Instability of breathers and periodic waves in NLS

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The rogue wave of the cubic NLS equation

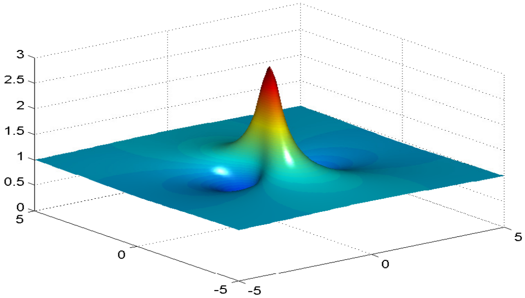
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits the exact solution

$$\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{it}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.



Modulational instability of the constant-amplitude wave

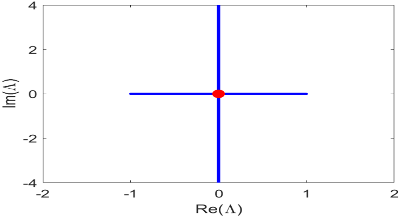
The rogue wave solution is related to the modulational instability of the constant-amplitude wave:

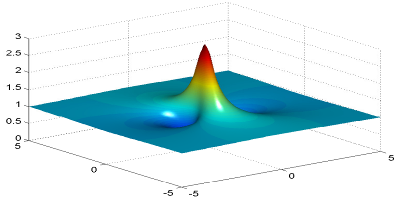
$$\psi(x,t) = e^{it}\left[1 + (k^2 + 2i\Lambda)e^{\Lambda t + ikx} + (k^2 + 2i\bar{\Lambda})e^{\bar{\Lambda} t - ikx}\right],$$

where $k \in \mathbb{R}$ is the wave number and Λ is given by

$$\Lambda^2 = k^2\left(1 - \frac{1}{4}k^2\right).$$

The wave is unstable for $k \in (0,2)$.



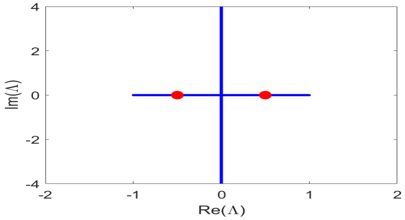


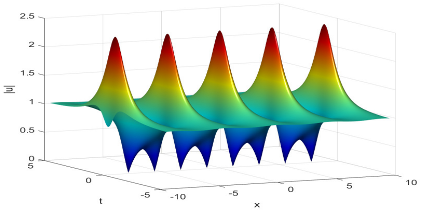
Other rogue waves - Akhmediev breathers (AB)

Spatially periodic homoclinic solution was constructed by N.N. Akhmediev, V.M. Eleonsky, and N.E. Kulagin (1985):

$$\psi(x,t) = e^{it}\left[1 - \frac{2(1-\lambda^2)\cosh(k\lambda t) + ik\lambda\sinh(k\lambda t)}{\cosh(k\lambda t) - \lambda\cos(kx)}\right],$$

where $k = 2\sqrt{1-\lambda^2} \in (0,2)$ and $\lambda \in (0,1)$ is the only free parameter. There is a unique solution for each spatial period $L = \frac{2\pi}{k} = \frac{\pi}{\sqrt{1-\lambda^2}} > \pi$.



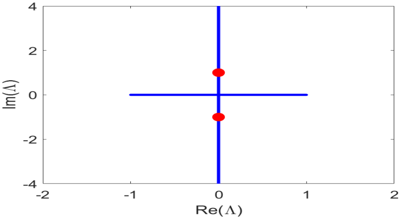


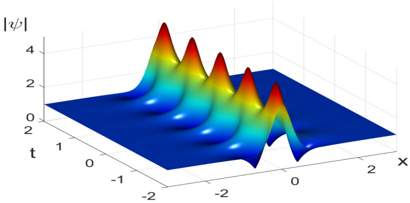
Other rogue waves - Kuznetsov-Ma breathers

