Stability of solitary waves in the massive Thirring model

Dmitry Pelinovsky
McMaster University, Canada

February 1, 2025

Section 1. The massive Thirring model (MTM)

The coupled mode equations

$$\left\{\begin{array}{l} \mathrm{i}(u_t + u_x) + v = (\gamma_1 |u|^2 + \gamma_2 |v|^2) u \\ \mathrm{i}(v_t - v_x) + u = (\gamma_2 |u|^2 + \gamma_1 |v|^2) v \end{array}\right.$$

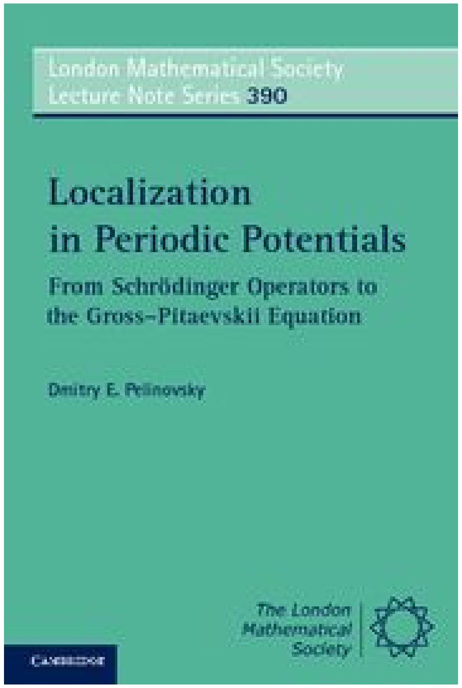
is a popular model for dynamics of two counter-propagating resonant waves across the periodic systems (photonics, Bose–Einstein condensation).

For instance, the Gross–Pitaevskii equation with 2π-periodic, bounded, real-valued potential $V(x)$:

$$i\partial_t \psi = -\partial_x^2 \psi + \varepsilon V(x) \psi \pm |\psi|^2 \psi,$$

is reduced asymptotically as $\epsilon \to 0$ to the coupled mode equations for the superposition of two 2π-antiperiodic waves (under Bragg's parameteric $1:2$ resonance)

$$\psi(x,t) \sim \sqrt{\varepsilon} \left[u(\varepsilon x, \varepsilon t) e^{\frac{i}{2}x} + v(\varepsilon x, \varepsilon t) e^{-\frac{i}{2}x} \right] e^{\frac{i}{4}t},$$



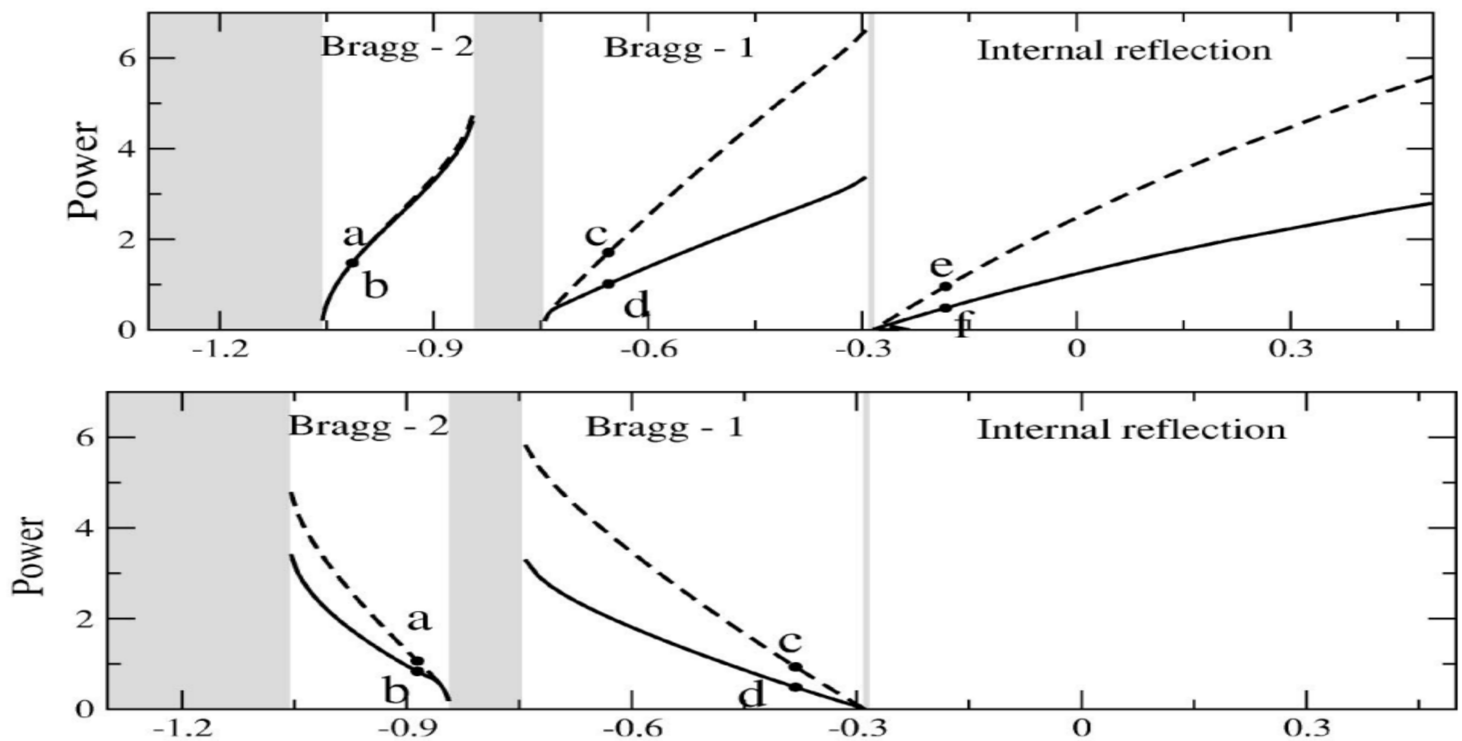
If $\gamma_1 = 0$, the system was introduced in quantum field theory by Thirring in 1958 as the relativistically invariant Dirac equation in one dimension, known now as the MTM in laboratory coordinates.

