Oscillations of dark BEC solitons in a parabolic trap

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Reference:
D.P., D. Franzeskakis, and P. Kevrekidis,
Physical Review E 72, 016615 (2005)

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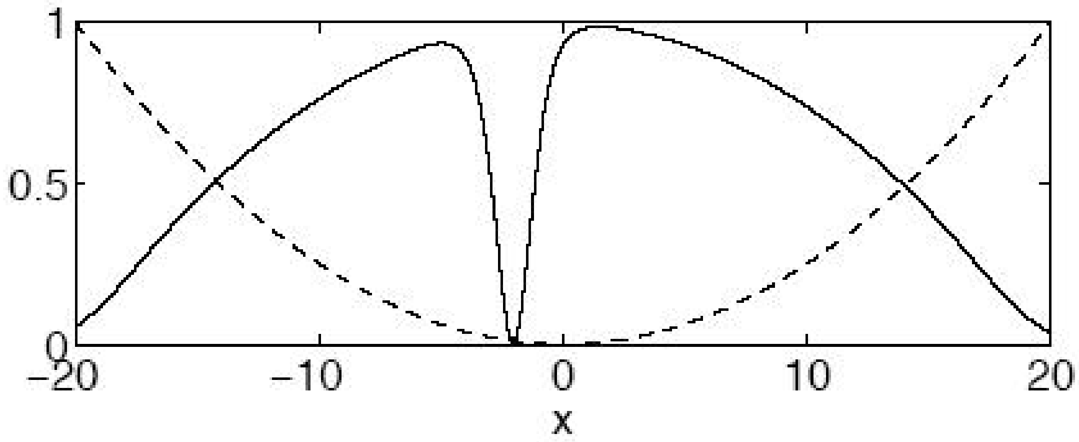
The Problem

The Gross–Pitaevsky equation:

$$iu_t = -\frac{1}{2}u_{xx} + \epsilon^2 x^2 u + |u|^2 u, \quad \epsilon \ll 1$$



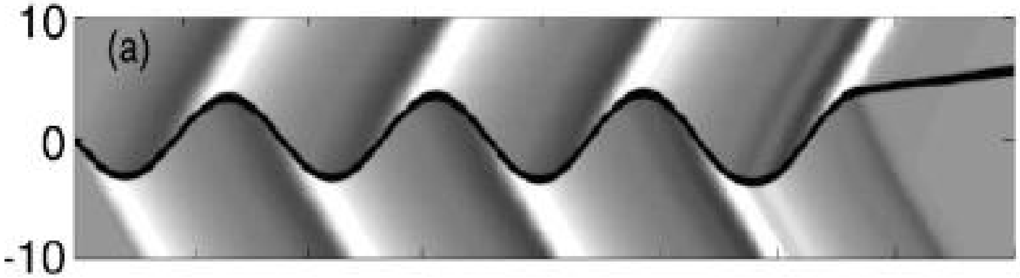
- Frequency of oscillations (adiabatic dynamics of dark solitons)
- Amplitude of oscillations (radiative effects of dark solitons)

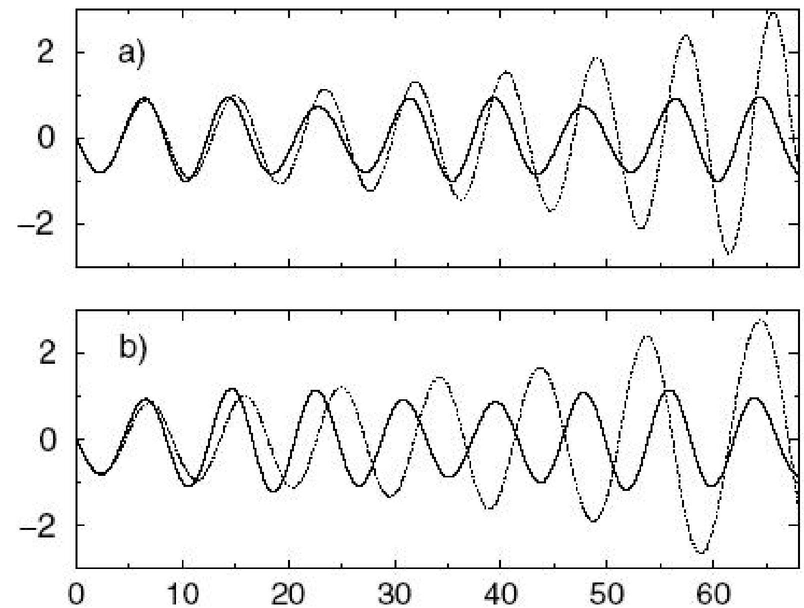
Different Solutions of the Problem

- Collective coordinates (the Ehrenfest Theorem)
  (1997: Reinhardt & Clark, Morgan et al.)
- Boundary-layer integrals (hydrodynamic formulation)
  (2000: Busch & Anglin)
- Shallow-soliton theory (KdV formulation)
  (2002: Huang et al.)
- Renormalized momentum (perturbation theory)
  (2002-2004: Frantzeskakis et al.)
- Renormalized powers (perturbation theory)
  (2003-2004 : Brazhnyi & Konotop, Konotop & Pitaevsky)
- Numerical simulations
  (2003-2004 : Parker, Proukakis, et al.)

Main Empiric Results

- The frequency of oscillations is independent of dark soliton amplitude.
- The amplitude of oscillations increases due to radiative losses.





