Ground state of the conformal flow on $\mathbb{S}^3$

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Outline of the talk

1. Overview of resonant normal forms
2. Resonant normal flow for conformal flow
3. Stationary states for conformal flow
4. Spectral stability of the ground state
5. Orbital stability of the ground state
6. Conclusion

Resonant normal forms

In many infinite-dimensional Hamiltonian systems with spatial confinement,

- The system can be written in canonical coordinates;
- The resonant energy transfer can be isolated from the rest.

If the resonant energy transfer also involves infinitely many modes, this reductive technique leads to the infinite-dimensional resonant normal form.

Hamiltonian systems with ∞ degrees of freedom

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

Resonant normal with ∞ canonical coordinates

Old examples include:

- Weak turbulence of quasi-periodic water waves (V. Zakharov, 1968)
- Bragg resonance in the $1D$ wave equation with a periodic potential (G. Simpson, M. Weinstein, 2013).

New example 1. Rotating Bose–Einstein condensates

The Gross–Pitaevskii equation with a harmonic potential in 2D:

$$i\partial_t \psi = -\Delta\psi + |x|^2\psi + |\psi|^2\psi - i\Omega\partial_\theta\psi,$$

where $x \in \mathbb{R}^2$, θ is an angle in the polar coordinates, and Ω is the angular frequency of rotation. The associated energy

$$E(\psi) = \int\int_{\mathbb{R}^2} \left[|\nabla\psi|^2 + |x|^2|\psi|^2 + \frac{1}{2}|\psi|^4 - i\Omega\psi\partial_\theta\bar{\psi} \right] dx.$$

Steadily rotating states are critical point of E subject to the fixed mass $Q(\psi) = \|\psi\|^2_{L^2}$.

- If $\Omega = 0$, the ground state of E is sign-definite (Thomas–Fermi cloud).
- When Ω increases, the ground state of E becomes a vortex of charge one, a pair of two vortices of charge one, . . . , an Abrikosov lattice.
- The case $\Omega = 2$ is marginal (balance of trapping and centrifugal forces).

Resonant normal form

General solution of the linear problem with $\Omega = 0$:

$$\psi = \sum_{n,m} \alpha_{n,m} \chi_{n,m}(r) e^{im\theta} e^{-iE_n t}$$

where $\chi_{nm}(r)e^{im\theta}$ is an eigenstate of the 2D quantum harmonic oscillator with the energy $E_n = n + 1$ and angular momentum $m \in \{-n, -n+2, ..., n-2, n\}$.

The eigenstates with $m = \pm n$ are resonant:

$$\Psi(t,z) = \sum_{n=0}^{\infty} \alpha_n(t)\chi_n(z), \quad \chi_n(z) \sim z^n e^{-\frac{1}{2}|z|^2}, \quad z = x + iy,$$

They satisfy the resonant normal form (labeled as Lowest Landau Level)

$$i\dot{\Psi} = \Pi(|\Psi|^2\Psi), \quad \Pi(\Psi)(z') = e^{-\frac{1}{2}|z'|^2} \int_{\mathbb{C}} e^{\bar{z}z' - \frac{1}{2}|z|^2} \Psi(z) dz.$$

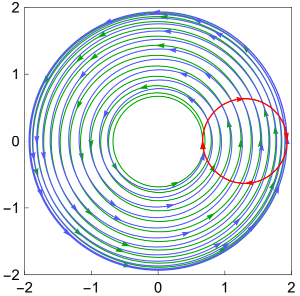
- Faou, Germain, & Hani (2016); Germain, Gerard, Thomann (2017).
- Biasi, Bizon, Craps, & Evnin (2017)

Vortices in BEC

- Bifurcations of vortices can be described when the condensate is stirred above a certain critical angular velocity,
  $\tilde{\Psi}(t,z) := e^{i\mu t}\Psi(t, e^{i\Omega t} z)$.
- There exists a 3-dimensional invariant manifold for the single-vortex configurations

$$\Psi(t,z) = (b(t) + a(t)z)\, e^{p(t)z}\, e^{-\frac{1}{2}|z|^2}$$

- This solution represents modulated precession of a vortex
- Such vortices have been seen in BEC experiments



Biasi-B-Craps-Evnin, 2017

New example 2. Cubic Szegö equation

The unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : \quad |z| = 1\}$ is parameterized by $\theta \in [0, 2\pi]$. Consider the Fourier series on $\mathbb{S}^1$:

$$u(\theta) = \sum_{n \in \mathbb{Z}} \alpha_n e^{in\theta}$$

and project it to the subspace $L^2_+ = \{u \in L^2(\mathbb{S}^1) : \quad \alpha_n = 0, \; n < 0\}$. L^2_+ is a Hardy space of L^2 functions which are extended to the unit disc as holomorphic functions.

