Bäcklund transformation and L^2-stability of NLS solitons

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Introduction

Consider a 1D NLS equation,

$$iu_t = -u_{xx} + V(x)u - |u|^{2p}u, \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}.$$

where $V : \mathbb{R} \to \mathbb{R}$ is a trapping potential and $p > 0$ is the nonlinearity power.

Assume existence of solitons $u(x,t) = \phi_\omega(x)e^{-i\omega t - i\theta}$ with some $\omega \in \mathbb{R}$ and arbitrary $\theta \in \mathbb{R}$.

Main questions:

- Orbital stability in $H^1(\mathbb{R})$: for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$, such that if $\|u(0) - \phi_\omega\|_{H^1} \leq \delta(\epsilon)$ then

$$\inf_{\theta \in \mathbb{R}} \|u(t) - e^{-i\theta}\phi_\omega\|_{H^1} \leq \epsilon, \quad \text{for all } t > 0.$$

- Asymptotic stability in $L^\infty(\mathbb{R})$ (scattering to solitons): there is ω_∞ near ω such that

$$\lim_{t \to \infty} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{-i\theta}\phi_{\omega_\infty}\|_{L^\infty} = 0.$$

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Previous literature

Orbital stability is well understood since the 1980s [Cazenave and Lions, 1982; Shatah and Strauss, 1985; Weinstein, 1986; Grillakis, Shatah and Strauss, 1987, 1990]. Regarding asymptotic stability,

- Buslaev and Sulem (2003) proved asymptotic stability of solitary waves in 1D NLS for the case $p \geq 4$ using dispersive decay estimates from Buslaev and Perelman (1993).
- Cuccagna (2008) and Mizumachi (2008) improved the results with Stritcharz analysis for the case $p \geq 2$.
- No results are available for $p = 1$ even if $V(x) \equiv 0$ (integrable case).

The difficulty comes from the slow decay of solutions in the L^∞ norm which makes it difficult to control convergence of modulation parameters.

Scattering near zero

More results are available on asymptotic stability of zero solution for

$$iu_t + u_{xx} + |u|^{2p} u = 0.$$

- For $p > 1$, scattering near zero follows from the dispersive decay

$$\|e^{it\partial_x^2}\|_{L^1 \to L^\infty} \leq \frac{C}{t^{1/2}}, \quad t > 0.$$

  because $\|u(t,\cdot)\|_{L^\infty}^{2p}$ is absolutely integrable for $p > 1$ (Ginibre & Velo, 1985; Ozawa, 1991; Cazenave & Weissler, 1992).

- Hayashi & Naumkin proved scattering for $p = 1$ (1998) and $p = 1/2$ (2008). In particular, for $p = 1$, they showed that if $u_0 \in H^1(\mathbb{R})$ and $xu \in L^2(\mathbb{R})$, then

$$\|u(t,\cdot)\|_{H^1} \leq C\epsilon, \quad \|u(t,\cdot)\|_{L^\infty} \leq C\epsilon(1+|t|)^{-1/2}, \quad t \in \mathbb{R}.$$

Cubic NLS equation

We shall consider the cubic NLS equation,

$$iu_t + u_{xx} + 2|u|^2 u = 0 \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{NLS}$$

- (NLS) is an integrable Hamiltonian system and has infinitely many conservation laws (Zakharov and Shabat, 1972):

$$N := \|u(t,\cdot)\|_{L^2}, \qquad E := \frac{1}{2}\int_{\mathbb{R}} (|u_x(t,x)|^2 - |u(t,x)|^4)dx$$

- (NLS) is locally well-posed in L^2 (Tsutsumi, 1987). Thanks to L^2 conservation, it is globally well-posed in L^2.
- (NLS) is also well-posed in H^k for any $k \in \mathbb{N}$ (Ginibre & Velo, 1984; Kato, 1987).

Soliton solutions

- (NLS) has a **4**-parameter family of **1**-solitons

$$Q_{k,v}(t+t_0, x+x_0) = Q_k(x-vt)\, e^{ivx/2+i(k^2-v^2/4)t} \,,$$

  where

$$Q_k(x) = k \operatorname{sech}(kx), \quad k > 0\,,\ v \in \mathbb{R},\ x_0 \in \mathbb{R},\ t_0 \in \mathbb{R}\,.$$

- Q_k is a minimizer of $E|_{\mathcal{M}}$, where

$$\mathcal{M} = \{u \in H^1(\mathbb{R})\,,\ \|u\|_{L^2} = \|Q_k\|_{L^2}\},$$

  hence, it is orbitally stable (Cazenave and Lions, 1982).

