Stability of solitary waves in the NLS equation with a regularized and intensity-dependent dispersion

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I. NLS models with a regularized and intensity-dependent dispersion

Generalized NLS models

The classical NLS equation realizes a balance between nonlinearity and dispersion for propagation of nonlinear dispersive waves.

$$i\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi = 0. \tag{NLS}$$

Taking into account higher-order nonlinearity and dispersion gives an extended version of the NLS equation:

$$\begin{aligned} & i\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi + i\alpha_1\psi_{xxx} + \alpha_2\psi_{xxxx} \\ & \quad + i\beta_1|\psi|^2\psi_x + i\beta_2(|\psi|^2\psi)_x + \gamma|\psi|^4\psi = 0. \end{aligned} \tag{gNLS}$$

Well-posedness of initial-value problem, stability of nonlinear waves, global dynamics (scattering versus blowup in a finite time), ...

NLS equation with regularized dispersion

The NLS with regularized dispersion was derived from Maxwell equations:

$$i(1-\mu^2\partial_x^2)\psi_t + \alpha\psi_{xx} + \beta|\psi|^2\psi = 0. \qquad \text{(rNLS)}$$

M. Colin, D. Lannes SIMA **41** (2009) 708–732

D. Lannes, Proc. R. Soc. Edinburgh Ser A **141** (2011) 253–286

The dispersion relation is bounded as

$$\omega(k) = \frac{\alpha k^2}{1+\mu^2 k^2}, \qquad k \in \mathbb{R},$$

similar to the BBM regularization for the KdV equation.

Well-posedness and stability of solitary waves was recently addressed in
P. Antonelli, J. Arbunich, and C. Sparber, SIMA **51** (2019), 110–130
J. Arbunich, C. Klein, and C. Sparber, ESAIM Math. Model. **53** (2019), 1477-1505
D.P. and M. Plum, Proc. AMS **152** (2024) 1217–1231

J. Albert and J. Arbunich, Stud. Appl. Math. (2024), early view.

NLS models with intensity-dependent dispersion

The dispersion coefficient may depend on the wave intensity:

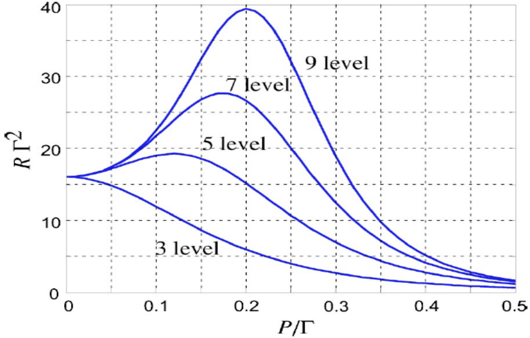


FIG. 3. Graphs showing dispersion (times linewidth squared), $R\Gamma^2$ as a function of P/Γ with $C/\Gamma = 0.25$ for Chain Λ systems of 3, 5, 7, and 9 states.

A.D. Greentree, D. Richards, J.A. Vaccaro, et al., Phys. Rev. A **67** (2003), 023818

Two NLS equations with intensity-dependent dispersion

The NLS equation where the dispersion vanishes at a selected intensity:

$$i\psi_t + \alpha(1 - |\psi|^2)\psi_{xx} + \beta|\psi|^2\psi = 0. \qquad \text{(NLS-IDD-1)}$$

C.Y. Lin, J.H. Chang, G. Kurizki, and R.K. Lee, Optics Letters **45** (2020) 1471–1474
R.M. Ross, P.G. Kevrekidis, and D.P., Quart. Appl. Math. **79** (2021) 641-665
D.P, R.M. Ross, and P.G. Kevrekidis, J. Phys. A: Math. Theor. **54** (2021) 445701
P.G. Kevrekidis, D.P, and R.M. Ross, arXiv:2408.11192 (2024)

The NLS equation where the dispersion diverges at a selected intensity:

$$i\psi_t + \alpha(1 - |\psi|^2)^{-1}\psi_{xx} + \beta|\psi|^2\psi = 0. \qquad \text{(NLS-IDD-2)}$$

D.P. and M. Plum, SIMA **56** (2024) 2521-2568

Quasilinear NLS equation due to nonlocal Kerr effects

The two NLS-IDD models are variants of the nonlocal Kerr model:

$$i\psi_t + \alpha\psi_{xx} + \beta\psi(\mathcal{K}_\epsilon * |\psi|^2) = 0,$$

with

$$\hat{K}_\epsilon(k) = 1 - \epsilon^2 k^2, \quad k \in \mathbb{R},$$

which yields the quasilinear NLS model

$$i\psi_t + \alpha\psi_{xx} + \beta\psi|\psi|^2 + \gamma\psi(|\psi|^2)_{xx} = 0. \qquad \text{(NLS-QL)}$$

I. Iliev and K. Kirchev, Differential Integral Equations **6** (1993) 685–703
W. Krolikowski and O. Bang, Phys. Rev. E **63** (2000) 016610
M. Colin, L. Jeanjean, and M. Squassina, Nonlinearity **23** (2010) 1353–1385
A. de Laire and E. Le Quiniou, arXiv:2311.08918 (2023)

Solitary waves in NLS models

Bright soliton $\psi(t,x) = e^{it}\mathrm{sech}(x)$ of the focusing NLS equation

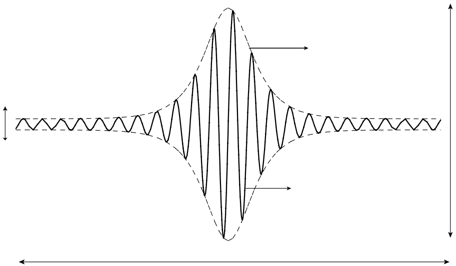
$$i\partial_t\psi + \partial_x^2\psi + 2|\psi|^2\psi = 0$$

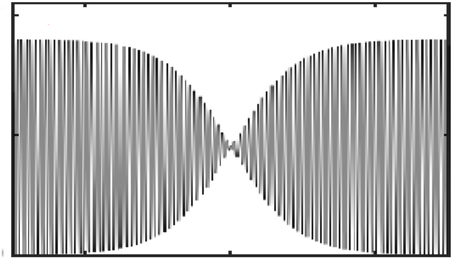
satisfying $|\psi(t,x)| \to 0$ as $|x| \to \infty$

Dark soliton $\psi(t,x) = e^{-2it}\tanh(x)$ of the defocusing NLS equation

$$i\partial_t\psi + \partial_x^2\psi - 2|\psi|^2\psi = 0$$

satisfying $|\psi(t,x)| \to 1$ as $|x| \to \infty$





II. Stability of the black soliton in the NLS equation with the regularized dispersion

Reformulation of the regularized NLS equation

We consider the regularized NLS equation

$$i(1-\mu^2\partial_x^2)\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0.$$

where $\mu \neq 0$ is the regularizing parameter.