With the NLS-type evolution, the function

$$U(t,z) = \sum_{n=0}^{\infty} \alpha_n(t) z^n, \quad z = x + iy,$$

satisfies the resonant normal form (labeled as Cubic Szegö equation)

$$i\dot{U} = \Pi(|U|^2 U), \quad \Pi\left(\sum_{n \in \mathbb{Z}} \alpha_n e^{in\theta}\right) = \sum_{n=0}^{\infty} \alpha_n e^{in\theta}.$$

- Gerard, Grellier (2010, 2012, 2015)

Properties of the cubic Szegö equation

Cubic Szegö equation

$$i\dot{U} = \Pi(|U|^2 U), \quad \Pi\left(\sum_{n\in\mathbb{Z}} \alpha_n e^{in\theta}\right) = \sum_{n=0}^{\infty} \alpha_n e^{in\theta}.$$

- Toy model for other more physically relevant resonant normal forms.
- It has basic conserved quantities

$$\begin{aligned} &\text{Energy:} && E(u) = \|u\|^4_{L^4(\mathbb{S}^1)} \\ &\text{Mass:} && Q(u) = \|u\|^2_{L^2(\mathbb{S}^1)} \\ &\text{Momentum:} && M(u) = \langle -i\partial_\theta u, u\rangle_{L^2(\mathbb{S}^1)}. \end{aligned}$$

- It admits a rich family of exact solutions:

$$U(t,z) = \frac{a(t)z + b(t)}{1 - p(t)z}, \quad U(t,z) = \prod_{j=1}^{N} \frac{z - \bar{p}_j(t)}{1 - p_j(t)z}.$$

- It admits a Lax pair and higher-order conserved quantities.

Resonant normal flow for conformal flow on $\mathbb{S}^3$

- Background geometry: the Einstein cylinder $\mathcal{M} = \mathbb{R} \times \mathbb{S}^3$ with metric

$$g = -dt^2 + dx^2 + \sin^2 x \, d\omega^2, \qquad (t, x, \omega) \in \mathbb{R} \times [0, \pi] \times \mathbb{S}^2$$

  This spacetime has constant scalar curvature $R(g) = 6$.

- On $\mathcal{M}$ we consider a real scalar field ϕ satisfying

$$\Box_g \phi - \phi - \phi^3 = 0\,.$$

- We assume that $\phi = \phi(t, x)$. Then, $\nu(t, x) = \sin(x)\phi(t, x)$ satisfies

$$\nu_{tt} - \nu_{xx} + \frac{\nu^3}{\sin^2 x} = 0$$

  with Dirichlet boundary conditions $\nu(t, 0) = \nu(t, \pi) = 0$.

- Linear eigenstates: $e_n(x) \sim \sin(\omega_n x)$ with $\omega_n = n + 1$ $(n = 0, 1, 2, ...)$

Time averaging

- Expanding $\nu(t,x) = \sum_{n=0}^{\infty} c_n(t) e_n(x)$ we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = -\sum_{jkl} S_{njkl}\, c_j c_k c_l, \quad S_{jkln} = \int_0^{\pi} \frac{dx}{\sin^2 x} e_n(x) e_j(x) e_k(x) e_l(x)$$

- Using variation of constants

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \qquad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t} \right)$$

  we factor out fast oscillations

$$2i\omega_n \frac{d\beta_n}{dt} = -\sum_{jkl} S_{njkl}\, c_j c_k c_l\, e^{-i\omega_n t}$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. The terms with $\Omega = 0$ correspond to resonant interactions.
- Let $\tau = \varepsilon^2 t$ and $\beta_n(t) = \varepsilon \alpha_n(\tau)$. For $\varepsilon \to 0$ the non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are highly oscillatory and therefore negligible.

Resonant system

- Keeping only the resonant terms (and rescaling), we obtain (Bizon-Craps-Evnin-Hunik-Luyten-Maliborski, 2016)

$$i(n+1)\frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j\alpha_k\alpha_{n+j-k}\,,$$

where $S_{njk,n+j-k} = \min\{n,j,k,n+j-k\}+1$.

