Dark solitons in external potentials

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References:

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Conventional dark solitons

Dark solitons are localized solutions of nonlinear PDEs with non-zero boundary conditions and non-zero phase shift.

Dark solitons in nonlinear optics

$$iu_t = -\frac{1}{2}u_{xx} + f(|u|^2)u,$$

where $f(s)$ is a smooth function with $f'(s) > 0$.

Example: cubic NLS with $f = |u|^2$ and dark solitons

$$u = e^{-it}\left[k\tanh(k(x - vt)) + iv\right],$$

where $k = \sqrt{1 - v^2}$ and $|v| < 1$. When $v = 0$, the solution $u = \tanh x \, e^{-it}$ is called the black soliton.

Recent results in mathematical literature

- Zhidkov (1992) - local existence of the Cauchy problem and stability of kink solutions in the cubic NLS
- de Bouard (1995) - spectral and nonlinear instability of black solitons with zero velocity and zero phase shift
- Lin (2002) - criterion for orbital stability and instability of dark solitons for non-zero velocities
- Maris (2003) - bifurcations of dark solitons for non-zero velocities in the delta-function potential
- Di Menza and Gallo (2006) - stability criterion for kinks with zero velocity and non-zero phase shift

New problems for dark solitons

Dark solitons in Bose–Einstein condensates

$$iu_t = -\frac{1}{2}u_{xx} + f(|u|^2)u + \epsilon V(x)u,$$

where ϵ is small and $V(x) : \mathbb{R} \mapsto \mathbb{R}$ is a smooth, exponentially decaying function such that

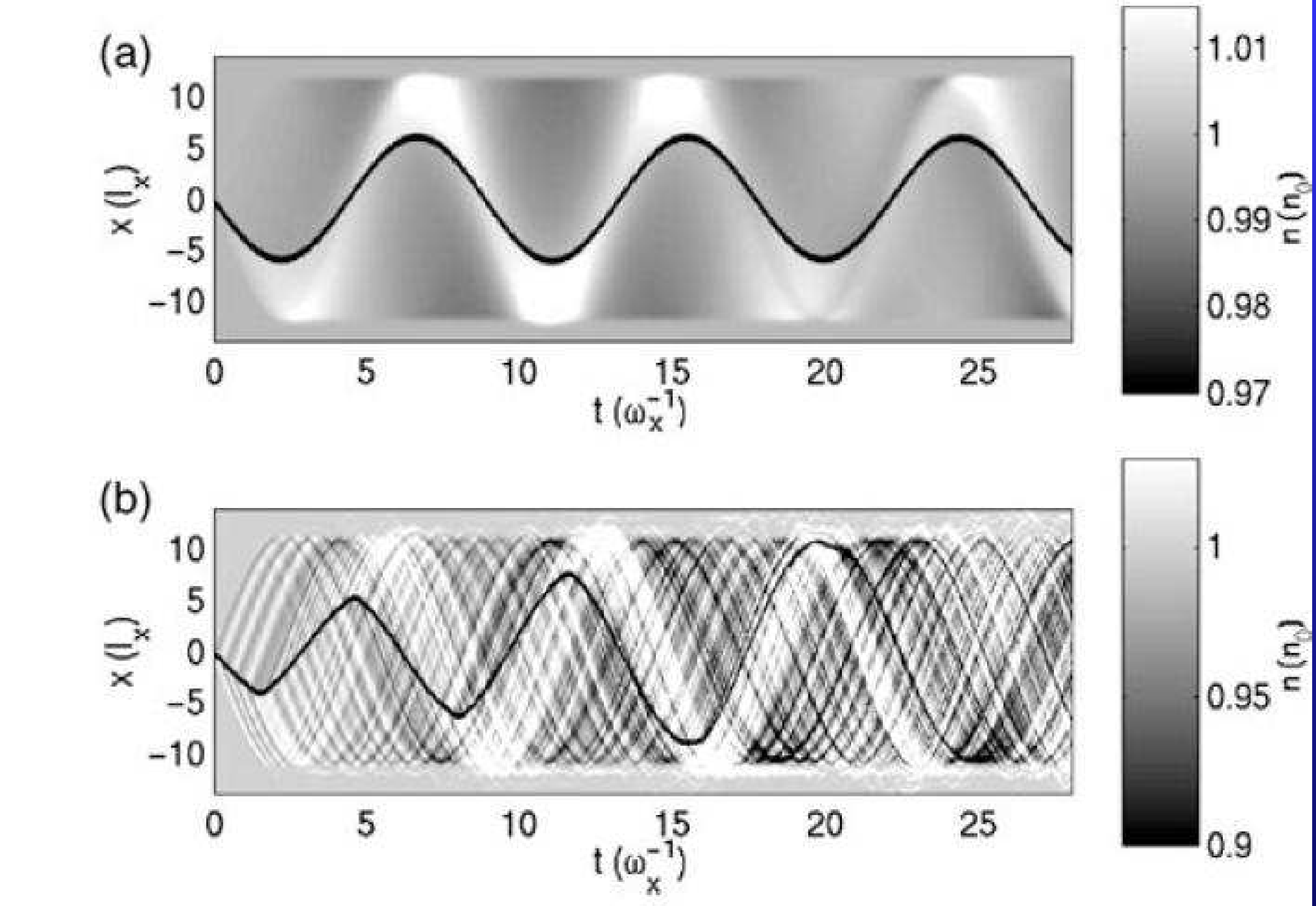
$$\exists C > 0,\ \kappa > 0 : \quad |V(x)| \leq Ce^{-\kappa|x|}, \quad \forall x \in \mathbb{R}$$

Example: symmetric external potentials

$$V_1(x) = -\text{sech}^2\left(\frac{\kappa x}{2}\right), \qquad V_2(x) = x^2 e^{-\kappa|x|}, \qquad x \in \mathbb{R}.$$

More general context: periodic and confining potentials $V(x)$.