Temporally periodic soliton was constructed by E. A. Kuznetsov (1977) and Y.-C. Ma (1979):

$$\psi(x,t) = \left[1 - \frac{2(\lambda^2-1)\cos(\beta\lambda t) + i\beta\lambda\sin(\beta\lambda t)}{\lambda\cosh(\beta x) - \cos(\beta\lambda t)}\right] e^{it},$$

where $\beta = 2\sqrt{\lambda^2-1}$ and $\lambda \in (1,\infty)$ is the only free parameter. There is a unique solution for each temporal period $T = \frac{2\pi}{\beta\lambda} = \frac{\pi}{\lambda\sqrt{\lambda^2-1}} > 0$ with $k = i\beta$.





The main task is to understand the linear and nonlinear instability of breathers.

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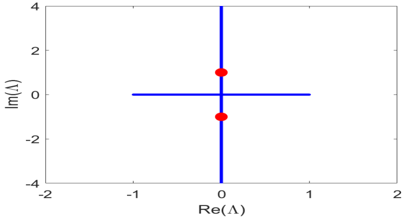
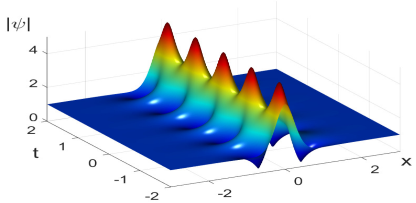




The main task is to understand the linear and nonlinear instability of breathers.

Selected results on Kuznetsov–Ma breathers (time-periodic solitons):

- Numerical computations of linearized NLS with Floquet theory in time (Cuevas–Maraver, Haragus, *et al.*, PRE, 2017)
- Attempt to prove the spectral mapping theorem:

$$\sigma_{\rm ess}(M) = \sigma_{\rm ess}(e^{T\mathcal{L}_\infty})$$

  (Zweck, Latushkin, Marzuola, Jones, J. Evol. Eqs., 2020)
- Proof of energetic instability in $H^s(\mathbb{R})$ for $s > 1/2$ from saddle point geometry associated with the fourth-order Lax-Novikov equation (Alejo, Fanelli, Munoz, arXiv, 2019)

Selected results on Akhmediev breathers (spatially-periodic homoclinics):

- Claim of linear stability based on eigenvectors of ZS problem (Calini, Schober, Nonlinearity, 2012)
- Claim of exponential instability based on computations of x-growing solutions of ZS problem (Grinevich, Santini, arXiv, 2021)
- Proof of energetic instability in $H^s_{\rm per}(\mathbb{R})$ for $s > 1/2$ from saddle point geometry associated with the fourth-order Lax-Novikov equation (Alejo, Fanelli, Munoz, Sao Paulo J. Math. Sci., 2019)

Selected results on Peregrine rogue wave:

- Perturbation theory via inverse scattering
  (Garnier, Kalimeris, J. Phys. A, 2012) (Biondini, Kovacic, 2014)
- Completeness of eigenfunctions via Riemann–Hilbert problem
  (Bilman, Miller, Comm. Pure Appl. Math., 2019)
- Numerical simulations of Peregrine breathers
  (Klein, Haragus, Ann. Math. Sci. Appl., 2017)
  (Calini, Schober, Stawn, Appl. Numer. Math., 2019)

The spectral context of breathers

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

is related to the integrable hierarchy, which admits the following stationary equations (Lax–Novikov equations):

$$\begin{aligned} & u'(x) + 2icu = 0, \\ & u''(x) + 2|u|^2 u + 2icu' + 4bu = 0, \\ & u'''(x) + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0, \end{aligned}$$

where c, b, a are constants.

The third-order Lax–Novikov equation with $a = c = 0$ admits the exact solutions by separation of variables (Akhmediev, Eleonskii, Kulagin, 1987):

$$\psi(x,t) = [q(x,t) + i\delta(t)]\, e^{i\theta(t)},$$

where $q(x + L, t) = q(x, t + T) = q(x, t)$ and $\delta(t + T) = \delta(t)$.

