Nonlinear instability of half-solitons on star graphs

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Outline of the talk

Nonlinear Schrödinger equation on a star graph

Ground state on the unbounded graphs

Half-solitons on the star graph

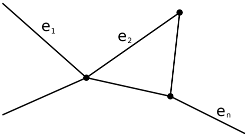
Half-solitons under reflectionless boundary conditions

Nonlinear Schrödinger equation on metric graphs

Nonlinear Schrödinger equation is considered on a graph Γ:

$$i\Psi_t = -\Delta\Psi - (p+1)|\Psi|^{2p}\Psi, \quad x \in \Gamma, \tag{1}$$

where Δ is the graph Laplacian and $\Psi(t,x)$ is defined componentwise on edges subject to boundary conditions at vertices.

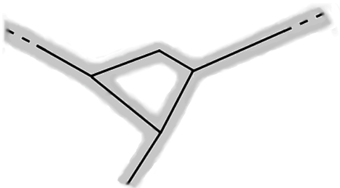


A metric graph Γ is given by a set of edges and vertices, with a metric structure on each edge. Proper boundary conditions are needed on the vertices to ensure that Δ is self-adjoint in $L^2(\Gamma)$.

Graph models are widely used in the modeling of quantum dynamics of thin graph-like structures (quantum wires, nanotechnology, large molecules, periodic arrays in solids, photonic crystals...).

Metric Graphs

Graphs are one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.



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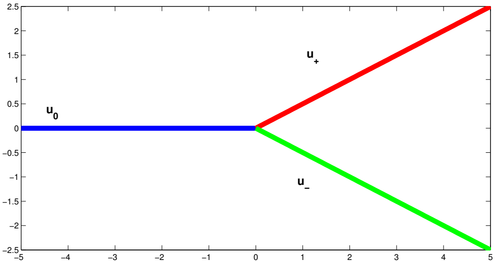
Example: a star graph

A star graph is the union of N half-lines (edges) connected at a vertex. For $N = 2$, the graph is the line $\mathbb{R}$. For $N = 3$, the graph is a Y-junction.



Kirchhoff boundary conditions:

- Components are continuous across the vertex.
- The sum of fluxes (signed derivatives of functions) is zero at the vertex.

Adami–Cacciapuoti–Finco–Noja (2012, 2014, 2016).

Graph Laplacian on the star graph

The Laplacian operator on the star graph Γ is defined by

$$\Delta\Psi = (\psi_1'', \psi_2'', \cdots, \psi_N'')$$

acting on functions in $L^2(\Gamma) = \oplus_{j=1}^N L^2(\mathbb{R}^+)$.

Weak formulation of Δ on Γ is in

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0)\},$$

Strong formulation of Δ on Γ is in

$$H^2_\Gamma := \left\{ \Psi \in H^2(\Gamma) : \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0), \quad \sum_{j=1}^N \psi_j'(0) = 0 \right\}.$$

Lemma

The graph Laplacian $\Delta : H^2_\Gamma \to L^2(\Gamma)$ is self-adjoint.

The Kirchhoff boundary conditions are symmetric:

$$\begin{aligned} \langle \Phi, \Delta\Psi \rangle - \langle \Delta\Phi, \Psi \rangle &= \sum_{j=1}^{N} \phi_j'(0)\psi_j(0) - \phi_j(0)\psi_j'(0) \\ &= 0, \end{aligned}$$

if Kirchhoff boundary conditions are satisfied by $\Phi, \Psi \in H^2_\Gamma$.

Lemma

The graph Laplacian $\Delta : H^2_\Gamma \to L^2(\Gamma)$ *is self-adjoint.*

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if Kirchhoff boundary conditions are satisfied by $\Phi, \Psi \in H^2_\Gamma$.

The graph Laplacian $\Delta : \tilde{H}^2_\Gamma \to L^2(\Gamma)$ is self-adjoint under generalized Kirchhoff boundary conditions in $\tilde{H}^2_\Gamma$:

$$\left\{ \begin{array}{l} \alpha_1\psi_1(0) = \alpha_2\psi_2(0) = \cdots = \alpha_N\psi_N(0) \\ \alpha_1^{-1}\psi_1'(0) + \alpha_2^{-1}\psi_2'(0) + \cdots + \alpha_N^{-1}\psi_N'(0) = 0, \end{array} \right.$$

where $(\alpha_1, \alpha_2, \dots, \alpha_N)$ are arbitrary nonzero parameters.