- Perturbations near solitons in H^s for $0 < s < 1$ may grow at most polynomially in time (Colliander-Keel-Staffilani-Takaoka-Tao, 2003).

Soliton solutions

Main Questions:

- Is **1**-soliton orbitally stable in L^2?
- Is **1**-soliton asymptotically stable in H^1 or L^2?

We aim to show the Lyapunov stability of **1**-solitons in L^2.
We use the Bäcklund transformation to define an isomorphism which maps solutions in an L^2-neighborhood of the zero solution to those in an L^2-neighborhood of a **1**-soliton.

A Bäcklund transformation is a mapping between two solutions of the same (or different) equations. It was originally found for the sine-Gordon equation by Bianchi (1879) and Bäcklund (1882) but was extended to KdV, KP, Benjamin-Ono, Toda, and other integrable equations in 1970s.

Bäcklund transformation of (NLS)

For (NLS), consider the Lax operator system,

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \eta & q \\ -\bar{q} & -\eta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \text{(Lax1)}$$

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 2\eta^2 + |q|^2 & \partial_x q + 2\eta q \\ \partial_x \bar{q} - 2\eta\bar{q} & -2\eta^2 - |q|^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \text{(Lax2)}$$

where η is the spectral parameter.

(Lax1) and (Lax2) are compatible if $iq_t + q_{xx} + 2|q|^2 q = 0$.
Let $q(t,x)$ be a solution of (NLS) and (ψ_1, ψ_2) be a solution of (Lax1)–(Lax2) for $\eta \in \mathbb{R}$. Define

$$Q := -q - \frac{4\eta\psi_1\bar{\psi}_2}{|\psi_1|^2 + |\psi_2|^2}.$$

Then $Q(t,x)$ is a solution of (NLS). (Chen'74, Konno and Wadati '75)

Bäcklund transformation $0 \longrightarrow 1$ soliton

- Let $\eta = \frac{1}{2}$ and $q \equiv 0$. Then,

$$\psi_1 = e^{(x+it)/2}, \quad \psi_2 = -e^{-(x+it)/2} \quad \Rightarrow \quad Q = e^{it}\,\mathrm{sech}(x).$$

- Let $\Psi_1 = \dfrac{\bar{\psi}_2}{|\psi_1|^2 + |\psi_2|^2}$ and $\Psi_2 = \dfrac{\psi_1}{|\psi_1|^2 + |\psi_2|^2}$. Then (Ψ_1, Ψ_2) satisfy the Lax operator system:

$$\partial_x \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \eta & Q \\ -\bar{Q} & -\eta \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{Lax'1}$$

$$\partial_t \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 2\eta^2 + |Q|^2 & \partial_x Q + 2\eta Q \\ \partial_x \bar{Q} - 2\eta \bar{Q} & -2\eta^2 - |Q|^2 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \tag{Lax'2}$$

- If $Q = e^{it}\,\mathrm{sech}(x)$, then $\eta = \frac{1}{2}$ is an eigenvalue of (Lax'1) with

$$\Psi_1 = -e^{(-x+it)/2}\,\mathrm{sech}(x), \quad \Psi_2 = e^{(x+it)/2}\,\mathrm{sech}(x).$$

Applications of Bäcklund transformation

We show Lyapunov stability of **1**-solitons in the L^2 class.

$$\begin{array}{ccc} Q(0,x) & \xrightarrow{NLS} & Q(t,x) \\ {\scriptstyle BT}\downarrow & & \uparrow{\scriptstyle BT} \\ q_0(x) & \xrightarrow{NLS} & q(t,x) \end{array}$$

$\|Q(0,\cdot) - Q_1\|_{L^2}$ is small,

$\|q(t)\|_{L^2} = \|q(0)\|_{L^2}$ is small.

