

Stability of breathers in nonlinear lattices

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Stability of relative equilibria in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J H'(u), \quad u(t) \in X$$

where $X \subset L^2$ is a phase space, $J^+ = -J$ is a bounded invertible operator for the symplectic structure, and $H : X \rightarrow \mathbb{R}$ is the Hamilton function.

- Assume existence of the stationary state $u_0 \in X$ such that $H'(u_0) = 0$.
- Perform linearization $u(t) = u_0 + ve^{\lambda t}$, where λ is the spectral parameter and $v \in X$ satisfies the spectral problem

$$JH''(u_0)v = \lambda v,$$

where $H''(u_0) : X \rightarrow L^2$ is a self-adjoint Hessian operator.

Spectral stability

Consider the spectral problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

Assumptions:

- The spectrum of $H''(u_0)$ is positive except for finitely many negative and zero eigenvalues of finite multiplicity.
- The continuous wave spectrum of $JH''(u_0)$ is purely imaginary.
- Multiplicity of the zero eigenvalue of $JH''(u_0)$ is given by the number of parameters in u_0 (symmetries).

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Example: discrete NLS equation

Consider the discrete nonlinear Schrödinger equation in 1D,

$$i \frac{du_n}{dt} = (\Delta u)_n + 2|u_n|^2 u_n, \quad n \in \mathbb{Z}.$$

The stationary state (discrete soliton) is

$$u(t) = U_\omega e^{-i\omega t}, \quad \omega > 0, \quad U_\omega \in \ell^2(\mathbb{Z}).$$

- U_ω is a critical point of $H_\omega(u) = H(u) + \omega Q(u)$,

$$H(u) = \sum_{n \in \mathbb{Z}} |u_{n+1} - u_n|^2 - |u_n|^4, \quad Q(u) = \sum_{n \in \mathbb{Z}} |u_n|^2.$$

- The self-adjoint Hessian operator $H''_\omega(U_\omega)$ is given by

$$H''_\omega(U_\omega) = \begin{bmatrix} -\Delta + \omega - 4|U_\omega|^2 & -2U_\omega^2 \\ -2\overline{U_\omega}^2 & -\Delta + \omega - 4|U_\omega|^2 \end{bmatrix}.$$

- $J = \text{diag}(i, -i)$ is a bounded invertible symplectic operator.

Main question

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

For a gradient system:

$$\frac{du}{dt} = -F'(u) \quad \Rightarrow \quad \lambda v = -F''(u_0)v,$$

Theorem

The number of unstable eigenvalues of $-F''(u_0)$ equals the number of negative eigenvalues of $F''(u_0)$.

The relation is less straightforward in a Hamiltonian system

$$\lambda v = JH''(u_0)v, \quad v \in X.$$

Quadruple Symmetry: If λ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2)$$

The quadratic form for H has **four positive** eigenvalues.

The two oscillators are **stable**:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = -\omega_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{cases}$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for H has **three positive** and **one negative** eigenvalues.

One of the two oscillators is **unstable**:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\text{re}}(JH) = 1 = N_{\text{neg}}(H)$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(-\lambda_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for H has **two positive** and **two negative** eigenvalues.

Both oscillators are **unstable**:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = \lambda_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 - \lambda_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\text{re}}(JH) = 2 = N_{\text{neg}}(H)$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \omega_2^2 x_2^2)$$

The quadratic form for H has **two positive** and **two negative** eigenvalues.

The two oscillators are nevertheless **stable**:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \omega_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$2N_{\text{im}}^-(JH) = 2 = N_{\text{neg}}(H)$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \omega^2 x_1 x_2$$

The quadratic form for H has **two positive** and **two negative** eigenvalues.

The two oscillators are **unstable** with a quartet of complex eigenvalues:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega^2 x_2, \\ \dot{y}_2 = -\omega^2 x_1, \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + \omega^2 x_2 = 0, \\ \ddot{x}_2 - \omega^2 x_1 = 0, \end{cases} \Rightarrow x_1^{(4)} + \omega^4 x_1 = 0.$$

Negative index count:

$$2N_c(JH) = 2 = N_{\text{neg}}(H)$$

Spectral stability theorems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

For simplicity, assume a zero-dimensional kernel of $H''(u_0)$.