Numerical simulations by N. Proukakis (2003)

Plan of the Lecture

- Definition of the ground state, the first excited state, and the dark soliton
- Failure of the formal adiabatic theory
- Adiabatic theory with dynamical scaling techniques
- Radiation of dark solitons with the asymptotic multi-scale expansions
- Comparison of asymptotic and numerical results
- Other ideas and prospects

Ground state of the GP equation

- Separation of variables

$$u_{\text{gs}}(x,t) = U_\epsilon(x) e^{-i\mu_\epsilon t + i\theta_0},$$

where $\mu_\epsilon \in D \subset \mathbb{R}$, $\theta_0 \in \mathbb{R}$, and $(U_\epsilon, \mu_\epsilon)$ are found from

$$\frac{1}{2}U'' - \epsilon^2 x^2 U - U^3 + \mu U = 0.$$

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- Linear ground state

$$U_\epsilon = \exp\left(-\frac{\epsilon x^2}{\sqrt{2}}\right), \quad \mu_\epsilon = \mu_0(\epsilon) = \frac{\epsilon}{\sqrt{2}}$$

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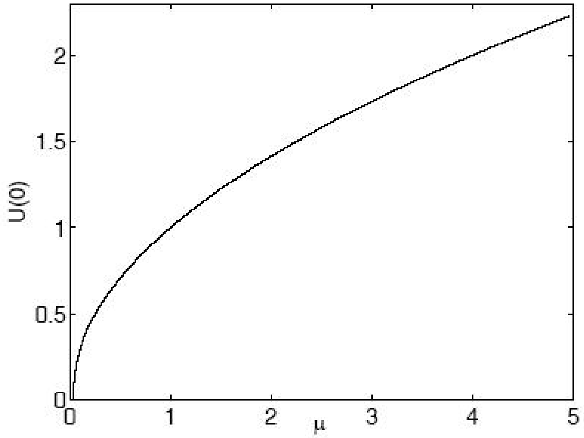
- Local bifurcation (by Lyapunov-Schmidt reduction)

$$\mu > \mu_0(\epsilon) : \quad U'(0) = 0, \quad \lim_{|x|\to\infty} U(x) = 0.$$

Ground state: numerical approximation

- There exists a smooth one-parameter family of $U(x)$ for a fixed value of $\epsilon > 0$, such that $U(0)$ is increasing function of μ



Normalization

$$U_\epsilon(0) = 1$$

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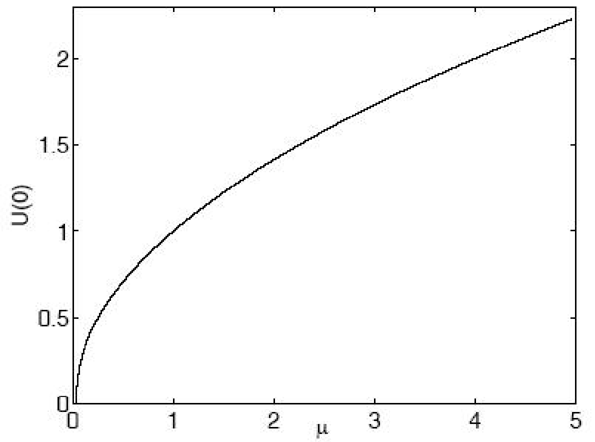

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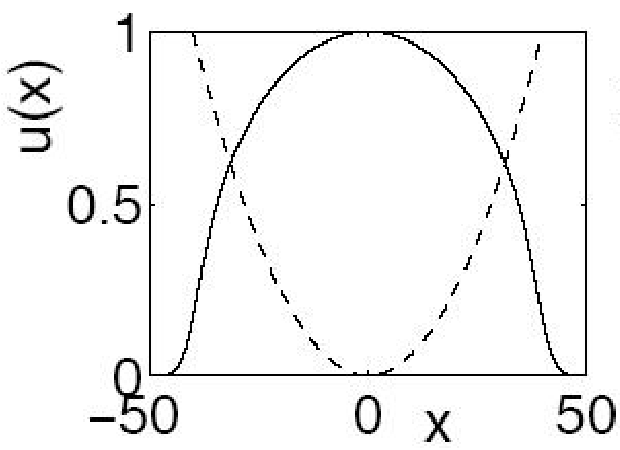
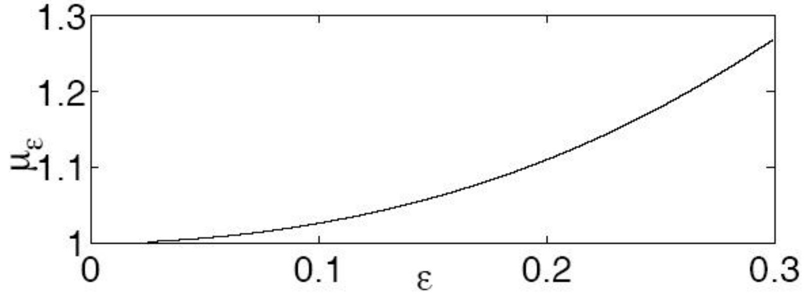
- Numerical approximations of ground state solutions





Ground state: WKB approximation

- Reformulation of the ODE for $Q(x) = U^2(x)$:

$$Q(x) = \mu - \epsilon^2 x^2 + \frac{2QQ'' - (Q')^2}{8Q^2}$$

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- WKB asymptotic series

$$Q = \mu^2 - X^2 + \sum_{k=1}^{\infty} \epsilon^{2k} Q_k(X), \qquad X = \epsilon x,$$

which converges for $|\epsilon x| < \sqrt{\mu}$

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- Normalization condition

$$Q(0) = \mu - \frac{\epsilon^2}{2\mu} + \mathrm{O}(\epsilon^4) = 1,$$

such that $\mu_\epsilon = 1 + \mathrm{O}(\epsilon^2)$.