- Merle and Vega (2003) used the Miura transformation to prove asymptotic stability of KdV solitons in L^2.
- Mizumachi and Tzvetkov (2011) applied the same transformation to prove L^2-stability of line solitons in the KP-II equation under periodic transverse perturbations.
- Mizumachi and Pego (2008) used Backlund transformation to prove asymptotic stability of Toda lattice solitons.
- Hoffman and Wayne (2009) extended this result to two and N Toda lattice solitons.

Main result

Theorem

(Mizumachi, P., 2012) Fix $k_0 > 0$*. Let* $u(t,x)$ *be a solution of* (NLS) *in the class*

$$u \in C(\mathbb{R}; L^2(\mathbb{R})) \cap L^8_{loc}(\mathbb{R}; L^4(\mathbb{R})).$$

There exist C, $\varepsilon > 0$ *such that if* $\|u(0,\cdot) - Q_{k_0}\|_{L^2} < \varepsilon$*, then there exist* k, v, t_0, x_0 *such that*

$$\sup_{t\in\mathbb{R}} \|u(t+t_0, \cdot+x_0) - Q_{k,v}\|_{L^2} + |k-k_0| + |v| + |t_0| + |x_0| \leq C\|u(0,\cdot) - Q_{k_0}\|_{L^2}.$$

Remark: In KdV, perturbations of **1**-solitons can cause logarithmic growth of the phase shift due to collisions with small solitary waves (Martel and Merle, 2005). For the cubic NLS, a solution remains in the neighborhood of a **1**-soliton for all the time.

Outline of the proof

For the sake of simplicity, we consider $k_0 = 1$ ($\eta = \frac{1}{2}$).

$$\begin{array}{ccc} Q(0,x) & \xrightarrow{NLS} & Q(t,x) \\ BT \downarrow & & \uparrow BT \\ q_0(x) & \xrightarrow{NLS} & q(t,x) \end{array}$$

$\|Q(0,\cdot) - Q_1\|_{L^2}$ is small,

$\|q(t)\|_{L^2} = \|q(0)\|_{L^2}$ is small.

- Step 1: From a nearly 1-soliton to a nearly zero solution at $t = 0$.
- Step 2: Time evolution of the nearly zero solution for $t \in \mathbb{R}$.
- Step 3: From the nearly zero solution to the nearly 1-soliton for $t \in \mathbb{R}$.
- Step 4: Approximation arguments in $H^3(\mathbb{R})$ to control modulations of parameters of 1-solitons for all $t \in \mathbb{R}$.

Step 1: From 1-soliton to 0-soliton at $t = 0$.

At $t = 0$, Q is close to $Q_1 = \mathrm{sech}(x)$ and η is close to $\frac{1}{2}$.

If $Q = Q_1$ and $\eta = \frac{1}{2}$, then the Lax operator

$$\partial_x \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \eta & Q \\ -\bar{Q} & -\eta \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},$$

has two linearly independent solutions

$$\begin{bmatrix} -e^{-x/2} \\ e^{x/2} \end{bmatrix} \mathrm{sech}(x), \quad \begin{bmatrix} (e^x + 2(1+x)e^{-x})e^{x/2} \\ (e^{-x} - 2xe^x)e^{-x/2} \end{bmatrix} \mathrm{sech}(x).$$

Define

$$q := -Q_1 - \frac{4\eta \Psi_1 \overline{\Psi_2}}{|\Psi_1|^2 + |\Psi_2|^2}.$$

Then $q = 0$ follows from the first (decaying) solution and

$$q(x) = \frac{2xe^{2x} + (4x^2 + 4x - 1) - 2x(1+x)e^{-2x}}{\cosh(3x) + 4(1 + x + x^2)\cosh(x)} - \mathrm{sech}(x)$$

follows from the second (growing) solution.

Step 1

By perturbation theory (Lyapunov–Schmidt reduction method), we prove:

- If $\|Q - Q_1\|_{L^2}$ is small, then there exists $\eta = (k + iv)/2$ and $\Psi \in H^1(\mathbb{R})$ such that

$$|k-1| + |v| + \|\Psi - \Psi_1\|_{H^1} \le C\|Q - Q_1\|_{L^2}.$$

- If

$$q := -Q - \frac{2k\Psi_1\bar{\Psi}_2}{|\Psi_1|^2 + |\Psi_2|^2},$$

then $q \in L^2(\mathbb{R})$ and

$$\begin{aligned}\|q_0\|_{L^2} \le & \|Q - Q_1\|_{L^2} + \left\| Q_1 + \frac{2k\Psi_1\overline{\Psi_2}}{|\Psi_1|^2 + |\Psi_2|^2} \right\|_{L^2} \\ \lesssim & \|Q - Q_1\|_{L^2}.\end{aligned}$$

- Moreover, if $Q \in H^3(\mathbb{R})$, then $q \in H^3(\mathbb{R})$.