Lax system

If u is a solution of the NLS, then it is a potential of the linear Lax system:

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \left(\begin{array}{cc} \lambda & u \\ -\bar{u} & -\lambda \end{array} \right)$$

and

$$\varphi_t = V(\lambda, u)\varphi, \qquad V(\lambda, u) = i \left(\begin{array}{cc} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda \bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{array} \right).$$

The x-part is referred to as the Zakharov–Shabat spectral problem.

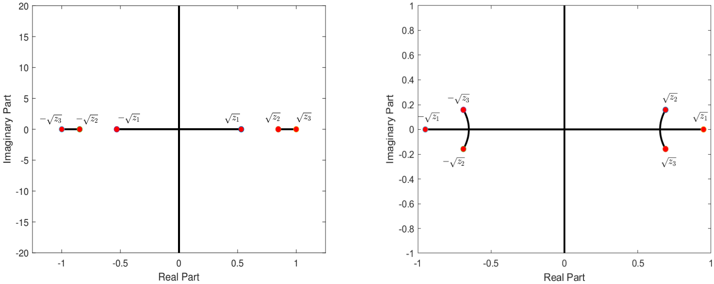
Since u is L-periodic in x for fixed t, by Floquet theorem, λ belongs to the Lax spectrum if $\varphi(x) = p(x)e^{ikx}$ is bounded with L-periodic p and $k \in [-\frac{\pi}{L}, \frac{\pi}{L}]$.

Double-periodic solutions

There are only two families of solutions of the third-order Lax-Novikov equation:

$$u'''(x) + 6|u|^2 u'(x) + 4bu'(x) = 0.$$

Their Lax spectrum is shown below.



Akhmediev breathers: the gap shrinks to a point.
Kuznetsov-Ma- breather: the outer band shrinks to a point.

Analytical method: linearized NLS equation

Recall the linear Lax system:

$$\varphi_x = U(\lambda, u)\varphi, \qquad \varphi_t = V(\lambda, u)\varphi,$$

where u is a solution of the NLS equation

$$iu_t + \frac{1}{2}u_{xx} + (|u|^2 - 1)u = 0.$$

If φ and ϕ are two linearly independent solutions of the Lax system, then

Pair I	Pair II	Pair III
$v = \varphi_1^2 - \bar{\varphi}_2^2$	$v = \varphi_1\phi_1 - \bar{\varphi}_2\bar{\phi}_2$	$v = \phi_1^2 - \bar{\phi}_2^2$
$v = i\varphi_1^2 + i\bar{\varphi}_2^2$	$v = i\varphi_1\phi_1 + i\bar{\varphi}_2\bar{\phi}_2$	$v = i\phi_1^2 + i\bar{\phi}_2^2$

are solutions of the linearized NLS equation

$$iv_t + \frac{1}{2}v_{xx} + (2|u|^2 - 1)v + u^2\bar{v} = 0.$$

Example: constant-amplitude background $u = 1$

Two linearly independent solutions

$$\varphi = \left[\begin{array}{c} \sqrt{\lambda - \frac{i}{2}k(\lambda)} \\ -\sqrt{\lambda + \frac{i}{2}k(\lambda)} \end{array} \right] e^{-\frac{1}{2}k(\lambda)(ix-\lambda t)}, \quad \phi = \left[\begin{array}{c} \sqrt{\lambda + \frac{i}{2}k(\lambda)} \\ -\sqrt{\lambda - \frac{i}{2}k(\lambda)} \end{array} \right] e^{\frac{1}{2}k(\lambda)(ix-\lambda t)},$$

where $k(\lambda) = 2\sqrt{1-\lambda^2}$ and $\lambda \in \mathbb{C}$ is the spectral parameter. Solutions are bounded if $\lambda \in i\mathbb{R} \cup [-1, 1]$.