NLS on the Y junction graph

Consider the cubic NLS on the Y junction graph:

$$
\begin{aligned}
i\partial_t \psi_0 + \partial_x^2 \psi_0 + 2|\psi_0|^2 \psi_0 = 0, \quad & x < 0, \\
i\partial_t \psi_\pm + \partial_x^2 \psi_\pm + 2|\psi_\pm|^2 \psi_\pm = 0, \quad & x > 0,
\end{aligned}
$$

subject to the Kirchhoff boundary conditions at $x = 0$.

The mass functional

$$
Q = \int_{-\infty}^{0} |\psi_0|^2 dx + \int_{0}^{+\infty} |\psi_+|^2 dx + \int_{0}^{+\infty} |\psi_-|^2 dx
$$

is constant in time t (related to the gauge symmetry).

The energy functional

$$
E = \int_{-\infty}^{0} \left(|\partial_x \psi_0|^2 - |\psi_0|^4 \right) dx + \text{similar terms for } \psi_\pm,
$$

is constant in time t (related to the time translation symmetry).

Momentum conservation

The momentum functional

$$P = i\int_{-\infty}^{0} \left(\bar{\psi}_0\partial_x\psi_0 - \psi_0\partial_x\bar{\psi}_0\right) dx + \text{similar terms for } \psi_{\pm},$$

is no longer constant in time t because the spatial translation is broken.

Momentum conservation

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is no longer constant in time t because the spatial translation is broken.

Let $(\alpha_0, \alpha_+, \alpha_-)$ be defined by the generalized Kirchhoff conditions:

$$\left\{ \begin{array}{l} \alpha_0 \psi_0(0) = \alpha_+ \psi_+(0) = \alpha_- \psi_-(0) \\ \alpha_0^{-1} \partial_x \psi_0(0) = \alpha_+^{-1} \partial_x \psi_+(0) + \alpha_-^{-1} \partial_x \psi_-(0). \end{array} \right.$$

The NLS equation is now modified with the account of $(\alpha_0, \alpha_+, \alpha_-)$:

$$\begin{aligned} i\partial_t \psi_0 + \partial_x^2 \psi_0 + \alpha_0^2 |\psi_0|^2 \psi_0 = 0, &\quad x < 0, \\ i\partial_t \psi_\pm + \partial_x^2 \psi_\pm + \alpha_\pm^2 |\psi_\pm|^2 \psi_\pm = 0, &\quad x > 0, \end{aligned}$$

Q and E are still constants of motion in time t.

Momentum conservation

Lemma

If $(\alpha_0, \alpha_+, \alpha_-)$ *satisfy the constraint:*

$$\frac{1}{\alpha_0^2} = \frac{1}{\alpha_+^2} + \frac{1}{\alpha_-^2},$$

then P *is a decreasing function of time* t *with*

$$\frac{dP}{dt} = -\frac{2\alpha_0^2}{\alpha_+^2\alpha_-^2}\left|\alpha_+\partial_x\psi_+(t,0) - \alpha_-\partial_x\psi_-(t,0)\right|^2 \leq 0.$$

If in addition,

$$\alpha_+\partial_x\psi_+(t,0) = \alpha_-\partial_x\psi_-(t,0),$$

is invariant with respect to t, then the momentum P is constant in time.

Reflectionless scattering of solitary waves

In the case of the invariant reduction

$$\alpha_+\psi_+(t,x) = \alpha_-\psi_-(t,x), \quad x \in \mathbb{R}^+,$$

we can set the following function on the infinite line:

$$\Psi(t,x) = \begin{cases} \alpha_0\psi_0(t,x), & x < 0, \\ \alpha_\pm\psi_\pm(t,x), & x > 0. \end{cases}$$

The function Ψ satisfies the integrable cubic NLS equation

$$i\partial_t\Psi + \partial_x^2\Psi + |\Psi|^2\Psi = 0, \quad x \in \mathbb{R},$$

where the vertex $x = 0$ does not appear as an obstacle in the time evolution.

