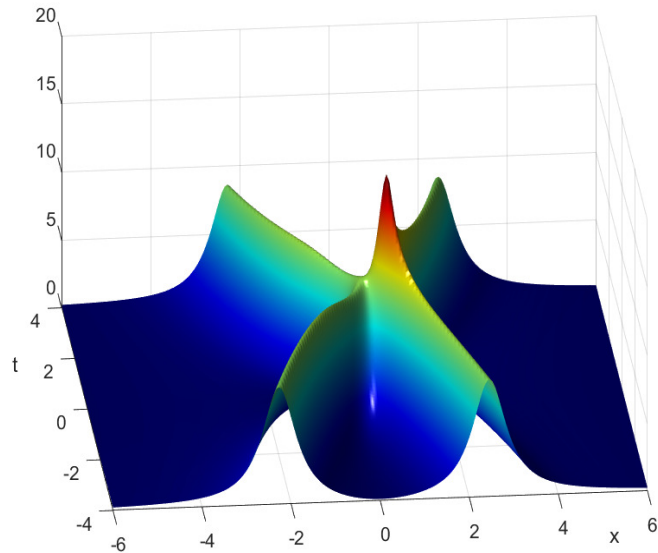
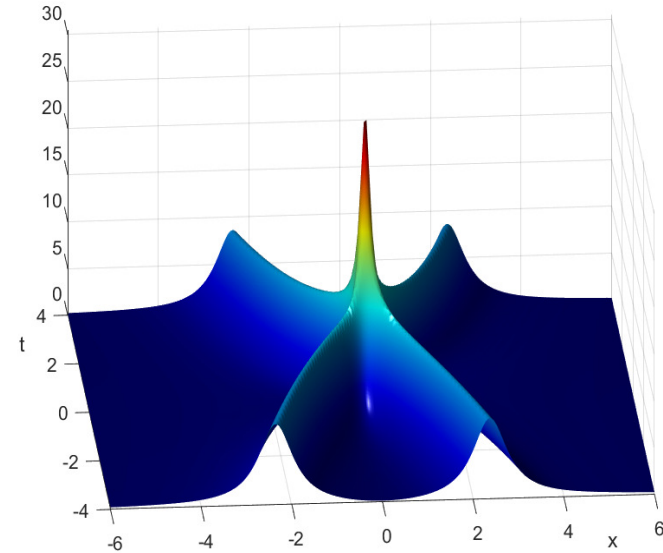
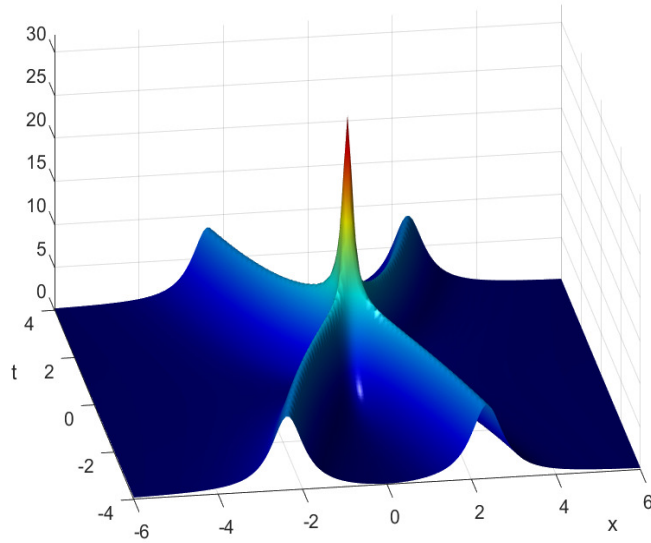
- f has no zeros on $\mathbb{R}$ in x for every $t \in \mathbb{R}$ and $\beta \in \mathbb{R}$.
- f has one root in $\mathbb{C}_+$ and three roots in $\mathbb{C}_-$: $N_+ = 1$, $N_- = 3$.
- $f_x/f - \bar{f}_x/\bar{f} = \mathcal{O}(|x|^{-2})$ as $|x| \rightarrow \infty$.