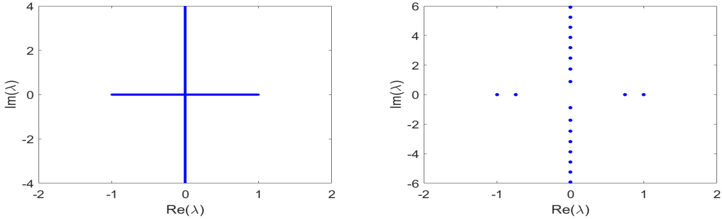


Figure: Left: basis in $L^2(\mathbb{R})$ (Fourier transform). Right: basis in $L^2_{\rm per}$ (Fourier series).

Case of periodic boundary conditions

Two linearly independent solutions

$$\varphi = \left[\begin{array}{c} \sqrt{\lambda_m - \frac{i}{2} k_m} \\ -\sqrt{\lambda_m + \frac{i}{2} k_m} \end{array} \right] e^{-\frac{1}{2} k_m (ix - \lambda_m t)}, \quad \phi = \left[\begin{array}{c} \sqrt{\lambda_m + \frac{i}{2} k_m} \\ -\sqrt{\lambda_m - \frac{i}{2} k_m} \end{array} \right] e^{\frac{1}{2} k_m (ix - \lambda_m t)},$$

where $k_m = \frac{2\pi m}{L}$ and $m \in \mathbb{N}$.

Pairs I and III give four linearly independent solutions for $m \in \mathbb{N}$:

$$v_m^+(x,t) = (2i\lambda_m + k_m) \sin(k_m x) e^{\lambda_m k_m t}, \quad v_{-m}^+(x,t) = (2i\lambda_m + k_m) \cos(k_m x) e^{\lambda_m k_m t},$$

$$v_m^-(x,t) = (2i\lambda_m - k_m) \sin(k_m x) e^{-\lambda_m k_m t}, \quad v_{-m}^-(x,t) = (2i\lambda_m - k_m) \cos(k_m x) e^{-\lambda_m k_m t},$$

Pair II gives $v = 0$ and $v_0 = 2i$ and the second solution is missing...

Recovering the missing element of the basis

In order to recover the second solution, we find two linearly independent solutions for $\lambda = 1$ ($m = 0$):

$$\varphi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \phi = \begin{bmatrix} x + it + 1 \\ -x - it \end{bmatrix},$$

Pairs I gives

$$v = 0, \quad v_0 = 2i.$$

Pair II gives

$$\tilde{v}_0 = 2it + 1, \quad v = i(2x + 1).$$

Pair III gives

$$v = 4ixt + 2x + 2it + 1, \quad v = 2i(x^2 - t^2 + x + it) + i.$$

General solution of the linearized NLS equation is

$$v = c_0 v_0 + \tilde{c}_0 \tilde{v}_0 + \sum_{m \in \mathbb{N}} c_m^+ v_m^+ + c_m^- v_m^- + c_{-m}^+ v_{-m}^+ + c_{-m}^- v_{-m}^-.$$

Analytical method: breather solutions

Recall the linear Lax system:

$$\varphi_x = U(\lambda, u)\varphi, \qquad \varphi_t = V(\lambda, u)\varphi.$$

For fixed $\lambda_0 \in \mathbb{C}$, let $\varphi = (p_0, q_0)^T$ be a particular solution of the Lax system with $u = u_0$ and $\lambda = \lambda_0$. Then, $\varphi = (\hat{p}_0, \hat{q}_0)^T$ is a particular solution of the Lax system with $u = \hat{u}_0$ and $\lambda = \lambda_0$, where

$$\left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right] = \frac{\lambda_0 + \bar{\lambda}_0}{|p_0|^2 + |q_0|^2} \left[\begin{array}{c} -\bar{q}_0 \\ \bar{p}_0 \end{array} \right]$$

and

$$\hat{u}_0 = u_0 + \frac{2(\lambda_0 + \bar{\lambda}_0) p_0 \bar{q}_0}{|p_0|^2 + |q_0|^2}.$$

