Justification of coupled-mode equations for optical lattices

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Motivations

Gap solitons are localized stationary solutions of nonlinear PDEs with space-periodic coefficients which reside in the spectral gaps of associated linear operators.

Examples: Complex-valued Maxwell equation

$$\nabla^2 E - E_{tt} + \left(V(x) + \sigma|E|^2\right) E_{tt} = 0$$

and the Gross–Pitaevskii equation

$$iE_t = -\nabla^2 E + V(x)E + \sigma|E|^2 E,$$

where $E(x,t) : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{C}$, $V(x) = V(x + 2\pi e_j) : \mathbb{R}^N \mapsto \mathbb{R}$, and $\sigma = \pm 1$.

Existence of stationary solutions

Stationary solutions $E(x,t) = U(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfies a nonlinear elliptic problem with a periodic potential

$$\nabla^2 U + \omega U = V(x)U + \sigma |U|^2 U$$

Theorem:[Pankov, 2005] Let $V(x)$ be a real-valued bounded periodic potential. Let ω be in a finite gap of the spectrum of $L = -\nabla^2 + V(x)$. There exists a non-trivial weak solution $U(x) \in H^1(\mathbb{R}^N)$, which is (i) real-valued, (ii) continuous on $x \in \mathbb{R}^N$ and (iii) decays exponentially as $|x| \to \infty$.

Remark: Additionally, there exists a localized solution $U(x) \in H^1(\mathbb{R}^N)$ in the semi-infinite gap for $\sigma = -1$ (NLS soliton).

Asymptotic reductions

The nonlinear elliptic problem with a periodic potential can be reduced asymptotically to the following problems:

- Coupled-mode (Dirac) equations for small potentials

$$\begin{cases} ia'(x) + \Omega a + \alpha b = \sigma(|a|^2 + 2|b|^2)a \\ -ib'(x) + \Omega b + \alpha a = \sigma(2|a|^2 + |b|^2)b \end{cases}$$

- Envelope (NLS) equations for finite potentials near band edges

$$a''(x) + \Omega a + \sigma |a|^2 a = 0$$

- Lattice (dNLS) equations for large or long-period potentials

$$\alpha \left(a_{n+1} + a_{n-1}\right) + \Omega a_n + \sigma |a_n|^2 a_n = 0.$$

Localized solutions of reduced equations exist in the analytic form.

Full versus asymptotic solutions

Main Question: Can we justify the use of the three approximations to classify localized solutions for $U(x)$?

Remark: We avoid consideration of time-dependent problems. For justification of Dirac and NLS equations on a finite time interval, see Schneider-Uecker (2001) and Busch *et al.* (2006).

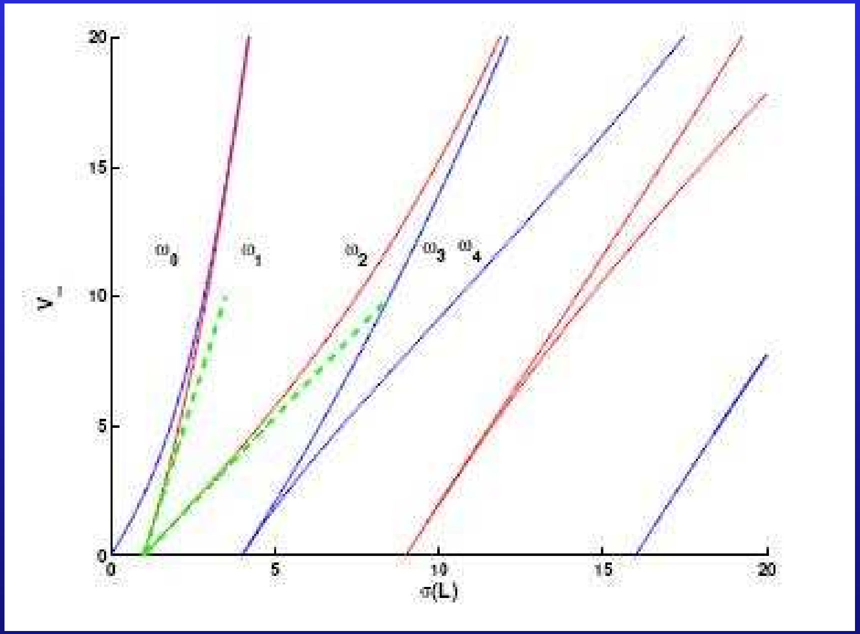
Theorem:[Goodman,Weinstein,Holmes, 2001] Let $(a,b) \in C([0,T_0], H^3(\mathbb{R},\mathbb{C}^2))$ be solutions of the time-dependent coupled-mode system for a fixed $T_0 > 0$. There exists $\epsilon_0, C > 0$ such that for all $\epsilon \in (0,\epsilon_0)$ the Gross–Pitaevskii equation has a local solution $E(x,t)$ and