Standing wave solutions are of the normalized form

$$\psi(t,x) = e^{-2it}u(t,\xi), \quad \xi = \frac{x}{\sqrt{1-2\mu^2}},$$

where u satisfies

$$i(1-\epsilon^2\partial_\xi^2)u_t + u_{\xi\xi} + 2(1-|u|^2)u = 0, \quad \epsilon := \frac{\mu}{\sqrt{1-2\mu^2}}.$$

The mapping $\mu \to \epsilon \in \mathbb{R}$ is monotonically increasing for $\mu \in (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Black soliton as a solution of the regularized NLS equation

Time-independent solutions of the regularized NLS equation

$$i(1-\epsilon^2\partial_\xi^2)u_t + u_{\xi\xi} + 2(1-|u|^2)u = 0$$

include the black soliton $u(t,\xi) = \varphi(\xi) := \tanh(\xi)$.

Cauchy problem near the black soliton is well-posed in Sobolev spaces.

C. Gallo, Comm. PDEs **33** (2008) 729–771

P. Gérard, Ann. Inst. H. Poincaré **23** (2006) 765–779

Theorem

For every $v_0 \in H^s(\mathbb{R})$ with $s > \frac{1}{2}$, there exists the maximal existence time $\tau_0 \in (0,\infty]$ and a unique solution in the form $u = \varphi + v$, where $\varphi(\xi) = \tanh(\xi)$ and $v \in C^1([0,\tau_0), H^s(\mathbb{R}))$ such that $v(0,\cdot) = v_0$. Moreover, for any $\tau \in (0,\tau_0)$, the solution $v \in C^1([0,\tau], H^s(\mathbb{R}))$ depends Lipschitz continuously on the initial data $v_0 \in H^s(\mathbb{R})$.

Analysis of well-posedness

With the substitution $u = \varphi + v$, where $\varphi(\xi) = \tanh(\xi)$, the evolution problem is

$$v_t = i(1 - \epsilon^2 \partial_\xi^2)^{-1} F(v),$$

where

$$F(v) := v_{\xi\xi} + 2(1 - 2\varphi^2)v - 2\varphi^2 \bar{v} - 2\varphi(v^2 + 2|v|^2) - 2|v|^2 v$$

- Since H^s, $s > \frac{1}{2}$ is a Banach algebra, $F(v) : H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$ maps any fixed ball $B(v_0) \subset H^s$ into a bounded set in H^{s-2}.
- $(1 - \epsilon^2 \partial_\xi^2)^{-1}$, $\epsilon \neq 0$ is a bounded operator from H^{s-2} back to H^s.
- The rest goes from the contraction mapping principle for

$$v(t, \cdot) = v_0 + i \int_0^t (1 - \epsilon^2 \partial_\xi^2)^{-1} F(v(t', \cdot)) dt'.$$

Stability of the black soliton via conserved quantities

The regularized NLS equation

$$i(1-\epsilon^2\partial_\xi^2)u_t + u_{\xi\xi} + 2(1-|u|^2)u = 0$$

admits the following conserved quantities:

- energy for $u = \varphi + v$ with $v \in H^s(\mathbb{R})$, $s \geq 1$

$$E(u) = \int_{\mathbb{R}} \left[|u_\xi|^2 + (1-|u|^2)^2 \right] d\xi$$

- momentum for $u = \varphi + v$ with $v \in H^s(\mathbb{R})$, $s > \frac{3}{2}$

$$P(u) = i \int_{\mathbb{R}} \left[(\bar{u}u_\xi - \bar{u}_\xi u) + \epsilon^2(\bar{u}_\xi u_{\xi\xi} - \bar{u}_{\xi\xi} u_\xi) \right] d\xi.$$

- mass for $u = \varphi + v$ with $v \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, $s \geq 1$

$$M(u) = \int_{\mathbb{R}} \left[\epsilon^2 |u_\xi|^2 + |u|^2 - 1 \right] d\xi,$$

Mismatch of conserved quantities

For the black soliton of the cubic NLS equation ($\epsilon = 0$),

- φ is a constrained minimizer of energy E for fixed momentum P.
- Mass M plays no role in the stability analysis.
- The energy expanded near φ provides control of the perturbation in the weighted $H^1(\mathbb{R})$ space with the exponential weight.

P. Gravejat and D. Smets. Proc. London Math. Soc. **111** (2015), 305–353.
T. Gallay and D.E. Pelinovsky, J. Diff. Eqs. **258** (2015), 3639–3660
M. A. Alejo and A. J. Corcho, arXiv: 2003.09994 (2020)

Since P is only defined in $H^s(\mathbb{R})$, $s > \frac{3}{2}$, the orbital stability of black solitons is open, with the exception of spatially odd perturbations (below).

Spectral stability of the black soliton

Using perturbation $u = \varphi + v$ with $v := U + iV$ and linearizing, we get the linear evolution equation:

$$(1 - \epsilon^2 \partial_\xi^2) U_t = L_- V, \qquad (1 - \epsilon^2 \partial_\xi^2) V_t = -L_+ U,$$

where $L_\pm : \mathrm{Dom}(L_\pm) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are the same as for the cubic NLS equation:

$$L_+ = -\partial_\xi^2 + 6\varphi^2 - 2 = -\partial_\xi^2 + 4 - 6\mathrm{sech}^2(\xi) \geq 0,$$
$$L_- = -\partial_\xi^2 + 2\varphi^2 - 2 = -\partial_\xi^2 - 2\mathrm{sech}^2(\xi).$$

This yields the spectral stability problem

$$\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \lambda (1 - \epsilon^2 \partial_\xi^2) \begin{bmatrix} U \\ V \end{bmatrix}$$

for $(U, V) \in H^1_\epsilon(\mathbb{R}) \times H^1_\epsilon(\mathbb{R})$ equipped with

$$(f, g)_\epsilon := \int_{\mathbb{R}} \left[\bar{f} g + \epsilon^2 \bar{f}' g' \right] d\xi, \qquad \|f\|_\epsilon = \sqrt{(f, f)_\epsilon} \simeq \|f\|_{H^1}.$$

Threshold on the spectral stability/instability

Theorem

Let $\epsilon_0 = (5/8)^{1/4}$. *The black soliton is spectrally stable for* $\epsilon \in (0, \epsilon_0]$ *with every* $\lambda \in i\mathbb{R}$ *and spectrally unstable for* $\epsilon \in (\epsilon_0, \infty)$ *with exactly one* $\lambda_0 \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda_0) > 0$.