First excited state of the GP equation

- Separation of variables

$$u_{\text{exc}}(x,t) = U_\epsilon(x)e^{-i\mu_\epsilon t + i\theta_0},$$

where $\mu_\epsilon \in D \subset \mathbb{R}$, $\theta_0 \in \mathbb{R}$, and $(U_\epsilon, \mu_\epsilon)$ are found from

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$$\mu > \mu_1(\epsilon): \quad U(0) = 0, \quad \lim_{|x|\to\infty} U(x) = 0,$$

such that it exists for $\mu \geq 1$.

Dark solitons on the ground state

- Analytical representation for $\epsilon = 0$

$$u_{\text{ds}}(x,t) = [k \tanh(k(x - vt - s_0)) + iv]\, e^{-it+i\theta_0},$$

where $k = \sqrt{1 - v^2} < 1$ and $(s_0, \theta_0) \in \mathbb{R}^2$.

- Boundary conditions for $\epsilon = 0$

$$|u_{\text{ds}}|^2 = 1 - k^2 \text{sech}^2(k(x - vt - s)) \to 1 \quad \text{as} \quad |x| \to \infty$$

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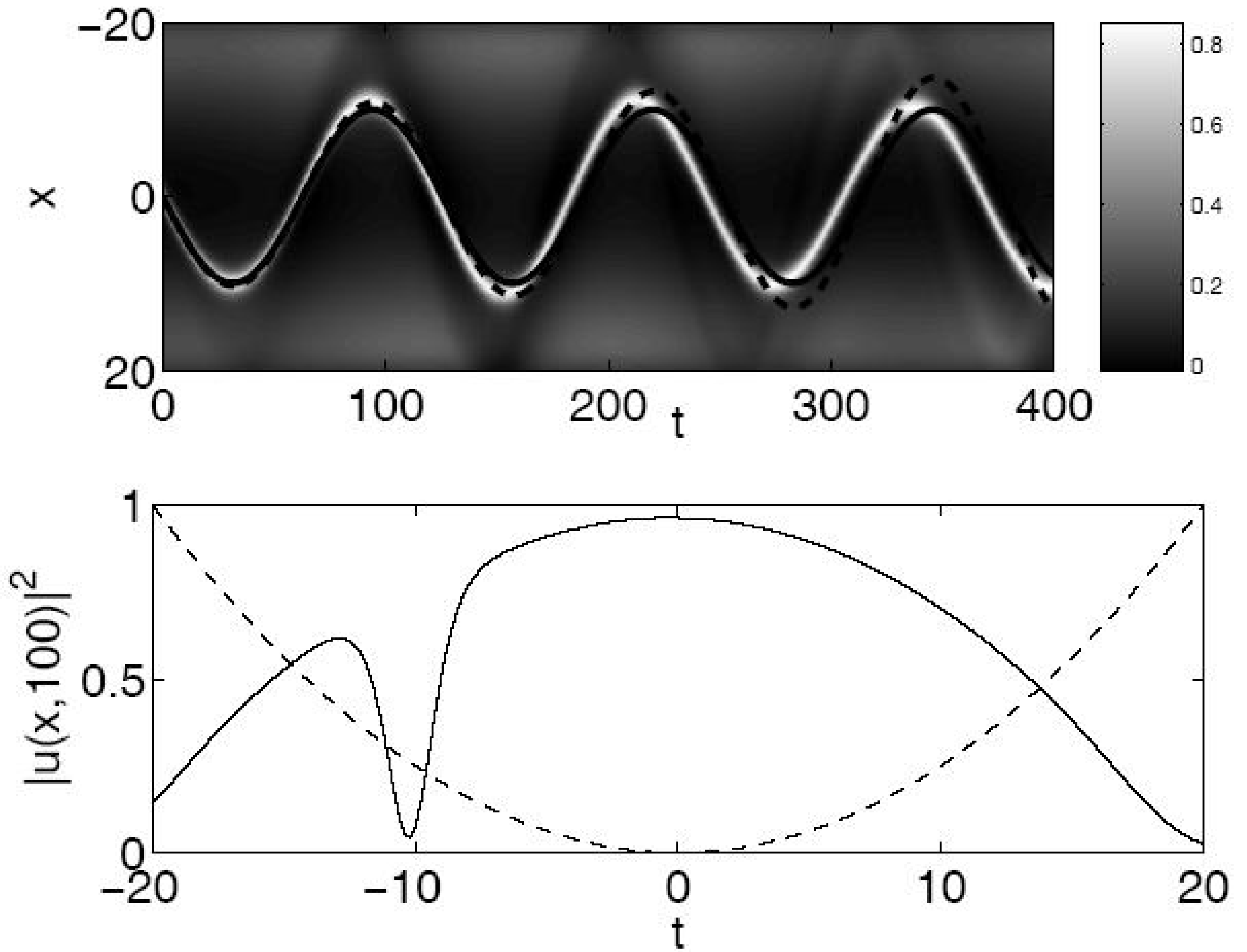
- When $\epsilon \neq 0$, the stationary solution persists only for $v = 0$ and $s_0 = 0$, when $u_{\mathrm{ds}}(x,t) = u_{\mathrm{exc}}(x,t)$ with zero boundary conditions as $|x| \to \infty$. Dark soliton solutions with $v \neq 0$ and $s_0 \neq 0$ undertake *nonstationary* dynamics in the parabolic trap.