Gap solitons

For every $\gamma_1, \gamma_2 > 0$, there exist spatially decaying (localized) solutions called gap solitons (solitary waves) in the gaps of purely continuous spectrum of

$$\mathcal{L} := -\partial_x^2 + \varepsilon V(x), \quad \mathrm{Dom}(\mathcal{L}) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}), \qquad V(x + 2\pi) = V(x).$$

[Y.Kivshar, D.P., A. Sukhorukov (2004)] [A. Pankov (2005)] [M. Weinstein (2006)]



The decay rate is exponential except for the end points in each finite gap. Algebraic solitons attain the maximal power $\int_{\mathbb{R}} |\psi|^2 dx$ in each family and the solutions are expressed explicitly within the coupled-mode theory.

About exceptionality of the MTM

The MTM in laboratory coordinates

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = |v|^2 u, \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v, \end{cases}$$

is the only example of the coupled-mode system which is relativistically invariant:

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \mapsto \begin{bmatrix} \left(\frac{1-c}{1+c}\right)^{1/4} u\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \\ \left(\frac{1+c}{1-c}\right)^{1/4} v\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \end{bmatrix}, \qquad c \in (-1,1).$$

It also has standard symmetries of translations in x, t, and $\arg(u) = \arg(v)$.

MTM is integrable due to existence of the Lax pair [Mikhailov, 1976]:

$$\partial_x \varphi = L(u,v,\lambda)\varphi, \qquad \partial_t \varphi = A(u,v,\lambda)\varphi, \qquad \varphi(x,t) \in \mathbb{C}^2, \qquad \lambda \in \mathbb{C}.$$

See also [Orfanidis, 1976], [Kaup & Newell, 1977], [Barashenkov & Getmanov, 1989], [Villaroel, 1991], [Lee, 1994], [Zhou, 1995].

About exceptionality of the algebraic solitons

One-soliton solutions are expressed in 1-parameter form:

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \sin\gamma \begin{bmatrix} \mathrm{sech}\left(x\ \sin\gamma + \frac{\mathrm{i}\gamma}{2}\right) \\ \mathrm{sech}\left(x\ \sin\gamma - \frac{\mathrm{i}\gamma}{2}\right) \end{bmatrix} e^{\mathrm{i}t\cos\gamma}, \quad \gamma \in (0,\pi)$$

with zero limit as $\gamma \to 0$ and nonzero limit as $\gamma \to \pi$:

$$\gamma = \pi : \qquad \begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} \frac{2}{1+2\mathrm{i}x} \\ \frac{2}{1-2\mathrm{i}x} \end{bmatrix} e^{-\mathrm{i}t}, \qquad (u,v) \in L^2(\mathbb{R}).$$

The frequency of standing waves $\omega = \cos\gamma \in (-1,1)$ takes values in the gap of the Dirac operator $\mathcal{D} := \begin{bmatrix} \mathrm{i}\partial_x & 1 \\ 1 & -\mathrm{i}\partial_x \end{bmatrix}$ in $L^2(\mathbb{R})$, where $\sigma(\mathcal{D}) = (-\infty,-1] \cup [1,\infty)$.

The algebraic soliton ($\gamma = \pi$) attains the maximal power
$Q(u,v) = \int_{\mathbb{R}} (|u|^2 + |v|^2) dx = 4\pi$ along the family.

Section 2. Stability of algebraic solitons

For stability of exponential solitons, several analytical results are available:

- Orbital stability in $H^1(\mathbb{R})$ via constrained minimization of energy: P. & Shimabukuro (2014)
- Orbital stability in $L^2(\mathbb{R})$ via Bäcklund transformation: Contreras, P., & Shimabukuro (2016)
- Asymptotic stability via IST: P. & Saalmann (2019), He, Liu, & Qu (2024)

In the limit of algebraic solitons, all methods fail!

- There is no coercivity of the second variation from higher-order energy.
- Bäcklund transformation become trivial and generalizations do not help: [Guo, Ling, & Liu (2013)]
- Solvability of the IST is not justified for slowly decaying potentials.

The only available result is from Klaus, P, & Rothos (2006)

Consider the Kaup–Newell spectral problem

$$\partial_x \varphi = \begin{bmatrix} -i\lambda^2 & \lambda w(x) \\ -\lambda \bar{w}(x) & i\lambda^2 \end{bmatrix} \varphi,$$

where $|w(x)| = |u(x)| = |v(x)|$ in defined from the solution of the MTM. Assume that

$$|w(x)| \sim \frac{b}{|x|} \quad \text{as } |x| \to \infty \quad \text{for some } \ b > 0.$$

- $\lambda_0 = i$ is an embedded eigenvalue only if $b > \frac{1}{2}$. [$b = 1$ for algebraic soliton.]
- If $\lambda_0 = i$ is an embedded eigenvalue, then its geometric multiplicity is one. Its algebraic multiplicity is $N + 1$ only if $b > N + \frac{1}{2}$. [No examples were given.]
- Let w_0 be the algebraic soliton and $\epsilon w_1(x)$ be a perturbation with fixed profile w_1. For every $\epsilon \neq 0$, there exists a simple eigenvalue of the Lax spectrum in each quadrant of $\mathbb{C}$ independently of the sign of ϵ provided that w_1 satisfies a non-degeneracy condition. [This suggests stability of an algebraic soliton.]

Why is this surprising?

The Gardner (modified KdV) equation

$$u_t + 12uu_x + 6u^2u_x + u_{xxx} = 0$$

also has the algebraic soliton

$$u_0(x) = -\frac{4}{1+4x^2}$$

associated with the Zakharov–Shabat spectral problem

$$\partial_x \varphi = \begin{bmatrix} i\lambda & -1-u(x) \\ 1+u(x) & -i\lambda \end{bmatrix} \varphi.$$