By the argument principle,

$$Q(u, v) = 4\pi(N_- - N_+) = 8\pi.$$

Properties of the algebraic double-soliton

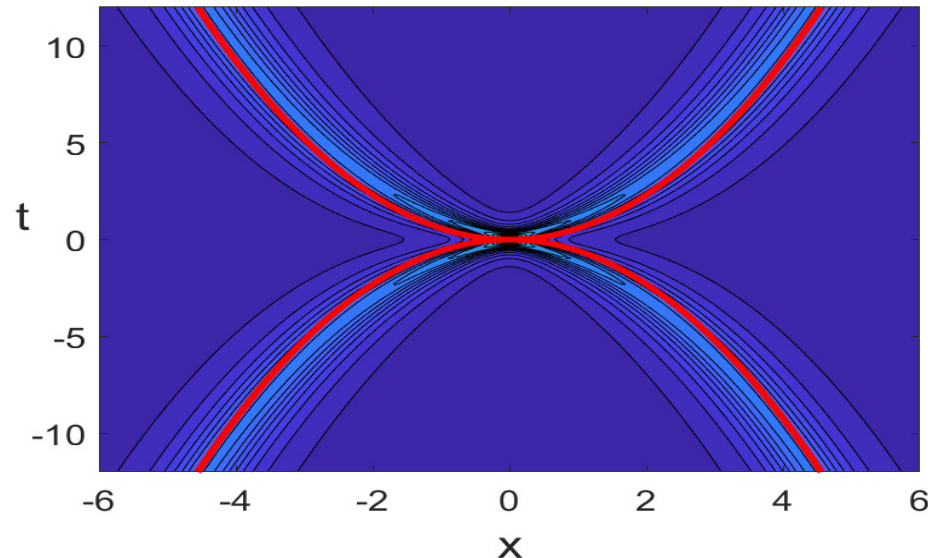
3. The solution suggests slow scattering of two identical algebraic solitons.



This suggests the nonlinear stability of a single algebraic soliton.

Properties of the algebraic double-soliton

4. The two solitons move along the parabolas $x^2 = \pm\sqrt{3}t$.



5. The algebraic double-soliton corresponds to the double embedded eigenvalue $\lambda_0 = i$ in the Kaup–Newell spectral problem.

[Zhi-Qiang Li, P., Shou-Fu Tian (2025)]

This is in agreement with the result from [Klaus,P.,Rothos (2006)]:

Algebraic multiplicity of $\lambda_0 = i$ is $N + 1$ only if $b > N + \frac{1}{2}$, where

$$|u(x)| \sim \frac{b}{|x|} \quad \text{as } |x| \rightarrow \infty \quad \text{for some } b > 0.$$

The algebraic double-soliton corresponds to $b = 2$.

Section 4. Construction of higher-order rational solutions.

The construction is done by the double-Wronskian formulas. In characteristic variables $t = 2(\xi + \eta)$ and $x = 2(\xi - \eta)$, we have

$$\begin{cases} \mathrm{i}u_\xi + 2v = 2|v|^2u, \\ \mathrm{i}v_\eta + 2u = 2|u|^2v. \end{cases}$$

The bilinear transformation

$$u = \frac{g}{\bar{f}}, \quad v = \frac{h}{f},$$

gives the bilinear equations

$$\begin{cases} \mathrm{i}D_\xi(g \cdot f) + 2h\bar{f} = 0, \\ \mathrm{i}D_\eta(h \cdot \bar{f}) + 2gf = 0, \\ \mathrm{i}D_\xi(f \cdot \bar{f}) - 2h\bar{h} = 0, \\ \mathrm{i}D_\eta(\bar{f} \cdot f) - 2g\bar{g} = 0. \end{cases}$$

Double-Wronskian solutions

Let $A \in \mathbb{M}^{2N \times 2N}$ be a complex-valued invertible matrix for $N \in \mathbb{N}$. Define two vectors $\phi, \psi \in \mathbb{C}^{2N}$ from solutions of the linear equations

$$\begin{cases} \partial_\xi \phi = iA\phi, \\ \partial_\eta \phi = iA^{-1}\phi, \end{cases} \quad \text{and} \quad \begin{cases} \partial_\xi \psi = -iA\psi, \\ \partial_\eta \psi = -iA^{-1}\psi, \end{cases}$$