If $\Phi(\lambda)$ is a general solution for u_0, then $\hat{\Phi}(\lambda)$ is a general solution for $\hat{u}_0$:

$$\hat{\Phi}(\lambda) = D(\lambda)\Phi(\lambda), \quad D(\lambda) = I + \frac{1}{\lambda - \lambda_0} \left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right] \left[-q_0 \;\; p_0 \right].$$

Case of Kuznetsov–Ma breathers

Fix $\lambda_0 \in (1, \infty)$ and choose the solution $\varphi = (p_0, q_0)^T$ with $u_0 = 1$ in the form:

$$\left\{ \begin{array}{l} p_0(x,t) = \sqrt{\lambda_0 + \frac{1}{2}\beta_0}\; e^{\frac{1}{2}\beta_0(x+i\lambda_0 t)} - \sqrt{\lambda_0 - \frac{1}{2}\beta_0}\; e^{-\frac{1}{2}\beta_0(x+i\lambda_0 t)}, \\ q_0(x,t) = -\sqrt{\lambda_0 - \frac{1}{2}\beta_0}\; e^{\frac{1}{2}\beta_0(x+i\lambda_0 t)} + \sqrt{\lambda_0 + \frac{1}{2}\beta_0}\; e^{-\frac{1}{2}\beta_0(x+i\lambda_0 t)}. \end{array} \right.$$

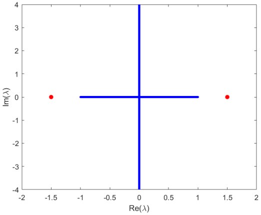
Then $\hat{u}_0$ is the Kuznetsov–Ma breather (the time-periodic soliton).

It is found from

$$\hat{\Phi}(\lambda) = D(\lambda)\Phi(\lambda), \quad D(\lambda) = I + \frac{1}{\lambda - \lambda_0} \left[\begin{array}{c} \hat{p}_0 \\ \hat{q}_0 \end{array} \right] [-q_0 \;\; p_0]$$

and the spectral analysis in $L^2(\mathbb{R})$ that

- The Lax spectrum consists of $i\mathbb{R} \cup [-1, 1]$ and $\{-\lambda_0, \lambda_0\}$ with $\lambda_0 > 1$.
- Eigenvalues at $\{-\lambda_0, \lambda_0\}$ are algebraically simple.
- Continuous spectrum at $i\mathbb{R} \cup [-1, 1]$ has same properties as for $u_0 = 1$.



Pair I	Pair II	Pair III
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$v = i\varphi_1^2 + i\bar{\varphi}_2^2$	$v = i\varphi_1 \phi_1 + i\bar{\varphi}_2 \bar{\phi}_2$	$v = i\phi_1^2 + i\bar{\phi}_2^2$

- Continuous spectrum gives what (Zweck *et al.*, 2020) wanted:

$$\sigma_{\rm ess}(M) = \sigma_{\rm ess}(e^{T\mathcal{L}_\infty})$$

- Eigenvalues at $\{-\lambda_0, \lambda_0\}$ give neutral modes

$$v = \frac{\partial \hat{u}_0}{\partial t}, \quad v = \frac{\partial \hat{u}_0}{\partial x}.$$

- Completeness of eigenfunctions of the linearized NLS equation ???

Case of Akhmediev breathers

Fix $\lambda_0 \in (0,1)$ and choose the solution $\varphi = (p_0, q_0)^T$ with $u_0 = 1$ in the form:

$$\begin{cases} p_0(x,t) = \sqrt{\lambda_0 - \frac{i}{2}k_0}\, e^{-\frac{1}{2}ik_0(x+i\lambda_0 t)} - \sqrt{\lambda_0 + \frac{i}{2}k_0}\, e^{\frac{1}{2}ik_0(x+i\lambda_0 t)}, \\ q_0(x,t) = -\sqrt{\lambda_0 + \frac{i}{2}k_0}\, e^{-\frac{1}{2}ik_0(x+i\lambda_0 t)} + \sqrt{\lambda_0 - \frac{i}{2}k_0}\, e^{\frac{1}{2}ik_0(x+i\lambda_0 t)}. \end{cases}$$

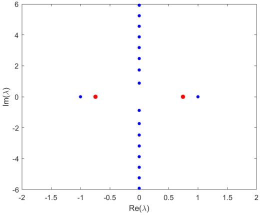
Then $\hat{u}_0$ is the Akhmediev breather (the space-periodic homoclinic orbit).