- **Grillakis, Shatah, Strauss, 1990** **Orbital Stability Theory:**

- ▶ If $H''(u_0)$ has no negative eigenvalues, then $JH''(u_0)$ has no unstable eigenvalues and u_0 is nonlinearly stable.
- ▶ If $H''(u_0)$ has an odd number of negative eigenvalues, then $JH''(u_0)$ has at least one real unstable eigenvalue.

- **Kapitula, Kevrekidis, Sandstede, 2004; Pelinovsky, 2005**

Negative Index Theory:

$$N_{\text{re}}(JH''(u_0)) + 2N_{\text{c}}(JH''(u_0)) + 2N_{\text{im}}^-(JH''(u_0)) = N_{\text{neg}}(H''(u_0)).$$

What is Krein signature for eigenvalues?

- Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JH''(u_0)$ with the eigenvector v . Then, the sign of

$$E''(v) = \langle H''(u_0)v, v \rangle_{\ell^2}$$

is called the Krein signature of the eigenvalue λ .

- If λ is a multiple isolated eigenvalue of $JH''(u_0)$, then the number $N_{\text{im}}^-(JH''(u_0))$ is introduced as the number of negative eigenvalues of the quadratic form $E''(v)$ restricted at the invariant subspace of $JH''(u_0)$ associated with the eigenvalue λ .

What if $H''(u_0)$ has a kernel?

In the dNLS example

$$i \frac{du_n}{dt} = (\Delta u)_n + 2|u_n|^2 u_n, \quad n \in \mathbb{Z}.$$

with the discrete soliton

$$u(t) = U_\omega e^{-i\omega t}, \quad \omega > 0, \quad U_\omega \in \ell^2(\mathbb{Z}),$$

the kernel is one-dimensional:

$$H''_\omega(U_\omega) \begin{bmatrix} U_\omega \\ -U_\omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $H_\omega(u) = H(u) + \omega Q(u)$.

- Let $d(\omega) := H_\omega(U_\omega)$, then $d'(\omega) = Q(U_\omega) = \|U_\omega\|_{\ell^2}^2$.

If $d''(\omega) = \frac{d}{d\omega} Q(U_\omega) > 0$, the negative index theory applies with

$$N_{\text{neg}}(H''_\omega(U_\omega)) \rightarrow N_{\text{neg}}(H''_\omega(U_\omega)) - 1.$$

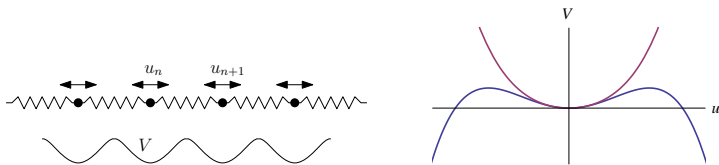
The soliton is nonlinearly stable if $d''(\omega) > 0$ and $N_{\text{neg}}(H''_\omega(U_\omega)) = 1$.

Klein-Gordon lattice

Klein-Gordon (KG) lattice models a chain of coupled anharmonic oscillators with nearest-neighbour interactions

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \epsilon(u_{n+1} - 2u_n + u_{n-1}),$$

where $\{u_n(t)\}_{n \in \mathbb{Z}} : \mathbb{R} \rightarrow \mathbb{R}^{\mathbb{Z}}$, ϵ is the coupling constant, and $V : \mathbb{R} \rightarrow \mathbb{R}$ is an on-site potential, e.g.



Applications:

- dislocations in crystals (e.g. Frenkel & Kontorova '1938)
- oscillations in biological molecules (e.g. Peyrard & Bishop '1989)

Relation to the discrete nonlinear Schrödinger equation

Discrete nonlinear Schrödinger equation (dNLS) corresponds to the small-amplitude weakly coupled limit of the KG lattice with $V'(u) = u \pm u^3$:

$$2i \frac{da_n}{d\tau} \pm 3|a_n|^2 a = a_{n+1} - 2a_n + a_{n-1},$$

where $\{a_n(\tau)\}_{n \in \mathbb{Z}} : \mathbb{R} \rightarrow \mathbb{C}^{\mathbb{Z}}$ and τ is new time variable.