Furthermore, relate $\psi = S\bar{\phi}$, where $S = \sqrt{-A} \in \mathbb{M}^{2N \times 2N}$. Then, the bilinear equations are solved by

$$\begin{cases} f = |\phi', \phi'', \dots, \phi^{(N)}, \psi, \psi', \dots, \psi^{(N-1)}| \\ g = |\phi, \phi', \dots, \phi^{(N)}, \psi', \psi'', \dots, \psi^{(N-1)}| \\ h = i^{N+1} |S| |\phi, \phi', \dots, \phi^{(N)}, \psi, \psi', \dots, \psi^{(N-2)}|. \end{cases}$$

Higher-order rational solutions are obtained from the N -multiple e-value $\lambda_0 = i$:

$$A = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & \dots & 0 & 0 \\ -\frac{1}{2^3} & -\frac{1}{2} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & -\frac{1}{2} & 1 \end{bmatrix}.$$

Properties of the higher-order rational solutions

For each integer N , we have

$$u = \frac{Q_N(x, t)}{\bar{P}_N(x, t)} e^{-it}, \quad v = \frac{R_N(x, t)}{P_N(x, t)} e^{-it},$$

where P_N is a polynomial in x of degree N^2 and Q_N, R_N are of degree $N^2 - 1$.

This follows from the leading-order behavior of f, g , and h , if we can prove that

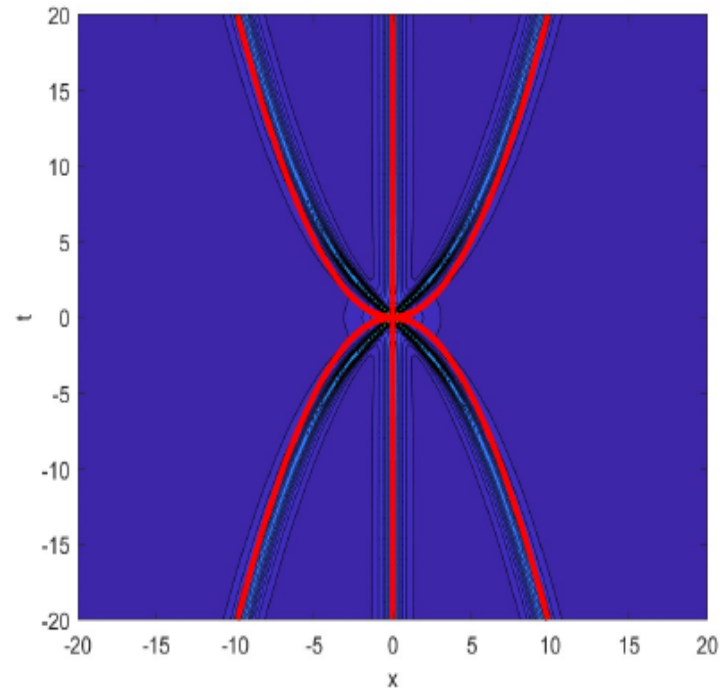
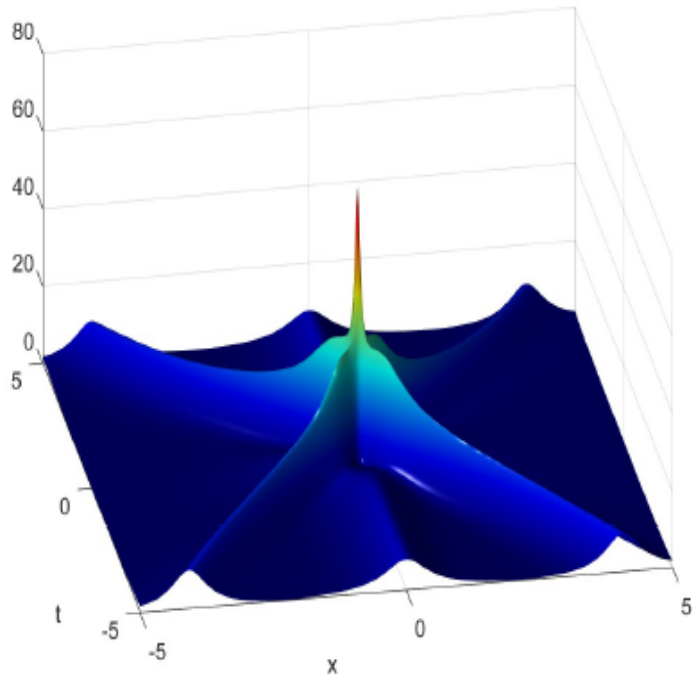
$$\Delta_N := \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots & -1 & 1 & 0 & 0 & \dots \\ \frac{1}{2!} & 1 & 1 & 0 & \dots & \frac{1}{2!} & -1 & 1 & 0 & \dots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & \dots & -\frac{1}{3!} & \frac{1}{2!} & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} \neq 0.$$