- This system (labeled as conformal flow) provides an accurate approximation to the cubic wave equation on the timescale $\sim \varepsilon^{-2}$.
- This is a Hamiltonian system

$$i(n+1)\frac{d\alpha_n}{d\tau} = \frac{1}{2}\frac{\partial H}{\partial \bar{\alpha}_n}$$

with

$$H = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\bar{\alpha}_n\bar{\alpha}_j\alpha_k\alpha_{n+j-k}$$

Properties of conformal flow

- Symmetries

$$
\begin{aligned}
\text{Scaling:} \quad & \alpha_n(t) \to c\alpha_n(c^2 t), \quad c \in \mathbb{R} \\
\text{Global phase shift:} \quad & \alpha_n(t) \to e^{i\theta}\alpha_n(t), \quad \theta \in \mathbb{R} \\
\text{Local phase shift:} \quad & \alpha_n(t) \to e^{in\mu}\alpha_n(t), \quad \mu \in \mathbb{R}
\end{aligned}
$$

- Conserved quantities

$$
Q = \sum_{n=0}^{\infty} (n+1)|\alpha_n|^2, \qquad E = \sum_{n=0}^{\infty} (n+1)^2|\alpha_n|^2
$$

- The Cauchy problem is locally (and therefore also globally) well-posed for initial data in $\ell^{2,1}(\mathbb{Z})$ where H, Q, E are finite and conserved.

Energy inequality

Energy

$$H = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk,n+j-k} \bar{\alpha}_n \bar{\alpha}_j \alpha_k \alpha_{n+j-k}$$

Two mass quantities:

$$Q = \sum_{n=0}^{\infty} (n+1)|\alpha_n|^2, \qquad E = \sum_{n=0}^{\infty} (n+1)^2 |\alpha_n|^2$$

Theorem

For every $\alpha \in \ell^{2,1/2}(\mathbb{N})$, it is true that $H(\alpha) \leq Q(\alpha)^2$. Moreover, $H(\alpha) = Q(\alpha)^2$ if and only if $\alpha_n = cp^n$ for some $c, p \in \mathbb{C}$ with $|p| < 1$.

- Local well-posedness holds in $\ell^{2,s}(\mathbb{N})$ for every $s > 1/2$.
  Open: if local well-posedness holds in the critical space $\ell^{2,1/2}(\mathbb{N})$.

Some definitions for stationary states

A solution of the conformal flow is called a stationary state if $|\alpha(t)| = |\alpha(0)|$.

A stationary state is called a standing wave if $\alpha(t) = Ae^{-i\lambda t}$, where $(A_n)_{n\in\mathbb{N}}$ are time-independent and λ is real.

The amplitudes of the standing wave satisfy

$$(n+1)\lambda A_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{n,j,k,n+j-k} \bar{A}_j A_k A_{n+j-k} \, .$$

or, in the variational form:

$$\lambda \frac{\partial Q}{\partial \bar{A}_n} = \frac{1}{2} \frac{\partial H}{\partial \bar{A}_n},$$

as critical points of the action functional $K(\alpha) = \frac{1}{2} H(\alpha) - \lambda Q(\alpha)$.

Standing waves are critical points of energy H for fixed mass Q.

Ground state is the global maximizer of H for fixed Q, since $H(\alpha) \leq Q(\alpha)^2$.

The list of stationary states

- Single-mode states:

$$\alpha_n(t) = c\delta_{nN} e^{-i|c|^2 t},$$

where $N \in \mathbb{N}$ is fixed and $c \in \mathbb{C}$ is arbitrary (due to scaling invariance).

- Ground state family:

$$\alpha_n(t) = cp^n e^{-i\lambda t}, \quad \lambda = \frac{|c|^2}{(1-|p|^2)^2},$$

where $c \in \mathbb{C}$ is arbitrary and $p \in \mathbb{C}$ is another parameter with $|p| < 1$. It bifurcates from the single-mode state with $N = 0$ as $p \to 0$.

- Twisted state family:

$$\alpha_n(t) = cp^{n-1}((1-|p|^2)n - 2|p|^2) e^{-i\lambda t}, \quad \lambda = \frac{|c|^2}{(1-|p|^2)^2},$$

where $c \in \mathbb{C}$ is arbitrary and $p \in \mathbb{C}$ is another parameter with $|p| < 1$. It bifurcates from the single-mode state with $N = 1$ as $p \to 0$.