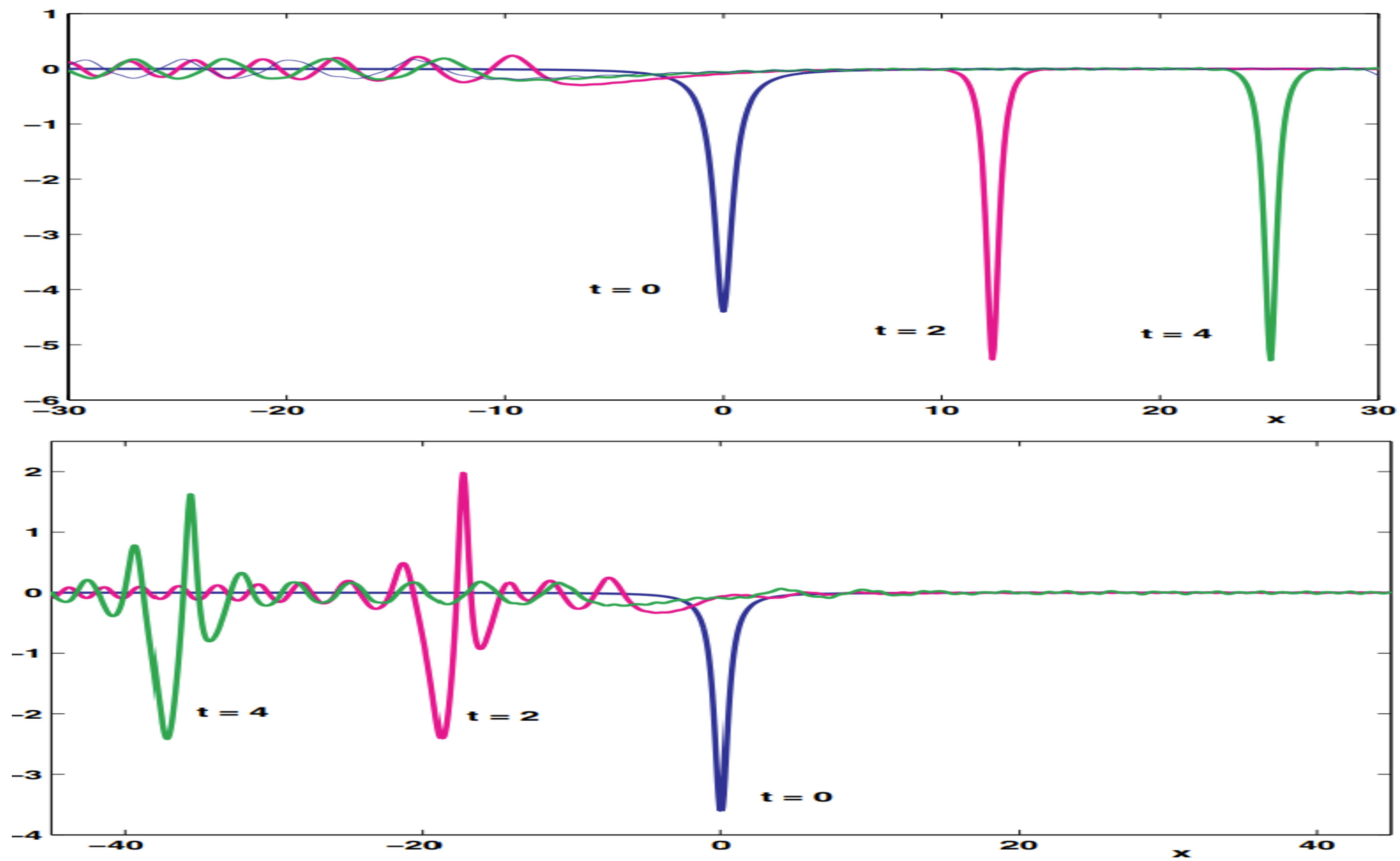
However, the algebraic soliton is a nonzero minimum of conserved momentum $Q(u) = \int_{\mathbb{R}} u^2 dx$ among exponential solitons and hence it is nonlinearly unstable.

Instability for similar mKdV and NLS models was shown in analysis papers: Fukaya & Hayashi (2021), Kfoury, Le Coz & Tsai (2022).

More results on instability of algebraic solitons in mKdV

- $\lambda_0 = i$ is an embedded eigenvalue of the Zakharov–Shabat spectral problem.

- Let $\epsilon u_1(x)$ be a perturbation with fixed profile u_1. For every $\epsilon \neq 0$, we have either a simple eigenvalue $\lambda = i + i\mathcal{O}(|\epsilon|^{2/3})$ or a symmetric pair of eigenvalues $\lambda = i \pm \mathcal{O}(|\epsilon|^{2/3})$ depending on the sign of ϵ. [P & Grimshaw (1997)]



More results on instability of algebraic solitons in mKdV

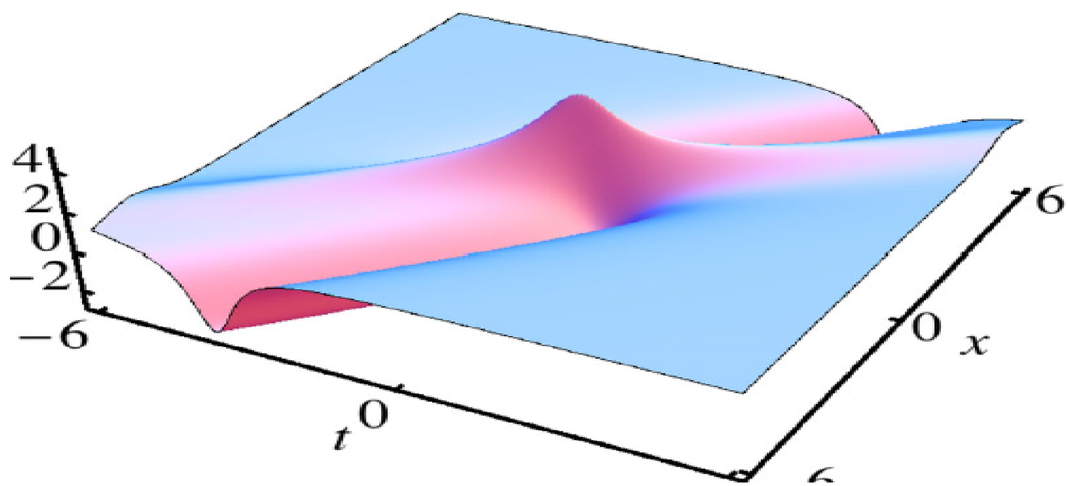
The instability of algebraic solitons can be shown from the rational solutions of the Gardner equation

$$u_t + 12uu_x + 6u^2u_x + u_{xxx} = 0.$$

A hierarchy of rational solutions is available in the form:

$$u_1(x) = -\frac{4}{1+4x^2}, \quad u_2(x,t) = \frac{P_4(x,t)}{P_6(x,t)}, \quad \dots.$$

[Chowdury, Ankiewicz & Akhmediev (2016)], [Xing *et al.* (2017)]



The solution u_2 suggests the instability of u_1: $\|u_2(\cdot,t) - u_1\|_{L^2}$ grows in time t.

Section 3. Rational solutions of MTM

Consider the MTM in laboratory coordinates

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = |v|^2 u, \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v. \end{cases}$$

Rational solutions on nonzero background (rogue waves) were already constructed:
Guo, Wang, Cheng, & He (2017)
Ye, Bu, Pan, Chen, Mihalche, & Baronio (2021)
Chen, Yang, & Feng (2023).