D. Matrasulov–K. Sabirov–Z. Sobirov (2012,2016)

Ground state on the unbounded graphs

Ground state is a standing wave of smallest energy E at fixed mass Q,

$$\mathcal{E} = \inf\{E(u) : \quad u \in H^1_\Gamma, \quad Q(u) = \mu\}.$$

Euler–Lagrange equation in the cubic case $p = 1$ is

$$-\Delta\Phi - 2|\Phi|^2\Phi = -\omega\Phi \qquad \Phi \in H^2_\Gamma$$

where $\omega \in \mathbb{R}$ ($\omega > 0$ in the focusing case) defines $\Psi(t,x) = \Phi(x)e^{i\omega t}$.

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Infimum of $E(u)$ exists due to Gagliardo–Nirenberg inequality in 1D.

If G is unbounded and contains at least one half-line, then

$$\min_{\phi \in H^1(\mathbb{R}^+)} E(u; \mathbb{R}^+) \leq \mathcal{E} \leq \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

Infimum may not be achieved by any of the standing waves Φ.

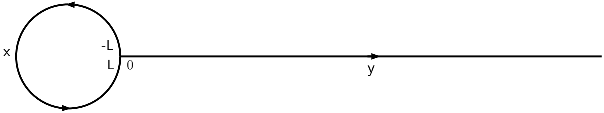
Adami–Serra–Tilli (2015, 2016)

Ground state on the unbounded graphs

If G consists of either one half-line or two half-lines and a bounded edge, then

$$\mathcal{E} < \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is achieved.



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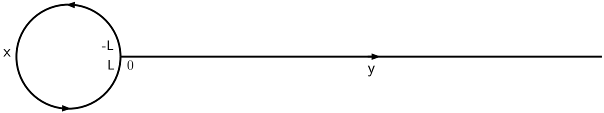


If G consists of more than two half-lines and is *connective to infinity*, then

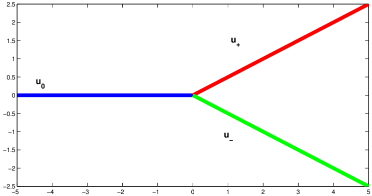
$$\mathcal{E} = \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is not achieved. The reason is topological. By the symmetry rearrangements,

$$E(u; \Gamma) > E(\hat{u}; \mathbb{R}) \geq \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R}) = \mathcal{E}.$$

At the same time, a sequence of solitary waves escaping to infinity along one edge yields a sequence of functions that minimize $E(u; \Gamma)$ until it reaches $\mathcal{E}$.

Ground state on the Y junction graph: $N = 3$



No ground state exists due to the same topological reason.

There exists a half-soliton to the Euler–Lagrange equation:

$$-\Delta\Phi - 2|\Phi|^2\Phi = -\omega\Phi \qquad \phi \in \mathcal{D}(\Gamma),$$

in the form

$$\Phi(x) = \left[\begin{array}{ll} \phi_0(x) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}x), & x \in (-\infty, 0) \\ \phi_\pm(x) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}x), & x \in (0, \infty) \end{array} \right].$$

Half-soliton is a saddle point of energy E at fixed mass Q.
(Adami *et al.*, 2012)

Half-solitons on the star graph with any $N \geq 3$

By using the scaling transformation

$$\Phi_\omega(x) = \omega^{\frac{1}{2p}} \Phi(z), \quad z = \omega^{\frac{1}{2}} x,$$

we can consider the Euler–Largrange equation:

$$-\Delta\Phi + \Phi - (p+1)|\Phi|^{2p}\Phi = 0, \qquad \Phi \in H^2_\Gamma,$$

The half-soliton state

$$\Phi(x) = \phi(x) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{with} \quad \phi(x) = \operatorname{sech}^{\frac{1}{p}}(px)$$

is a critical point of the action functional

$$\Lambda(\Psi) = E(\Psi) + Q(\Psi).$$

Second variation

Substituting $\Psi = \Phi + U + iW$ with real-valued $U, W \in H^1_\Gamma$ into $\Lambda(\Psi)$ yield