Numerical solution : nearly shallow soliton



$\epsilon = 0.05, \; s(0) = 0, \; v(0) = 0.5$

Starting transformation

- The original GP equation

$$iu_t = -\frac{1}{2}u_{xx} + \epsilon^2 x^2 u + |u|^2 u, \quad \epsilon \ll 1$$

- Transformation of the GP equation

$$u(x,t) = U_\epsilon(x) w(x,t) e^{-i\mu_\epsilon t},$$

where $(U_\epsilon, \mu_\epsilon)$ is the ground state pair with $U_\epsilon(0) = 1$

- Perturbed NLS equation (Frantzeskakis et al, 2002):

$$iw_t + \frac{1}{2}w_{xx} + U_\epsilon^2(x)(1 - |w|^2)w = -\frac{U_\epsilon'(x)}{U_\epsilon(x)} w_x$$

where $U_\epsilon^2 = 1 - \epsilon^2 x^2 + \mathrm{O}(\epsilon^2)$ for $\epsilon|x| = \mathrm{O}(1)$ and $\epsilon|x| < 1$.

Failure of formal adiabatic theory

- Formal perturbed NLS equation

$$iw_t + \frac{1}{2}w_{xx} + (1 - |w|^2)w = R(w, \bar{w}),$$

where

$$R(w, \bar{w}) = \epsilon^2 x^2 (1 - |w|^2)w + \frac{\epsilon^2 x}{1 - \epsilon^2 x^2} w_x$$

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- First-order balance for renormalized momentum

$$\frac{ds}{dt} = v, \qquad P_r'(v)\frac{dv}{dt} = -\int_{-\infty}^{\infty} w_0'(x)\left(R + \bar{R}\right)(w_0, \bar{w}_0)dx,$$

where $w_0 = w_0(x - s)$ is the exact dark soliton for $\epsilon = 0$ and $P_r'(v) = 4k$ is the renormalized momentum.

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- Formal computations give a *wrong* dynamical equation:

$$\ddot{s} + \frac{(3 - s^2)(1 - \dot{s}^2)}{3(1 - s^2)}s = \mathrm{O}(\epsilon^2)$$

The main equation for perturbation theory

- Scaling of dark solitons for adiabatic dynamics

$$T = \epsilon t, \qquad v = v(T) = \dot{s}(T),$$

implies that $w_0 = w_0(x - s/\epsilon) \equiv w_0(\eta)$, such that

$$\epsilon^2 x^2 = s^2 + 2\epsilon s\eta + \epsilon^2\eta^2, \quad \eta = \mathrm{O}(1).$$

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- Let $w = w(\eta, t)$ with $\eta = x - s(T)/\epsilon$ and rewrite the perturbed NLS equation in the form

$$iw_t - ivw_\eta + \frac{1}{2}w_{\eta\eta} + U_\epsilon^2(s)(1 - |w|^2)w = R(w, \bar{w}),$$

where

$$R = -\epsilon\left(\frac{U_\epsilon'(s)}{U_\epsilon(s)}w_\eta + 2U_\epsilon(s)U_\epsilon'(s)\eta(1 - |w|^2)w\right) + \mathrm{O}(\epsilon^2)$$

Dynamical rescaling of the main equation

- Let $w = w(z,t)$ with $z = \eta U_\epsilon(s(T))$ and let
$$\dot{s}(T) = v(T) = \nu(T) U_\epsilon(s(T)),$$
such that the final perturbed NLS equation is
$$iw_t + U_\epsilon^2(s)\left[-i\nu w_z + \frac{1}{2}w_{zz} + (1-|w|^2)w\right] + \epsilon R_1(w,\bar{w}) = \mathrm{O}(\epsilon^2),$$
where
$$R_1 = U_\epsilon'(s)\left[i\nu z w_z + w_z + 2z(1-|w|^2)w\right].$$

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- An asymptotic solution is sought in the form:
$$w(z,t) = \left[w_0(z) + \epsilon w_1(z,t) + \mathrm{O}(\epsilon^2)\right] e^{i\theta},$$
where $w_0(z) = \kappa \tanh(\kappa z) + i\nu$, $\kappa = \sqrt{1 - \nu^2}$, and parameters $\theta(T)$ and $s(T)$ are independent.

The first-order correction: the inhomogeneous pr

- First-order linearization problem

$$i\partial_t \sigma_3 \mathbf{w}_1 + U_\epsilon^2(s)\mathcal{H}\mathbf{w}_1 = \dot{\theta}\mathbf{w}_0 - i\partial_T \sigma_3 \mathbf{w}_0 - \mathbf{R}_1(w_0, \bar{w}_0),$$

where

$$\mathcal{H} = -i\nu\sigma_3\partial_z + \sigma_0\left(\frac{1}{2}\partial_z^2 + 1\right) - \begin{pmatrix} 2|w_0|^2 & w_0^2 \\ \bar{w}_0^2 & 2|w_0|^2 \end{pmatrix}$$

and

$$\mathbf{w}_0 = \begin{pmatrix} w_0 \\ \bar{w}_0 \end{pmatrix}, \qquad \mathbf{w}_1 = \begin{pmatrix} w_1 \\ \bar{w}_1 \end{pmatrix},$$

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and

$$\mathbf{w}_0 = \begin{pmatrix} w_0 \\ \bar{w}_0 \end{pmatrix}, \qquad \mathbf{w}_1 = \begin{pmatrix} w_1 \\ \bar{w}_1 \end{pmatrix},$$