Surprisingly, rational solutions on the zero background have not been constructed for MTM, although they were constructed for derivative NLS [Wang & Wu (2022)]

We have constructed the second-order rational solution to the MTM:
Jiaqi Han, Cheng He, & D.P. (2024)

Construction of the second-order rational solution

We use the bilinear formulation of the MTM from [Chen & Feng (2023)]:

$$u = \frac{g}{\bar{f}}, \qquad v = \frac{h}{f},$$

where

$$\left. \begin{array}{r} \mathrm{i}f(g_t + g_x) - \mathrm{i}g(f_t + f_x) + h\bar{f} = 0, \\ \mathrm{i}\bar{f}(h_t - h_x) - \mathrm{i}h(\bar{f}_t - \bar{f}_x) + gf = 0, \\ \mathrm{i}\bar{f}(f_x + f_t) - \mathrm{i}f(\bar{f}_t + \bar{f}_x) - |h|^2 = 0, \\ \mathrm{i}f(\bar{f}_t - \bar{f}_x) - \mathrm{i}\bar{f}(f_t - f_x) - |g|^2 = 0. \end{array} \right\}$$

- The exponential 2-solitons are obtained with 8 parameters.
- Four parameters yields two eigenvalues $\lambda_1 = \delta_1 e^{i\gamma_1}$ and $\lambda_2 = \delta_2 e^{i\gamma_2}$ in the first quadrant of $\mathbb{C}$. Four more parameters are translational parameters.
- A limit to the algebraic 2-soliton solutions yields a soluton with 6 parameters:

$$\delta_1 \neq \delta_2, \qquad \gamma_1, \gamma_2 \to \pi.$$

- The limit $\delta_1 \to \delta_2$ gives the algebraic double-soliton with 5 parameters.

The algebraic double-soliton

The algebraic double-soliton is expressed as the second-order rational solution

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{4(-3+6\mathrm{i}x-12x^2-8\mathrm{i}x^3-12t(2x-\mathrm{i})-\mathrm{i}\beta)}{3+24\mathrm{i}x-24x^2+32\mathrm{i}x^3-16x^4+48t^2+2\beta(2x-\mathrm{i})} \\ \dfrac{4(-3-6\mathrm{i}x-12x^2+8\mathrm{i}x^3+12t(2x+\mathrm{i})+\mathrm{i}\beta)}{3-24\mathrm{i}x-24x^2-32\mathrm{i}x^3-16x^4+48t^2+2\beta(2x+\mathrm{i})} \end{bmatrix} e^{-\mathrm{i}t},$$

where $\beta \in \mathbb{R}$ is a parameter in addition to $c \in (-1,1)$ and $x_0, t_0, \theta_0 \in \mathbb{R}$.

The existence of the second-order rational solution suggestes the existence of a hierarchy of rational solutions in the form:

$$u_1(x,t) = \frac{2}{1+2ix}e^{-it}, \quad u_2(x,t) = \frac{P_3(x,t)}{P_4(x,t)}e^{-it}, \quad \ldots.$$

[Baofeng Feng, Jiaqi Han, Cheng He, & D.P. (2025) in progress]

Properties of the algebraic double-soliton

1. $u(\cdot,t), v(\cdot,t) \in C^\omega(\mathbb{R})$ for every $t \in \mathbb{R}$ and $\beta \in \mathbb{R}$
2. $Q(u,v) = \int_{\mathbb{R}}(|u|^2 + |v|^2)dx = 8\pi = 2Q_{AS}$.

For the proofs, the bilinear formulation is very useful:

$$|u|^2 + |v|^2 = \frac{|g|^2 + |h|^2}{|f|^2} = 2\mathrm{i}\left(\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}}\right),$$

where

$$f = 16x^4 + 32\mathrm{i}x^3 + 24x^2 + 24\mathrm{i}x - 3 - 48t^2 - 2\beta(2x + \mathrm{i})$$

satisfies

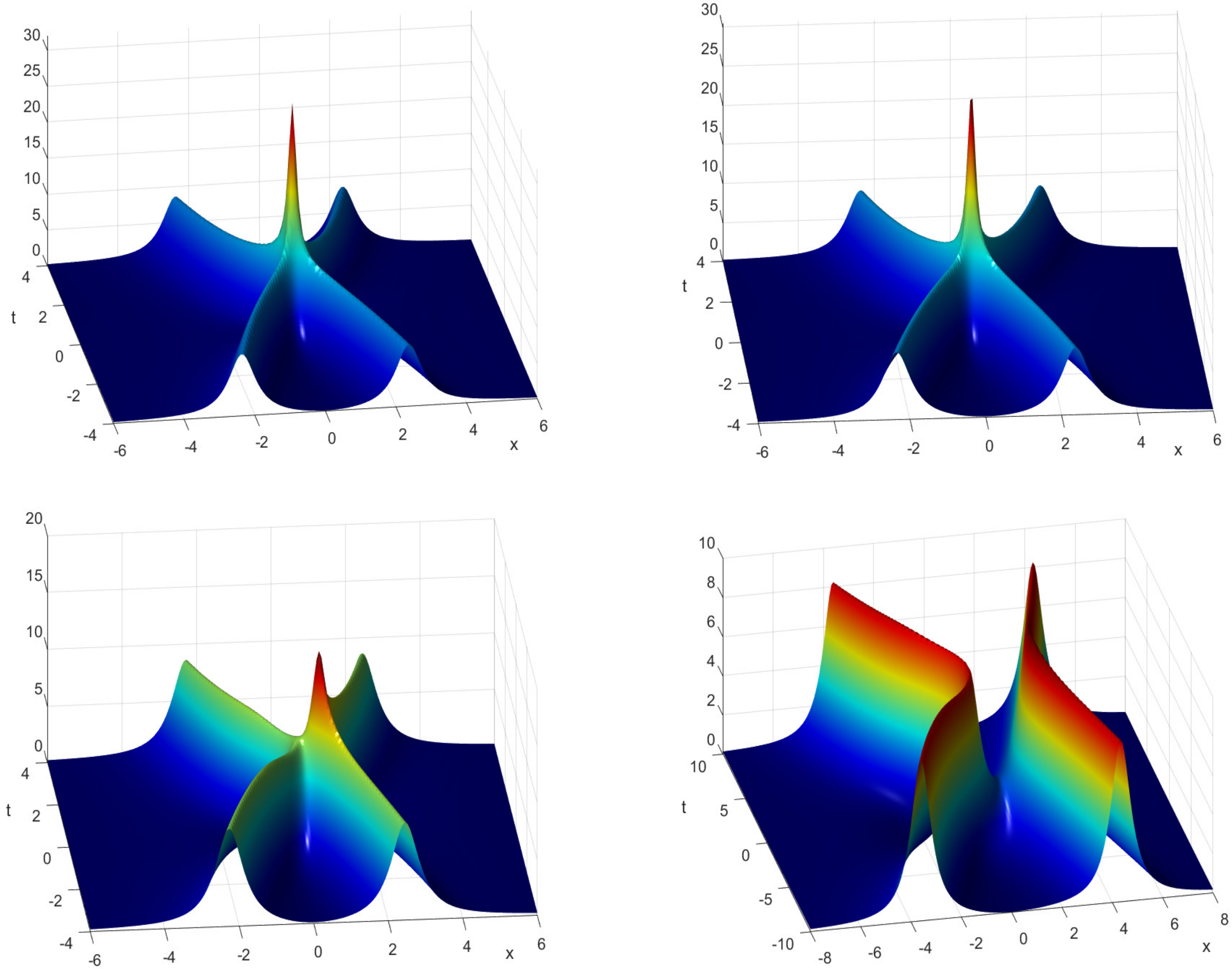
- f has no zeros on $\mathbb{R}$ in x for every $t \in \mathbb{R}$ and $\beta \in \mathbb{R}$.
- f has one root in $\mathbb{C}_+$ and three roots in $\mathbb{C}_-$: $N_+ = 1$, $N_- = 3$.
- $f_x/f - \bar{f}_x/\bar{f} = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$.