Three-dimensional invariant manifold

The conformal flow can be closed at the three-parameter solution:

$$\alpha_n = (b(t)p(t) + a(t)n)\, p(t)^{n-1},$$

where a, b, p are functions of t.

The dynamics of the invariant manifold is described by the reduced Hamiltonian system

$$\frac{da}{dt} = f_1(a,b,p), \quad \frac{db}{dt} = f_2(a,b,p), \quad \frac{dp}{dt} = f_3(a,b,p)$$

Three conserved quantities H, Q, and E are in involution, so that the reduced system is completely integrable.

Both the ground-state and twisted-state families are critical points of the reduced Hamiltonian system and they are stable in the time evolution.
Are they stable in the full resonant system?

Main result: $p = 0$

Normalized ground state with $\lambda = 1$

$$A_n(p) = (1 - p^2)p^n, \quad p \in (0, 1)$$

defines the ground state orbit

$$\mathcal{A}(p) = \left\{ \left(e^{i\theta + i\mu n} A_n(p) \right)_{n \in \mathbb{N}} : (\theta, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1 \right\}.$$

As $p \to 0$, the ground state $A_n(0)$ reduces to the single-mode state δ_{n0} and the orbit $\mathcal{A}(0)$ becomes one-dimensional.

Theorem

For every small $\epsilon > 0$*, there is* $\delta > 0$ *such that for every initial data* $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ *with* $\|\alpha(0) - A(0)\|_{\ell^{2,1}} \leq \delta$*, the corresponding unique solution* $\alpha(t) \in C(\mathbb{R}, \ell^{2,1})$ *of the conformal flow satisfies for all* t

$$\mathrm{dist}_{\ell^{2,1}}(\alpha(t), \mathcal{A}(0)) \leq \epsilon.$$

Main result: $p \in (0,1)$

Theorem

For every $p_0 \in (0,1)$ and every small $\epsilon > 0$, there is $\delta > 0$ such that for every initial data $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ satisfying $\|\alpha(0) - A(p_0)\|_{\ell^{2,1}} \leq \delta$, the corresponding unique solution $\alpha(t) \in C(\mathbb{R}_+, \ell^{2,1})$ of the conformal flow satisfies for all t

$$\mathrm{dist}_{\ell^{2,1/2}} \left(\alpha(t) - \mathcal{A}(p(t))\right) \leq \epsilon,$$

and

$$\mathrm{dist}_{\ell^{2,1}} \left(\alpha(t) - \mathcal{A}(p(t))\right) \lesssim \epsilon + (p_0 - p(t))^{1/2}$$

for some continuous function $p(t) \in [0, p_0]$.

(i) the distance between the solution and the ground state orbit is bounded in the norm $\ell^{2,1/2}$;

(ii) the parameter $p(t)$ may drift in time towards smaller values compensated by the increasing $\ell^{2,1}$ distance between the solution and the orbit.

Open: if the drift towards $\mathcal{A}(0)$ can actually occur.

Be wise and linearize

The standing wave $\alpha = A$ is a critical point of the action functional

$$K(\alpha) = \frac{1}{2} H(\alpha) - \lambda Q(\alpha).$$

If $\alpha = A + a + ib$ with real a, b, then

$$K(A + a + ib) - K(A) = \langle L_+ a, a \rangle + \langle L_- b, b \rangle + \mathcal{O}(\|a\|^3 + \|b\|^3),$$

where

$$(L_\pm a)_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{njk,n+j-k} \left[2 A_j A_{n+j-k} a_k \pm A_k A_{n+j-k} a_j \right] - (n+1) \lambda a_n.$$

The linearized evolution system is

$$M \frac{da}{dt} = L_- b, \quad M \frac{db}{dt} = -L_+ a,$$

where $M = \mathrm{diag}(1, 2, ...)$.

Linear operators for the ground state

Taking the normalized ground state with $\lambda = 1$

$$A_n(p) = (1 - p^2)p^n, \quad p \in (0,1)$$

yields

$$(L_\pm a)_n = \sum_{j=0}^{\infty} [B_\pm(p)]_{nj} a_j - (n+1) a_n,$$

where $B_\pm(p) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ are bounded operators:

$$[B_\pm(p)]_{nj} = 2p^{|n-j|} - 2p^{2+n+j} \pm (1-p^2)^2 (j+1)(n+1) p^{n+j}.$$

Lemma

For every $p \in [0,1)$, $[L_+(p), L_-(p)] = 0$ and $[M^{-1} L_+(p), M^{-1} L_-(p)] = 0$.