It is found from

$$\hat{\Phi}(\lambda) = D(\lambda)\Phi(\lambda), \quad D(\lambda) = I + \frac{1}{\lambda - \lambda_0} \begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix} [-q_0 \;\; p_0]$$

and the spectral analysis in $L^2_{\rm per}$ that

- Spectrum for $\hat{u}_0$ consists of the same double eigenvalues on $i\mathbb{R}$ as for u_0.
- Eigenvalues at $\pm\lambda_0$ are algebraically double and geometrically simple.
- Eigenvalues at ± 1 are algebraically simple.



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$v = i\varphi_1^2 + i\bar{\varphi}_2^2$	$v = i\varphi_1 \phi_1 + i\bar{\varphi}_2 \bar{\phi}_2$	$v = i\phi_1^2 + i\bar{\phi}_2^2$

- Double eigenvalues give four modes v_m^+, v_m^-, v_{-m}^+, v_{-m}^-.
- Eigenvalues at $\pm\lambda_0$ give neutral modes

$$v = \frac{\partial \hat{u}_0}{\partial t}, \quad v = \frac{\partial \hat{u}_0}{\partial x},$$

which become as $t \to \pm\infty$

$$v_1^-(x,t) = (2i\lambda_0 - k_0)\sin(k_0 x)e^{-\lambda_0 k_0 t}, \; v_{-1}^-(x,t) = (2i\lambda_0 - k_0)\cos(k_0 x)e^{-\lambda_0 k_0 t}.$$

- Simple eigenvalues at ± 1 give modes similar to $v_0 = 2i$ and $\tilde{v}_0 = 2it + 1$.
- Two modes are missing as $t \to \pm\infty$.

Missing modes - found

Expanding

$$\hat{\Phi}(\lambda) = D(\lambda)\Phi(\lambda), \quad D(\lambda) = I + \frac{1}{\lambda - \lambda_0}\begin{bmatrix} \hat{p}_0 \\ \hat{q}_0 \end{bmatrix}[-q_0 \;\; p_0]$$

gives

$$\hat{\Phi}(\lambda) = \frac{\hat{\Phi}_{-1}}{\lambda - \lambda_0} + \hat{\Phi}_0 + \hat{\Phi}_1(\lambda - \lambda_0) + \mathcal{O}((\lambda - \lambda_0)^2),$$

where $\hat{\Phi}_{-1} = [\varphi_0, 0]$, $\hat{\Phi}_0 = [\varphi_1, \phi_0]$, and $\hat{\Phi}_1 = [\varphi_2, \phi_1]$ with L-antiperiodic φ_0, φ_1 and x-growing φ_2, ϕ_0, ϕ_1. Combinations

$$\begin{aligned} v_+ &= \varphi_{0,1}\phi_{1,1} - \bar{\varphi}_{0,2}\bar{\phi}_{1,2} + \varphi_{1,1}\phi_{0,1} - \bar{\varphi}_{1,2}\bar{\phi}_{0,1}, \\ v_- &= i\varphi_{0,1}\phi_{1,1} + i\bar{\varphi}_{0,2}\bar{\phi}_{1,2} + i\varphi_{1,1}\phi_{0,1} + i\bar{\varphi}_{1,2}\bar{\phi}_{0,1}, \end{aligned}$$

give the two missing modes which become as $t \to \pm\infty$

$$v_1^+(x,t) = (2i\lambda_0 + k_0)\sin(k_0 x)e^{\lambda_0 k_0 t}, \;\; v_{-1}^+(x,t) = (2i\lambda_0 + k_0)\cos(k_0 x)e^{\lambda_0 k_0 t}.$$

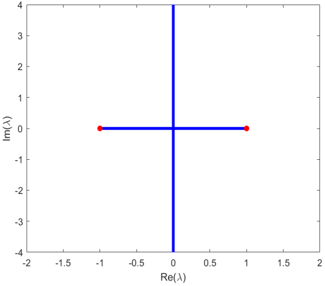
This confirms computations in (Grinevich, Santini, 2021). Completeness ???