Numerical simulations



Questions:

Find approximations of the frequency of oscillations of a dark soliton and study long-time changes in the amplitude of oscillations.

Approaches to the solution at glance

$$iu_t = -\frac{1}{2}u_{xx} + f(|u|^2)u + \epsilon V(x)u,$$

- $\epsilon = 0$ - existence and stability of dark solitons is known
- $\epsilon \ll 1$ - persistence of solutions by using the method of Lyapunov–Schmidt reductions
- $\epsilon \ll 1$ - stability of solutions by using the methods of Evans function and the theory of negative indices
- $\epsilon \neq 0$ - long-time dynamics by using the Newton's law of motion and central manifold reductions

Main results

1. A black soliton $u = \phi_0(x - s)e^{-it}$ with $\phi_0 \to \pm 1$ as $x \to \pm\infty$ persists for small $\epsilon \neq 0$ if $M'(s) = 0$ and $M''(s) \neq 0$, where

$$M'(s) = \int_{\mathbb{R}} V'(x) \left[1 - \phi_0^2(x - s)\right] dx.$$

2. If a black soliton is spectrally stable for $\epsilon = 0$, then it is spectrally unstable for small $\epsilon \neq 0$ with one real positive eigenvalue if $M''(s) < 0$ and two complex-conjugate eigenvalues if $M''(s) > 0$.

3. If $u(x, 0)$ is close to $\phi_\epsilon(x - s(0))$, then $u(x, t)$ remains close to $\phi_\epsilon(x - s(t))e^{-it}$, where $s(t)$ solves for $0 < t < C/\epsilon$

$$\mu_0 \ddot{s} - \epsilon \lambda_0 M''(s) \dot{s} + \epsilon M'(s) = \mathrm{O}(\epsilon^2), \qquad \lambda_0, \mu_0 > 0.$$

Persistence of black solitons

Black soliton in the form $u = \phi(x)e^{-it}$ satisfies the ODE

$$\frac{1}{2}\phi''(x) + \left[1 - \phi(x)^2\right]\phi(x) = \epsilon V(x)\phi(x),$$

subject to the boundary conditions $\lim_{x\to\pm\infty}\phi(x) = \pm 1$.

If $\phi(x) = \phi_0(x-s) + \varphi(x,s)$, then φ is found from the operator-valued equation

$$F(\varphi, s) = L_+\varphi + N(\varphi, s) + \epsilon V(x)\left[\phi_0(x-s) + \varphi\right] = 0,$$

where $N : H^1(\mathbb{R}) \mapsto H^1(\mathbb{R})$ and $L_+ : H^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$, such that

$$L_+ = -\frac{1}{2}\partial_x^2 + 3\phi_0^2 - 1 = -\frac{1}{2}\partial_x^2 + 2 - 3\,\mathrm{sech}^2 x.$$

Lyapunov–Schmidt reductions

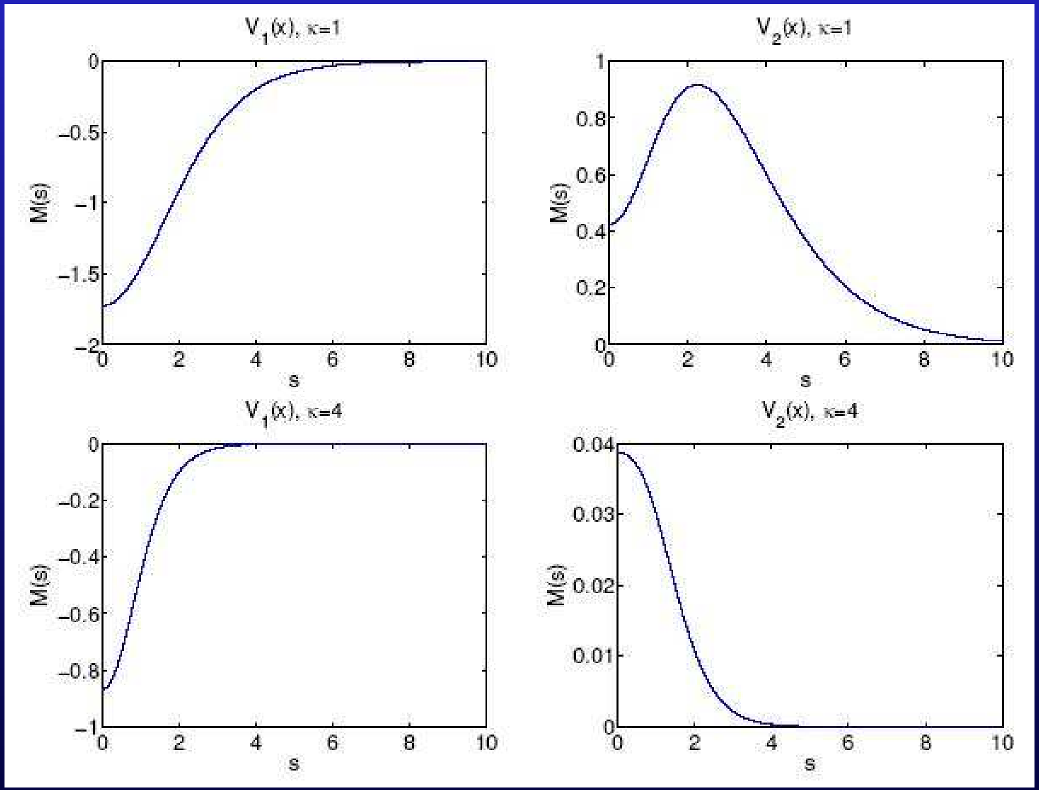
- The essential spectrum of L_+ is bounded from below by 2, $\mathrm{Ker}(L_+) = \{\phi'(x)\}$, and other isolated eigenvalues are located in $(0, 2)$.
- Let $\varphi \in H^1(\mathbb{R})$, such that $(\phi_0', \varphi) = 0$ and