$$\|E(x,t) - \sqrt{\epsilon}\left[a(\epsilon x,\epsilon t)e^{i(kx-\omega t)} + b(\epsilon x,\epsilon t)e^{i(-kx-\omega t)}\right]\|_{H^1(\mathbb{R})} \leq C\epsilon$$

for some (k,ω) and any $t \in [0, T_0/\epsilon]$.

Formal coupled-mode theory

Let $N = 1$ and $V(x) = \epsilon(1 - \cos x)$. The finite-band spectrum of $L = -\partial_x^2 + V(x)$ is



Asymptotic multi-scale expansion:

$$U(x) = \sqrt{\epsilon}\left[a(\epsilon x)e^{\frac{ix}{2}} + b(\epsilon x)e^{-\frac{ix}{2}} + \mathrm{O}(\epsilon)\right], \quad \omega = \frac{1}{4} + \epsilon\Omega + \mathrm{O}(\epsilon^2)$$

Gap solitons in coupled-mode equations

The vector $(a,b) : \mathbb{R} \mapsto \mathbb{C}^2$ satisfies asymptotically the coupled-mode system with parameter $\Omega \in \mathbb{R}$:

$$\left\{ \begin{array}{r} ia' + \Omega a + V_2 b = \sigma(|a|^2 + 2|b|^2)a, \\ -ib' + \Omega b + V_{-2} a = \sigma(2|a|^2 + |b|^2)b, \end{array} \right.$$

where $V_2 = \bar{V}_{-2}$ are Fourier coefficients of $V(x)$ and derivatives are taken with respect to $y = \epsilon x$. Gap solitons of the coupled-mode system are obtained in the explicit analytic form, e.g. for $\sigma = 1$,

$$a(y) = \bar{b}(y) = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|V_2|^2 - \Omega^2}}{\sqrt{|V_2| - \Omega}\cosh(\kappa y) + i\sqrt{|V_2| + \Omega}\sinh(\kappa y)},$$

where $\kappa = \sqrt{|V_2|^2 - \Omega^2}$ and $|\Omega| < |V_2|$.

Definitions for the main theorem

Assumption: Let $V(x)$ be a smooth 2π-periodic real-valued function with zero mean and symmetry $V(x) = V(-x)$ on $x \in \mathbb{R}$, such that

$$V(x) = \sum_{m \in \mathbb{Z}} V_{2m} e^{imx} : \quad \sum_{m \in \mathbb{Z}} (1 + m^2)^s |V_{2m}|^2 < \infty,$$

for some $s \geq 0$, where $V_0 = 0$ and $V_{2m} = V_{-2m} = \bar{V}_{-2m}$.

Definition: The gap soliton of the coupled-mode system is said to be a reversible non-degenerate homoclinic orbit if $a(y) = \bar{a}(-y) = \bar{b}(y)$ and $a(y)$ decays to zero as $|y| \to \infty$ exponentially fast.

Remark: If $V(x) = V(-x)$ and $U(x)$ is a solution of the nonlinear elliptic problem, then $U(-x)$ is also a solution.

Spaces for the main theorem

Let $U(x)$ be represented by the Fourier transform

$$U(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{U}(k) e^{ikx} dk, \qquad \hat{U}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} U(x) e^{-ikx} dx,$$

in the vector space

$$\hat{U} \in L^1_q(\mathbb{R}) : \quad \|\hat{U}\|_{L^1_q(\mathbb{R})} = \int_{\mathbb{R}} (1+k^2)^{q/2} |\hat{U}(k)| dk < \infty.$$

By the Riemann–Lebesque Lemma, if $\hat{U} \in L^1(\mathbb{R})$, then $U(x)$ decays to zero at infinity as $|x| \to \infty$ and $U(x)$ is n-times continuously differentiable on $x \in \mathbb{R}$ for $0 \leq n \leq [q]$.