Remarks:

- The essential spectrum has no spectral gap, e.g. $\sigma_{\rm ess} := i[-\epsilon^{-2}, \epsilon^{-2}]$.
- Isolated eigenvalues may exist on $i\mathbb{R}\backslash\sigma_{\rm ess}$ and embedded eigenvalues may exist inside $\sigma_{\rm ess}$, which we do not control.
- The stability threshold $\epsilon_0 \approx 0.89$ corresponds to $\mu_0 \approx 0.55 \in \left(0, \frac{1}{\sqrt{2}}\right)$ of the original model $i(1 - \mu^2\partial_x^2)\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0$.

Analysis of spectral stability 1

We have

$$L_+ = -\partial_\xi^2 + 4 - 6\text{sech}^2(\xi) \geq 0$$

with $L_+\varphi' = 0$ and a spectral gap. There is $C > 0$ such that

$$\langle L_+ U, U\rangle \geq C\|U\|_\epsilon^2, \quad \text{for every } U \in H^1(\mathbb{R}) : \ (U, \varphi')_\epsilon = 0.$$

Define a bounded operator for $\epsilon \neq 0$:

$$\mathcal{L}_+ = (1 - \epsilon^2\partial_\xi^2)^{-1/2} L_+ (1 - \epsilon^2\partial_\xi^2)^{-1/2} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Then we have

$$(\mathcal{L}_+ W, W) \geq C\|W\|^2, \quad \text{for every } W \in L^2(\mathbb{R}) : \ (W, W_0) = 0,$$

due to correspondence with $U = (1 - \epsilon^2\partial_\xi^2)^{-1/2} W$, $\|U\|_\epsilon^2 = \|W\|^2$, and $(U, \varphi')_\epsilon = (W, W_0)$, where $W_0 := (1 - \epsilon\partial_\xi^2)^{1/2}\varphi'$.

Analysis of spectral stability 2

Thus we have an invertible and strictly positive operator

$$\mathcal{T}_+ := \mathcal{L}_+|_{\{W_0\}^\perp} \; : \; L^2(\mathbb{R})|_{\{W_0\}^\perp} \mapsto L^2(\mathbb{R})|_{\{W_0\}^\perp}.$$

For any eigenvalue $\lambda_0 \neq 0$, $\boxed{-L_+ U = \lambda_0(1 - \epsilon^2\partial_\xi^2)V}$ is rewritten as

$$\mathcal{L}_+(1 - \epsilon^2\partial_\xi^2)^{1/2}U = -\lambda_0(1 - \epsilon^2\partial_\xi^2)^{1/2}V.$$

We have $(V, \varphi')_\epsilon = 0$ due to momentum conservation and we can ensure that $(U, \varphi')_\epsilon = 0$ by adding φ' to $U \in H^1(\mathbb{R})$. This yields uniquely

$$(1 - \epsilon^2\partial_\xi^2)^{1/2}U = -\lambda_0\mathcal{T}_+^{-1}(1 - \epsilon^2\partial_\xi^2)^{1/2}V.$$

Substituting it to $\boxed{L_- V = \lambda_0(1 - \epsilon^2\partial_\xi^2)U}$ yields

$$L_- V = -\lambda_0^2(1 - \epsilon^2\partial_\xi^2)^{1/2}\mathcal{T}_+^{-1}(1 - \epsilon^2\partial_\xi^2)^{1/2}V,$$

where L_- admits exactly one simple negative eigenvalue since $L_-\varphi = 0$.

Analysis of spectral stability 3

Since $(1-\epsilon^2\partial_\xi^2)^{1/2}\mathcal{T}_+^{-1}(1-\epsilon^2\partial_\xi^2)^{1/2}$ is strictly positive, we have $\lambda_0^2 \in \mathbb{R}$. $\lambda_0 \in \mathbb{R}\backslash\{0\}$ exists if and only if

$$\inf_{\substack{V \in H^1_\epsilon(\mathbb{R})\backslash\{0\} \\ (\varphi', V)_\epsilon = 0}} \frac{\langle L_- V, V\rangle}{\langle (1-\epsilon^2\partial_\xi^2)^{1/2}\mathcal{T}_+^{-1}(1-\epsilon^2\partial_\xi^2)^{1/2} V, V\rangle} < 0,$$

or equivalently, if and only if

$$-\mu_0^2 := \inf_{\substack{V \in H^1_\epsilon(\mathbb{R})\backslash\{0\} \\ (\varphi', V)_\epsilon = 0}} \frac{\langle L_- V, V\rangle}{\|V\|^2} < 0.$$

The stability criterion is the same as for the cubic NLS equation!

L. Di Menza and C. Gallo, Nonlinearity **20** (2007) 461–496

Analysis of spectral stability 4

According to the stability criterion, $-\mu_0^2 < 0$ if and only if

$$(L_-^{-1}\varphi', \varphi')_\epsilon = (V_\varphi, \varphi')_\epsilon = -1 + \frac{8}{5}\epsilon^4 > 0,$$

where V_φ is the unique even and bounded solution of

$$L_- V_\varphi = (1 - \epsilon^2 \partial_\xi^2)\varphi'$$

obtained explicitly as

$$V_\varphi(\xi) = -\frac{1}{2}(1 + 2\epsilon^2) + \frac{3}{2}\epsilon^2 \mathrm{sech}^2(\xi).$$

Thus, $-\mu_0^2 < 0$ if $\epsilon^4 > \frac{5}{8}$ (instability), whereas $-\mu_0^2 = 0$ if $\epsilon^4 \leq \frac{5}{8}$.