$$\Lambda(\Phi+U+iW) = \Lambda(\Phi)+\langle L_+U, U\rangle_{L^2(\Gamma)}+\langle L_-W, W\rangle_{L^2(\Gamma)}+\mathrm{o}(\|U+iW\|^2_{H^1(\Gamma)}),$$

where

$$\begin{aligned}
\langle L_+U, U\rangle_{L^2(\Gamma)} &:= \int_\Gamma \left[(\nabla U)^2 + U^2 - (2p+1)(p+1)\Phi^{2p}U^2\right] dx, \\
\langle L_-W, W\rangle_{L^2(\Gamma)} &:= \int_\Gamma \left[(\nabla W)^2 + W^2 - (p+1)\Phi^{2p}W^2\right] dx,
\end{aligned}$$

Theorem (Kairzhan–P, 2017)

For every $p \in (0,2)$, $\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)} \geq 0$ for every $V \in H^1_\Gamma \cap L^2_c$, where

$$L^2_c := \left\{V \in L^2(\Gamma) : \quad \langle V, \Phi\rangle_{L^2(\Gamma)} = 0\right\}.$$

Moreover, $\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)} = 0$ if and only if $V \in \ker(L_+)$ of dimension $(N-1)$. Consequently, $V = 0$ is a degenerate minimizer of $\langle \Lambda''(\Phi)V, V\rangle_{L^2(\Gamma)}$ in $H^1_\Gamma \cap L^2_c$.

Spectral information

The second variation is a sum of two quadratic forms:

$$\begin{aligned}\langle L_+ U, U\rangle_{L^2(\Gamma)} &:= \int_\Gamma \left[(\nabla U)^2 + U^2 - (2p+1)(p+1)\Phi^{2p} U^2\right] dx, \\ \langle L_- W, W\rangle_{L^2(\Gamma)} &:= \int_\Gamma \left[(\nabla W)^2 + W^2 - (p+1)\Phi^{2p} W^2\right] dx,\end{aligned}$$

where $L_\pm : H^2_\Gamma \to L^2(\Gamma)$ with $\sigma_c(L_\pm) \in [1,\infty)$.

- $L_- \geq 0$ and $\ker(L_+) = \mathrm{span}\{\Phi\}$.
- L_+ has one simple negative eigenvalue and $\ker(L_+) = \mathrm{span}\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$ with

$$N = 3: \quad U^{(1)} = \phi'(x)\begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad U^{(2)} = \phi'(x)\begin{pmatrix}1\\1\\-2\end{pmatrix}.$$

- $L_+|_{L^2_c} \geq 0$ if $p \in (0,2)$, where

$$L^2_c := \left\{U \in L^2(\Gamma): \quad \langle U, \Phi\rangle_{L^2(\Gamma)} = 0\right\}.$$

Saddle-point geometry

Theorem (Kairzhan–P, 2017)

Consider the orthogonal decomposition in H^1_Γ,

$$\Psi = \Phi + c_1 U^{(1)} + c_2 U^{(2)} + \cdots + c_{N-1} U^{(N-1)} + U^{\perp},$$

where $X_c = \mathrm{span}\{U^{(1)}, U^{(2)}, \dots, U^{(N-1)}\}$ *and* $U^{\perp} \in H^1_\Gamma \cap L^2_c \cap [X_c]^{\perp}$.

For every $p \in \left[\frac{1}{2}, 2\right)$, *there exists* $\delta > 0$ *such that for every* $c = (c_1, c_2, \dots, c_{N-1})^T \in \mathbb{R}^{N-1}$ *satisfying* $\|c\| \leq \delta$, *there exists a unique minimizer* $U^{\perp} \in H^1_\Gamma \cap L^2_c \cap [X_c]^{\perp}$ *of the variational problem*

$$M(c) := \inf_{U^{\perp} \in H^1_\Gamma \cap L^2_c \cap [X_c]^{\perp}} \left[\Lambda(\Psi) - \Lambda(\Phi)\right]$$

such that $\|U^{\perp}\|_{H^1(\Gamma)} \leq A\|c\|^2$ *for a c-independent constant* $A > 0$.