By the argument principle,

$$Q(u,v) = 4\pi(N_- - N_+) = 8\pi.$$

Properties of the algebraic double-soliton

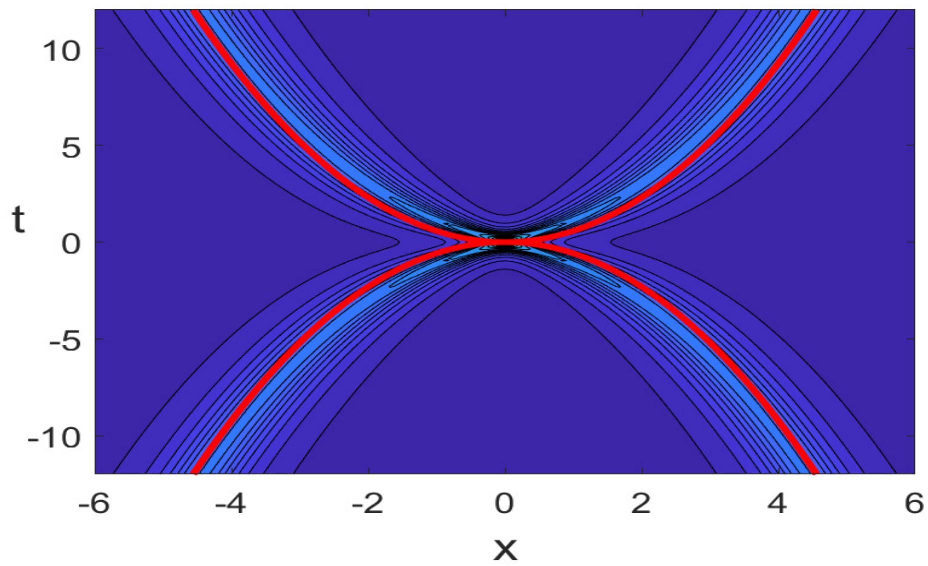
3. The solution suggests slow scattering of two identical algebraic solitons.



This suggests orbital stability of a single algebraic soliton (an open problem).

Properties of the algebraic double-soliton

4. The two solitons move along the parabolas $x^2 = \pm\sqrt{3}t$.



5. The algebraic double-soliton corresponds to the double embedded eigenvalue $\lambda_0 = i$ in the Kaup–Newell spectral problem. [Li, P., Tian (2025)]

This is in agreement with the result from [Klaus,P.,Rothos (2006)]:

Algebraic multiplicity of $\lambda_0 = i$ is $N+1$ only if $b > N + \frac{1}{2}$, where

$$|u(x)| \sim \frac{b}{|x|} \quad \text{as } |x| \to \infty \quad \text{for some} \quad b > 0.$$

The algebraic double-soliton corresponds to $b = 2$.

Section 4. Stability of exponential solitons

Consider the initial-value problem for the MTM in laboratory coordinates

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = |v|^2 u, \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v, \end{cases}$$

starting with initial data $(u, v)|_{t=0} = (u_0, v_0)$.

- Local and global solutions in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$. [Goodman & Weinstein (2001)]
- Local and global solutions in $L^2(\mathbb{R})$ [Candy (2011)], [Huh & Moon (2015)]

Conservation of mass, momentum and energy:

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v})\, dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v})\, dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx.$$

Orbital stability of exponential solitons

Recall the exponential solitons with frequency $\omega := \cos\gamma \in (-1,1)$:

$$\mathbf{u}(x,t) = \mathbf{U}_\omega(x) e^{i\omega t}, \quad \mathbf{U}_\omega(x) = \sin\gamma \begin{bmatrix} \mathrm{sech}\left(x \ \sin\gamma + \frac{\mathrm{i}\gamma}{2}\right) \\ \mathrm{sech}\left(x \ \sin\gamma - \frac{\mathrm{i}\gamma}{2}\right) \end{bmatrix}.$$

Definition 1. *We say that the exponential soliton is orbitally stable in* X *if for any* $\epsilon > 0$ *there is a* $\delta > 0$*, such that if* $\|\mathbf{u}(\cdot,0) - \mathbf{U}_\omega(\cdot)\|_X \leq \delta$ *then*

$$\inf_{\theta,a\in\mathbb{R}} \|\mathbf{u}(\cdot,t) - e^{-i\theta}\mathbf{U}_\omega(\cdot + a)\|_X \leq \epsilon,$$

for all $t > 0$*. Here* $X = H^1(\mathbb{R})$ *or* $X = L^2(\mathbb{R})$*.*

First derivative test: $\mathbf{U}_\omega$ is a critical point of $H + \omega Q$.
Second derivative test: the quadratic part of energy H is not bounded from neither above or below since $\omega \in (-1,1)$ is in the gap of the spectrum $\sigma(\mathcal{D}) = (-\infty,-1] \cup [1,\infty)$ of the Dirac operator

$$\mathcal{D} = \begin{bmatrix} \mathrm{i}\partial_x & 1 \\ 1 & -\mathrm{i}\partial_x \end{bmatrix} : H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