Orbital stability of the black soliton for odd perturbations

Recall the energy conservation

$$E(u) = \int_{\mathbb{R}} \left[|u_\xi|^2 + (1 - |u|^2)^2 \right] d\xi$$

and consider $u = \varphi + v$ with spatially odd $v := U + iV \in H^1(\mathbb{R})$. Then

$$E(\varphi + U + iV) - E(\varphi) = (L_+ U, U) + (L_- V, V) + \mathcal{O}(\|U + iV\|_{H^1}^3).$$

No momentum conservation is needed since $(L_+ U, U) \geq C \|U\|_{H^1}^2$ and $(L_- V, V) \geq 0$ if $U + iV$ is spatially odd.

The lack of coercivity for $(L_- V, V)$ is compensated in the exponentially weighted space

$$\mathcal{H} := \{ f \in H^1_{\rm loc}(\mathbb{R}) : \quad f' \in L^2(\mathbb{R}), \quad \sqrt{1 - \varphi^2} f \in L^2(\mathbb{R}) \},$$

subject to the only orthogonality condition $(\varphi, V)_{\mathcal{H}} = 0$ due to the orbit $\{e^{-i\theta} u\}_{\theta \in \mathbb{R}/2\pi\mathbb{Z}}$.

P. Gravejat and D. Smets. Proc. London Math. Soc. **111** (2015), 305–353.

Numerical illustration

We numerically simulate evolution of the regularized NLS equation

$$i(1-\epsilon^2\partial_\xi^2)u_t + u_{\xi\xi} + 2(1-|u|^2)u = 0$$

subject to the initial data:

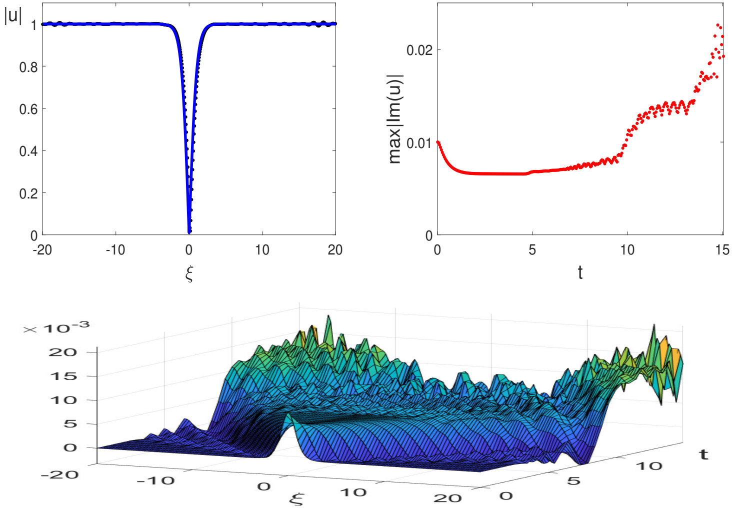
$$u(0,\xi) = \varphi(\xi) + iV(0,\xi) = \tanh(\xi) + ia\,\mathrm{sech}^2(\xi)$$

with $a = 0.01$.

A finite-difference (Crank–Nicholson) method has been employed.

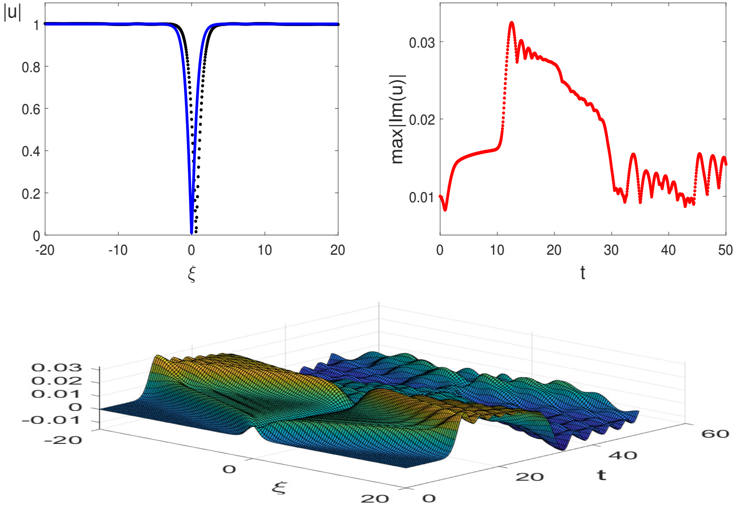
Numerical illustration

$\epsilon = 0$



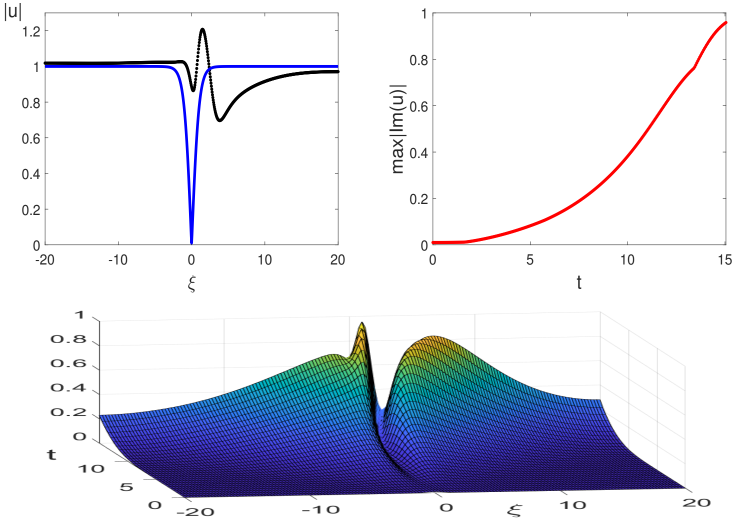
Numerical illustration

$\epsilon = 0.5$



Numerical illustration

$\epsilon = 1$



III. Stability of the bright soliton in the NLS equation with the intensity-dependent dispersion

NLS-IDD model

We consider the NLS equation with intensity-dependent dispersion

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} + \gamma|\psi|^2\psi = 0.$$

It admits the following conserved quantities:

- energy for $\psi \in H^1(\mathbb{R})$

$$E(\psi) = \int_{\mathbb{R}} (|\psi_x|^2 + \gamma|\psi|^2) dx.$$

- mass for $\psi \in H^1(\mathbb{R})$ with small $\|\psi\|_{L^\infty} \leq C < 1$:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx$$

- momentum for $\psi \in H^1(\mathbb{R})$ such that $\psi \neq 0$.