Step 2: Time evolution near $\mathbf{0}$-soliton for $t \in \mathbb{R}$.

If $q(0,\cdot) \in H^3(\mathbb{R})$ and $\|q(0,\cdot)\|_{L^2}$ is small, then $q \in C(\mathbb{R}, H^3(\mathbb{R}))$ and

$$\|q(t,\cdot)\|_{L^2} = \|q(0,\cdot)\|_{L^2}$$

remains small for all $t \in \mathbb{R}$.

This result completes step 2 for the NLS equation.

Step 3: From $\mathbf{0}$-soliton to $\mathbf{1}$-soliton for $t \in \mathbb{R}$.

If $q \equiv 0$, then $\{(e^{x/2}, 0)\,,\ (0, e^{-x/2})\}$ is a fundamental system of (Lax1).

If $q = q(0,x)$ is small in L^2, there exist bounded solutions of (Lax1):

$$e^{x/2}\vec{\varphi}(x) = e^{x/2}(\varphi_1(x), \varphi_2(x)), \quad e^{-x/2}\vec{\chi}(x) = e^{-x/2}(\chi_1(x), \chi_2(x)),$$

where

$$\begin{cases} \varphi_1' = q\varphi_2\,, \\ \varphi_2' = -\bar{q}\varphi_1 - \varphi_2\,, \end{cases}, \quad \begin{cases} \chi_1' = \chi_1 + q\chi_2\,, \\ \chi_2' = -\bar{q}\chi_1\,, \end{cases}$$

$$\lim_{x\to\infty} \varphi_1(x) = 1\,, \quad \lim_{x\to-\infty} \chi_2(x) = -1\,.$$

A bounded solution (φ_1, φ_2) satisfies

$$\begin{cases} \varphi_1(x) = 1 - \int_x^\infty q(y)\varphi_2(y)dy =: T_1(\varphi_1, \varphi_2)(x), \\ \varphi_2(x) = -\int_{-\infty}^x e^{-(x-y)}\overline{q(y)}\varphi_1(y)dy =: T_2(\varphi_1, \varphi_2)(x). \end{cases}$$

Step 3

- Note that we are not using here any smallness of $\|q\|_{L^1}$, a typical assumption in inverse scattering to guarantee no solitons in $q(t,x)$.
- If $\|q\|_{L^2}$ is small, then $T = (T_1, T_2)$ is a contraction mapping on $L^\infty \times (L^\infty \cap L^2)$ and

$$\begin{aligned} \|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^\infty \cap L^2} &\leq C\|q\|_{L^2}, \\ \|\chi_1\|_{L^\infty \cap L^2} + \|\chi_2 + 1\|_{L^\infty} &\leq C\|q\|_{L^2}. \end{aligned}$$

- If $q(t,x)$ is an $H^3(\mathbb{R})$-solution of (NLS), then

$$\sup_t (\|\varphi_1(t,\cdot) - e^{it/2}\|_{L^\infty} + \|\varphi_2(t,\cdot)\|_{L^2 \cap L^\infty}) \leq C\|q(0,\cdot)\|_{L^2},$$

$$\sup_t (\|\chi_1(t,\cdot)\|_{L^2 \cap L^\infty} + \|\chi_2(t,\cdot) + e^{-it/2}\|_{L^\infty}) \leq C\|q(0,\cdot)\|_{L^2}.$$

Step 3

Let $q \in C(\mathbb{R}, H^3(\mathbb{R}))$ is a solution of (NLS) such that $\|q(0,\cdot)\|_{L^2}$ is small.