Linear operators for the ground state

Operators $L_\pm(p) : \ell^{2,1}(\mathbb{N}) \to \ell^2(\mathbb{N})$ are:

$$(L_\pm a)_n = \sum_{j=0}^{\infty} [B_\pm(p)]_{nj} a_j - (n+1) a_n,$$

where $B_\pm(p) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ are bounded operators. Operators $L_\pm(p) : \ell^{2,1}(\mathbb{N}) \to \ell^2(\mathbb{N})$ commute and have a common basis of eigenvectors.

Lemma

For every $p \in [0,1)$*,*

$$\sigma(L_-) = \{\dots, -3, -2, -1, 0, 0\},$$

and

$$\sigma(L_+) = \{\dots, -3, -2, -1, 0, \lambda_*(p)\},$$

where $\lambda_*(p) = 2(1+p^2)/(1-p^2) > 0$*.*

- $L_-(p)A(p) = 0$ and $L_-(p)MA(p) = 0$
- $L_+(p)A'(p) = 0$ and $L_+(p)MA(p) = \lambda_*(p)MA(p)$.

Spectral stability of the ground state

Spectral stability problem:

$$\begin{bmatrix} 0 & L_-(p) \\ -L_+(p) & 0 \end{bmatrix} \begin{bmatrix} \mathtt{a} \\ \mathtt{b} \end{bmatrix} = \Lambda \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \mathtt{a} \\ \mathtt{b} \end{bmatrix}.$$

Bounded operators $M^{-1}L_\pm(p) : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ commute and have a common basis of eigenvectors.

Lemma

For every $p \in [0,1)$, eigenvalues are purely imaginary $\Lambda_m = \pm i\Omega_m$ with

$$\Omega_0 = \Omega_1 = 0, \quad \Omega_m = \frac{m-1}{m+1}, \quad m \geq 2.$$

- Geometric kernel is three-dimensional.
- One generalized eigenvector exists $L_+(p)A(p) = 2MA(p)$.
- All eigenvalues are simple except for the double zero eigenvalue related to the phase symmetry $\alpha_n(t) \to e^{i\theta}\alpha_n(t)$, $\theta \in \mathbb{R}$.

Orbital stability for $p = 0$

Single-mode state with $\lambda = 1$

$$A_n(0) = \delta_{n0}$$

defines the single-mode state orbit

$$\mathcal{A}(0) = \left\{ \left(e^{i\theta} \delta_{n0} \right)_{n \in \mathbb{N}} : \quad \theta \in \mathbb{S}^1 \right\}.$$

Theorem

For every small $\epsilon > 0$, there is $\delta > 0$ such that for every initial data $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ with $\|\alpha(0) - A(0)\|_{\ell^{2,1}} \leq \delta$, the corresponding unique solution $\alpha(t) \in C(\mathbb{R}, \ell^{2,1})$ of the conformal flow satisfies for all t

$$\mathrm{dist}_{\ell^{2,1}}(\alpha(t), \mathcal{A}(0)) \leq \epsilon.$$

Decomposition near the single-mode state orbit

Lemma

There exist $\delta_0 > 0$ *such that for every* $\alpha \in \ell^2$ *satisfying*

$$\delta := \inf_{\theta \in \mathbb{S}} \|\alpha - e^{i\theta} A(0)\|_{\ell^2} \leq \delta_0,$$

there exists a unique choice of real-valued numbers (c, θ) *and real-valued sequences* $a, b \in \ell^2$ *in the orthogonal decomposition*

$$\alpha_n = e^{i\theta} \left(cA_n(0) + a_n + ib_n \right),$$

subject to the orthogonality conditions

$$\langle MA(0), a \rangle = \langle MA(0), b \rangle = 0,$$

satisfying the estimate

$$|c - 1| + \|a + ib\|_{\ell^2} \lesssim \delta.$$

Control of the decomposition as the time evolves

Lemma

Assume that initial data $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ *satisfy*

$$\|\alpha(0) - A(0)\|_{h^1} \leq \delta$$

for some sufficiently small $\delta > 0$. *Then, the corresponding unique global solution* $\alpha(t) \in C(\mathbb{R}, \ell^{2,1})$ *of the conformal flow can be represented by the decomposition*

$$\alpha_n(t) = e^{i\theta(t)} \left(c(t) A_n(0) + a_n(t) + i b_n(t) \right), \quad \langle MA(0), a(t) \rangle = \langle MA(0), b(t) \rangle = 0,$$

satisfying for all t

$$|c(t) - 1| \lesssim \delta, \quad \|a(t) + ib(t)\|_{\ell^{2,1}} \lesssim \delta^{1/2}.$$