I. Orbital stability of exponential solitons in $H^1(\mathbb{R})$

A higher-order energy exists for the MTM due to its integrability:

$$R = \int_{\mathbb{R}} \Big[|u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x\overline{u} - \overline{u}_x u)(|u|^2 + 2|v|^2) \\ -(u\overline{v} + \overline{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big] \, dx.$$

Theorem 2 (P–Shimabukuro (2014)). *We have*

- *First derivative test:* $\mathbf{U}_\omega$ *is a critical point of* $\Lambda_\omega := R + (1-\omega^2)Q$.
- *Second derivative test:* $\mathbf{U}_\omega$ *is a local non-degenerate minimizer of* R *in* $H^1(\mathbb{R})$ *under the constraints of fixed mass* Q *and fixed momentum* P *up to the translational and rotational symmetries.*

The proof is based on the coercivity of the conserved Lyapunov functional

$$\Lambda_\omega(\mathbf{U}_\omega + \mathbf{U}) - \Lambda_\omega(\mathbf{U}_\omega) \geq C \left(\|\mathbf{U}\|_{H^1}^2 - \|\mathbf{U}\|_{H^1}^4 \right),$$

subject to four constraints and control of four modulation parameters in time.

What goes wrong for stability of algebraic solitons in $H^1(\mathbb{R})$

Recall the Lyapunov functional $\Lambda_\omega := R + (1-\omega^2)Q$ with

$$R = \int_{\mathbb{R}} \Big[|u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x \overline{u} - \overline{u}_x u)(|u|^2 + 2|v|^2) \\ -(u\overline{v} + \overline{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big] \, dx$$

and

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2 \right) dx.$$

For the algebraic soliton with frequency $\omega = -1$:

$$\mathbf{u}(x,t) = \mathbf{U}_{\omega=-1}(x) e^{-it}, \quad \mathbf{U}_{\omega=-1}(x) = \begin{bmatrix} \frac{2}{1+2ix} \\ \frac{2}{1-2ix} \end{bmatrix},$$

coercivity of the Lyapunov functional $\Lambda_{\omega=-1} = R$ is lost. The continuous spectrum of R'' has no gap from the zero eigenvalue due to symmetries.

II. Orbital stability of exponential solitons in $L^2(\mathbb{R})$

The Bäcklund transformation $\mathcal{B}$ is a map that takes one solution (u, v) of the MTM system to another solution $(\tilde{u}, \tilde{v})$ of the MTM system:

$$\mathcal{B} : (u, v) \mapsto (\tilde{u}, \tilde{v}),$$

In particular, the Bäcklund transformation relates zero $\leftrightarrow$ one soliton:

$$(0, 0) \overset{\mathcal{B}}{\longleftrightarrow} (u_\lambda, v_\lambda)$$

Theorem 3 (Contreras–P–Shimabukuro (2016)). *Let $\mathbf{u}(\cdot, t) \in C(\mathbb{R}; L^2(\mathbb{R}))$ be a solution of the MTM system and $\lambda_0 \in \mathbb{C}$ be an eigenvalue in the first quadrant. There exist a real positive constant ϵ such that if the initial value $\mathbf{u}_0 \in L^2(\mathbb{R})$ satisfies*

$$\|\mathbf{u}_0 - \mathbf{u}_{\lambda_0}(0, \cdot)\|_{L^2} \leq \epsilon,$$

then for every $t \in \mathbb{R}$, there exists $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| \leq C\epsilon$,

$$\inf_{a,\theta \in \mathbb{R}} \|\mathbf{u}(\cdot + a, t) - e^{-i\theta}\mathbf{u}_\lambda(\cdot, t)\|_{L^2} \leq C\epsilon,$$

where the constant C is independent of ϵ and t.

Bäcklund transformation for the MTM system

- Let (u, v) be a C^1 solution of the MTM system.
- Let $\vec{\phi} = (\phi_1, \phi_2)^t$ be a C^2 nonzero solution of the linear system

$$\vec{\phi}_x = L(u, v, \lambda)\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A(u, v, \lambda)\vec{\phi},$$

for $\lambda = e^{\frac{i\gamma}{2}}$, $\gamma \in (0, \pi)$.

A new C^1 solution of the MTM system is given by

$$\tilde{u} = -u \frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i \sin\gamma \overline{\phi}_1 \phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}$$

$$\tilde{v} = -v \frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i \sin\gamma \overline{\phi}_1 \phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2},$$

If $(u, v) = (0, 0)$ and

$$\begin{cases} \phi_1 = e^{\frac{i}{4}(\lambda^2 - \lambda^{-2})x + \frac{i}{4}(\lambda^2 + \lambda^{-2})t}, \\ \phi_2 = e^{-\frac{i}{4}(\lambda^2 - \lambda^{-2})x - \frac{i}{4}(\lambda^2 + \lambda^{-2})t}. \end{cases}$$

then $(\tilde{u}, \tilde{v})$ is 1-soliton.

The proof of orbital stability consists of three steps

Fix $\lambda_0 \in \mathbb{C}_I$ for a MTM soliton $\mathbf{u}_{\lambda_0}$. Take $\mathbf{u}_0 \in L^2(\mathbb{R})$ s.t. $\|\mathbf{u}_0 - \mathbf{u}_{\lambda_0}(\cdot, 0)\|_{L^2} < \epsilon$.

1 From a perturbed one-soliton to a small solution at $t = 0$:

There exists $\lambda \in \mathbb{C}$ and $\vec{\psi} \in H^1(\mathbb{R})$ of $\partial_x \vec{\psi} = L(\mathbf{u}_0; \lambda)\vec{\psi}$ such that $|\lambda - \lambda_0| \lesssim \epsilon$. The Bäcklund transformation $\mathcal{B}(\vec{\psi}, \lambda) : \mathbf{u}_0 \mapsto \widetilde{\mathbf{u}}_0$ yields the estimate

$$\|\widetilde{\mathbf{u}}_0\|_{L^2} \lesssim \|\mathbf{u}_0 - \mathbf{u}_{\lambda_0}(\cdot, 0)\|_{L^2}.$$

2 Time evolution: $\|\widetilde{\mathbf{u}}(\cdot, t)\|_{L^2} = \|\widetilde{\mathbf{u}}_0\|_{L^2}$, $t \in \mathbb{R}$.

3 From the small solution to the perturbed one-soliton for $t \in \mathbb{R}$:

There exists two linearly independent solutions of

$$\vec{\phi}_x = L(\widetilde{\mathbf{u}}(\cdot, t), \lambda)\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A(\widetilde{\mathbf{u}}(\cdot, t), \lambda)\vec{\phi},$$

The Bäcklund transformation $\mathcal{B}(\vec{\phi}, \lambda) : \widetilde{\mathbf{u}}(\cdot, t) \mapsto \mathbf{u}(\cdot, t)$ yields the estimate

$$\|\mathbf{u}(\cdot, t) - e^{-i\theta(t)}\mathbf{u}_\lambda(\cdot + a(t), t)\|_{L^2} \lesssim \|\widetilde{\mathbf{u}}(\cdot, t)\|_{L^2} \quad \forall t \in \mathbb{R}.$$

where $a(t)$ and $\theta(t)$ are defined in the linear combination of two solutions.