$$
\begin{aligned}
G(s) &= \epsilon\,(\phi_0', V(x)(\phi_0 + \varphi)) + (\phi_0', N(\varphi, s)) \\
&= \frac{\epsilon}{2} M'(s) + \tilde{G}(s) = 0.
\end{aligned}
$$

- By the Implicit Function Theorem, $\|\varphi\|_{H^1} = \mathrm{O}(\epsilon)$ subject to $G(s) = 0$. If $M'(s_0) = 0$ and $M''(s_0) \neq 0$, the root of $G(s) = 0$ persists as $s = s_0 + \mathrm{O}(\epsilon)$.

Applications

- If $V(-x) = V(x)$, then $M'(0) = 0$ and the black soliton with $s = 0$ persists for $\epsilon \neq 0$.
- Additional roots $s = \pm s_0$ may exist if $\text{sign}(M(0)M''(0)) = 1$ since $M(s) \to 0$ as $s \to \infty$.



Stability of black solitons

Linearization at a black soliton $\phi_0(x)e^{-it}$ is defined by

$$u = e^{-it}\left[\phi_0(x) + (u(x) + iw(x))e^{\lambda t} + (\bar{u}(x) + i\bar{w}(x))e^{\bar{\lambda} t}\right]$$

Spectral stability problem:

$$L_+ u = -\lambda w, \qquad L_- w = \lambda u,$$

where

$$\begin{aligned} L_+ &= -\frac{1}{2}\partial_x^2 + 2 - 3\mathrm{sech}^2 x, \\ L_- &= -\frac{1}{2}\partial_x^2 - \mathrm{sech}^2 x. \end{aligned}$$

Spectra of $L_\pm$ in $L^2(\mathbb{R})$

Continuous spectra σ_c:

$$\sigma_c(L_+) \geq 2 > 0, \qquad \sigma_c(L_-) \geq 0, \quad L_-\phi_0 = 0$$

Kernel and negative eigenvalues:

- $L_+\phi_0' = 0 \Rightarrow L_+$ has no negative eigenvalues
- L_- has exactly one negative eigenvalue

Define the constrained space

$$X_c = \left\{ w \in L^2(\mathbb{R}) : \quad (\phi_0', w) = 0 \right\}$$

Operator L_- has no negative eigenvalues in X_c if $P_r'|_{v=0} > 0$ and exactly one negative eigenvalue if $P_r'|_{v=0} < 0$, where

$$P_r'|_{v=0} = (\phi_0', \psi_0), \qquad \psi_0 = L_-^{-1}\phi_0' \in L^\infty(\mathbb{R}).$$

Constrained L^2-space

Consider the spectral problem for $|\lambda| \neq 0$:

$$L_+ u = -\lambda w, \qquad L_- w = \lambda u,$$

If $w \in X_c$, then the stability problem is equivalent to the generalized eigenvalue problem

$$L_- w = \gamma L_+^{-1} w, \qquad \gamma = -\lambda^2, \;\; w \in X_c.$$

- If $P_r'|_{v=0} > 0$, then $\gamma = \frac{(w, L_- w)}{(w, L_+^{-1} w)} \geq 0$, such that $\lambda \in i\mathbb{R}$.
- If $P_r'|_{v=0} < 0$, then there exists exactly one $w \in X_c$ such that $\gamma < 0$ with $\lambda \in \mathbb{R}_+$.

Pontryagin Invariant Subspace Theorem

Definition 1: Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $(\cdot,\cdot)$ and the sesquilinear form $[\cdot,\cdot]$. The Hilbert space $\mathcal{H}$ is called the Pontryagin space (denoted as Π_κ) if it can be decomposed into the sum $\mathcal{H} \doteq \Pi_\kappa = \Pi_+ \oplus \Pi_-$, which is orthogonal with respect to $[\cdot,\cdot]$, where $\kappa = \dim(\Pi_-) < \infty$.