By using the leading-order approximation

$$U_j(t) = \epsilon^{1/2} [a_j(\epsilon t)e^{it} + \bar{a}_j(\epsilon t)e^{-it}],$$

in dKG, one can obtain dNLS and estimate the residual terms

$$\text{Res}_j(t) := \pm \epsilon^{3/2} (a_j^3 e^{3it} + \bar{a}_j^3 e^{-3it}) + \epsilon^{5/2} (\ddot{a}_j e^{it} + \ddot{\bar{a}}_j e^{-it}),$$

For every $|t| \leq \tau_0 \epsilon^{-1}$, there is $C > 0$ such that

$$\|\mathbf{u}(t) - \mathbf{U}(t)\|_{l^2} + \|\dot{\mathbf{u}}(t) - \dot{\mathbf{U}}(t)\|_{l^2} \leq C\epsilon^{3/2}.$$

D.P., T. Penati, S. Paleari, Rev. Math. Phys. (2016), in press.

Relation to the anti-continuum limit

In the **anti-continuum limit** ($\epsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H_{per}^2(0, T)$.

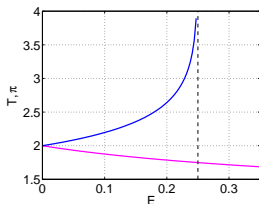


Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.

The period of the oscillator is

$$T(E) = \sqrt{2} \int_{-a(E)}^{a(E)} \frac{dx}{\sqrt{E - V(x)}},$$

where $a(E)$, the amplitude, is the smallest root of $V(a) = E$.

Multi-breathers near the anti-continuum limit

Breathers are spatially localized time-periodic solutions to the Klein-Gordon lattice. Multi-breathers are constructed by parameter continuation in ϵ from the limiting configuration:

$$\mathbf{u}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k \in H_{per}^2((0, T); l^2(\mathbb{Z})),$$

where $S \subset \mathbb{Z}$ is a finite set of excited sites and \mathbf{e}_k is the unit vector in $l^2(\mathbb{Z})$ at the node k . The oscillators are in-phase if $\sigma_k = +1$ and anti-phase if $\sigma_k = -1$.

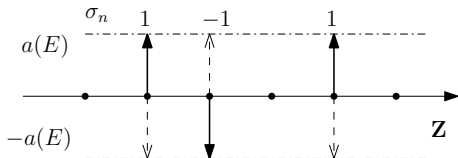


Figure: An example of a multi-site discrete breather at $\epsilon = 0$.

Spectral stability of breathers in the anti-continuum limit

- Archilla, Cuevas, Sánchez-Rey, Alvarez '2003
- Koukouloyannis, Kevrekidis '2009
- Pelinovsky, Sakovich '2012
- Youshimura '2012

Short summary of stability results near the anti-continuum limit:

- Single-site breather - spectrally stable
- Two-site breathers at two adjacent sites:
 - ▶ spectrally unstable if in-phase (soft) or anti-phase (hard)
 - ▶ spectrally stable if anti-phase (soft) or in-phase (hard)

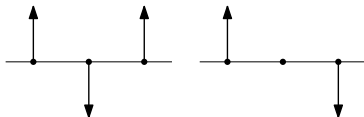
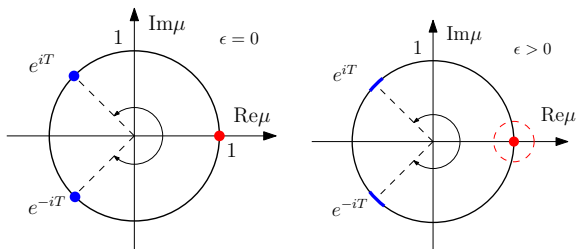


Figure: Stable configuration in soft potential: $T'(E) > 0$.