Local solutions exist in $H^\infty(\mathbb{R})$.

M. Poppenberg, Nonlinear Anal. **45** (2001) 723

Bright solitons

These are the standing wave solutions of the form $\psi(x,t) = e^{i\omega t}\varphi_\omega(x)$ in

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} + \gamma|\psi|^2\psi = 0,$$

where φ_ω is a solution of

$$\frac{d^2\varphi}{dx^2} = \frac{(\omega - \gamma\varphi^2)}{1 - \varphi^2}\varphi = -\frac{dV}{d\varphi},$$

which is integrable as

$$\frac{1}{2}(\varphi')^2 + V(\varphi) = C, \quad V(\varphi) := \frac{\omega - \gamma}{2}\log|1 - \varphi^2| - \frac{\gamma}{2}\varphi^2.$$

Solitary waves with $\varphi_\omega(x) \to 0$ as $|x| \to \infty$ exist if and only if $\omega > 0$.

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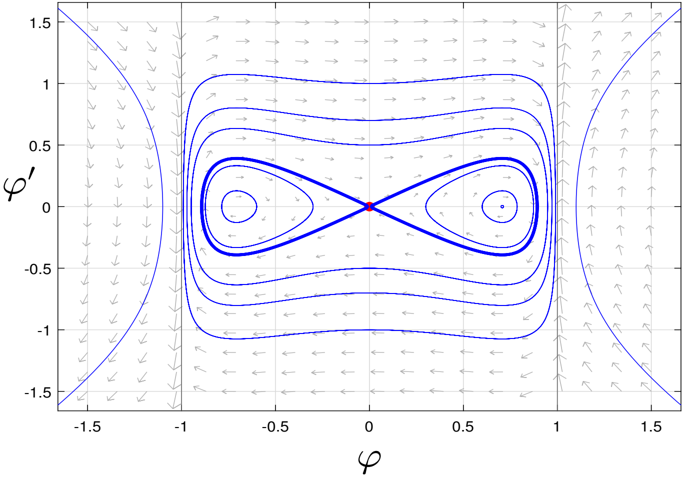
Solitary waves with $\varphi_\omega(x) \to 0$ as $|x| \to \infty$ exist if and only if $\omega > 0$.

Theorem

Fix $\gamma > 0$. There exists a smooth soliton with $\varphi_\omega \in H^\infty(\mathbb{R})$ if and only if $\omega \in (0,\gamma)$. Moreover, the family $\{\varphi_\omega\}_{\omega\in(0,\gamma)}$ is also smooth in ω.

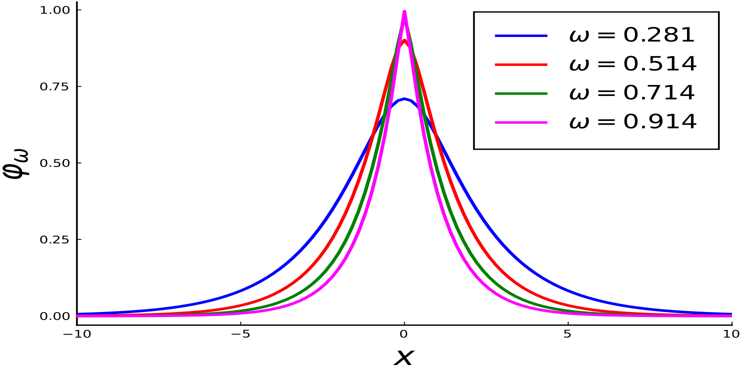
Bright solitons

The phase portrait for $\omega \in (0, \gamma)$ takes the form ($\gamma = 1$):



Bright solitons

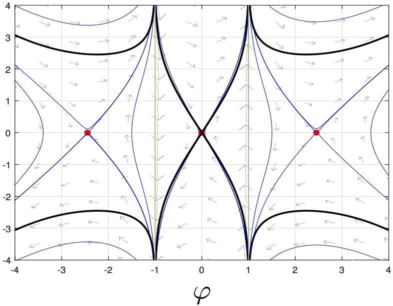
The soliton profile $\varphi_\omega \in H^\infty(\mathbb{R})$ is shown here:



For $\omega \to 0$, the profile φ_ω is approximated by the sech-soliton.
For $\omega = \gamma$, the profile is peaked as $\varphi_{\omega=\gamma}(x) = e^{-\sqrt{\gamma}|x|}$.

Bright solitons

Nonsmooth (cusped) solitons exist formally for $\omega \in (\gamma, \infty)$:



Existence and stability of cusped solitons for $\gamma = 0$ and $\omega > 0$ in

R.M. Ross, P.G. Kevrekidis, and D.P., Quart. Appl. Math. **79** (2021) 641-665

D.P, R.M. Ross, and P.G. Kevrekidis, J. Phys. A: Math. Theor. **54** (2021) 445701

Stability of the bright soliton via conserved quantities

Theorem

Let $\varphi_\omega \in H^\infty(\mathbb{R})$ *be the spatial profile for* $\omega \in (0, \gamma)$. *Then, it is a local nondegenerate (up to two symmetries) minimizer of the augmented energy* $\Lambda_\omega := E + \omega Q$ *subject to fixed mass* Q *in* $H^1(\mathbb{R})$ *if and only if the mapping* $\omega \mapsto Q(\varphi_\omega)$ *is monotonically increasing.*

P.G. Kevrekidis, D.P, and R.M. Ross, arXiv:2408.11192 (2024)

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P.G. Kevrekidis, D.P, and R.M. Ross, arXiv:2408.11192 (2024)

Remark:

This result yields spectral stability of bright solitons. It does not imply the orbital stability along the orbit $\{\varphi_\omega(\cdot - \xi)e^{i\theta}\}_{\xi,\theta\in\mathbb{R}}$ because the local well-posedness holds in $H^\infty(\mathbb{R})$ but perturbations are controlled in $H^1(\mathbb{R})$.