What goes wrong for stability of algebraic solitons in $L^2(\mathbb{R})$

Recall Bäcklund transformation:

$$\tilde{u} = -u\frac{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2} + \frac{2i\sin\gamma\bar{\phi}_1\phi_2}{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}$$

$$\tilde{v} = -v\frac{e^{i\gamma/2}|\phi_1|^2 + e^{-i\gamma/2}|\phi_2|^2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2} - \frac{2i\sin\gamma\bar{\phi}_1\phi_2}{e^{-i\gamma/2}|\phi_1|^2 + e^{i\gamma/2}|\phi_2|^2},$$

If $\gamma = \pi$, then $(\tilde{u}, \tilde{v}) = (u, v)$ and the Bäcklund transformation fails to provide the mapping: zero $\leftrightarrow$ one soliton.

A generalized Bäcklund transformation is available [Guo, Ling, & Liu (2013)]. However, the estimate

$$\|\widetilde{\mathbf{u}}_0\|_{L^2} \lesssim \|\mathbf{u}_0 - \mathbf{u}_{\lambda_0}(\cdot, 0)\|_{L^2}$$

is based on estimates of solutions $\vec{\psi} \in H^1(\mathbb{R})$ of $\partial_x \vec{\psi} = L(\mathbf{u}_0; \lambda)\vec{\psi}$ in the exponentially weighted spaces. It is not clear how to introduce similar algebraically weighted spaces for a generalized Bäcklund transformation which would work for algebraically decaying potentials $\mathbf{u}_{\lambda_0}$ with $\lambda_0 = i$.

III. Asymptotic stability of exp. solitons in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$

The MTM system is a compatibility condition of the linear system

$$\vec{\phi}_x = L(u,v,\lambda)\vec{\phi} \quad \text{and} \quad \vec{\phi}_t = A(u,v,\lambda)\vec{\phi},$$

where

$$L = \frac{i}{2}(|v|^2 - |u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2} - \lambda^2\right)\sigma_3$$

and

$$A = -\frac{i}{4}(|u|^2 + |v|^2)\sigma_3 - \frac{i\lambda}{2}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{2\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\sigma_3$$

Theorem 4 (P–Saalmann (2019); He–Liu–Qu (2023))**.** *Let $\lambda_0 \in \mathbb{C}$ be the only eigenvalue in the first quadrant for $\mathbf{u}_0 \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. There exist $\epsilon > 0$ such that if $\|\mathbf{u}_0 - \mathbf{u}_{\lambda_0}(0,\cdot)\|_{L^2} \leq \epsilon$, then there exist functions $a(t), \theta(t) \in C^0(\mathbb{R})$ such that the solution $\mathbf{u}(\cdot,t) \in C^0(\mathbb{R}, H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R}))$ satisfies*

$$\lim_{t\to+\infty} \|\mathbf{u}(\cdot,t) - e^{-i\theta(t)}\mathbf{u}_{\lambda_0}(\cdot - a(t),t)\|_{L^\infty} = 0.$$

Direct scattering

Assuming $(u,v)\to(0,0)$ as $|x|\to\infty$ fast enough, there exist matrix Jost functions satisfying the asymptotic values

$$\phi^{(\pm)} \to \begin{pmatrix} e^{\frac{i}{4}(\lambda^2-\lambda^{-2})x+\frac{i}{4}(\lambda^2+\lambda^{-2})t} & 0 \\ 0 & e^{-\frac{i}{4}(\lambda^2-\lambda^{-2})x-\frac{i}{4}(\lambda^2+\lambda^{-2})t} \end{pmatrix} \quad \text{as} \quad x\to\pm\infty$$

and the scattering relations

$$\phi^{(-)} = \phi^{(+)} \left(\begin{array}{cc} \overline{a}(\lambda) & b(\lambda) \\ -\overline{b}(\lambda) & a(\lambda) \end{array} \right),$$

Fixed point arguments are not uniform in λ as $|\lambda|\to\infty$ and $|\lambda|\to 0$ because of the singularity of $L(u,v,\lambda)$:

$$L = \frac{i}{2}(|v|^2-|u|^2)\sigma_3 - \frac{i\lambda}{\sqrt{2}}\begin{pmatrix} 0 & \overline{v} \\ v & 0 \end{pmatrix} - \frac{i}{\sqrt{2}\lambda}\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} + \frac{i}{4}\left(\frac{1}{\lambda^2}-\lambda^2\right)\sigma_3$$

Renormalized direct scattering

By introducing the transformation

$$\left\{ \begin{array}{l} n_1^{(\pm)} := T(v,\lambda)\phi_1^{(\pm)} e^{-\frac{i}{4}(\lambda^2-\lambda^{-2})x-\frac{i}{4}(\lambda^2+\lambda^{-2})t}, \\ n_2^{(\pm)} := \lambda^{-1}T(v,\lambda)\phi_2^{(\pm)} e^{\frac{i}{4}(\lambda^2-\lambda^{-2})x+\frac{i}{4}(\lambda^2+\lambda^{-2})t}, \end{array} \right. \qquad T(v,\lambda) := \left(\begin{array}{cc} 1 & 0 \\ v & \lambda \end{array} \right)$$

we get renormalized Jost functions satisfying the asymptotic values

$$n_1^{(\pm)} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n_2^{(\pm)} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as} \quad x \to \pm\infty.$$

and the scattering relation

$$n^{(-)} = n^{(+)} \left(\begin{array}{cc} \overline{\alpha}(z) & \beta_-(z)e^{2i\theta(z)} \\ -\overline{\beta}_+(z)e^{-2i\theta(z)} & \alpha(z) \end{array} \right),$$

where $z := \lambda^2$, $\alpha(z) = a(\lambda)$, $\beta_+(z) = \lambda b(\lambda)$, $\beta_-(z) = \lambda^{-1}b(\lambda)$, and

$$\theta(z) := \frac{1}{4}(z - z^{-1})x + \frac{1}{4}(z + z^{-1})t.$$

Direct scattering result [P–Saalmann (2019)]