- Completeness of eigenfunctions of $\mathcal{H}$ (Chen et al, 1998)
  - Continuous spectrum on $\lambda \in i\mathbb{R}$
  - Embedded kernel at $\lambda = 0$ with

$$\mathcal{H}\mathbf{w}_0' = \mathbf{0}, \qquad \mathcal{H}\left(i\sigma_3\mathbf{w}_0\right) = \mathbf{0}$$

The first-order correction: inner part

- Orthogonality of $\mathbf{R}_1$ to $\mathbf{w}_0'(z)$ produces the main equation for adiabatic dynamics of a dark soliton:

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- The first-order solution $w_1(z,t)$ is decomposed into eigenfunctions of the continuous spectrum of $\mathcal{H}$. By the stationary phase method, the first-order solution $w_1(z,t)$ becomes stationary as $t \to \infty$:

$$w_{1s} = \frac{q(T)}{U_\epsilon^2(s)}\left(izw_0 - \partial_\nu w_0\right) + \frac{3\nu q(T) - \dot{\theta}(T)}{2\kappa U_\epsilon^2(s)}\partial_\kappa w_0 + \tilde{w}_{1s}(z,T),$$

where $q(T)$ is arbitrary parameter.

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where $q(T)$ is arbitrary parameter.

- The stationary solution $w_{1s}(z,T)$ grows linearly in z as $|z| \to \infty$.

The first-order correction: outer part

- Matching conditions from $z = O(1)$ to $\epsilon x = O(1)$:

$$\lim_{z \to \pm\infty} w_s(z, T) = \left(1 + \epsilon W^{\pm}(X, T)\right) e^{i\Theta^{\pm}(X,T)},$$

where $X = \epsilon x$, $T = \epsilon t$, and

$$W^{\pm}\Big|_{X=s(T)}, \frac{\partial \Theta}{\partial X}\Big|_{X=s(T)} \quad \text{are given.}$$

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- Radiation problem outside the dark soliton:

$$\Theta^{\pm}_{TT} - \left(U_{\epsilon}^2(X)\Theta^{\pm}_X\right)_X = 0,$$

where $U_{\epsilon}^2(X) = 1 - X^2$ and

$$W^{\pm} = -\frac{\Theta^{\pm}_T}{2U_{\epsilon}^2(X)}$$

Solution of the radiation problem

- Solution along the characteristics

$$\frac{d\xi_\pm}{dT} = \pm U_\epsilon(\xi_\pm), \qquad R_\pm = W^\pm \pm \frac{\Theta_X^\pm}{2U_\epsilon(X)},$$

where

$$\begin{aligned}\frac{dR_+}{dT} &= -\frac{1}{2}U'_\epsilon(\xi_+(T;\tau_0))\left(5R_+ - R_-\right), \\ \frac{dR_-}{dT} &= -\frac{1}{2}U'_\epsilon(\xi_-(T;\tau_0))\left(R_+ - 5R_-\right).\end{aligned}$$

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$$\frac{dR_+}{dT} = -\frac{1}{2}U'_\epsilon(\xi_+(T;\tau_0))\,(5R_+ - R_-)\,,$$
$$\frac{dR_-}{dT} = -\frac{1}{2}U'_\epsilon(\xi_-(T;\tau_0))\,(R_+ - 5R_-)\,.$$

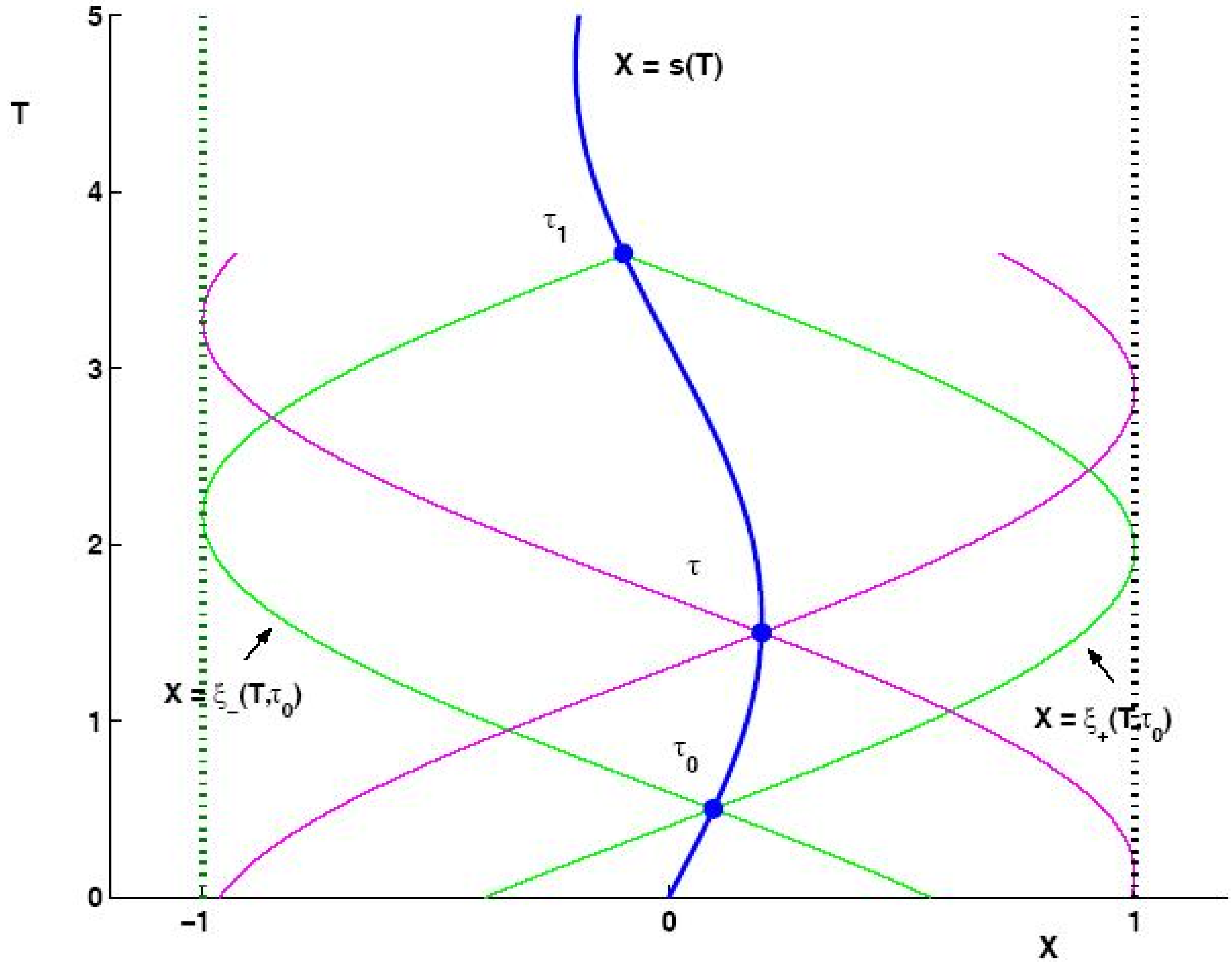
- Let us assume no radiation from the outer domain:

$$X > s(T): \quad R_- = 0 \qquad\qquad X < s(T): \quad R_+ = 0$$

The system of equations for the first-order correction is then closed. The orthogonality of $\mathbf{R}_2$ to $\mathbf{w}'_0(z)$ extends the main equation for dynamics of a dark soliton:

$$\ddot{s} + s = \frac{\epsilon\dot{s}}{2\sqrt{(1-s^2)^3}\sqrt{1-s^2-\dot{s}^2}} + \mathrm{O}(\epsilon^2).$$

Family of characteristics for radiation problem



Families of characteristics in the parabolic trap

Outcomes of the dynamical equation

$$\ddot{s} + s = \frac{\epsilon \dot{s}}{2\sqrt{(1-s^2)^3}\sqrt{1-s^2-\dot{s}^2}} + \mathrm{O}(\epsilon^2).$$

- The equilibrium point $(0, 0)$ recovers the first excited state $u_{\text{exc}}(x)$.

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- Lyapunov function

$$E = \frac{1}{2}\left(\dot{s}^2 + s^2\right)$$

  shows that all trajectories are outgoing spirals:

$$\dot{E} = \frac{\epsilon \dot{s}^2}{2\sqrt{(1-s^2)^3}\sqrt{1-s^2-\dot{s}^2}} + \mathrm{O}(\epsilon^2) > 0.$$

Conclusions on the asymptotic analysis

- The main equation for dynamics of a dark soliton is valid in the case of no incoming radiation, e.g.

$$iu_t = -\frac{1}{2}u_{xx} + V(\epsilon x)u + |u|^2 u, \quad \epsilon \ll 1,$$

where

  - $V(X) = X^2 + \mathrm{O}(X^3)$ near $X = 0$
  - $V(X) \to 0$ as $|X| \to \infty$