Definition 2: We say that Π is a non-positive subspace of Π_κ if $[x,x] \leq 0 \; \forall x \in \Pi$. We say that the non-positive subspace Π has the maximal dimension κ if any subspace of Π_κ of dimension higher than κ is not a non-positive subspace of Π_κ.

Theorem: Let T be a self-adjoint bounded operator in Π_κ, such that $[T\cdot,\cdot] = [\cdot,T\cdot]$. There exists a T-invariant non-positive subspace of Π_κ of the maximal dimension κ.

Application of the Pontryagin Theorem

Reference: L. Pontryagin, Izv. Acad. Nauk SSSR **8**, 243-280 (1944); M. Chugunova and D.P., preprint (2006)

- Let operators $L_\pm$ have $n_\pm$ negative eigenvalues, empty kernels in $L^2(\mathbb{R})$, while $\sigma_c(L_+) > 0$ and $\sigma_c(L_-) \geq 0$.
- Let embedded eigenvalues of the spectral problem $L_+ u = -\lambda w$, $L_- w = \lambda u$ be algebraically simple.
- Then, the spectral problem has exactly N_c complex eigenvalues, $N_i^\pm$ imaginary eigenvalues and $N_r^\pm$ real eigenvalues with $(w, L_+^{-1} w) \geq 0$ and $(w, L_+^{-1} w) \leq 0$, such that

$$N_r^- + N_i^- + N_c = n_+, \qquad N_r^+ + N_i^- + N_c = n_-.$$

- If $n_+ = 0$ and $n_- = 1$, then $N_r^+ = 1$ (when $P_r'|_{v=0} < 0$) and the soliton is spectrally unstable.

Dark soliton in a potential

Linearized operators are

$$\begin{aligned}\mathcal{L}_+ &= -\frac{1}{2}\partial_x^2 + f(\phi_\epsilon^2) - f(q_0) + 2\phi_\epsilon^2 f'(\phi_\epsilon^2) + \epsilon V(x),\\ \mathcal{L}_- &= -\frac{1}{2}\partial_x^2 + f(\phi_\epsilon^2) - f(q_0) + \epsilon V(x).\end{aligned}$$

In particular,

$$(\phi_0', L_+\phi_0') = -\frac{\epsilon}{2}M''(s_0) + \mathrm{O}(\epsilon^2)$$

- If $M''(s_0) > 0$, then $n_+ = 1$ and $n_- = 1$, such that either $N_c + N_i^- = 1$ or $N_r^+ = N_r^- = 1$.
- If $M''(s_0) < 0$, then $n_+ = 0$ and $n_- = 1$, such that $N_r^+ = 1$ and the kink is spectrally unstable.

Fundamental solutions

Recall that

$$L_+ u = -\lambda w, \quad L_- w = \lambda u$$

Define four fundamental solutions

$$\begin{pmatrix} u_\pm \\ w_\pm \end{pmatrix} \to \begin{pmatrix} \kappa_\pm \\ -\kappa_\mp \end{pmatrix} e^{\kappa_\pm x} \quad \text{as} \quad x \to -\infty,$$

$$\begin{pmatrix} \tilde{u}_\pm \\ \tilde{w}_\pm \end{pmatrix} \to \begin{pmatrix} \kappa_\pm \\ -\kappa_\mp \end{pmatrix} e^{-\kappa_\pm x} \quad \text{as} \quad x \to +\infty,$$

where $\kappa_\pm$ with $\mathrm{Re}\kappa_\pm > 0$ are given by

$$\kappa_\pm^2 = 2c^2 \left(1 \pm \sqrt{1 - \frac{\lambda^2}{c^4}} \right).$$

where $\kappa_+^2 \neq \kappa_-^2$ $(\lambda \neq \pm c^2)$ and $\kappa_- \neq 0$ $(\lambda \neq 0)$.

Evans function

The *Evans* function $E(\lambda)$ is a 4-by-4 determinant of the four fundamental solutions. Its zero for $\mathrm{Re}(\lambda) > 0$ coincide with eigenvalues λ with the account of their algebraic multiplicities.

Example: $f(q) = q$ and $\phi_0 = \tanh x$, such that

$$E(\lambda) = \frac{4\kappa_+^3 \kappa_-^3 (\kappa_+^2 - \kappa_-^2)^2}{(\kappa_+ + 2)^2 (\kappa_- + 2)^2},$$

such that $E(\lambda) = 8\lambda^3(1 + \mathrm{O}(\lambda))$ near $\lambda = 0$.

Lemma: The Evans function is analytic function of κ_- and ϵ near $\kappa_- = 0$ ($\lambda = 0$) and $\epsilon = 0$.