Lemma 5. *Let* $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $(u_x, v_x) \in L^1(\mathbb{R})$. *For every* $z \in \mathbb{R} \setminus \{0\}$, *there exists unique bounded Jost functions* $n_1^{(\pm)}$ *and* $n_2^{(\pm)}$. *For every* $x \in \mathbb{R}$, $n_1^{(\pm)}$ *and* $n_2^{(\pm)}$ *are continued analytically in* $\mathbb{C}^\pm$ *and satisfy the following limits as* $|z| \to \infty$ *and* $|z| \to 0$ *along a contour in the domains of their analyticity:*

$$\lim_{|z|\to\infty} \frac{n_1^{(\pm)}}{n_1^{\pm\infty}} = e_1, \quad \lim_{|z|\to\infty} \frac{n_2^{(\pm)}}{n_2^{\pm\infty}} = e_2,$$

and

$$\lim_{|z|\to 0} \left[n_1^{\pm\infty} n_1^{(\pm)} \right] = e_1 + v e_2, \quad \lim_{|z|\to 0} \left[n_2^{\pm\infty} n_2^{(\pm)} \right] = \bar{u} e_1 + (1 + \bar{u} v) e_2,$$

where

$$n_1^{\pm\infty} := e^{\frac{i}{4} \int_{\pm\infty}^{x} (|u|^2 + |v|^2) dy}, \quad n_2^{\pm\infty} := e^{-\frac{i}{4} \int_{\pm\infty}^{x} (|u|^2 + |v|^2) dy}.$$

Similarly, α *is continued analytically into* $\mathbb{C}^+$ *with the following limits in* $\mathbb{C}^+$*:*

$$\lim_{|z|\to\infty} \alpha(z) = e^{-\frac{i}{4} \int_{\mathbb{R}} (|u|^2 + |v|^2) dy}, \quad \lim_{|z|\to 0} \alpha(z) = e^{\frac{i}{4} \int_{\mathbb{R}} (|u|^2 + |v|^2) dy}.$$

Choice of function spaces

We have the requirement of $(u,v) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $(u_x, v_x) \in L^1(\mathbb{R})$ based on solutions of Volterra's integral equations after the transformation uniformly as $|z| \to \infty$:

$$\vec{\phi}_x = \left[\widehat{Q}_1(u,v) + \frac{1}{z}\widehat{Q}_2(u,v) + \frac{i}{4}\left(z - \frac{1}{z}\right)\sigma_3 \right] \vec{\phi},$$

where

$$\widehat{Q}_1(u,v) = \left(\begin{array}{cc} \frac{i}{4}(|u|^2 + |v|^2) & -\frac{i}{2}\overline{v} \\ v_x + \frac{i}{2}|u|^2 v + \frac{i}{2}u & -\frac{i}{4}(|u|^2 + |v|^2) \end{array} \right),$$

$$\widehat{Q}_2(u,v) = -\frac{i}{2}\left(\begin{array}{cc} \overline{u}v & -\overline{u} \\ v + \overline{u}v^2 & -\overline{u}v \end{array} \right).$$

To use Fourier theory, it is better to work in $H^{1,1}(\mathbb{R})$ with $\mathbf{u}, \partial_x \mathbf{u} \in L^{2,1}(\mathbb{R})$.

The time evolution also requires $\mathbf{u} \in H^2(\mathbb{R})$ for the reflection data to stay at the same spaces for $t \neq 0$. Thus, the stability result is formulated in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$.

Function spaces for reflection coefficients

Lemma 6 (P–Saalmann (2019)). *Let* $(u,v) \in H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$. *Then,* $r_+ \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R}) \cap L^{2,-2}(\mathbb{R})$ *and* $r_-(z) = r_+(z)/z$ *belongs to* $r_- \in H^{1,1}(\mathbb{R}) \cap L^{2,2}(\mathbb{R}) \cap L^{2,-1}(\mathbb{R})$*, where* $r_\pm(z) = \beta_\pm(z)/\alpha(z)$*.*

Fourier transform is an isomorphism between $H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$. However, we need further restriction in $L^{2,-2}(\mathbb{R})$ for r_+ because of the time evolution, which gives

$$r_\pm(z,t) = r_\pm(z,0)e^{-\frac{i}{2}t(z+z^{-1})}$$

with

$$\partial_z r_\pm(z,t) = \left[\partial_z r_\pm(z,0) - \frac{i}{2}t(1-z^{-2})r_\pm(z,0)\right]e^{-\frac{i}{2}t(z+z^{-1})}.$$

With the constraint $r_+(\cdot,0) \in L^{2,-2}(\mathbb{R})$, $r_\pm(\cdot,t)$ belongs to the same function space for every $t \neq 0$.

The asymptotic stability of exponential solitons is obtained by applications of the steepest descent method and reformulations of the Riemann–Hilbert problem in different regions of the (x,t) plane. [Cheng, Liu, Qu (2024)].

What goes wrong for stability of algebraic solitons in $H^2(\mathbb{R}) \cap H^{1,1}(\mathbb{R})$

Recall the algebraic soliton:

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{1+2\mathrm{i}x} \\ \dfrac{2}{1-2\mathrm{i}x} \end{bmatrix} e^{-\mathrm{i}t}.$$

- $(u,v) \in H^2(\mathbb{R})$ but $(u,v) \notin L^{2,1}(\mathbb{R})$ due to slow spatial decay at infinity.
- The embedded eigenvalue $\lambda = \mathrm{i}$ corresponds to $z = -1$ so that the scattering data $r_\pm(z)$ have the simple pole singularity on $\mathbb{R}$.
- Inverse scattering is not available.

Section 5. Conclusion

I have explained three methods in the proof of nonlinear (orbital and asymptotic) stability of exponential solitons in the MTM system:

- Lyapunov functional with the higher-order energy.
- Bäcklund transformation and the stability of the zero solution.
- Inverse scattering and steepest descent method.

Rational solutions of the MTM system suggest the nonlinear stability of the algebraic soliton but the proof of stability remains an open problem.

- Coercivity of the Lyapunov functional is lost at the algebraic soliton.
- Bäcklund transformation becomes trivial for embedded eigenvalues.
- Inverse scattering is not allowed due to slow spatial decay of algebraic solitons.

Many thanks for your attention! Questions?