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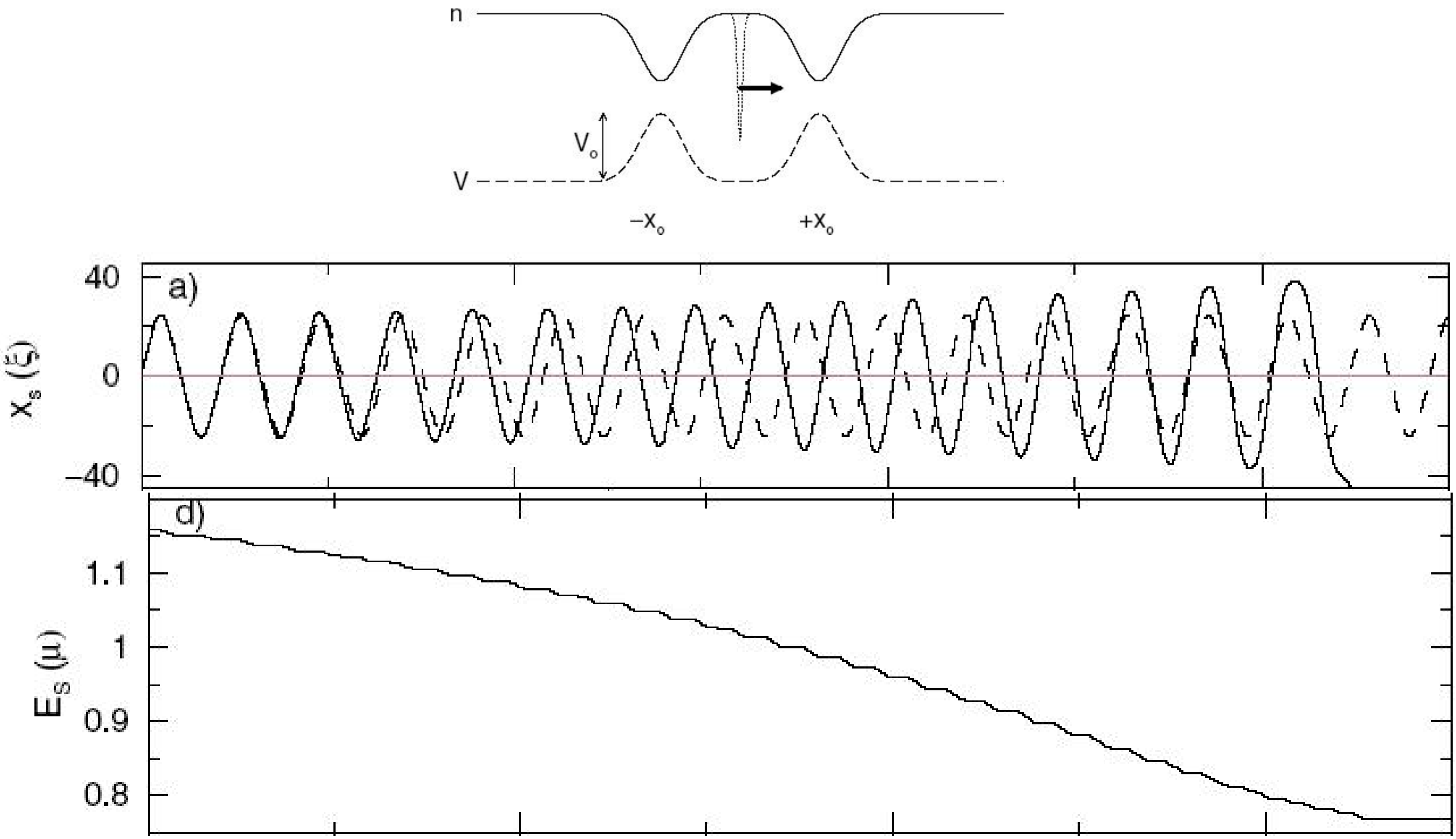
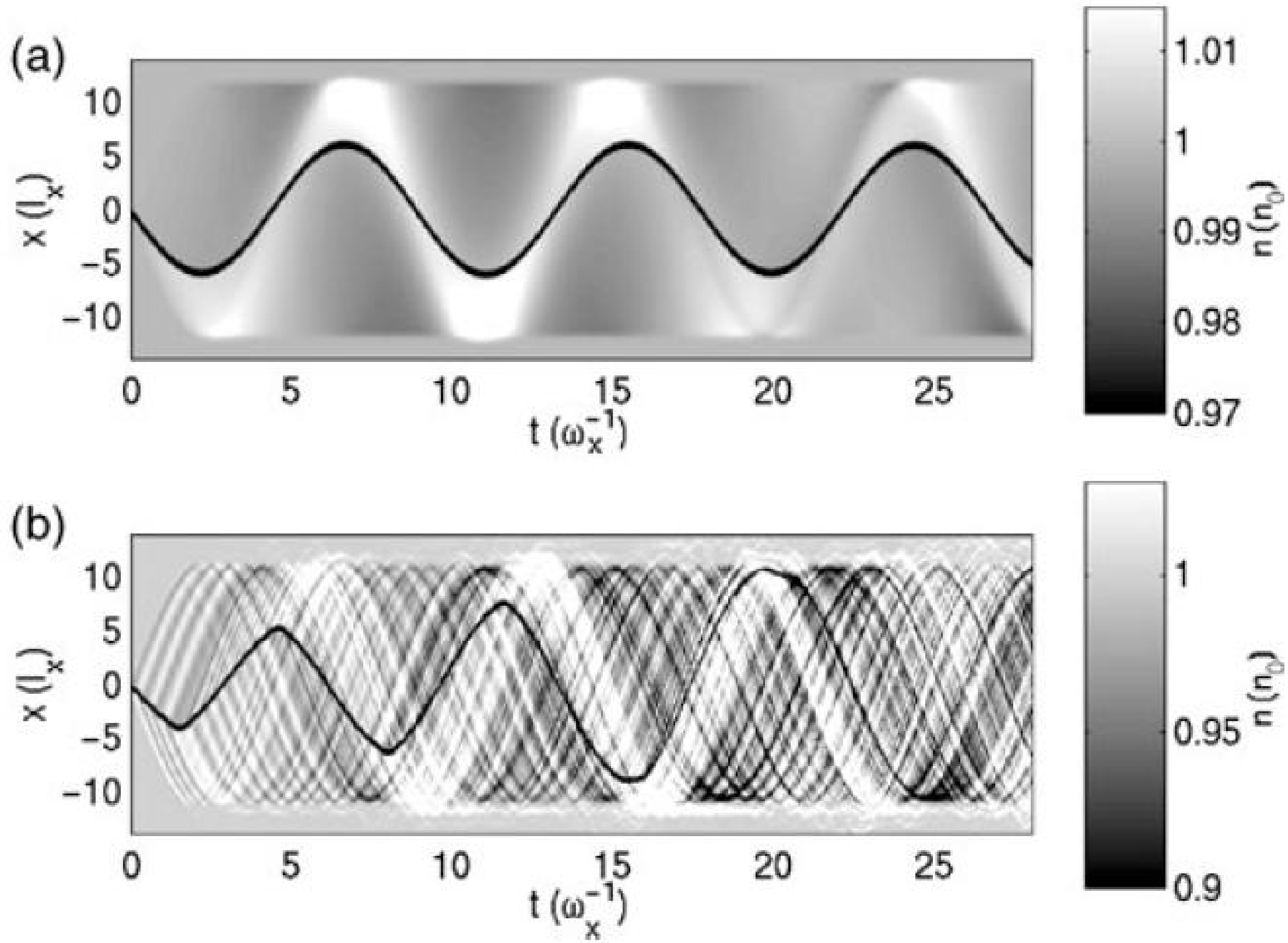
- In the case of a harmonic trap ($V = X^2$), the main equation is only valid for the first half-period of oscillations. For longer times, the radiative waves are expected to be in balance, so that oscillations of a dark soliton are expected to be synchronized.