Stability of the bright soliton via conserved quantities

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P.G. Kevrekidis, D.P, and R.M. Ross, arXiv:2408.11192 (2024)

Remark:

We show numerically that there exist ω_1, ω_2 satisfying $0 < \omega_1 < \omega_2 < \gamma$ such that the mapping $\omega \mapsto Q(\varphi_\omega)$ is monotonically increasing if $\omega \in (0,\omega_1) \cup (\omega_2,\gamma)$ and monotonically decreasing if $\omega \in (\omega_1,\omega_2)$. The former is energetically stable and the latter is energetically unstable.

Analysis of energetic stability 1

Recall the conserved quantities

$$E(\psi) = \int_{\mathbb{R}} (|\psi_x|^2 + \gamma|\psi|^2) dx, \quad Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx.$$

The Hamiltonian structure of the NLS equation is non-standard:

$$i\partial_t \psi = (1 - |\psi|^2) \frac{\delta(E - \gamma Q)}{\delta \bar{\psi}}, \qquad \frac{\delta(E - \gamma Q)}{\delta \bar{\psi}} = -\partial_x^2 \psi - \frac{\gamma|\psi|^2 \psi}{1 - |\psi|^2}$$

with the gauge symmetry due to

$$\psi = (1 - |\psi|^2) \frac{\delta Q}{\delta \bar{\psi}}, \qquad \frac{\delta Q}{\delta \bar{\psi}} = \frac{\psi}{1 - |\psi|^2}.$$

Hence $\psi(x,t) = e^{i\omega t}\varphi_\omega$ is defined by a critical point of $\Lambda_\omega := E + (\omega - \gamma)Q$ with

$$0 = (1 - |\varphi|^2) \left(\frac{\delta(E - \gamma Q)}{\delta \bar{\varphi}} + \omega \frac{\delta Q}{\delta \bar{\varphi}} \right).$$

Analysis of energetic stability 2

Expansion of Λ_ω at φ_ω yields

$$\Lambda_\omega(\varphi_\omega + u + iv) - \Lambda_\omega(\varphi_\omega) = \langle \mathcal{S}_+ u, u \rangle_{L^2} + \langle \mathcal{S}_- v, v \rangle_{L^2} + \mathcal{O}(\|u + iv\|^3_{H^1}),$$

where

$$\mathcal{S}_- := -\partial_x^2 + \frac{\omega - \varphi_\omega^2}{1 - \varphi_\omega^2} \geq 0,$$
$$\mathcal{S}_+ := -\partial_x^2 + \frac{\omega + 2\varphi_\omega \partial_x^2 \varphi_\omega - 3\varphi_\omega^2}{1 - \varphi_\omega^2}.$$

- Since $\mathcal{S}_- \varphi_\omega = 0$ and $\varphi_\omega > 0$, $\mathcal{S}_- : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is positive with a simple zero eigenvalue.
- Since $\mathcal{S}_+ \partial_x \varphi_\omega = 0$ and $\partial_x \varphi_\omega$ has a simpe zero on $\mathbb{R}$, $\mathcal{S}_+ : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has a simple negative and a simple zero eigenvalue.

Analysis of energetic stability 3

The criterion that $\mathcal{S}_+|_{\{v_\omega\}^\perp}$ is positive with a simple zero eigenvalue, where

$$v_\omega := \frac{\delta Q}{\delta \bar{\varphi}_\omega} = \frac{\varphi_\omega}{1 - \varphi_\omega^2}.$$

This is true if and only if

$$\langle \mathcal{S}_+^{-1} v_\omega, v_\omega \rangle_{L^2} = -\langle \partial_\omega \varphi_\omega, v_\omega \rangle_{L^2} = -\frac{1}{2} \partial_\omega Q(\varphi_\omega) < 0,$$

where even $\partial_\omega \varphi_\omega$ is uniquely defined from

$$\mathcal{S}_+ \partial_\omega \varphi_\omega = -v_\omega = -\frac{\delta Q}{\delta \bar{\varphi}_\omega}.$$

The stability criterion is a monotone increasing of $\omega \mapsto Q(\varphi_\omega)$.

A twist for the spectral stability argument

The energetic stability implies that the linear spectral problem

$$\begin{pmatrix} 0 & \mathcal{S}_- \\ -\mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

admits no eigenvalues $\lambda \in \mathbb{C} \backslash \{i\mathbb{R}\}$ with $(u, v) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$.

However, the correct spectral stability problem is

$$\begin{pmatrix} 0 & (1 - \varphi_\omega^2)\mathcal{S}_- \\ -(1 - \varphi_\omega^2)\mathcal{S}_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

defined in the weighted Hilbert space $\mathcal{H} \times \mathcal{H}$, $\mathcal{H} := L^2(\mathbb{R}, (1 - \varphi_\omega^2)^{-1} \, dx)$.

The same count of eigenvalues and the same spectral stability results applies to $(1 - \varphi_\omega^2)^{1/2}\mathcal{S}_-(1 - \varphi_\omega^2)^{1/2}$ by Sylvester's inertia law theorem since $(1 - \varphi_\omega^2)^{1/2}$ is bounded and invertible for smooth solitons.

Numerical explorations

A bubble of instability is detected for $\omega \in (\omega_1, \omega_2)$, where $0 < \omega_1 < \omega_2 < \gamma$.

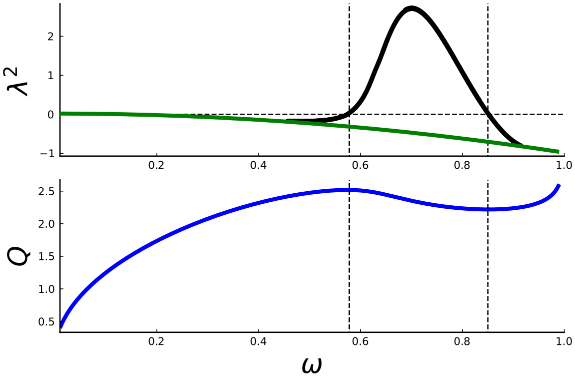
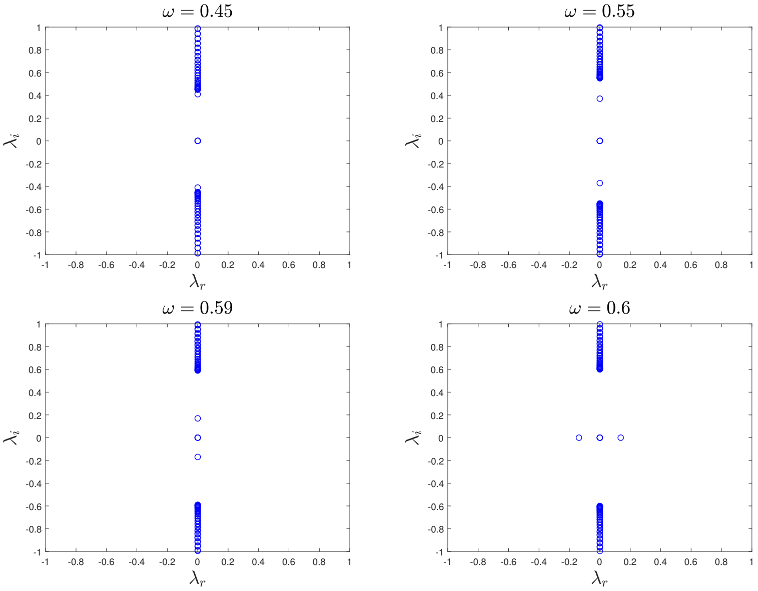