For $1 \leq N \leq 10$, we confirmed the formula

$$\Delta_N = \frac{2^N (-1)^N}{1^{2N-1} 3^{2N-3} \dots (2N-3)^3 (2N-1)^1}$$

but cannot prove it for now. This computation yields $|u(x)| \sim \frac{b}{|x|}$ with $b = N$ for N -multiple eigenvalue $\lambda_0 = i$ in agreement with [Klaus, P., Rothos (2006)].

The example of $N = 3$



The separation distance grows as $x^2 = \sqrt{9 + 6\sqrt{6}}|t|$ and the mass $Q(u, v) = 12\pi$.

Generally, we conjecture for every integer N that

$$Q(u, v) = \int_{\mathbb{R}} (|u|^2 + |v|^2) dx = 4\pi N$$

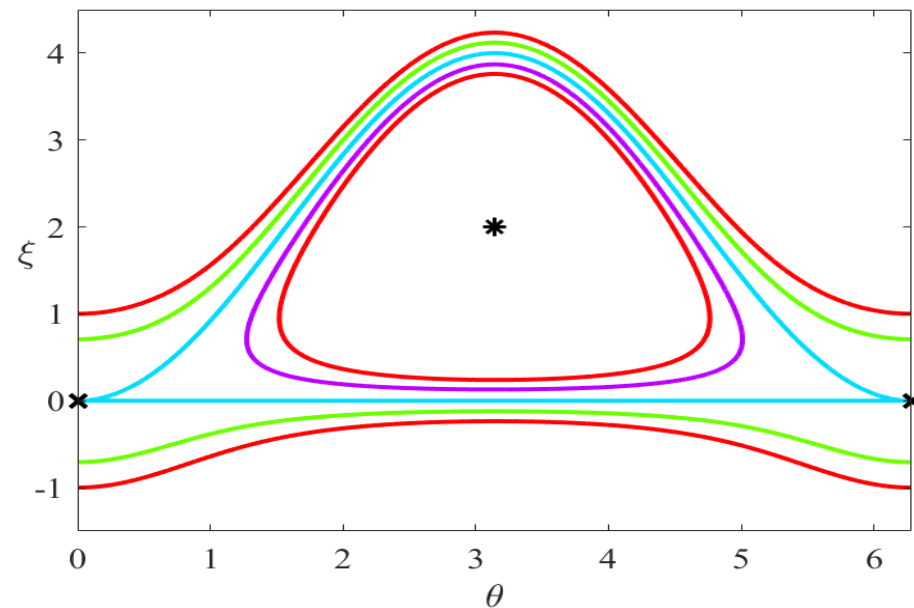
and that $x^2 \sim |t|$ as $|t| \rightarrow \infty$.

Section 5. Algebraic solitons in the limit of periodic solutions

Here is the phase plane for the standing wave solutions

$$u(x, t) = U(x)e^{-i\omega t}, \quad v(x, t) = \bar{U}(x)e^{-i\omega t},$$

with $U(x) = \sqrt{\xi(x)}e^{\frac{i}{2}\theta(x)}$.