Numerical solution (by N.G. Parker et al, 2003)



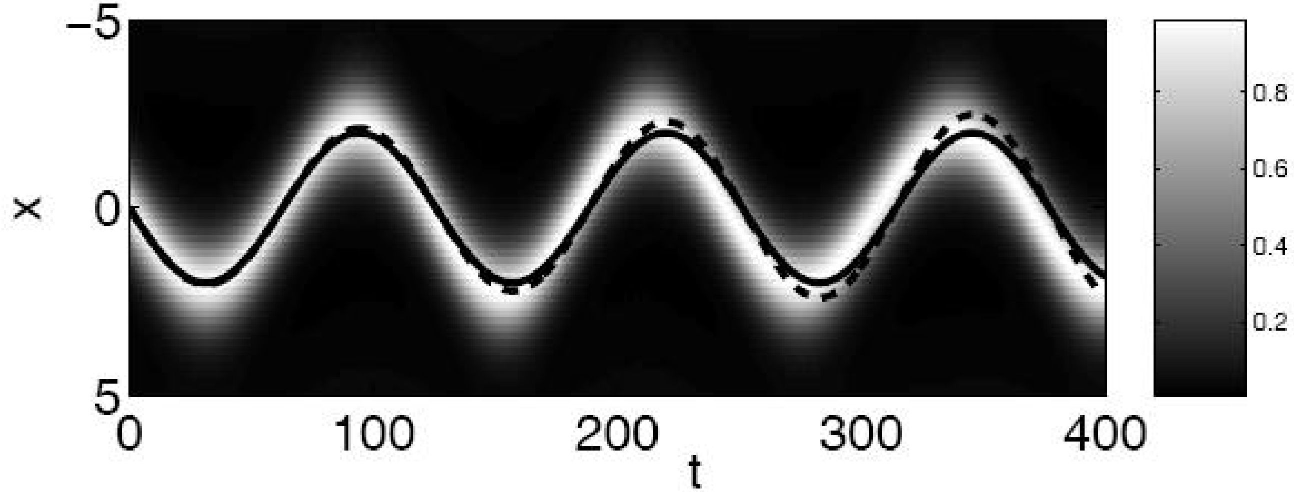
Position and energy of dark soliton in a double Gaussian trap.

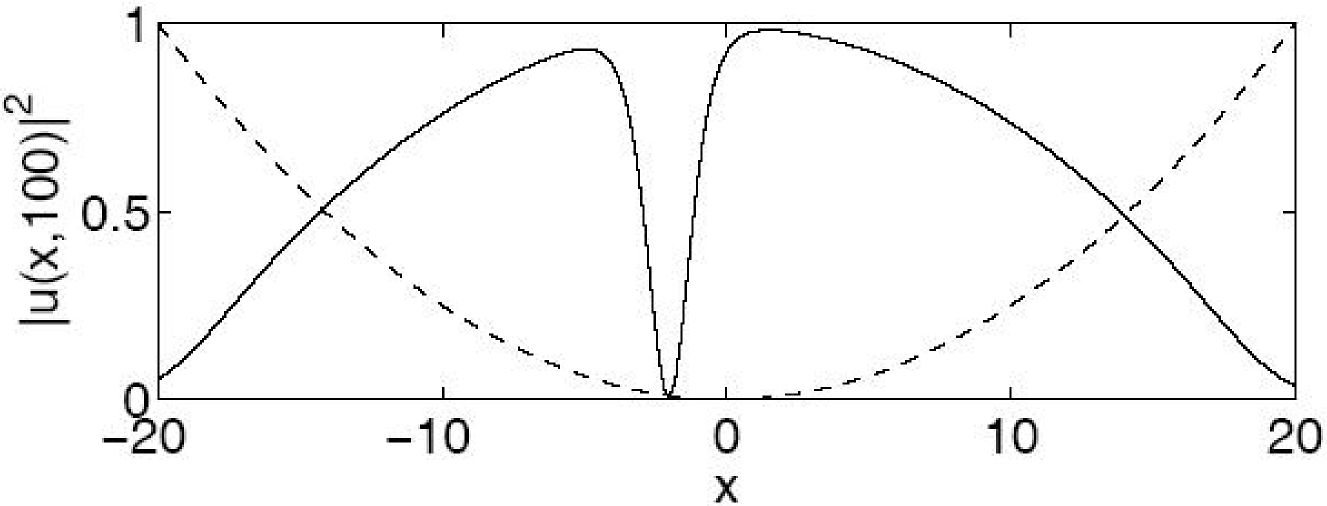
Numerical solution (by N.G. Parker et al, 2004)



Top: parabolic trap. Bottom: parabolic trap and optical lattice

Numerical solution : nearly black soliton





$\epsilon = 0.05,\ s(0) = 0,\ v(0) = 0.1$

Further directions

- Perturbation theory for complex eigenvalues of the linearized problem in the presence of external potentials
- Hermite function expansions for dynamics of dark solitons in the parabolic potentials (normal forms)
- Modeling of PDE problems along characteristics with incoming and outcoming radiation waves
- Derivation of the $O(\epsilon^2)$ error bound for the main equation describing dynamics of a dark soliton