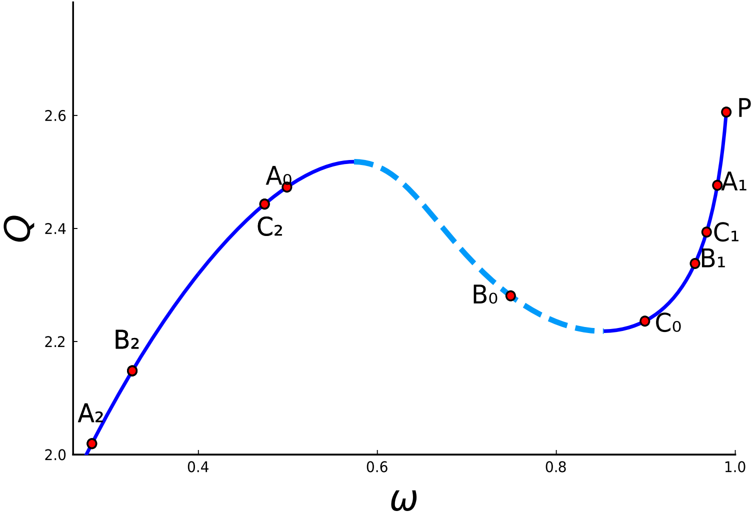
Figure: Top: Squared eigenvalue λ^2 and the map $\omega \mapsto Q(\varphi_\omega)$. The dashed vertical lines are drawn at ω_1 and ω_2.

Numerical explorations



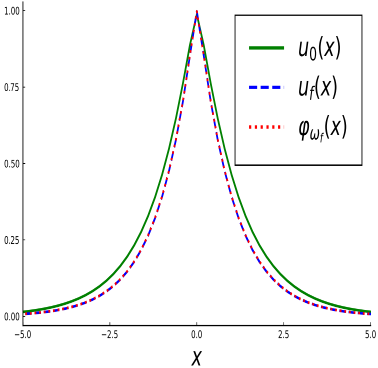
Numerical explorations

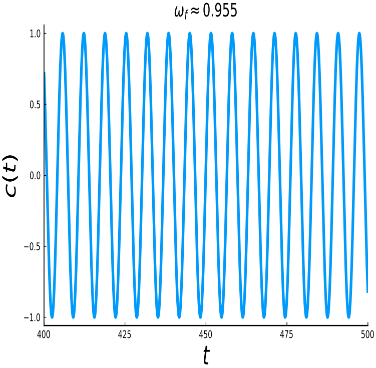
Numerically detected transitions from the unstable branch:



Numerical explorations

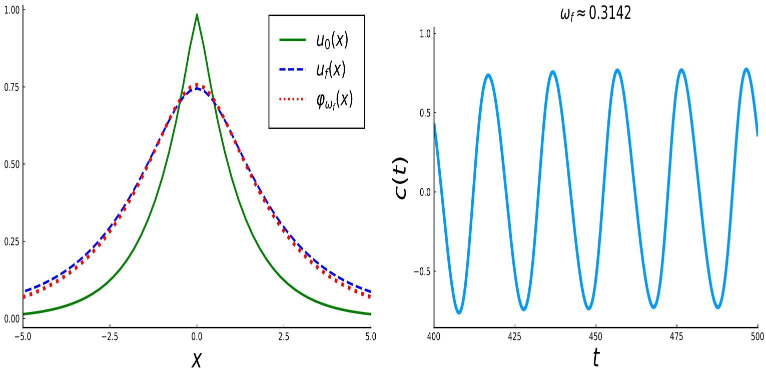
Transition $B_0 \to B_1$





Numerical explorations

Transition $B_0 \to B_2$



IV. Stability of the black soliton in the NLS equation with the intensity-dependent dispersion

Another NLS-IDD model

We consider the NLS model with increasing intensity-dependent dispersion:

$$i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0.$$

A standing wave transformation $\psi(t,x) = u(t,x)e^{2it}$ recovers the defocusing NLS equation

$$i(1 - |u|^2)u_t + u_{xx} + 2(1 - |u|^2)u = 0,$$

which admit the black soliton in the form $u(x) = \tanh(x)$.

Dark solitons $u(t,x) = U_c(x - 2ct)$ are found from

$$U_c'' - 2ic(1 - |U_c|^2)U_c' + 2(1 - |U_c|^2)U_c = 0,$$

for any $c \in \mathbb{R}$.

Time evolution

Solutions are to be considered in the set $\mathcal{F}$,

$$\mathcal{F} := \{f \in L^\infty(\mathbb{R}) : \; |f(x)| < 1, \; x \in \mathbb{R}, \; |f(x)| \to 1 \text{ as } |x| \to \infty\}.$$

Dark solitons exist with $U_c \in \mathcal{F}$.

Conjecture: the set $\mathcal{F}$ is invariant under the time evolution of the NLS-IDD for solutions satisfying $u(t, \cdot) - U_c \in H^\infty(\mathbb{R})$, $t \in [0, \tau_0)$.

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Dark solitons exist with $U_c \in \mathcal{F}$.