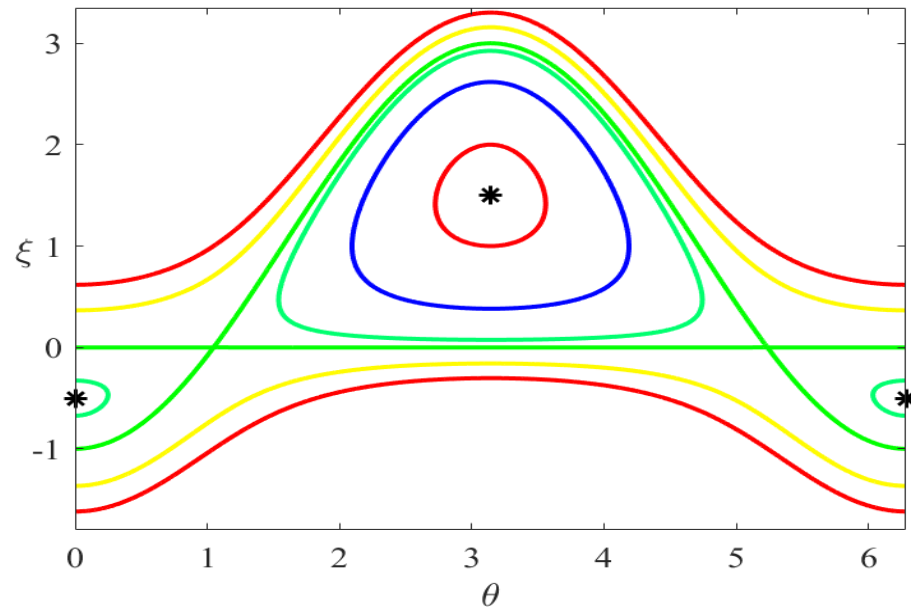


(a) $\omega = -1$.

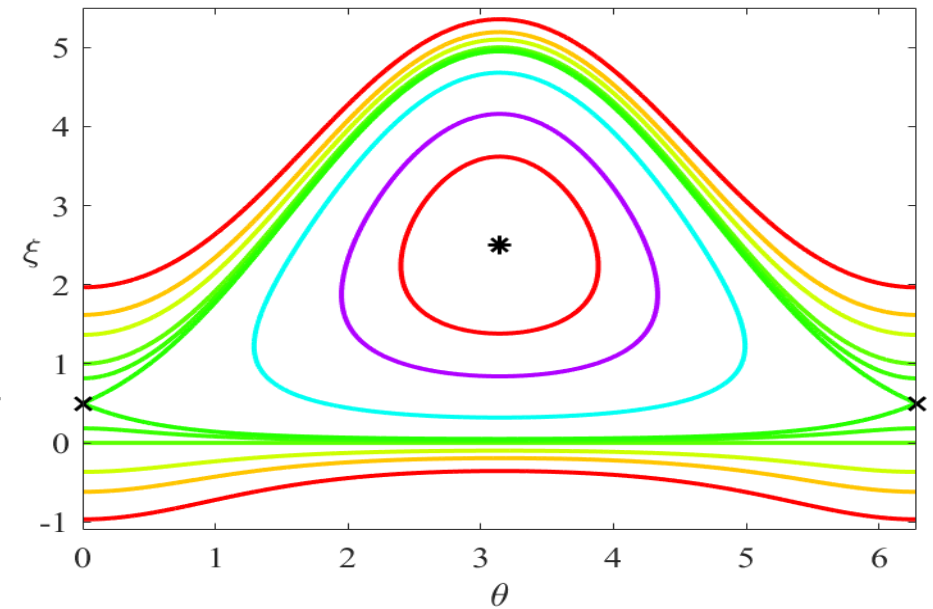
Algebraic soliton corresponds to the degenerate homoclinic orbit for $\omega = -1$.

Algebraic solitons between two different exponential solitons

Algebraic soliton at $\omega = -1$ is located in between exponential soliton for $\omega \in (-1, 1)$ and solitons on the constant background for $\omega < -1$.



(b) $\omega \in (-1, 0)$.



(c) $\omega < -1$.