Moreover, $M(c)$ *is sign-indefinite in c. Consequently,* Φ *is a nonlinear saddle point of* Λ *in* H^1_Γ *with respect to perturbations in* $H^1_\Gamma \cap L^2_c$.

Minimization of the remainder term

Expanding for real $U \in H^1_\Gamma$:

$$\Lambda(\Phi+U) = \Lambda(\Phi)+\langle L_+U, U\rangle_{L^2(\Gamma)}-\frac{2}{3}p(p+1)(2p+1)\langle \Phi^{2p-1}U^2, U\rangle_{L^2(\Gamma)}+\mathrm{o}(\|U\|^3_{H^1})$$

Looking at $M(c) := \inf_{U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp} [\Lambda(\Phi+U) - \Lambda(\Phi)]$ with

$$U = c_1 U^{(1)} + c_2 U^{(2)} + \cdots + c_{N-1} U^{(N-1)} + U^\perp,$$

we obtain $F(U^\perp, c) = 0$ with

$$F(U^\perp, c) : X \times \mathbb{R}^{N-1} \mapsto Y, \quad X := H^1_\Gamma \cap L^2_c \cap [X_c]^\perp, \quad Y := H^{-1}_\Gamma \cap L^2_c \cap [X_c]^\perp,$$

$$F(U^\perp, c) := L_+U^\perp - p(p+1)(2p+1)\Pi_c \Phi^{2p-1} \left(\sum_{j=1}^{N-1} c_j U^{(j)} + U^\perp \right)^2 + \mathrm{o}(\|U\|^2_{H^1}).$$

(i) F is a C^2 map from $X \times \mathbb{R}^{N-1}$ to Y;
(ii) $F(0,0) = 0$;
(iii) $D_{U^\perp} F(0,0) = \Pi_c L_+ \Pi_c : X \mapsto Y$ is invertible with a bounded inverse from Y to X;
(iv) $\Pi_c L_+ \Pi_c$ is strictly positive;
(v) $D_c F(0,0) = 0$.

Normal form argument

By the minimization problem, we obtain

$$\begin{aligned} M(c) &= \inf_{U^\perp \in H^1_\Gamma \cap L^2_c \cap [X_c]^\perp} [\Lambda(\Phi + U) - \Lambda(\Phi)] \\ &= M_0(c) + o(\|c\|^3), \end{aligned}$$

where

$$M_0(c) := -\frac{2}{3}p(p+1)(2p+1)\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} c_i c_j c_k \langle \Phi^{2p-1} U^{(i)} U^{(j)}, U^{(k)} \rangle_{L^2(\Gamma)}.$$

$M_0(c)$, and hence $M(c)$, is sign-indefinite near $c = 0$:

$$N = 3: \quad M_0(c) = 2p^2(c_1^2 - c_2^2)c_2.$$

Nonlinear instability

Theorem (Kairzhan–P, 2017)

For every $p \in \left[\frac{1}{2}, 2\right)$, there exists $\epsilon > 0$ such that for every $\delta > 0$ (sufficiently small) there exists $V \in H^1_\Gamma$ with $\|V\|_{H^1_\Gamma} \leq \delta$ such that the unique global solution $\Psi(t) \in C(\mathbb{R}, H^1_\Gamma) \cap C^1(\mathbb{R}, H^{-1}_\Gamma)$ to the NLS equation starting with the initial datum $\Psi(0) = \Phi + V$ satisfies

$$\inf_{\theta \in \mathbb{R}} \|e^{-i\theta}\Psi(t_0) - \Phi\|_{H^1(\Gamma)} > \epsilon \quad \text{for some} \;\; t_0 > 0.$$

Consequently, the orbit $\{\Phi e^{i\theta}\}_{\theta \in \mathbb{R}}$ is unstable in the time evolution of the NLS equation in H^1_Γ.

Nonlinear instability of saddle points of action functionals does not hold generally for Hamiltonian systems.
Example: negative Krein signature of stable eigenvalues.