NLS-IDD admits conserved mass and energy

$$M(\psi) = \int (1 - |\psi|^2)^2 dx, \quad E(\psi) = \int |\psi_x|^2 dx$$

as well as momentum

$$P(\psi) = \frac{1}{2i} \int \frac{(1 - |\psi|^2)^2}{|\psi|^2} (\bar{\psi}\psi_x - \bar{\psi}_x \psi) dx.$$

Their conservation is proven for smooth solutions in $\mathcal{F}$ satisfying $\psi(t,x) = e^{i\theta_\pm}(1 + \mathcal{O}(e^{-\alpha_\pm |x|}))$ as $x \to \pm\infty$.

Linearization and spectral stability of the black soliton

Using the decomposition $\psi(t,x) = e^{-2it}[\varphi(x) + u(t,x) + iv(t,x)]$, where $\varphi(x) = \tanh(x)$ and $u + iv$ is the perturbation, we obtain the linearized equations of motion

$$(1-\varphi^2)u_t = L_- v, \quad (1-\varphi^2)v_t = -L_+ u,$$

where $L_+ = -\partial_x^2 + 4 - 6\mathrm{sech}^2(x)$ and $L_- = -\partial_x^2 - 2\mathrm{sech}^2(x)$ are the same as in the NLS equation.

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The spectral problem

$$L_- v = \lambda(1-\varphi^2)u, \quad L_+ u = -\lambda(1-\varphi^2)v$$

is defined in the Hilbert space $\mathcal{H}$ with the inner product

$$(f,g)_{\mathcal{H}} := \int (1-\varphi^2)\bar{f} g dx = \int \mathrm{sech}^2(x)\bar{f}(x)g(x)dx.$$

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Theorem

- *The spectrum of L_+ in $\mathcal{H}$ consists of simple eigenvalues $\mu_n = n(n+5)$, $n \geq 0$.*
- *The spectrum of L_- in $\mathcal{H}$ consists of simple eigenvalues $\nu_n = n(n+1) - 2$, $n \geq 0$.*
- *The spectrum of the stability problem in $\mathcal{H} \times \mathcal{H}$ consists of pairs of isolated eigenvalues $\{\pm i\omega_1, \pm i\omega_2, \cdots\}$ and zero eigenvalue.*

Energetic stability of the black soliton

Expanding the energy functional

$$\Lambda(\psi) := \int [|\psi_x|^2 + (1 - |\psi|^2)^2] dx$$

at the black soliton $\varphi(x) = \tanh(x)$ yields

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_+(u) + Q_-(v) + R(u, v),$$

where $Q_+(u) = (L_+ u, u)_{L^2}$, $Q_-(v) = (L_- v, v)_{L^2}$, and

$$R(u, v) = \int [(2\varphi u + u^2 + v^2)^2 - 4\varphi^2 u^2] dx$$

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Black soliton is energetically stable w.r.t. perturbations in H^1 if

$$\Lambda(\psi) - \Lambda(\varphi) \geq C(\|u\|_{H^1}^2 + \|v\|_{H^1}^2) - C(\|u\|_{H^1}^3 + \|v\|_{H^1}^3).$$

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However, two obstacles arise due to nonzero boundary conditions

- $L_- = -\partial_x^2 - 2\mathrm{sech}^2(x)$ is not coercive in $H^1(\mathbb{R})$
- $R(u, v)$ is not cubic if $(u, v) \notin H^1(\mathbb{R})$.

Energetic stability of the black soliton

For the cubic NLS, these issues were handled in [Gravejat–Smets, 2015] by using the revised decomposition

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_-(u) + Q_-(v) + \|\eta\|_{L^2}^2$$

where $Q_-(v) = (L_- v, v)_{L^2}$ and $\eta := |\psi|^2 - \varphi^2 = 2\varphi u + u^2 + v^2$. The distance for perturbations in Banach space X was chosen to be

$$\mathcal{D}_X(\psi_1, \psi_2) := \sqrt{\|\psi_1' - \psi_2'\|_{L^2}^2 + \||\psi_1|^2 - |\psi_2|^2\|_{L^2}^2 + \|\psi_1 - \psi_2\|_{\mathcal{H}}^2}.$$

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For the NLS–IDD, we have several advantages:

- $\mathcal{H}$ appears naturally in the time evolution
- $Q_-(u)$ and $Q_-(v)$ are coercive in $\mathcal{H}$ if
  - $u \in \mathcal{H}$ satisfies orthogonality $(\varphi', u)_{\mathcal{H}} = (\varphi, u)_{\mathcal{H}} = 0$
  - $v \in \mathcal{H}$ satisfies orthogonality $(\varphi', v)_{\mathcal{H}} = (\varphi, v)_{\mathcal{H}} = 0$

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For the four orthogonality conditions, we use the decomposition

$$\psi(t,x) = e^{i\theta(t)} \left[U_{c(t),\omega(t)}(x + \zeta(t)) + u(t, x + \zeta(t)) + iv(t, x + \zeta(t)) \right],$$

where the additional parameter ω is due to the scaling invariance $\psi(t,x) \mapsto \psi(\omega^2 t, \omega x)$ of the NLS equation $i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0$.

Energetic stability of the black soliton

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where $Q_-(v) = (L_- v, v)_{L^2}$ and $\eta := |\psi|^2 - \varphi^2 = 2\varphi u + u^2 + v^2$. The distance for perturbations in Banach space X was chosen to be

$$\mathcal{D}_X(\psi_1, \psi_2) := \sqrt{\|\psi_1' - \psi_2'\|_{L^2}^2 + \||\psi_1|^2 - |\psi_2|^2\|_{L^2}^2 + \|\psi_1 - \psi_2\|_{\mathcal{H}}^2}.$$

Theorem

Assume that the initial-value problem is well-posed in $\mathcal{F} \subset X$ with the distance $\mathcal{D}_X$. Then, the values of $M(\psi)$, $E(\psi)$, and $P(\psi)$ are conserved in the time evolution and the black soliton is orbitally stable in X.

V. Conclusion

Conclusion

- We have considered new variations of the cubic NLS model with a regularized dispersion and the intensity-dependent dispersion.
- We have spectral and energetic stability of the bright and black solitons, which present twisted versions of the stability problem for the cubic NLS equation.
- The NLS model with the regularized dispersion is well-posed in the energy space but the energy space does not coincide with the momentum space.
- The NLS models with the intensity-dependent dispersion presents challenges in the existence of time-dependent solutions in the energy space, where solitons are energetically stable.

MANY THANKS FOR YOUR ATTENTION!