Case of Peregrine's rogue wave

Choose the solution of Lax equations with $u_0 = 1$ and $\lambda = 1$:

$$\varphi = \left[\begin{array}{c} 1 \\ -1 \end{array} \right], \quad \phi = \left[\begin{array}{c} x + it + 1 \\ -x - it \end{array} \right],$$

Then $\hat{u}_0$ is the Peregrine's rogue wave, which is localized in (x, t) plane.



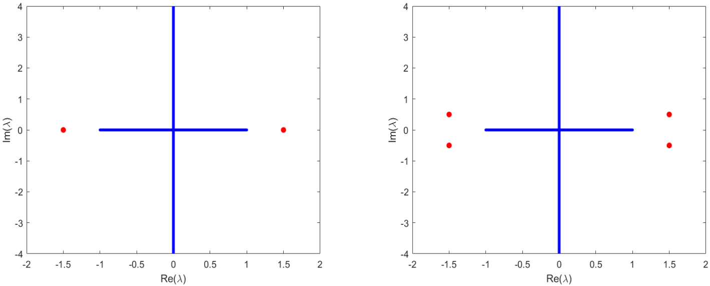
- Spectrum in $L^2(\mathbb{R})$ consists of $i\mathbb{R} \cup [-1, 1]$.
- Eigenvalues at $\{-1.1\}$ are embedded into the continuous spectrum.

Solutions of the linearized NLS equation only give instability of the background:

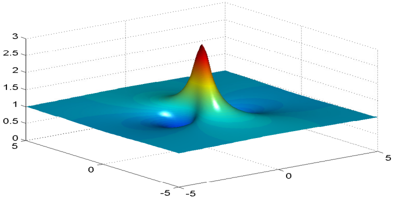
$$i v_t + \frac{1}{2} v_{xx} + (2|\hat{u}_0|^2 - 1) v + \hat{u}_0^2 \bar{v} = 0.$$

The embedded eigenvalue is structurally unstable under small perturbation (Klaus, Pelinovsky, Rothos, J. Nonlin.Sci., 2006):

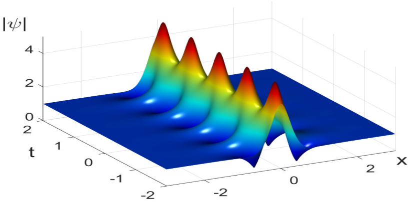
$$\int_{-\infty}^{\infty} \hat{u}_0(x,0) \left[\mathrm{Re}\, u(x,0) - \hat{u}_0(x,0) \right] dx$$

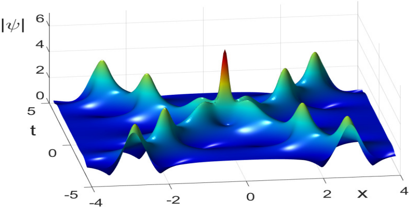


In addition to linear instability, Peregrine's rogue wave is structurally unstable



It transforms to either Kuznetsov–Ma breathers or to two counter-propagating Tajiri–Watanabi breathers (Zakharov, Gelash, Phys. Rev. Lett., 2013)