[Shikun Cui, P. (2025)]

Squared eigenfunction relation

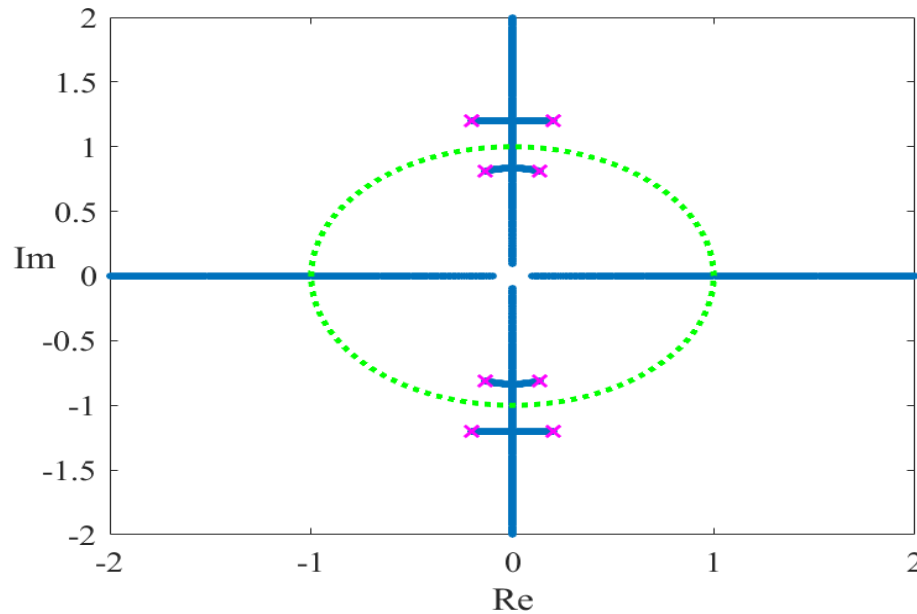
Linear stability is solved by the squared eigenfunctions [Kaup–Lakoba (1996)]

$$\Lambda = \pm i\sqrt{P(\lambda)}, \quad P(\lambda) = \frac{1}{4} \left(\lambda^2 + \frac{1}{\lambda^2} - 2\omega \right)^2 - b,$$

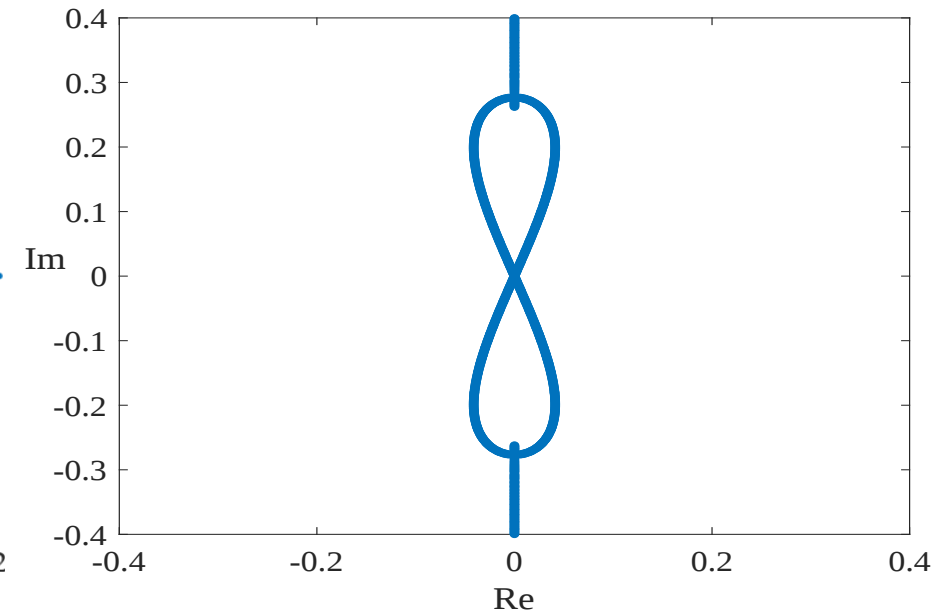
where b is the conserved energy for the spatial profile $U(x)$ and $\lambda \in \mathbb{C}$ belongs to the Lax spectrum with bounded eigenfunctions

$$\partial_x \varphi = L(u, v, \lambda) \varphi, \quad \varphi \in L^\infty(\mathbb{R}, \mathbb{C}^2)$$