Moreover, since $\|\hat{U}\|_{L^2_q} \leq \|\hat{U}\|_{L^1_q}$, then $U \in H^q(\mathbb{R})$.

Main Theorem in 1D

Theorem: Let $V(x)$ satisfy the assumption and $V_{2n} \neq 0$ for a fixed $n \in \mathbb{N}$. Let $\omega = \frac{n^2}{4} + \epsilon\Omega$ with $|\Omega| < |V_{2n}|$. Let (a, b) be a reversible homoclinic orbit of the coupled-mode system. Then, there exists $\epsilon_0, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the nonlinear elliptic problem has a non-trivial solution $U(x)$ and

$$\|U(x) - \sqrt{\epsilon}\left[a(\epsilon x)e^{\frac{inx}{2}} + b(\epsilon x)e^{-\frac{inx}{2}}\right]\|_{H^q(\mathbb{R})} \leq C\epsilon^{5/6},$$

for any $q \geq 0$. Moreover, the solution $U(x)$ is real-valued, continuous on $x \in \mathbb{R}$, and $\lim\limits_{|x|\to\infty} U(x) = 0$.

Remarks: 1) We do not prove that $U(x)$ decays exponentially at infinity. 2) The power of $\epsilon^{5/6}$ can be extended to any ϵ^p for $\frac{1}{2} < p < 1$.

Steps of the proof

1. Convert the problem to the integral equation

$$\left(\omega - k^2\right)\hat{U}(k) = \epsilon \sum_{m\in\mathbb{Z}} V_{2m}\hat{U}(k-m)$$

$$+\epsilon\sigma \int\int \hat{U}(k_1)\hat{\bar{U}}(k_2)\hat{U}(k-k_1+k_2)dk_1dk_2$$

2. If $\mathbf{V} \in l^2_{s+q}(\mathbb{Z})$ for any $s > \frac{1}{2}$ and $q \geq 0$, then the vector field of the integral equation is closed in $L^1_q(\mathbb{R})$ such that

$$\left\|\int_{\mathbb{R}} \hat{U}(k_1)\hat{W}(k-k_1)dk_1\right\|_{L^1_q(\mathbb{R})} \leq \|\hat{U}\|_{L^1_q(\mathbb{R})}\|\hat{W}\|_{L^1_q(\mathbb{R})}$$

$$\left\|\sum_{m\in\mathbb{Z}} V_{2m}\hat{U}(k-m)\right\|_{L^1_q(\mathbb{R})} \leq \|\hat{U}\|_{L^1_q(\mathbb{R})}\|\mathbf{V}\|_{l^2_{s+q}(\mathbb{Z})}.$$

Steps of the proof

3. Decompose the solution $\hat{U}(k)$ into three parts

$$\hat{U}(k) = \hat{U}_+(k)\chi_{\mathbb{R}'_+}(k) + \hat{U}_-(k)\chi_{\mathbb{R}'_-}(k) + \hat{U}_0(k)\chi_{\mathbb{R}'_0}(k)$$

with a compact support on

$$\mathbb{R}'_\pm = \left[\pm n/2 - \epsilon^{2/3}, \pm n/2 + \epsilon^{2/3}\right], \quad \mathbb{R}'_0 = \mathbb{R}\backslash(\mathbb{R}'_+ \cup \mathbb{R}'_-),$$

where $\inf_{k\in\mathbb{R}'_0} |n^2/4 - k^2| \geq C\epsilon^{2/3}$.

4. There exists a unique map $\hat{U}_\epsilon : L^1_q(\mathbb{R}'_+) \times L^1_q(\mathbb{R}'_-) \mapsto L^1_q(\mathbb{R}'_0)$ such that $\hat{U}_0(k) = \hat{U}_\epsilon(\hat{U}_+, \hat{U}_-)$ and

$$\forall|\epsilon| < \epsilon_0 : \quad \|\hat{U}_0(k)\|_{L^1_q(\mathbb{R}'_0)} \leq \epsilon^{1/3} C \left(\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)} + \|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)} \right).$$