- Let

$$\psi_1(t,x) = c_1 e^{x/2}\varphi_1(t,x) + c_2 e^{-x/2}\chi_1(t,x)\,,$$
$$\psi_2(t,x) = c_1 e^{x/2}\varphi_2(t,x) + c_2 e^{-x/2}\chi_2(t,x)\,.$$

  with $c_1 = ae^{(\gamma+i\theta)/2}$, $c_2 = ae^{-(\gamma+i\theta)/2}$, and $a \neq 0$.

- Let

$$Q(t,x) := -q(t,x) - \frac{2\psi_1(t,x)\overline{\psi_2(t,x)}}{|\psi_1(t,x)|^2 + |\psi_2(t,x)|^2}\,.$$

Then, $Q \in C(\mathbb{R}, H^3(\mathbb{R}))$ is a solution of (NLS) and

$$\|Q(t,\cdot) - e^{i(t+\theta)}Q_1(\cdot+\gamma)\|_{L^2} \leq C\|q(0,\cdot)\|_{L^2} \quad \text{for } \forall\, t\,.$$

Step 4: Proof of L^2-stability

Let $u_{n,0} \in H^3(\mathbb{R})$ be a sequence such that

$$\lim_{n\to\infty} \|u_{n,0} - u(0,\cdot)\|_{L^2} = 0.$$

Let $u_n(t,x)$ be a solution of (NLS) with $u_n(0,x) = u_{n,0}(x)$.
By the previous construction, there is an n-independent $C > 0$ such that

$$\sup_{t\in\mathbb{R}} \|u_n(t+t_n,\cdot+x_n) - Q_{k_n,v_n}\|_{L^2} + |k_n - 1| + |v_n| + |t_n| + |x_n| \leq C\|u_{n,0} - Q_1\|_{L^2}.$$

Therefore, there exists k, v, t_0, and x_0 such that

$$k_n \to k, \quad v_n \to v, \quad x_n \to x_0, \quad t_n \to t_0 \quad \text{as} \quad n \to \infty.$$

From L^2-well-posedness, it then follows that

$$\sup_{t\in\mathbb{R}} \|u(t+t_0,\cdot+x_0) - Q_{k,v}\|_{L^2} + |k - 1| + |v| + |t_0| + |x_0| \leq C\|u(0,\cdot) - Q_1\|_{L^2}.$$

Discussion: asymptotic stability

Hayashi and Naumkin (1998) proved that if $q_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that

$$\|q_0\|_{H^1} + \|q_0\|_{L^1} \leq \epsilon \quad \text{(small)},$$

there exists a unique global solution in $H^1(\mathbb{R})$ such that

$$\|q(\cdot,t)\|_{H^1} \leq C\epsilon, \quad \|q(\cdot,t)\|_{L^\infty} \leq C\epsilon(1+|t|)^{-1/2}, \quad t \in \mathbb{R}.$$

Note that $\|q(\cdot,t)\|_{L^1}$ may grow as $|t| \to \infty$.

However, we are not able to prove that if $\|Q - Q_1\|_{H^1} \leq \|q\|_{H^1}$ is small, then

$$\exists C > 0 : \quad \|Q - Q_1\|_{L^\infty} \leq C\|q\|_{L^\infty},$$

without assuming that $\|q\|_{L^1}$ is small.

Asymptotic stability of 1-solitons in (NLS) is still an open problem.

Discussion: Hasimoto transformation

The integrable Landau–Lifshitz (LL) model is

$$\mathbf{u}_t = \mathbf{u} \times \mathbf{u}_{xx}, \tag{LL}$$

where $\mathbf{u}(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2$ such that $\mathbf{u} \cdot \mathbf{u} = 1$. NLS and LL equations are connected by the Hasimoto (Miura-type) transformation.

- L^2-orbital stability of 1-solitons of (NLS) is related to H^1-orbital stability of the domain wall solutions of (MTM).
- H^1-asymptotic stability of 1-solitons of (NLS) is related to H^2-asymptotic stability of domain wall solutions of (MTM)

Discussion: nonlinear Dirac equation

The nonlinear Dirac equations (the massive Thirring model) is

$$\begin{cases} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{cases} \qquad \text{(MTM)}$$

where $(u, v) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$.

Orbital stability of **0**-solution or **1**-solitons is a difficult problem because the energy functional is sign-indefinite. Asymptotic stability approaches (if they work) give the orbital stability.

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