Expansion of the action functional

Expanding for real $U, W \in H^1_\Gamma \cap L^2_c$:

$$\begin{aligned} \Delta(t) &:= E(\Phi_{\omega(t)} + U(t) + iW(t)) - E(\Phi) \\ &\quad + \omega(t)\left[Q(\Phi_{\omega(t)} + U(t) + iW(t)) - Q(\Phi)\right] \\ &= D(\omega) + \langle L_+(\omega)U, U\rangle_{L^2(\Gamma)} + \langle L_-(\omega)W, W\rangle_{L^2(\Gamma)} + N_\omega(U, W), \end{aligned}$$

where

$$\begin{aligned} D(\omega) &:= E(\Phi_\omega) - E(\Phi) + \omega\left[Q(\Phi_\omega) - Q(\Phi)\right] \\ &= (\omega - 1)^2 \langle \Phi, \partial_\omega \Phi_\omega|_{\omega=1}\rangle_{L^2(\Omega)} + \mathcal{O}(|\omega - 1|^3) \end{aligned}$$

and

$$\Delta(t) = \Delta(0) + (\omega(t) - 1)\left[Q(\Phi + U(0) + iW(0)) - Q(\Phi)\right],$$

If $\|U(0) + iW(0)\|_{H^1_\Gamma} \leq \delta$, then

$$|\Delta(0)| + |Q(\Phi + U_0 + iW_0) - Q(\Phi)| \leq A\delta^2.$$

Secondary decomposition

Expand $U, W \in H^1_\Gamma \cap L^2_c$ as

$$U(t) = \sum_{j=1}^{N-1} c_j(t) U^{(j)}_{\omega(t)} + U^\perp(t), \quad W(t) = \sum_{j=1}^{N-1} b_j(t) W^{(j)}_{\omega(t)} + W^\perp(t),$$

and

$$\langle U^\perp(t), W^{(j)}_{\omega(t)} \rangle_{L^2(\Gamma)} = \langle W^\perp(t), U^{(j)}_{\omega(t)} \rangle_{L^2(\Gamma)} = 0, \quad 1 \leq j \leq N-1,$$

where $L_+(\omega) U^{(j)}_\omega = 0$ and $L_-(\omega) W^{(j)}_\omega = U^{(j)}_\omega$.

The action functional is further expanded as follows:

$$\begin{aligned} \Delta &= D(\omega) + \langle L_+(\omega) U^\perp, U^\perp \rangle_{L^2(\Gamma)} + \langle L_-(\omega) W^\perp, W^\perp \rangle_{L^2(\Gamma)} \\ &\quad + \sum_{j=1}^{N-1} \langle W^{(j)}_\omega, U^{(j)}_\omega \rangle_{L^2(\Gamma)} b_j^2 + M_0(c) + \widetilde{\Delta}(c, b, U^\perp, W^\perp), \end{aligned}$$

where $\tilde{\Delta}$ is a remainder term (of higher order).

Truncated Hamiltonian system

At the leading order, $\{c_j, b_j\}_{j=1}^{N-1}$ satisfy

$$\begin{cases} \dot{c}_j = b_j, \\ \dot{b}_j = \sum_{k=1}^{N-1} \sum_{n=1}^{N-1} \frac{\langle \Phi^{2p-1} U^{(k)} U^{(n)}, U^{(j)} \rangle_{L^2(\Gamma)}}{\langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)}} c_k c_n, \end{cases}$$

which is Hamiltonian system with the conserved energy

$$H_0(c,b) := \sum_{j=1}^{N-1} \langle W^{(j)}, U^{(j)} \rangle_{L^2(\Gamma)} b_j^2 + M_0(c).$$

For $N = 3$,

$$\begin{cases} \|\phi\|^2_{L^2(\mathbb{R}_+)} \ddot{c}_1 = -4 c_1 c_2, \\ 3\|\phi\|^2_{L^2(\mathbb{R}_+)} \ddot{c}_2 = -2(c_1^2 - 3c_2^2). \end{cases}$$

$c_1 = 0$ is an invariant reduction. Zero solution is nonlinearly unstable.