Steps of the proof

5. Write projections to the new amplitudes for the singular part

$$\hat{U}_+(k) = \frac{1}{\epsilon}\hat{A}\left(\frac{k-n/2}{\epsilon}\right), \quad \hat{U}_-(k) = \frac{1}{\epsilon}\hat{B}\left(\frac{k+n/2}{\epsilon}\right),$$

where $\hat{A}(p)$, $\hat{B}(p)$ are defined on $p \in \mathbb{R}_0 = [-\epsilon^{-1/3}, \epsilon^{-1/3}]$ and

$$\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)} \leq C\|\hat{A}\|_{L^1_q(\mathbb{R}_0)}, \quad \|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)} \leq C\|\hat{B}\|_{L^1_q(\mathbb{R}_0)}.$$

6. Prove persistence of gap soliton solutions in the coupled-mode system on $p \in \mathbb{R}_0$, e.g.

$$(\Omega - np)\,\hat{A}(p) + V_{2n}\hat{B}(p) - \sigma\text{Conv.Int.}$$
$$= \epsilon p^2\hat{A}(p) + \epsilon^{1/3}\hat{R}_a(\hat{A}, \hat{B}, \hat{U}_\epsilon(\hat{A}, \hat{B})).$$

Steps of the proof

7. Analyze the reminder terms, e.g.

$$\|\hat{R}_a\|_{L^1_q(\mathbb{R}_0)} \leq C_a \|\hat{A}\|_{L^1_q(\mathbb{R}_0)}, \quad \epsilon \|p^2 \hat{A}(p)\|_{L^1_q(\mathbb{R}_0)} \leq \epsilon^{1/3} \|\hat{A}(p)\|_{L^1_q(\mathbb{R}_0)},$$

8. Solve the system $\hat{\mathbf{N}}(\hat{\mathbf{A}}) = \hat{\mathbf{R}}(\hat{\mathbf{A}})$ for $\hat{\tilde{A}} = \hat{A} - \hat{a}$ by fixed-point iterations

$$\hat{L}\hat{\tilde{\mathbf{A}}} = \hat{\mathbf{R}}(\hat{\mathbf{a}} + \hat{\tilde{\mathbf{A}}}) - \left[\hat{\mathbf{N}}(\hat{\mathbf{a}} + \hat{\tilde{\mathbf{A}}}) - \hat{\mathbf{L}}\hat{\tilde{\mathbf{A}}}\right], \quad \hat{\mathbf{L}} = \mathbf{D}_{\hat{\mathbf{a}}}\hat{\mathbf{N}}(\hat{\mathbf{a}}),$$

where $\hat{L}$ is a linearized operator for the coupled-mode system.

9. Analyze the truncation terms, e.g.

$$\|\hat{A} - \hat{a}\|_{L^1_{q+1}(\mathbb{R}\backslash\mathbb{R}_0)} \leq \|\hat{A} - \hat{a}\|_{L^1_{q+1}(\mathbb{R})} \leq \epsilon^{1/3} C \|\hat{R}_a\|_{L^1_q(\mathbb{R})}.$$

Remarks

1. The method of the proof does not work in $N \geq 2$ since $|k|^2 - \omega$ is not invertible on the sphere of radius $|k| = \sqrt{\omega}$ while resonances occur in a finite number of points on $|k| = \sqrt{\omega}$.

2. Persistence of y-independent solutions of the coupled-mode system is proved with a simple application of Lyapunov–Schmidt reductions.

Theorem: The nonlinear elliptic problem has a non-trivial 2π-periodic (or 2π-antiperiodic) solution $U(x)$ in $H^s_{\rm per}(\mathbb{R})$ for any $s > \frac{1}{2}$ and sufficiently small ϵ if and only if there exists a non-trivial solution for $(a,b) \in \mathbb{C}^2$ of the y-independent coupled-mode system. In particular, there exists $\epsilon_0, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$

$$\|U(x) - \sqrt{\epsilon}\left[a e^{\frac{inx}{2}} + b e^{-\frac{inx}{2}}\right]\|_{H^s_{\rm per}(\mathbb{R})} \leq C \epsilon^{3/2}.$$

Extensions

- We have justified approximations of gap solitons by the coupled-mode equations for small one-dimensional potentials.
- Coupled-mode equations in two dimensions lead to coupled NLS equations, which are generalizations of the coupled NLS equations derived near band edges.
- Approximations of gap solitons in the coupled NLS equations near band edges can be justified using the Fourier–Bloch analysis.
- Similarly, we can justify approximations of gap solitons in the discrete NLS (lattice) equations for large potentials.
- The last two results remain valid in one, two, and three dimensions for a class of separable bounded periodic potentials.