Spectral stability via Floquet multipliers

For $\epsilon > 0$, Floquet multipliers split as follows:



One-site breathers have a double Floquet multiplier at $\mu = 1$.

Question: Do they remain stable far from the anti-continuum limit?

Two-site breathers have one split pair of multipliers:

- the pair is on the unit circle if the breathers are spectrally stable
- the pair is on the real line if the breathers are unstable

Question: Are spectrally stable two-site breathers also nonlinearly stable?

Energy stability criterion for breathers

The KG lattice

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \epsilon(u_{n+1} - 2u_n + u_{n-1})$$

has the conserved energy

$$H = \sum_{n \in \mathbb{Z}} \frac{1}{2} \left(\frac{du_n}{dt} \right)^2 + V(u_n) + \frac{1}{2} \epsilon (u_{n+1} - u_n)^2.$$

Breathers (time-periodic solutions) are NOT relative equilibria of the energy function H . They can be written in the normalized form:

$$u(t) = U(\tau), \quad \tau = \omega t, \quad U(\tau + 2\pi) = U(\tau),$$

where $\omega = 2\pi/T$ is breather frequency and $U(\tau) \in H_{\text{per}}^2((0, 2\pi), \ell^2(\mathbb{Z}))$.

Breathers with increasing (decreasing) energy-frequency dependence are generically unstable in soft (hard) nonlinear potentials.

P.G. Kevrekidis, J. Cuevas, D.P., Phys. Rev. Lett. **117**, 094101 (2016).

A simple argument for energy stability criterion

Normalized breather profile $u(t) = U(\tau) \in H_{\text{per}}^2((0, 2\pi), \ell^2(\mathbb{Z}))$ satisfies

$$\omega^2 U_n''(\tau) + V'(U_n(\tau)) = \epsilon(\Delta U)_n(\tau), \quad n \in \mathbb{Z}.$$

Linearized equations for small perturbations are given by

$$\ddot{w}_n + V''(u_n)w_n = \epsilon(\Delta w)_n, \quad n \in \mathbb{Z}. \quad (1)$$

With Floquet theory,

$$w(t) = W(\tau)e^{\lambda t}, \quad \tau = \omega t, \quad W(\tau + 2\pi) = W(\tau),$$

we obtain the spectral stability problem

$$(LW)(\tau) = 2\lambda\omega W'(\tau) + \lambda^2 W(\tau),$$

where $L = \epsilon\Delta - V''(U(\tau)) - \omega^2\partial_\tau^2$ acts on $H_{\text{per}}^2((0, 2\pi), \ell^2(\mathbb{Z}))$.

A simple argument for energy stability criterion

Spectral stability problem is

$$(LW)(\tau) = 2\lambda\omega W'(\tau) + \lambda^2 W(\tau).$$

$\lambda = 0$ is at least a double eigenvalue because of the translational invariance:

$$LU'(\tau) = 0, \quad L\partial_\omega U(\tau) = 2\omega U''(\tau).$$

$\lambda = 0$ is at least a quadruple eigenvalue if $TH'(\omega) = 0$.

Assumptions:

- The spectral bands of the spectral stability problem are bounded away from $\lambda = 0$,
- The kernel of L is exactly one-dimensional with the eigenvector $W(\tau) = U'(\tau)$.
- The energy H of the breather U is a C^1 function of frequency ω .

Energy stability criterion in the anti-continuum limit

In the **anti-continuum limit** ($\epsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H_{per}^2(0, T)$.

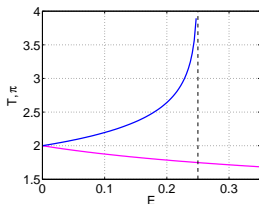


Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.

Since $|T'(E)| < \infty$ in

$$H'(\omega) = -\frac{T}{\omega T'(E)},$$

the stability threshold $H'(\omega) = 0$ cannot be achieved.