In other words, for all t

$$\inf_{\theta \in \mathbb{S}} \|\alpha(t) - e^{i\theta} A(0)\|_{\ell^2} \leq \epsilon$$

The proof with the use of conserved quantities

- The decomposition

$$\alpha_n(t) = e^{i\theta(t)} \left(c(t) A_n(0) + a_n(t) + i b_n(t) \right),$$

with $\langle MA(0), a(t) \rangle = \langle MA(0), b(t) \rangle = 0$ holds at least for small t.

- Since $A_n(0) = \delta_{n0}$, the orthogonality conditions yield $a_0 = b_0 = 0$.
- Expansions of the two mass conserved quantities

$$\begin{aligned} Q(\alpha(0)) &= Q(\alpha(t)) = c(t)^2 + \sum_{n=1}^{\infty} (n+1)(a_n^2 + b_n^2), \\ E(\alpha(0)) &= E(\alpha(t)) = c(t)^2 + \sum_{n=1}^{\infty} (n+1)^2 (a_n^2 + b_n^2). \end{aligned}$$

yields the bound

$$\sum_{n=1}^{\infty} n(n+1)(a_n^2 + b_n^2) = E(\alpha(0)) - 1 - Q(\alpha(0)) + 1 \lesssim \delta,$$

- Continuation in t yields the decomposition and the bounds for all t.

Orbital stability for $p \in (0, 1)$

Normalized ground state with $\lambda = 1$

$$A_n(p) = (1 - p^2)p^n, \quad p \in (0, 1)$$

defines the ground state orbit

$$\mathcal{A}(p) = \left\{ \left(e^{i\theta + i\mu n} A_n(p) \right)_{n \in \mathbb{N}} : (\theta, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1 \right\}.$$

Theorem

For every $p_0 \in (0, 1)$ and every small $\epsilon > 0$, there is $\delta > 0$ such that for every initial data $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ satisfying $\|\alpha(0) - A(p_0)\|_{\ell^{2,1}} \leq \delta$, the corresponding unique solution $\alpha(t) \in C(\mathbb{R}_+, \ell^{2,1})$ of the conformal flow satisfies for all t

$$\mathrm{dist}_{\ell^{2,1/2}} \left(\alpha(t) - \mathcal{A}(p(t)) \right) \leq \epsilon,$$

and

$$\mathrm{dist}_{\ell^{2,1}} \left(\alpha(t) - \mathcal{A}(p(t)) \right) \lesssim \epsilon + (p_0 - p(t))^{1/2}$$

for some continuous function $p(t) \in [0, p_0]$.

Coercivity of the energy in $\ell^{2,1/2}(\mathbb{N})$

Symplectically orthogonal subspace of $\ell^2(\mathbb{N})$:

$$[X_c(p)]^\perp := \left\{ a \in \ell^2(\mathbb{N}) : \quad \langle MA(p), a \rangle = \langle MA'(p), a \rangle = 0 \right\}.$$

Lemma

There exists $C > 0$ such that

$$-\langle L_\pm(p) a, a \rangle \gtrsim \|a\|^2_{\ell^{2,1/2}}$$

for every $a \in \ell^{2,1/2}(\mathbb{N}) \cap [X_c(p)]^\perp$.