Characteristic equation

Expansion near $\lambda = 0$ and $\epsilon = 0$:

$$E(\lambda, \epsilon) = \lambda \left(\alpha \lambda^2 + \beta \epsilon + \tilde{\alpha} \lambda^3 + \tilde{\beta} \lambda \epsilon + \mathrm{O}(\lambda^4, \lambda^2 \epsilon, \epsilon^2) \right)$$

where $\alpha \neq 0$ due to

$$L_+ \phi_0'(x) = 0, \quad L_- w_1 = \phi_0'(x), \quad L_- \phi_0 = 0$$

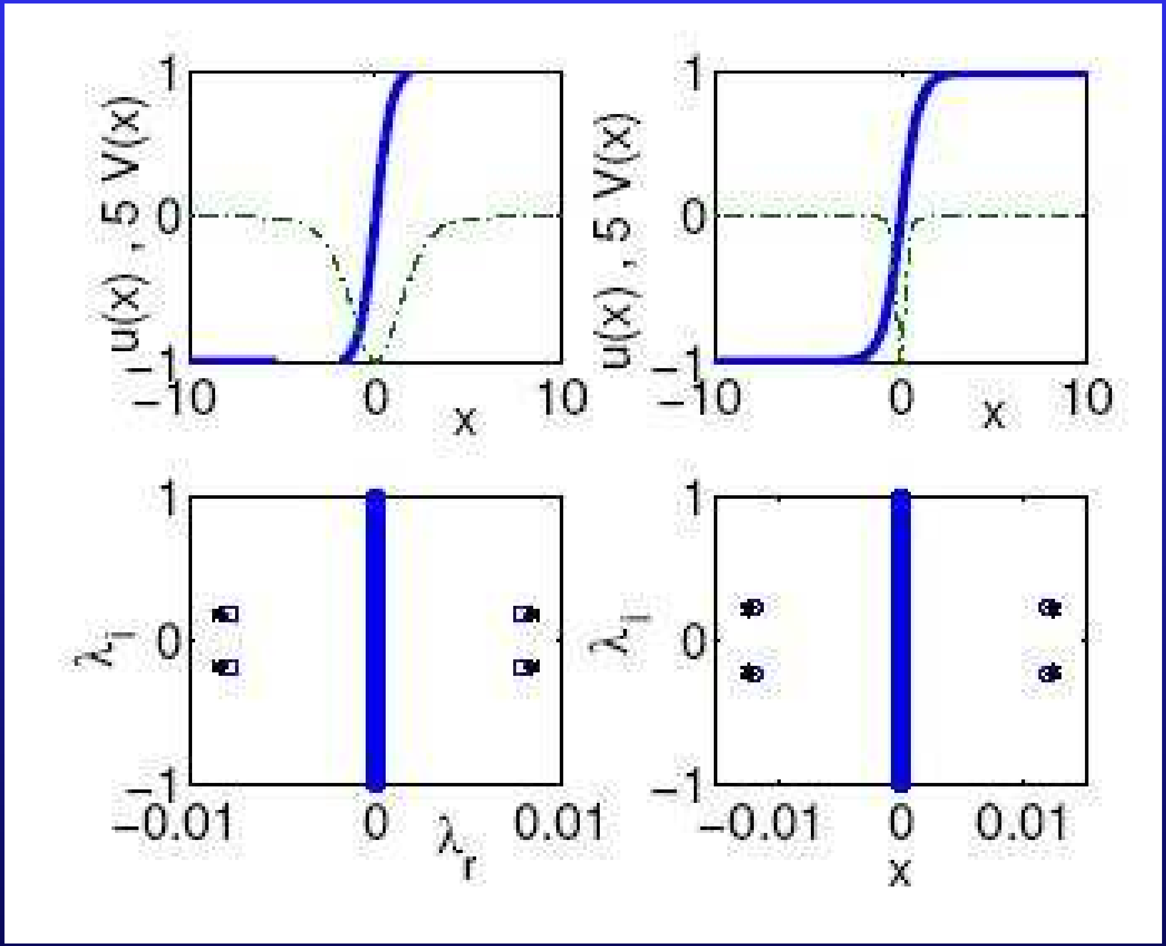
and $\beta = 0$ due to $M''(s_0) \neq 0$.
Explicit computation near $\lambda = 0$ and $\epsilon = 0$:

$$\mathrm{Re}\lambda > 0 : \quad \lambda^2 + \frac{\epsilon}{4} M''(s_0) \left(1 - \frac{\lambda}{2} \right) = \mathrm{O}(\epsilon^2).$$

If $M''(s_0) < 0$, there is a small root $\lambda \in \mathbb{R}_+$
If $M''(s_0) > 0$, there are two small roots with $\mathrm{Re}\lambda > 0$ and $\mathrm{Im}\lambda \neq 0$.