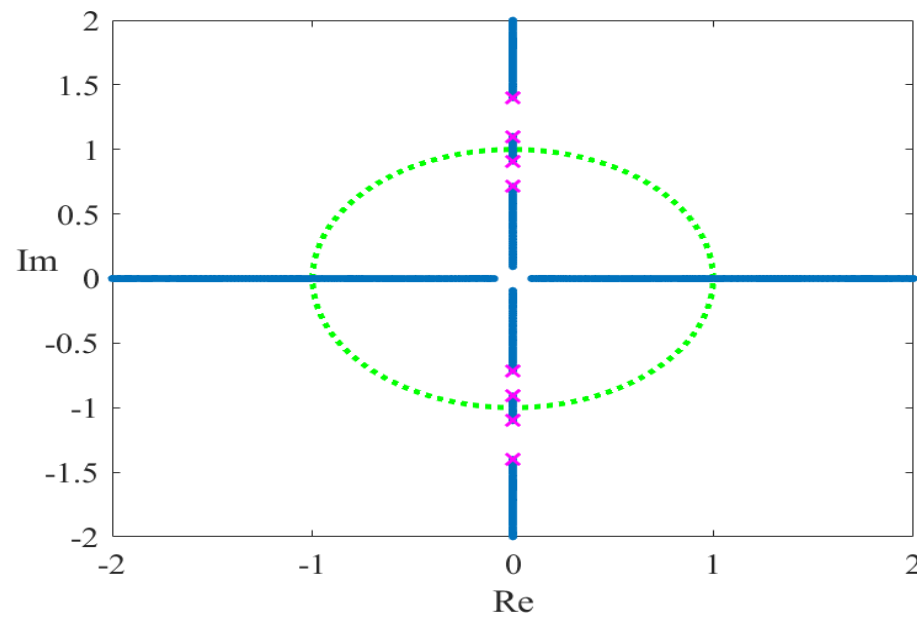
Algebraic soliton between unstable and stable periodic waves



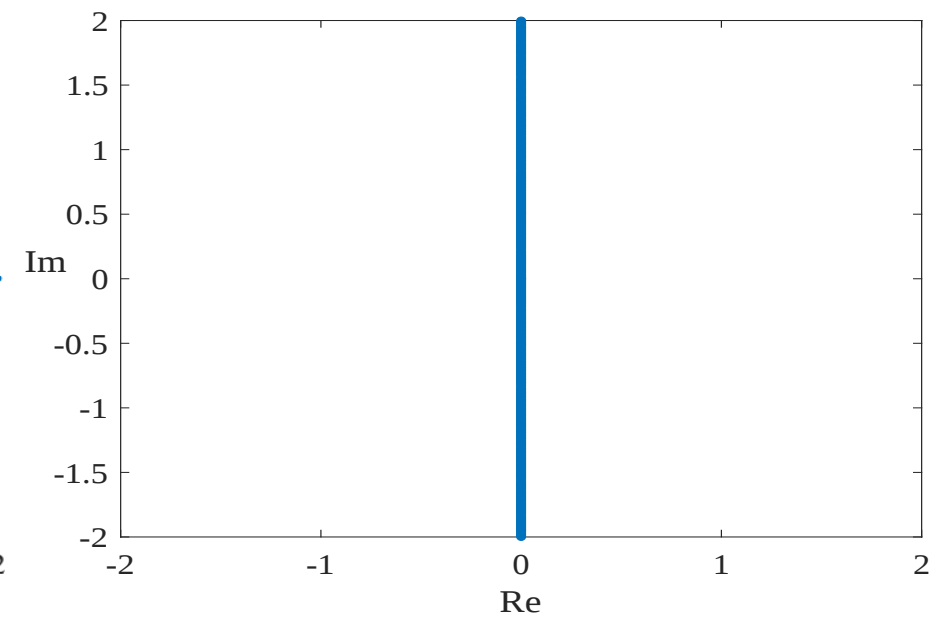
(d) Lax spectrum in λ -plane.



(e) Stability spectrum in Λ -plane.



(f) Lax spectrum in λ -plane.



(g) Stability spectrum in Λ -plane.

Section 6. Conclusion

There are multiple evidences of stability of algebraic solitons in the MTM system:

- Persistence with respect to perturbations. [Klaus, P, & Rothos (2006)]
- Persistence in rational solutions. [Cheng, Han, P. (2024)]
- Linear stability with squared eigenfunctions. [Cui, P. (2025)]

However, there exists no mathematically rigorous proof of the nonlinear stability of algebraic solitons for the MTM system.

A similar question is open for the derivative NLS equation.