Oscillators are always stable with $H'(\omega) > 0$ for hard potentials and $H'(\omega) < 0$ for soft potentials.

Further arguments for energy stability criterion

Expanding in powers of λ :

$$W(\tau) = U'(\tau) + \lambda \partial_\omega U(\tau) + \lambda^2 Y(\tau) + \lambda^3 Z(\tau) + \mathcal{O}(\lambda^4)$$

and using Fredholm conditions yields the dispersion relation

$$0 = \lambda^2 TH'(\omega) + \lambda^4 M(\omega) + \mathcal{O}(\lambda^6),$$

where $M(\omega)$ is computed in terms of U and Y .

The sign of $M(\omega)$ is not generally defined...

However, in the dNLS approximation limit, we can show that
 $M(\omega) > 0$ for hard potentials [breathers are stable for $H'(\omega) > 0$];
 $M(\omega) < 0$ for soft potentials [breathers are stable for $H'(\omega) < 0$].

Energy stability criterion in the dNLS approximation

Consider the KG lattice

$$\frac{d^2 u_n}{dt^2} + u_n \pm \epsilon u_n^{1+2p} = \epsilon(u_{n+1} - 2u_n + u_{n-1}), \quad n \in \mathbb{Z}.$$

By using the leading-order approximation (P., Penati, Paleari, 2016),

$$U_n(\tau) = A_n e^{i\tau} + \bar{A}_n e^{-i\tau} + \mathcal{O}(\epsilon),$$

one can derive and justify the stationary dNLS equation

$$(\Delta A)_n = \epsilon^{-1}(1 - \omega^2)A_n \pm \gamma |A_n|^{2p} A_n, \quad \gamma = \frac{(2p+1)!}{p!(p+1)!}.$$

Energy stability criterion in the dNLS approximation

Consider the KG lattice

$$\frac{d^2 u_n}{dt^2} + u_n \pm \epsilon u_n^{1+2p} = \epsilon(u_{n+1} - 2u_n + u_{n-1}), \quad n \in \mathbb{Z}.$$

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$$(\Delta A)_n = \epsilon^{-1}(1 - \omega^2)A_n \pm \gamma |A_n|^{2p} A_n, \quad \gamma = \frac{(2p+1)!}{p!(p+1)!}.$$

Breathers exist for hard potentials if $\omega^2 > 1 + 4\epsilon$ and for soft potentials if $\omega^2 < 1$. Hence, we can introduce $\Omega > 0$ in either $\omega^2 = 1 + 4\epsilon + \epsilon\Omega$ or $\omega^2 = 1 - \epsilon\Omega$. Then, $A \in \ell^2(\mathbb{Z})$ depends on Ω and is independent of ϵ .

$$H(\omega) = 2Q(\Omega) + \mathcal{O}(\epsilon).$$

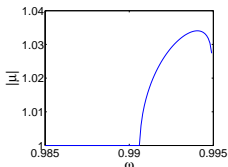
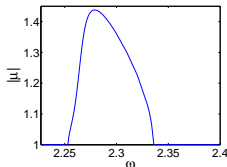
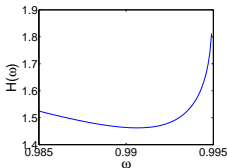
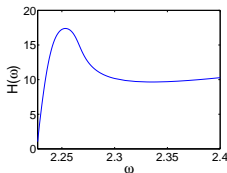
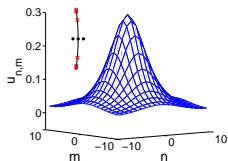
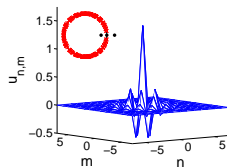
The energy stability criterion becomes the slope condition:

$$H'(\omega) = \pm 4\omega\epsilon^{-1}Q'(\Omega) + \mathcal{O}(1), \quad Q(\Omega) = \|A\|_{\ell^2}^2.$$

Numerical illustration: 2D KG lattice.

Left - hard ϕ^4 potential with $\epsilon = 0.5$.

Right - soft Morse potential with $\epsilon = 0.2$.