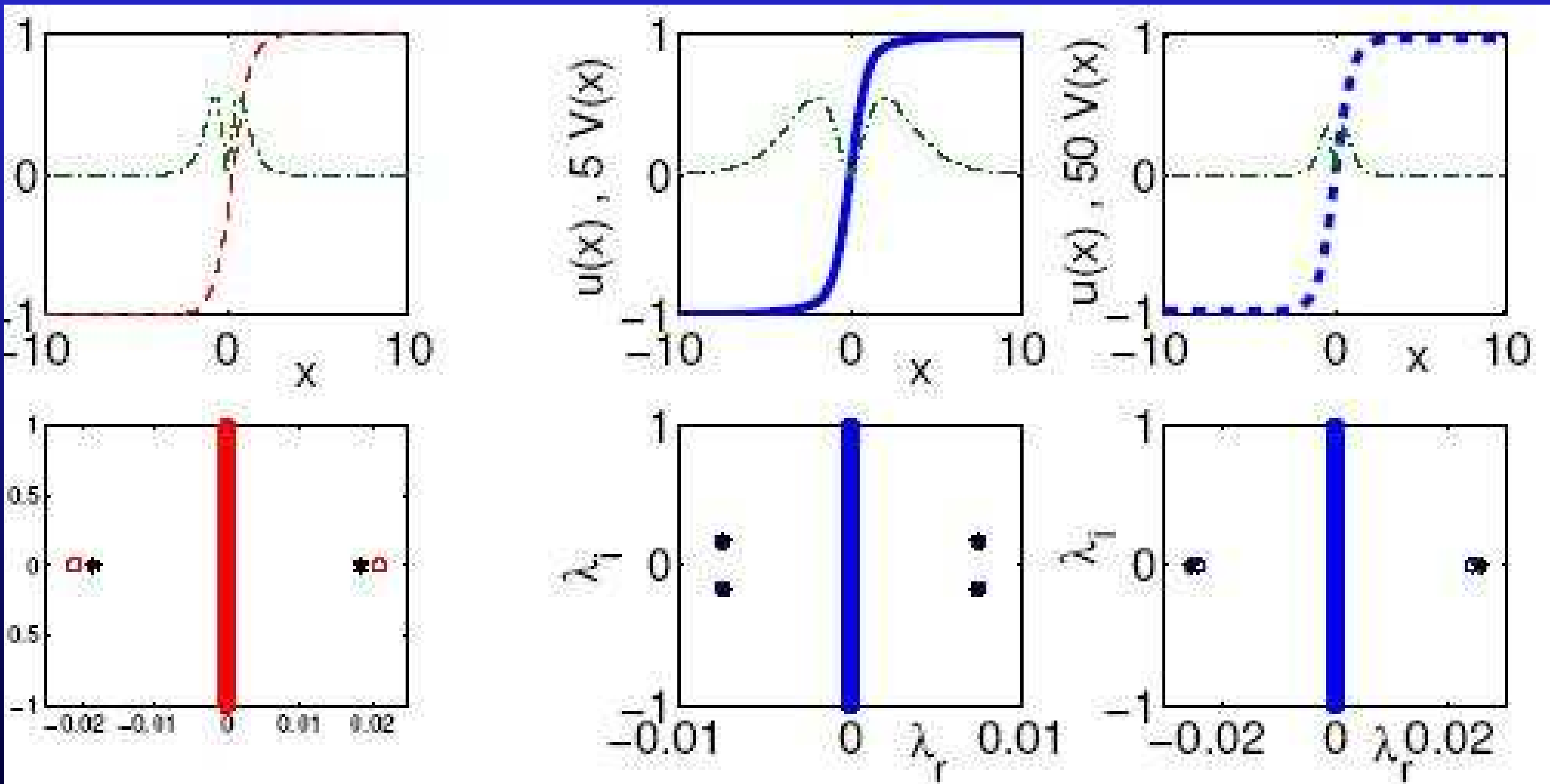
Example: $V_1 = -\text{sech}^2\left(\frac{\kappa x}{2}\right)$

Only one solution persists with $s_0 = 0$ and $M''(0) > 0$



Example: $V_2 = x^2 e^{-\kappa|x|}$

For $\kappa < 3.21$, three solutions persist with $s_0 = 0$ ($M''(0) > 0$) and $s_0 = \pm s_*$ ($M''(s_*) < 0$). For $\kappa > 3.21$, only one solution persists with $s_0 = 0$ and $M''(0) < 0$