Closing the energy estimates

Consider the region where nonlinear instability is developed in the Hamiltonian system:

$$\|c(t)\| \leq A\epsilon, \quad \|b(t)\| \leq A\epsilon^{3/2}, \quad t \in [0, t_0], \quad t_0 \leq A\epsilon^{-1/2},$$

By energy estimates, we have:

$$|\omega(t) - 1| + \|U^{\perp}(t) + iW^{\perp}(t)\|_{H^1(\Gamma)} \leq A\left(\delta + \epsilon^{3/2}\right), \quad t \in [0, t_0].$$

which is much smaller than the leading-order term if $\delta = \mathcal{O}(\epsilon^{3/2})$.

Solutions of the system for $\{c_j, b_j\}_{j=1}^{N-1}$ remain close to the (unstable) solutions of the truncated Hamiltonian system. Hence, there exists $t_0 = \mathcal{O}(\epsilon^{-1/2})$ such that

$$\|U(t_0) + iW(t_0)\|_{H^1(\Gamma)} > \epsilon.$$

NLS under generalized Kirchhoff conditions

Let $(\alpha_0, \alpha_+, \alpha_-)$ be defined by the generalized Kirchhoff conditions:

$$\begin{cases} \alpha_0 \psi_0(0) = \alpha_+ \psi_+(0) = \alpha_- \psi_-(0) \\ \alpha_0^{-1} \partial_x \psi_0(0) = \alpha_+^{-1} \partial_x \psi_+(0) + \alpha_-^{-1} \partial_x \psi_-(0), \end{cases}$$

and the cubic NLS equation

$$\begin{aligned} i\partial_t \psi_0 + \partial_x^2 \psi_0 + \alpha_0^2 |\psi_0|^2 \psi_0 = 0, \quad x < 0, \\ i\partial_t \psi_\pm + \partial_x^2 \psi_\pm + \alpha_\pm^2 |\psi_\pm|^2 \psi_\pm = 0, \quad x > 0. \end{aligned}$$

If $(\alpha_0, \alpha_+, \alpha_-)$ satisfy the constraint:

$$\frac{1}{\alpha_0^2} = \frac{1}{\alpha_+^2} + \frac{1}{\alpha_-^2},$$

then there exists an invariant reduction

$$\alpha_+ \psi_+(t,x) = \alpha_- \psi_-(t,x), \quad x \in \mathbb{R}^+,$$

to the integrable cubic NLS equation

$$i\partial_t \Psi + \partial_x^2 \Psi + |\Psi|^2 \Psi = 0, \quad x \in \mathbb{R}.$$

Translated stationary state

The half-soliton can now be translated along the graph Γ:

$$\Phi(x) = \left[\begin{array}{ll} \phi_0(x) = \alpha_0^{-1}\text{sech}(x-a), & x \in (-\infty, 0), \\ \phi_+(x) = \alpha_+^{-1}\text{sech}(x-a), & x \in (0, \infty), \\ \phi_-(x) = \alpha_-^{-1}\text{sech}(x-a), & x \in (0, \infty), \end{array} \right],$$

where $a \in \mathbb{R}$ is arbitrary parameter.

When $a = 0$, $L_+ : H^2_\Gamma \to L^2(\Gamma)$ has one simple negative eigenvalue and $\ker(L_+) = \text{span}\{U^{(1)}, U^{(2)}\}$ with

$$U^{(1)} = \begin{pmatrix} \phi_0'(x) \\ \phi_+'(x) \\ \phi_-'(x) \end{pmatrix}, \quad U^{(2)} = \begin{pmatrix} 0 \\ \alpha_+\phi_+'(x) \\ -\alpha_-\phi_-'(x) \end{pmatrix}.$$

The first mode is due to the translational invariance of the invariant reduction. The second mode breaks the invariant reduction.

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Half-soliton is still a nonlinear saddle point of the action functional.

Summary

- For the star graphs with Kirchhoff boundary conditions, we proved that the saddle points of action functional are nonlinearly unstable.

- For the star graphs with reflectionless boundary conditions, we proved that the half-solitons are still nonlinearly unstable due to symmetry-breaking perturbations.

- In the latter case, half-solitons are continued as shifted states along the parameter a and the shifted solitons with $a > 0$ are orbitally stable because they are local constrained minimizers of the action functional.