Energy stability criterion for FPU lattices?

The FPU lattice

$$\frac{d^2 u_n}{dt^2} = W'(u_{n+1} - u_n) - W'(u_n - u_{n-1}), \quad n \in \mathbb{Z},$$

has the conserved energy

$$H = \sum_{n \in \mathbb{Z}} \frac{1}{2} \left(\frac{du_n}{dt} \right)^2 + W(u_{n+1} - u_n).$$

In the strain variables $r_n = u_{n+1} - u_n$, the FPU lattice can be rewritten as

$$\frac{d^2 r_n}{dt^2} = W'(r_{n+1}) - 2W'(r_n) + W'(r_{n-1}), \quad n \in \mathbb{Z},$$

and the normalized breather profile $r_n(t) = R_n(\tau) \in H_{\text{per}}^2((0, 2\pi), \ell^2(\mathbb{Z}))$.

The derivations and conclusions apply verbatim... In monoatomic chains, the dNLS approximation is valid at the maximal optical frequency and leads to breathers in hard potentials (G.James, 2003).

Nonlinear instability of breathers

Consider the discrete KG equation

$$\frac{d^2 u_n}{dt^2} + V'(u_n) = \varepsilon(u_{n+1} - 2u_n + u_{n-1}), \quad n \in \mathbb{Z},$$

where V is smooth and $V = \frac{1}{2}u^2 + \mathcal{O}(u^3)$.

Assumptions:

- The double eigenvalue $\lambda = 0$ is isolated from the spectral bands.
- There exists a pair of eigenvalues at $\lambda = \pm i\Omega$ isolated from the spectral bands.
- The double eigenvalue $\lambda = \pm 2i\Omega$ belongs to the spectral bands with nonzero Fermi golden rule.

If Krein signature of eigenvalues at $\lambda = \pm i\Omega$ is opposite to that of the spectral bands, the breather is spectrally stable but nonlinearly unstable.

P.G. Kevrekidis, D.P., A. Saxena, Phys. Rev. Lett. **114** (2015), 214101.

J. Cuevas, P.G. Kevrekidis, D.P., Stud. Appl. Math. **137** (2016), 214.

Krein quantity

Linearized equations for small perturbations are given by

$$\ddot{w}_n + V''(u_n)w_n = \epsilon(\Delta w)_n, \quad n \in \mathbb{Z}. \quad (2)$$

With Floquet theory,

$$w(t) = W(t)e^{\lambda t}, \quad W(t+T) = W(t),$$

we obtain the spectral stability problem

$$\ddot{W}_n + 2\lambda\dot{W}_n + \lambda^2 W_n + V''(u_n)W_n = \epsilon(\Delta w)_n, \quad n \in \mathbb{Z}.$$

The symplectic structure is given by

$$\frac{dw_n}{dt} = \frac{\partial H}{\partial p_n}, \quad \frac{dp_n}{dt} = -\frac{\partial H}{\partial w_n}, \quad n \in \mathbb{Z}$$

The Krein quantity K is real and constant in time t :

$$K = i \sum_{n \in \mathbb{Z}} (\bar{p}_n w_n - p_n \bar{w}_n) = 2\Omega \sum_{n \in \mathbb{Z}} |W_n|^2 + i \sum_{n \in \mathbb{Z}} (\dot{W}_n W_n - \dot{W}_n \bar{W}_n).$$