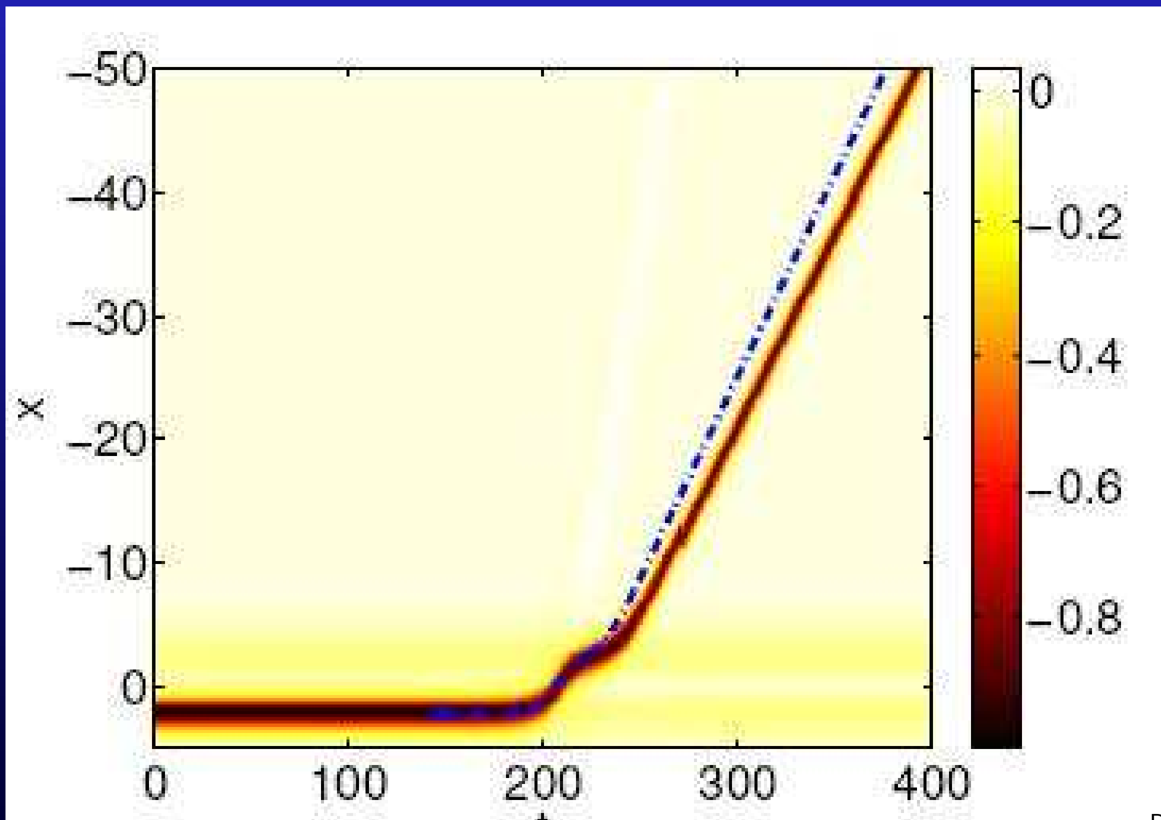


Nonlinear dynamics of instability

Newton's particle equation

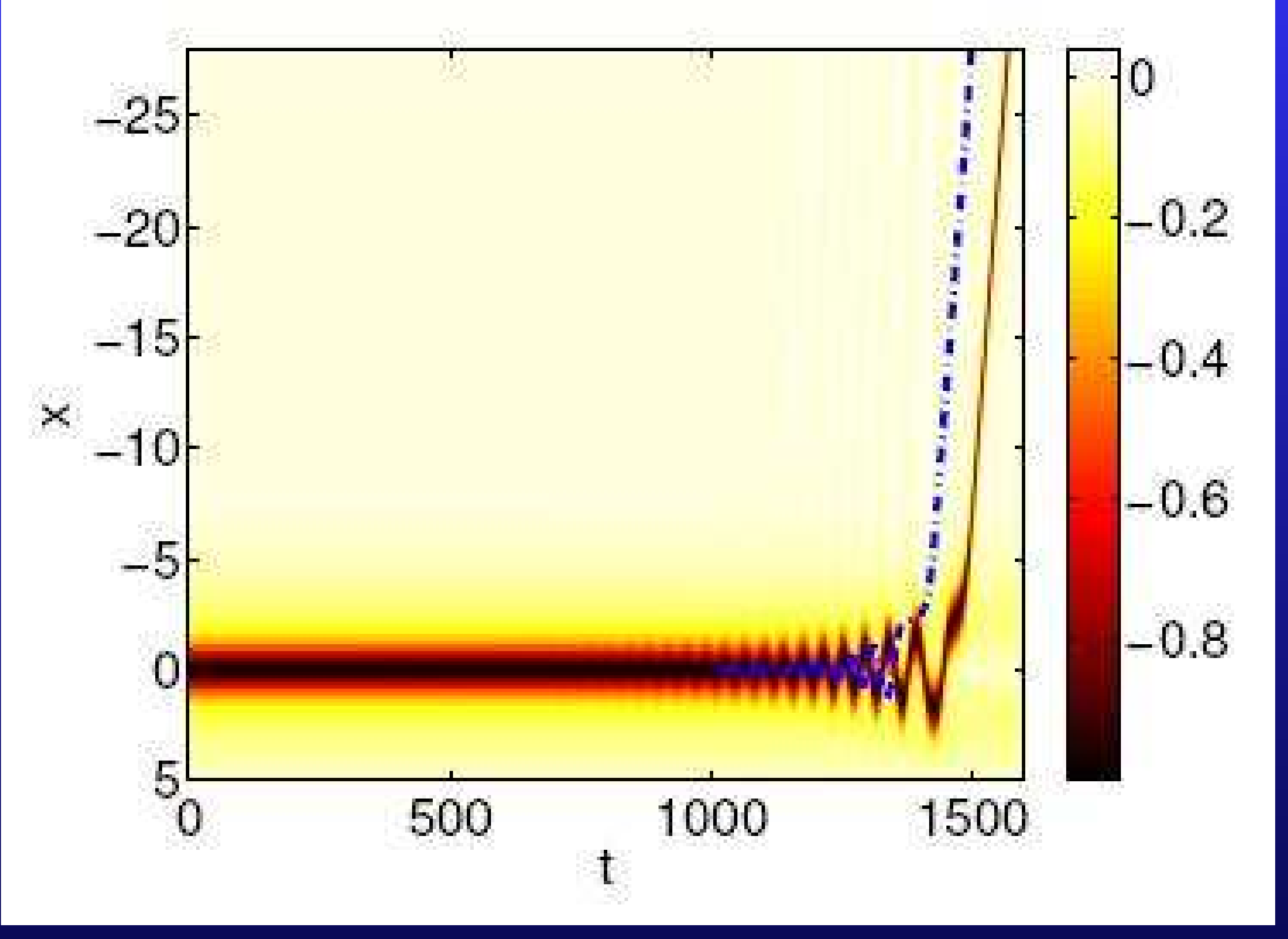
$$\mu_0 \ddot{s} - \epsilon \lambda_0 M''(s) \dot{s} + \epsilon M'(s) = \mathrm{O}(\epsilon^2), \qquad \lambda_0, \mu_0 > 0.$$