Decomposition near the ground state orbit

Lemma

For every $p_0 \in (0,1)$*, there exists* $\delta_0 > 0$ *such that for every* $\alpha \in \ell^2(\mathbb{N})$ *satisfying*

$$\delta := \inf_{\theta,\mu \in \mathbb{S}} \|\alpha - e^{i(\theta+\mu+\mu\cdot)} A(p_0)\|_{\ell^2} \leq \delta_0,$$

there exists a unique choice for real-valued numbers (c, p, θ, μ) *and real-valued sequences* $a, b \in \ell^2$ *in the orthogonal decomposition*

$$\alpha_n = e^{i(\theta+\mu+\mu n)} \left(cA_n(p) + a_n + ib_n \right),$$

subject to the orthogonality conditions

$$\langle MA(p), a \rangle = \langle MA'(p), a \rangle = \langle MA(p), b \rangle = \langle MA'(p), b \rangle = 0, \tag{1}$$

satisfying the estimate

$$|c-1| + |p-p_0| + \|a+ib\|_{\ell^2} \lesssim \delta.$$

Control of the decomposition as the time evolves

Lemma

Assume that the initial data $\alpha(0) \in \ell^{2,1}(\mathbb{N})$ *satisfy*

$$\|\alpha(0) - A(p_0)\|_{\ell^{2,1}} \leq \delta,$$

for some $p_0 \in [0,1)$ *and a sufficiently small* $\delta > 0$*. Then, the corresponding unique global solution* $\alpha(t) \in C(\mathbb{R}_+, \ell^{2,1})$ *of the conformal flow can be represented by the decomposition*

$$\alpha_n(t) = e^{i(\theta(t)+(n+1)\mu(t))} \left(c(t) A_n(p(t)) + a_n(t) + i b_n(t) \right),$$

$a, b \in [X_c(p)]^\perp$ *satisfying for all t*

$$|c(t) - 1| + \|a(t) + ib(t)\|_{\ell^{2,1/2}} \lesssim \delta.$$

- The proof is based on the Lyapunov function

$$\Delta(c) := c^2 \left(Q(\alpha) - 1 \right) - \frac{1}{2} \left(H(\alpha) - 1 \right).$$

Control of the drift of $p(t)$ as the time evolves

Lemma

Under the same assumptions,

$$p(t) \lesssim p_0 + \delta, \quad \|a(t) + ib(t)\|_{\ell^{2,1}} \lesssim \delta^{1/2} + |p_0 - p(t)|^{1/2}.$$

- The proof is based on the additional mass conservation:

$$E(\alpha(t)) = c(t)^2 \frac{1 + p(t)^2}{1 - p(t)^2} + \|a(t) + ib(t)\|^2_{\ell^{2,1}},$$

which yields

$$\frac{2(p(t)^2 - p_0^2)}{(1 - p(t)^2)(1 - p_0^2)} + \|a(t) + ib(t)\|^2_{\ell^{2,1}} \lesssim \delta,$$

Twisted state ?

Twisted state family

$$A_n(p) = (1 - p^2)((1 - p^2)n - 2p^2)p^{n-1}, \quad \lambda = 1,$$

bifurcates from $A_n(0) = \delta_{n1}$.

- Linearized operators $L_+(p)$ and $L_-(p)$ also commute.
- Spectral stability also holds.
- Coercivity is lost as $L_+(p)$ has two positive eigenvalues and $L_-(p)$ has one positive eigenvalue in addition to the triple zero eigenvalue.
- Nonlinear stability is opened.

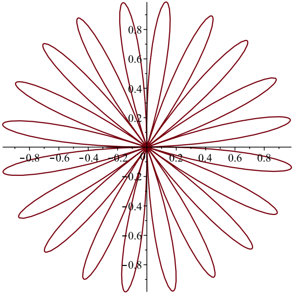
Twisted state for the cubic Szegő flow

- For cubic Szegő equation

$$p(\tau) = -\frac{i}{\sqrt{1+\varepsilon^2/4}} \sin(\omega\tau)\, e^{-\frac{1}{2} i \varepsilon^2 \tau}$$

with $\omega = \varepsilon\sqrt{1+\varepsilon^2/4}$.

- Thus, $|p(\tau_n)| \sim 1 - \varepsilon^2/8$ for a sequence of times $\tau_n = \frac{(2n+1)\pi}{2\omega}$.



Gérard-Grellier daisy

- This instability provided a hint for the existence of unbounded orbits (Gérard-Grellier, 2015)

Conclusion

- We considered a novel resonant normal form, which describes conformal flow on $\mathcal{S}^3$.
- We obtained a nice commutativity formula for linearized operators $L_+(p)$ and $L_-(p)$.
  Open: is this a coincidence or a sign of integrability?
- We obtained orbital stability results for the ground state family near the single-mode state.
  Open: is there an actual drift towards the single-mode state along the ground state family?
- Spectral stability also hold for other (twisted) states, e.g. $A_n = \delta_{nN}$.
  Open: are they stable in the nonlinear dynamics?