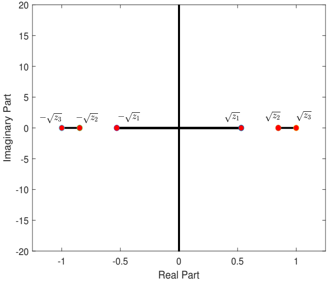
Double-periodic solutions

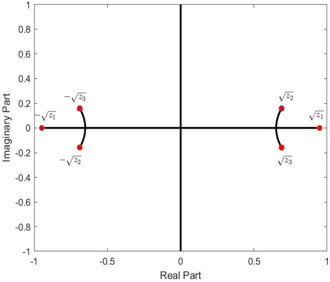
Recall two families of solutions

$$u(x,t) = [q(x,t) + i\delta(t)]\, e^{i\theta(t)}.$$

of the third-order Lax-Novikov equation:

$$u'''(x) + 6|u|^2 u'(x) + 4bu'(x) = 0.$$





Recall the Lax system

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda, u)\varphi, \qquad V(\lambda, u) = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda \bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{pmatrix}.$$

Since u is L-periodic in x and T-periodic in t, the Lax spectrum in λ is found from

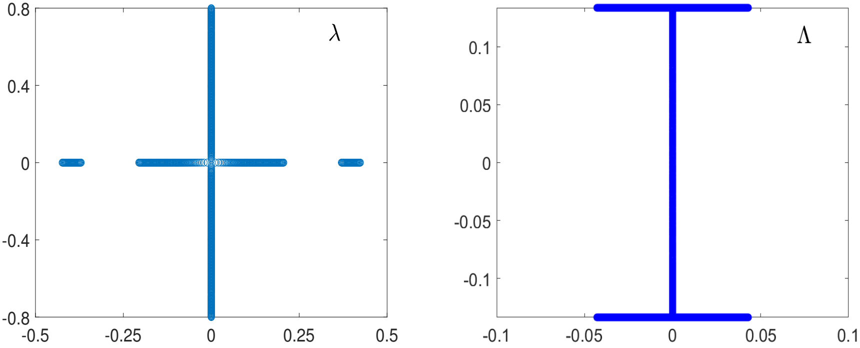
$$\varphi(x,t) = p(x,t)e^{ikx + t\Omega},$$

where $p(x+L,t) = p(x,t+T) = p(x,t)$, $k \in [-\frac{\pi}{L}, \frac{\pi}{L}]$, and $\Omega = \Omega(\lambda)$.

- Ω is uniquely determined in $\mathrm{Im}(\Omega) \in [-\frac{\pi}{T}, \frac{\pi}{T}]$.
- $2\mathrm{Re}(\Lambda)$ give the instability rate due to $v = \varphi_1^2$ solving the linearized NLS.

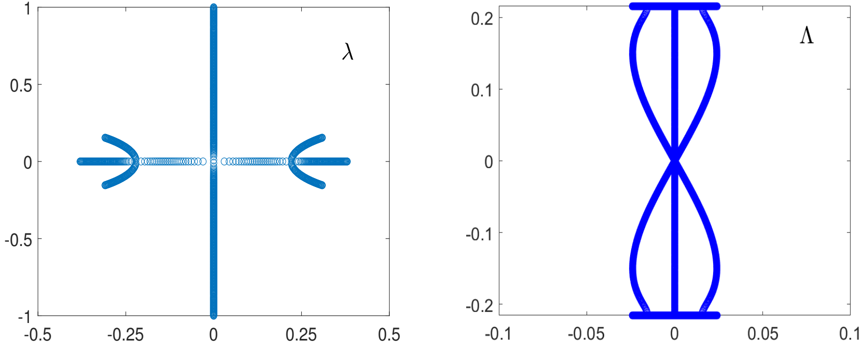
Is there an explicit formula for $\Omega(\lambda)$ from algebro-geometric construction???

Instabilities of one family



Left: Lax spectrum. Right: stability spectrum.

Instabilities of another family



Left: Lax spectrum. Right: stability spectrum.

Summary

- Breathers are particular cases of double-periodic solutions of the third-order Lax–Novikov equation.
- Breathers and their linear instabilities can be obtained by using Darboux transformations for both the NLS equation and the Lax linear system.
- Peregrine's rogue wave is structurally unstable in the time evolution.
- Double-periodic waves are also linearly unstable, their linear instability is computed from the Lax system numerically.