Real instability for $V_2(x)$, $\kappa < 3.21$ and $s_0 = s_* \neq 0$



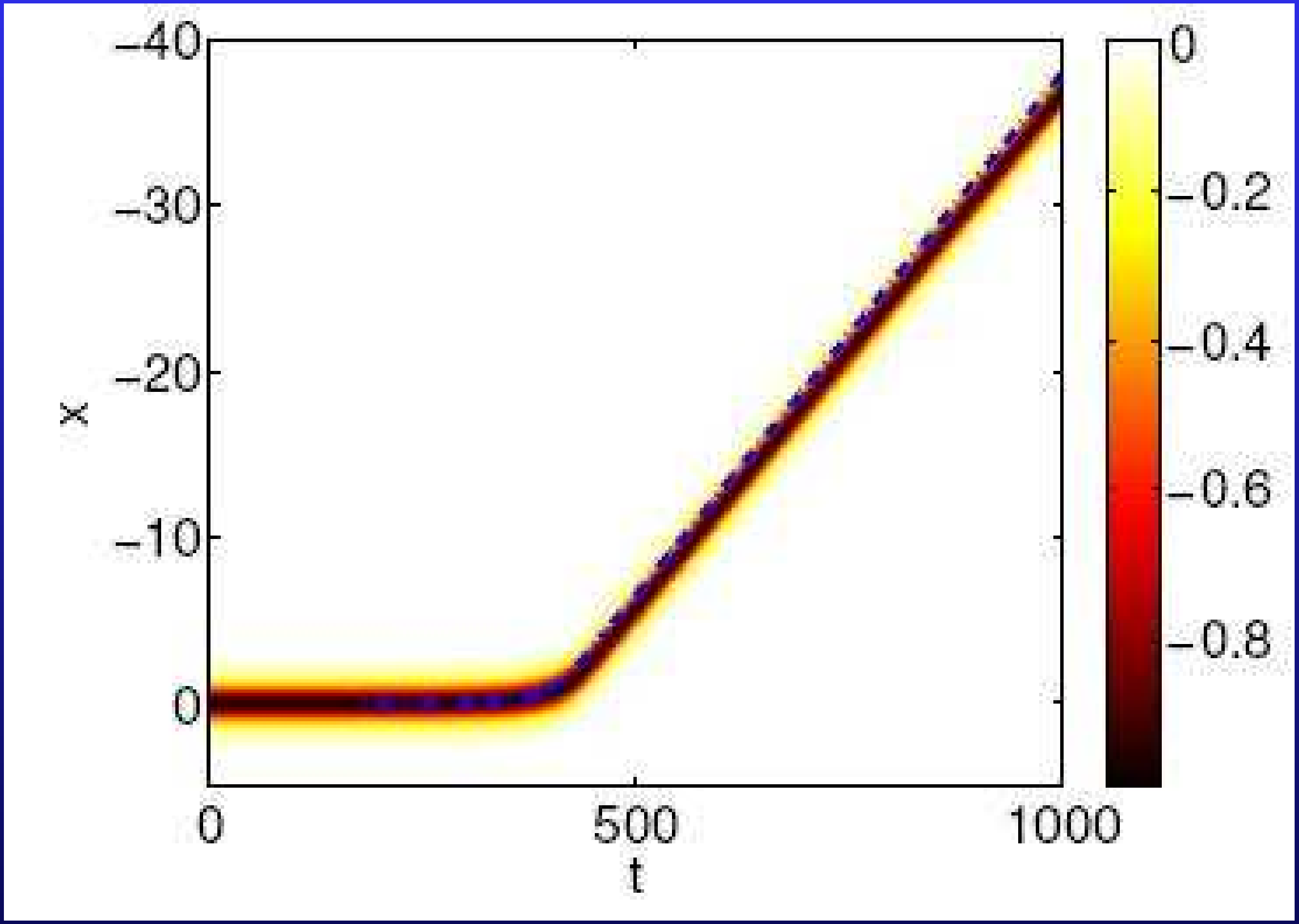
Nonlinear dynamics of instability

Complex instability for $V_2(x)$, $\kappa < 3.21$ and $s_0 = 0$



Nonlinear dynamics of instability

Real instability for $V_2(x)$, $\kappa > 3.21$ and $s_0 = 0$



Conclusion

- Persistence of dark solitons is studied for non-zero boundary conditions and decaying potentials
- Stability of dark solitons is studied for linearized problems without spectral gaps
- Numerical modeling suggests an adequate approximation of the nonlinear dynamics by the Newton's particle equation
- Extension of this work is needed for bounded (periodic) and unbounded (parabolic) potentials $V(x)$.