Krein quantity for two-site breathers

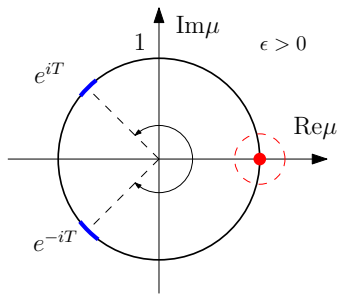
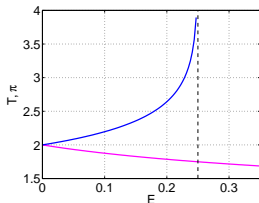
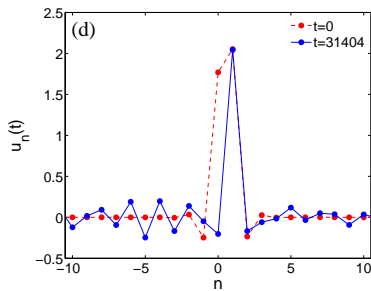
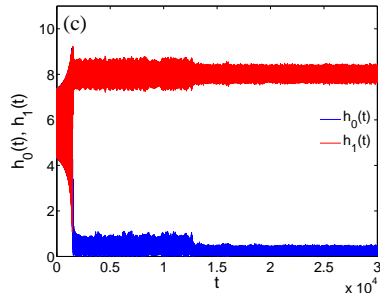
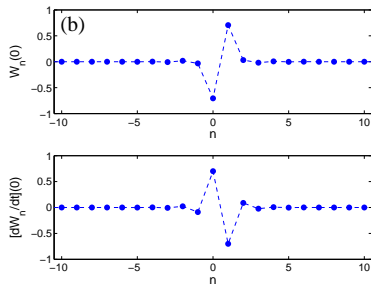
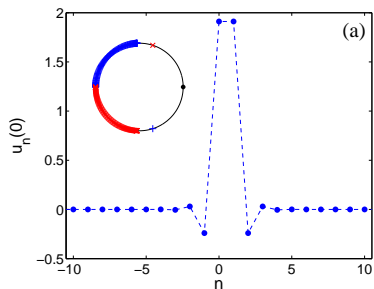


Figure: Period vs. energy in hard (magenta) and soft (blue) potential $V(u) = \frac{1}{2}u^2 \pm \frac{1}{4}u^4$.

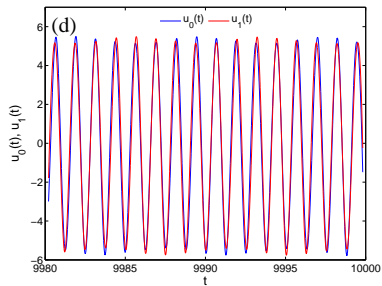
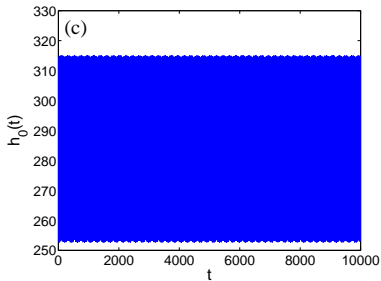
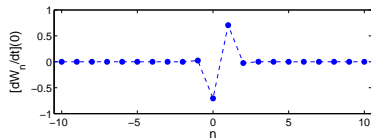
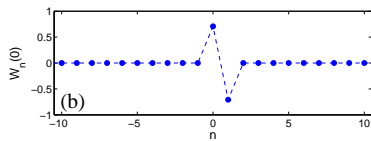
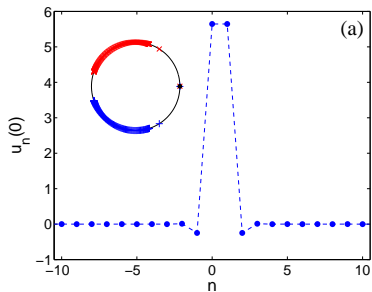
For the hard potential with $T'(E) < 0$ and $T(E) < 2\pi$,

- $0 < T < \pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle coincide;
- $\pi \leq T < 2\pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle are opposite to each other.

Numerical illustration: hard ϕ^4 potential $T = \pi$



Numerical illustration: hard ϕ^4 potential $T < \pi$



Numerical illustration in 1D

The dNLS equation

$$i\dot{u}_n + \epsilon(u_{n+1} - 2u_n + u_{n-1}) + |u_n|^2 u_n = 0, \quad n \in \mathbb{Z}.$$

For $\epsilon = 0.07$ and $\omega = 1$, we have $\Omega \approx 0.598$, so that $\Omega < \omega$ but $2\Omega > \omega$.

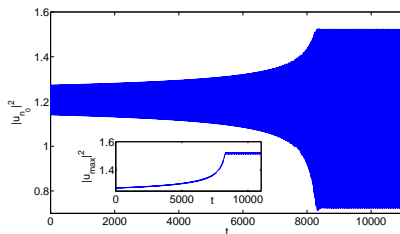


Figure: Evolution of a two-site discrete soliton in 1D dNLS.

Recall the **Negative Index Theory**:

$$N_{\text{re}}(JH''(u_0)) + 2N_{\text{c}}(JH''(u_0)) + 2N_{\text{im}}^-(JH''(u_0)) = N_{\text{neg}}(H''(u_0)) = 2$$

A simple argument for nonlinear instability

Using the asymptotic multi-scale expansion for solutions to the KG lattice,

$$u(t) = U(t) + \delta \left(c(\tau)W(t)e^{i\Omega t} + \bar{c}(\tau)\bar{W}(t)e^{-i\Omega t} \right) + \delta^2 Y(t) + \mathcal{O}(\delta^3),$$

yields

- the breather $U(t + T) = U(t)$,
- the Floquet mode $W(t + T) = W(t)$ for eigenvalues $\lambda = \pm i\Omega$,
- the slowly varying envelope $c(\tau)$, $\tau = \delta^2 t$,
- the correction terms at $\mathcal{O}(\delta^2)$,

$$Y(t) = c(\tau)^2 P(t)e^{2i\Omega t} + |c(\tau)|^2 Q(t) + \bar{c}(\tau)^2 \bar{P}(t)e^{-2i\Omega t},$$

where $P(t) \in H_{\text{per}}^2((0, T), \ell^\infty(\mathbb{Z}))$ and $Q(t) \in H_{\text{per}}^2((0, T), \ell^2(\mathbb{Z}))$ from the assumptions of the theory.

The correction term $P(t)$ satisfies Sommerfeld radiation boundary conditions at infinity due to coupling with the spectral bands.

A simple argument for nonlinear instability

Removing secular terms at $\mathcal{O}(\delta^3)$ yields the amplitude equation

$$iK \frac{dc}{d\tau} + \beta |c|^2 c = 0,$$

where $K \in \mathbb{R}$ is the Krein quantity of the eigenvalue $\lambda = i\Omega$ and $\text{Im}(\beta)$ encodes Sommerfeld conditions. By the Fermi Golden Rule, $\text{Im}(\beta) \neq 0$.

For the hard potential with $T'(E) < 0$ and $T(E) < 2\pi$,

- $K > 0$ for eigenvalue $\lambda = i\Omega$;
- $\text{Im}(\beta) > 0$ if $0 < T < \pi$ and $\text{Im}(\beta) < 0$ if $\pi \leq T < 2\pi$.

If $\text{sign}(K) = -\text{sign}(\text{Im}(\beta))$, then $|c|^2$ grows in τ ,

$$K \frac{d|c|^2}{d\tau} = -2\text{Im}(\beta)|c|^4,$$

hence, the breather is **nonlinearly unstable**.

For NLS-type models, see S. Cuccagna (2009); M. Maeda (2014).

Conclusions

- Spectral stability theory is well-developed for relative equilibria in Hamiltonian systems.
- Negative eigenvalues of the quadratic Hamiltonian show up in the spectral stability problem either as unstable eigenvalues or as stable eigenvalues of negative Krein signature.
- If no negative eigenvalues exist, nonlinear stability holds by Lyapunov method. In the presence of negative eigenvalues, nonlinear instability may destroy stationary states in spite of their spectral stability.

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- If no negative eigenvalues exist, nonlinear stability holds by Lyapunov method. In the presence of negative eigenvalues, nonlinear instability may destroy stationary states in spite of their spectral stability.
- Breathers are not relative equilibria of the Hamiltonian system. The generalization of the above results to breathers is not trivial.
- Energy stability criterion is presented for breathers for the first time.
- We have also shown that spectrally stable multi-site breathers may be either nonlinearly stable or unstable, depending on their period T .